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NOISE-INDUCED TRANSITIONS AND MEAN FIELD LIMITS FOR MULTISCALE DIFFUSIONS

Multiscale Methods for Stochastic Dynamics, Geneva

01.02.2017

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- *Mean field limits for interacting diffusions in a two-scale potential* (G.A. Pavliotis), Preprint (2017)
- *Brownian motion in an N-scale periodic potential* (A.B. Duncan and G.A. Pavliotis). Submitted to SIAM J MMS (2016).
- *Noise-induced transitions in rugged energy landscapes* (A.B. Duncan, S. Kalliadasis, G.A. Pavliotis, M. Pradas). Phys. Rev. E, 94, 032107 (2016).

Research Funded by the EPSRC, Grants EP/J009636/1, EP/L024926/1, EP/L020564/1 and EP/L025159/1

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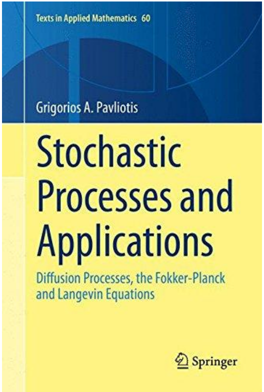
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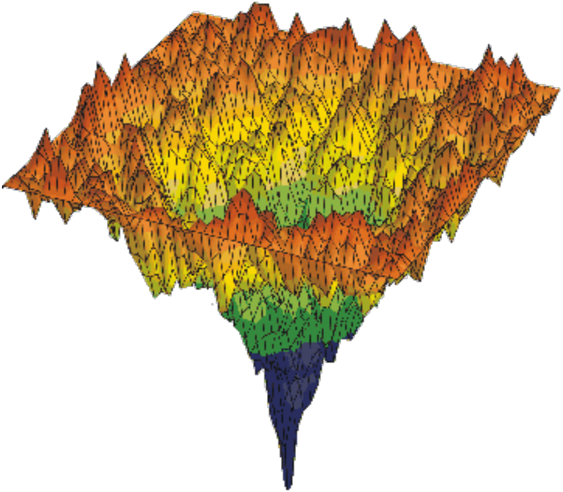
MOTIVATION 1: PROTEIN FOLDING

Transition between two protein configurations described by random walk in an energy landscape.



MOTIVATION 1: PROTEIN FOLDING

Quantities of interest:

- Average transition time/ binding time.
- Stability of a given conformation.
- Evolution of reaction coordinate.
- Most common transition path.

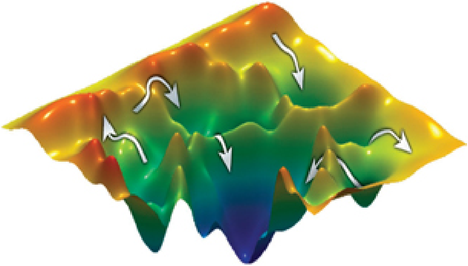


Figure: Transition between protein conformations

[Pirchi, Ziv et al. Nat. Comms. 2:493 (2011)]

MOTIVATION 2: COLLECTIVE TRANSPORT OF PARTICLES

System of d interacting particles moving in a multiscale potential.

$$\ddot{q}_i = -\nabla_{q_i} V^{\epsilon}(\mathbf{q}) - \theta\left(q_i - \frac{1}{N}\sum_{k=1}^{N} q_k\right) - \gamma q_i + \sqrt{2\gamma\beta^{-1}}\,\dot{W}_t^{(i)},$$

- V^{ϵ} denotes a rugged/multiscale potential.
- We can also consider localized coupling via e.g. a discrete Laplacian.
- Applications:
  1. Polymer Dynamics.
  2. MOdeling of Dislocation dynamics.
  3. Biophysics.

Question: How are dynamics of $\mathbf{q}$ affected by multiple scales within V^{ϵ}?

MOTIVATION 3: INFERENCE FOR MULTI–SCALE SYSTEMS

Inference problems involving multiscale structure are ubiquitous in science.

- Given observations from a multi-scaled model:

  $$dX_t^\epsilon = A^\epsilon(X_t^\epsilon;\theta)\,dt + B^\epsilon(X_t^\epsilon;\theta)\,dW_t,$$

  can we infer properties of the coarse grained model? **[Pavliotis & Stuart, J. Stat. Phys, 127, 2007]**.
- Related to Equation-free approach **[Kevrekidis et al, Comm. Math. Sci, 2006]** and H.M.M **[E et al, Phys. Rev. B 67, 2003]**.
- Variance reduction methods for sampling from a "rough" distribution π^ϵ **[Dupuis, Spiliopoulos & Hwang, 2011]**

PROBLEM OVERVIEW

Simplest model: Overdamped diffusion in a multiscale potential.

- Dynamics given by Itô SDE:

$$dX_t^\epsilon = -\nabla V^\epsilon(X_t^\epsilon)\, dt + \sqrt{2\beta^{-1}}\, dW_t.$$

- For $\epsilon \ll 1$, V^ϵ models a "rough" potential:

$$V^\epsilon(x) := V\left(x, \frac{x}{\epsilon}, \frac{x}{\epsilon^2}, \ldots, \frac{x}{\epsilon^N}\right),$$

  for a smooth function $V(x_0, y_1, \ldots, y_N)$.
  - x_0: slowly–varying structure of potential.
  - $y_1, \ldots, y_N$: multiscale periodic fluctuations occuring at different scales.

EXAMPLE

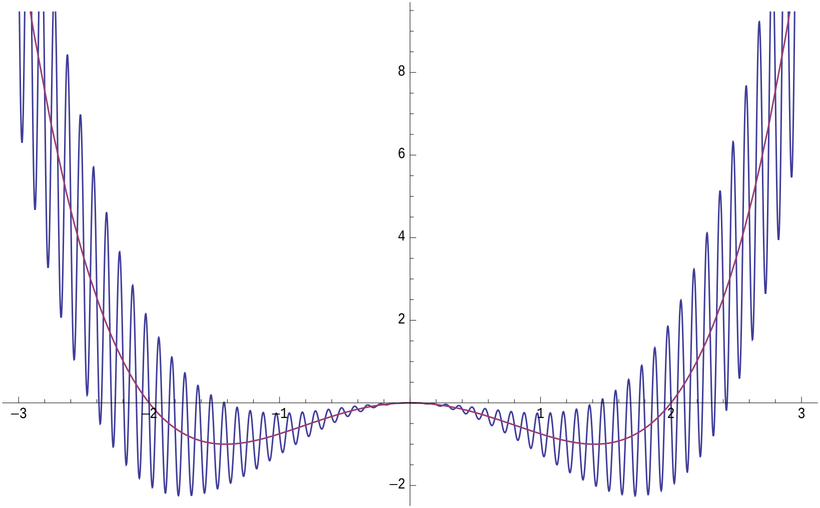


Figure: $V^{\epsilon}(x) = \frac{1}{4}\left(x^4 - 2\alpha x^2\right) - \frac{1}{2}\sin(2\pi x/\epsilon)x^2$

BACKGROUND

- Zwanzig, *Diffusion in a Rough Potential*, 1987
- Saven, Wang and Wolynes, *Kinetics of protein folding: The dynamics of globally connected rough energy landscapes with biases*, 1994
- Bryngelson et al, *Funnels, Pathways and the Energy Landscape of Protein Folding: A Synthesis*, 1995
- Hyeon, Thirumalai *Can energy landscape roughness of proteins and RNA be measured by using mechanical unfolding experiments?*, 2003
- Mondal, Ghosh and Ray, *Noise-induced transport in a rough ratchet potential*, 2009
- Depuis, Spiliopoulos and Wang, *Rare Event Simulation for Rough Energy Landscapes*, 2011.

INVARIANT BEHAVIOUR OF THE SLOW-FAST DYNAMICS

X_t^ϵ is a Markov diffusion process with infinitesimal generator defined by

$$\mathcal{L}^\epsilon f = \sigma^2 e^{V^\epsilon(x)/\sigma^2} \nabla \cdot \left(e^{-\beta V^\epsilon(x)} \nabla f(x) \right).$$

Stationary distribution satisfies the stationary Fokker-Planck equation:

$$\nabla \cdot \left(e^{-\beta V^\epsilon(x)} \nabla (\pi^\epsilon(x) e^{V^\epsilon(x)/\sigma^2} \right) = 0, \quad x \in \mathbb{R}^d.$$

Suppose $Z^\epsilon = \int_{\mathbb{R}^d} e^{-\beta V^\epsilon(x)}\, dx < \infty$,

- X_t^ϵ is ergodic, with stationary density $\pi^\epsilon(x) = \frac{1}{Z^\epsilon} e^{-\beta V^\epsilon(x)}$.
- X_t^ϵ satisfies detailed balance with respect to $\pi^\epsilon(x)$, i.e.

$$\text{Stationary Probability Flux } = \nabla \left(\pi^\epsilon(x) e^{\beta V^\epsilon(x)} \right) = 0, \quad \forall x \in \mathbb{R}^d.$$

QUESTIONS AND OBJECTIVES

Questions:

- Can behaviour of X_t^ϵ for small ϵ be approximated by some X_t^0?
- X_t^ϵ ergodic $\Rightarrow X_t^0$ ergodic?
- Relationship between $\pi^\epsilon(\cdot)$ and $\pi^0(\cdot)$?
- Asymptotic behaviour of other quantities related to X_t^ϵ,
  - Observables of X_t^ϵ, e.g. reaction coordinates.
  - Mean First Passage Time (MFPT), as $\epsilon \to 0$.

Approach:

- Formal approach: Asymptotic expansions of the Kolmogorov Backward Equation for X_t^ϵ in powers of $O(\epsilon^{-1})$.
- Rigorous Approach: probabilistic techniques for locally-periodic homogenization, **[Bensoussans, Lyons, Papanicolau, 1979], [Pardoux, 1999], [Pardoux, Veretennikov, 2001], [Bencherif-Madani, Pardoux, 2003]**.

PROVING THE HOMOGENIZATION RESULT

To prove the existence of the limit of X_t^ϵ as $\epsilon \to 0$, we make the following assumptions on V.

- There exist confining potentials $M_0(x)$ and $M_1(x)$ such that

$$M_0(x) \leq V(x, y_1, \dots, y_N) \leq M_1(x), \quad \forall x \in \mathbb{R}^d, y_1, \dots y_N \in \mathbb{T}^d$$

- $V(x, y_1, \dots, y_N)$ is smooth in all variables (can be relaxed).
- The gradient of the potential is Lipschitz in x, i.e.

$$\left|\nabla V(x, y_1, \dots, y_N) - \nabla V(x', y_1, \dots, y_N)\right| \leq C|x - x'|.$$

- $|\nabla V(x, y_1, \dots, y_N)| \leq C'|x|$, for some C, C' for all $x, x' \in \mathbb{R}$, $y_1, \dots, y_N \in \mathbb{T}^d$.

HOMOGENIZATION RESULT

As $\epsilon \to 0$, the process X_t^ϵ converges weakly in $C([0,T],\mathbb{R}^d)$ to a diffusion process X_t^0 having generator defined by

$$\mathcal{L}^0 f(x) = \frac{\beta^{-1}}{Z(x)}\nabla_x \cdot \left(Z(x)\mathcal{K}(x)\nabla_x f(x)\right), \quad f \in C_c^2(\mathbb{R}^d).$$

where $Z(x) = \int \cdots \int e^{-\beta V(x,\ldots)}\, dy_N \ldots dy_1$, and

$$\mathcal{K}(x) = I + \frac{1}{Z(x)} \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} (I + \nabla_{x_N}\theta_N^\top) \cdots (I + \nabla_{x_1}\theta_1^\top) e^{-\beta V}\, dy_N \ldots dy_1.$$

and θ_k are mean-zero solutions of the following Poisson equations on $\mathbb{T}^d$:

$$\nabla_{y_k} \cdot \left(\mathcal{K}_k(\nabla_{y_k}\theta_k + I)\right) = 0, \quad y \in \mathbb{T}^d$$

where $\mathcal{K}_N(x, y_1, \ldots, y_N) = e^{-\beta V(x, y_1, \ldots, y_N)} I$ and

$$\mathcal{K}_k(x, y_1, \ldots, y_k) = \int (I + \nabla_N \theta_N^\top) \cdots (I + \nabla_{k+1}\theta_{k+1}^\top) e^{-\beta V}\, dy_N \ldots dy_{k+1}.$$

HOMOGENIZATION RESULT

- The limiting dynamics can be characterised by the following Itô SDE:

  $$dX_t^0 = -\mathcal{K}(X_t^0)\nabla\Psi(X_t^0)\,dt + \sigma^2\nabla\cdot\mathcal{K}(X_t^0)\,dt + \sqrt{2\beta^{-1}\mathcal{K}(X_t^0)}\,dW_t,$$

  where the *effective potential* is given by

  $$\Psi(x) = -\beta^{-1}\log Z(x).$$

- X_t^0 satisfies detailed balance with respect to the invariant measure

  $$\pi^0(x) = \frac{1}{\mathcal{Z}}e^{-\Psi(x)} = \frac{Z(x)}{\mathcal{Z}}, \quad \mathcal{Z} = \int Z(x')\,dx'.$$

- The limiting SDE
- For all $e \in \mathbb{R}^d$:

  $$\frac{|e|^2}{Z(x)\hat{Z}(x)} \leq e\cdot\mathcal{K}(x)e \leq |e|^2,$$

  where $\hat{Z}(x) = \int\cdots\int e^{V(x,y_1,\ldots,y_N)/\sigma^2}\,dy_N\ldots dy_1$.

DYNAMICS IN THE LIMIT $\epsilon \to 0$

The generator of X_t^0:

$$\mathcal{L}^0 f(x) = \frac{\beta^{-1}}{K(x)\pi^0(x)} \nabla_x \cdot \left(\pi^0(x) K(x) \nabla_x f\right),$$

where

$$\pi^0(x) = \frac{1}{\mathcal{Z}} e^{-\beta\Psi(x)} = \frac{Z(x)}{\overline{Z}}.$$

- X_t^0 satisfies detailed balance with respect to $\pi^0(x)$.
- Limiting behaviour is described by overdamped diffusion in a potential $\Psi(x)$ with inhomogeneous diffusion coefficent $\mathcal{K}(x)$.

PROOF OF THE HOMOGENIZATION THEOREM

Slight generalisation of classical *martingale approach to homogenization*, applied to SDEs with locally-periodic coefficients having N-scales.

Rough idea:

1. The slow–fast system is the solution to the following martingale problem:

$$\mathbb{E}_x\left[\phi^\epsilon(X_t^\epsilon) - \int_s^t \mathcal{L}^\epsilon\phi^\epsilon(X_u^\epsilon)\,du \,\middle|\, \mathcal{F}_s\right] = \phi^\epsilon(X_s^\epsilon), \quad \forall\phi^\epsilon \in \mathcal{D}(\mathcal{L}^\epsilon).$$

Construct a test function

$$\phi^\epsilon(x) = \phi_0(x)+\epsilon\phi_1(x, x/\epsilon)+\ldots+\epsilon^N\phi_N(x, x/\epsilon, \ldots, x/\epsilon^N)+\ldots$$

such that

$$\mathcal{L}^\epsilon\phi^\epsilon(x) = \mathcal{L}^0\phi_0(x) + \epsilon R^\epsilon(x),$$

where $E_x[\epsilon R^\epsilon(X_u^\epsilon)] \to 0$, as $\epsilon \to 0$.

PROOF OF THE HOMOGENIZATION THEOREM CTD.

1. If the set of measures $\mathbb{P}^\epsilon$ on $C([0,T],\mathbb{R}^d)$ corresponding to the processes $\{X_t^\epsilon, t \in [0,T]\}$ possesses a limit point X_t^0 then it is the unique solution of the following martingale problem

$$\mathbb{E}_x \left[\phi_0(X^0) - \int_s^t \mathcal{L}^0 \phi_0(X_u^\epsilon)\, du \,\middle|\, \mathcal{F}_s \right] = \phi_0(X_s^\epsilon), \quad \forall \phi \in \mathcal{D}(\mathcal{L}^0).$$

2. Show that $\{X_t^\epsilon\}_{\epsilon>0}$ possesses an accumulation point. i.e. Establish tightness of the family of processes in $\{X_t^\epsilon\}_{\epsilon>0}$.

INVARIANT DISTRIBUTION OF COARSE GRAINED PROCESS

We can distinguish between two types of potential

- Separable Potential:

$$V^{\epsilon}(x) = V_0(x) + V_1(x/\epsilon, x/\epsilon^2, \dots x/\epsilon^N).$$

  In this case:

$$Z(x) \propto \int \cdots \int e^{-\beta V(x, y_1, \dots y_N)} dy_N \dots dy_1 \propto e^{-V_0(x)}.$$

  and $\mathcal{K}$ is independent of x. **Rapid fluctuations do not alter stationary behaviour, but only speed of convergence to equilibrium and effective diffusion tensor.**

- Nonseparable Potential. In this case

$$Z(x) \not\propto e^{-V_0(x)}, \qquad \text{in general.}$$

  Rapid fluctuations can affect the stationary behaviour.

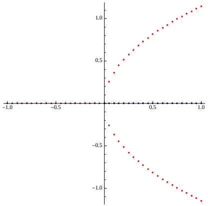
TOY EXAMPLE: 1D DOUBLE WELL POTENTIAL

Consider the ODE in $\mathbb{R}$:

$$\dot{x}(t) = -\frac{d}{dx} V_0(x;\alpha), \quad t > 0,$$

where $V_0(x;\alpha) = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4$.

- Normal form for supercritical pitchfork bifurcation.
- $\alpha < 0$: One stable equilibrium at $x = 0$.
- $\alpha > 0$: Stable equilibria at $x = \pm\sqrt{\alpha}$. Unstable equilibrium at $x = 0$.



1D DOUBLE WELL POTENTIAL

Consider the ODE in $\mathbb{R}$:

$$\dot{x}(t) = -\frac{d}{dx} V_0(x;\alpha), \quad t > 0,$$

where $V_0(x;\alpha) = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4$.

Add multiscale fluctuations $V^\epsilon(x;\alpha) = V(x, x/\epsilon;\alpha)$, where

$$V(x,y;\alpha) = \frac{1}{4}x^4 - \left(\frac{\alpha + \sin(2\pi y)}{2}\right)x^2.$$

Thermal motion in potential:

$$dX_t^\epsilon = -\frac{dV^\epsilon}{dx}(X_t^\epsilon)\,dt + \sqrt{2\beta^{-1}}\,dW_t.$$

1D DOUBLE WELL POTENTIAL

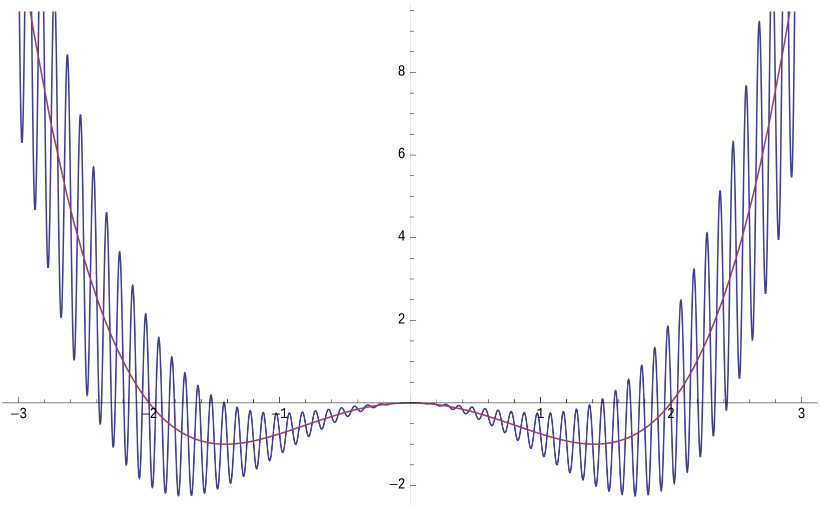


Figure: $V_0(x)$ and $V^\epsilon(x) = \frac{1}{4}\left(x^4 - 2\alpha x^2\right) - \frac{1}{2}\sin(2\pi x/\epsilon)x^2$

1D DOUBLE WELL POTENTIAL

By previous theory, $X_t^\epsilon \Rightarrow X_t^0$, as $\epsilon \to 0$, where X_t^0 is ergodic with stationary distribution

$$\pi^0(dx) \propto Z(x)\,dx$$

Can show that

$$\pi^0(x) \propto \underbrace{e^{\beta\left(\frac{\alpha^2 x^2}{2} - \frac{x^4}{4}\right)}}_{\pi_0(x)} \underbrace{I\left(0, \frac{x^2}{2\beta^{-1}}\right)}_{\text{correction}},$$

where I is the modified Bessel function of the first kind.

Varying the intensity of the noise can alter the equilibrium dynamics of the system

1D DOUBLE WELL POTENTIAL

The strength of the noise now plays an interesting role in the dynamics.

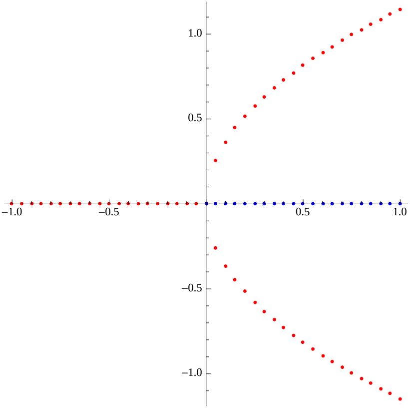


Figure: $\beta^{-1} = 1.0$

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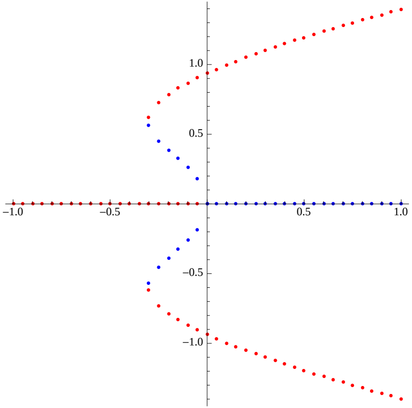


Figure: $\beta^{-1} = 10^{-1}$

1D DOUBLE WELL POTENTIAL

The strength of the noise now plays an interesting role in the dynamics.

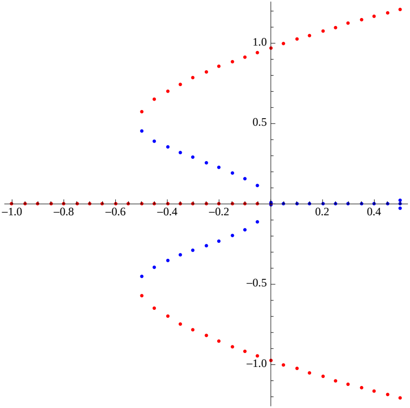


Figure: $\beta^{-1} = 5 \cdot 10^{-2}$

1D DOUBLE WELL POTENTIAL

More generally: consider an N-scale potential

$$V^{\epsilon}(x;\alpha) = V_0(x;\alpha) - \frac{1}{2}\sum_{n=1}^{N} \sin\left(\frac{2\pi\, x}{\epsilon^n}\right) x^2$$

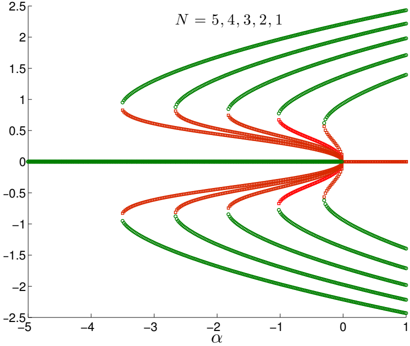


Figure: Bifurcation diagram for a different number N of microscopic scales in the potential

1D DOUBLE WELL POTENTIAL

Stationary PDF of homogenized dynamics is:

$$Z_N(x;\alpha) \propto e^{-\beta V_0(x;\alpha)} I\left(0, \frac{x^2}{2\beta^{-1/2}}\right)^N.$$

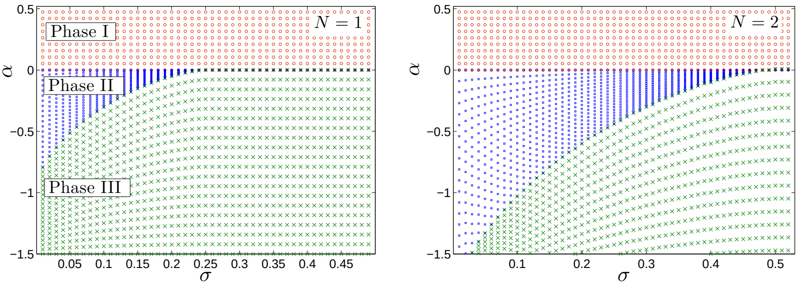
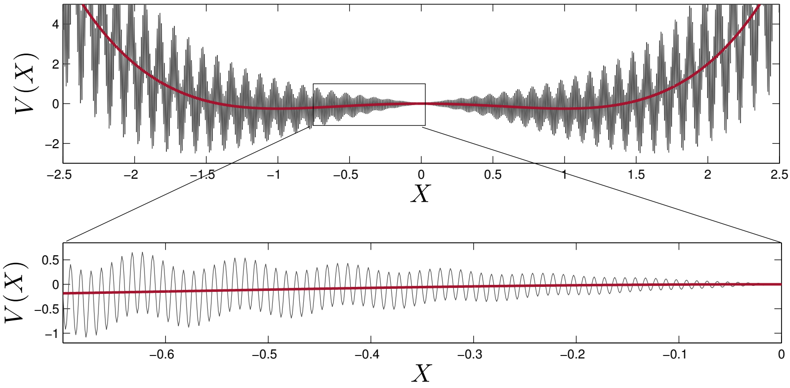


Figure: Phase diagram for α and σ

EXAMPLE: PITCHFORK BIFURCATION

Change in qualitative dynamics depends on how fast and slow scales interact. Consider

$$V^{\epsilon}(x;\alpha;\lambda) = \frac{1}{4}x^4 - \left[\frac{\alpha + \sin(2\pi x/\epsilon)}{2}\right]x^2 + \lambda \sin(2\pi x/\epsilon^2)x.$$



EXAMPLE: PITCHFORK BIFURCATION

Coarse-grained process has stationary distribution

$$\pi^0(x) = e^{-\beta V_0(x)} I_0(x^2\beta/2) I_0(\lambda x \beta).$$

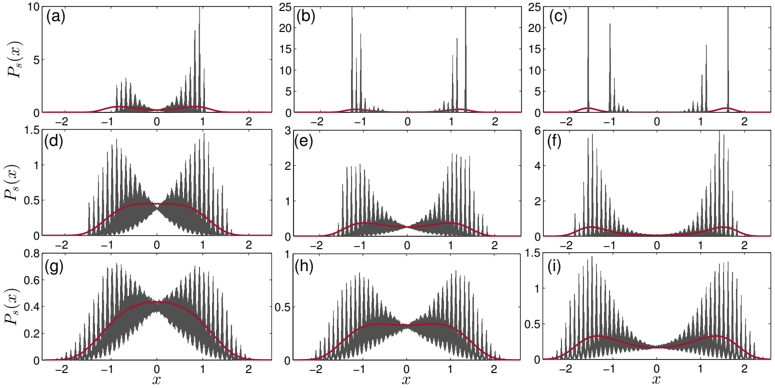


Figure: Plot $\pi^\epsilon(x)$ and $\pi^0(x)$. Horizontally, $\alpha = -1, -0.02, 1.0$, vertically $\beta^{-1} = 0.2, 0.5, 1.0$.

EXAMPLE: PITCHFORK BIFURCATION

Stable equilibrium points of the solution satisfy

$$-x_s^3 + x_s\left[\alpha + \frac{I_0'(x_s^2\beta/2)}{I_0(x_s^2\beta/2)}\right] + \lambda\frac{I_0'(\lambda x_s\beta)}{I_0(\lambda x_s\beta)} = 0.$$

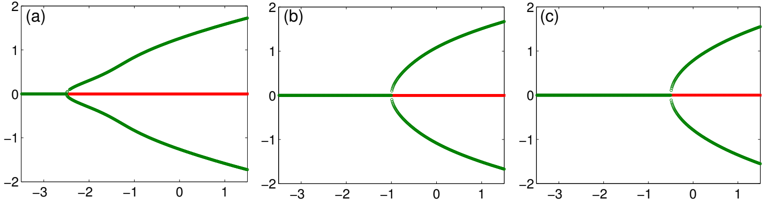


Figure: Bifurcation diagram for $\beta^{-1} = 0.2, 0.5$ and 1.

EFFECT OF NUMBER OF SCALES ON DIFFUSIVITY TENSOR

As the number of scales increase, the effective diffusivity $K(x)$ decreases.

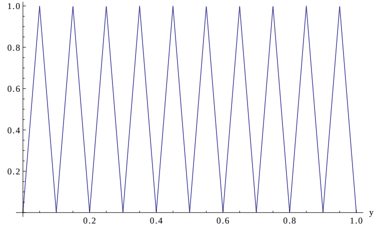
Must increase temperature β^{-1} to overcome "trapping effect" of regions of slow diffusivity. Consider separable N-scale potential

$$V^{\epsilon}(x) = S\left(x/\epsilon\right) + \ldots + S\left(x/\epsilon^{N}\right),$$

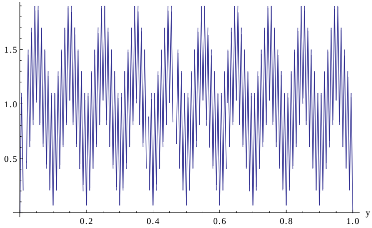
where

$$S(x) = \begin{cases} 2x & \text{if } x \bmod 1 \in [0, \frac{1}{2}) \\ 2 - 2x & \text{if } x \bmod 1 \in [\frac{1}{2}, 1) \end{cases}$$

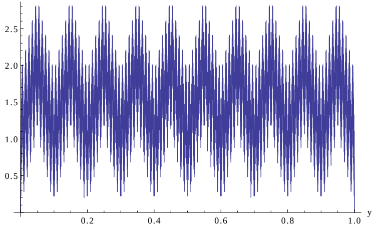
EFFECT OF NUMBER OF SCALES ON DIFFUSIVITY TENSOR



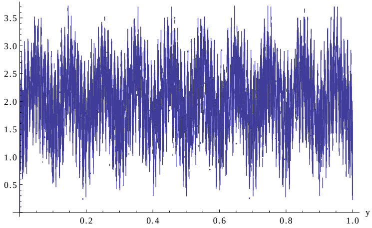
(a) $N = 1.$



(b) $N = 2.$



(c) $N = 3.$



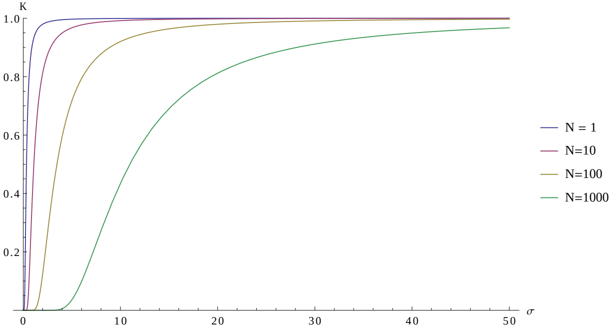
(d) $N = 3.$

EFFECT OF NUMBER OF SCALES ON DIFFUSIVITY TENSOR

As $\epsilon \to 0$, $X_t^\epsilon \Rightarrow X_t^0$, where

$$dX_t^0 = \frac{\sigma}{K(\sigma)^N} dW_t$$

where $\sigma = \beta^{-1}$, for $K(\sigma) = 2\sigma^2 \left(\cosh\left(\frac{1}{\sigma}\right) - 1\right)$.



MEAN FIELD LIMITS FOR INTERACTING DIFFUSIONS IN A TWO-SCALE POTENTIAL

- We consider a system of weakly interacting diffusions moving in a 2-scale locally periodic potential:

$$dX_t^i = -\nabla V^\epsilon(X_t^i)dt - \frac{1}{N}\sum_{j=1}^N \nabla F(X_t^i - X_t^j)dt + \sqrt{2\beta^{-1}}dB_t^i, \quad i = 1, .., N \tag{1}$$

- where

$$V^\epsilon(x) = V_0(x) + V_1(x, x/\epsilon). \tag{2}$$

- The full N-particle potential is

$$\begin{aligned} U(x_1, \dots, x_N, y_1, \dots, y_N) &= \sum_{i=1}^N V_0(x_i) + \frac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N F(x_i - x_j) \\ &\quad + \sum_{i=1}^N V_1(x_i, y_i). \end{aligned} \tag{3}$$

- The homogenization theorem applies to the N-particle system.

The homogenized equation is

$$dX_t^i = -M(X_t^i)\left(\nabla V_0(X_t^i) + \frac{1}{N}\sum_{i\neq j}\nabla F\left(X_t^j - X_t^i\right) + \nabla\Psi(X_t^i)\right)dt$$
$$+\beta^{-1}\nabla\cdot M(X_t^i)dt + \sqrt{2\beta^{-1}M(X_t^i)}dW_t^i, \tag{4}$$

for $i = 1, \dots, N$, where $M : \mathbb{R}^d \to \mathbb{R}^{d\times d}_{sym}$ is defined by

$$M(x) = \frac{1}{Z(x)}\int_{\mathbb{T}^d}\int (I + \nabla_y\theta(x,y))e^{-\beta V_1(x,y)}dy, \quad x \in \mathbb{R}^d, \tag{5}$$

and

$$\Psi(x) = -\beta\nabla\log Z(x), \tag{6}$$

for (this is the free energy **only** with respect to $V_1(x,y)$)

$$Z(x) = \int_{\mathbb{T}^d} e^{-\beta V_1(x,y)}\,dy,$$

and where, for fixed $x \in \mathbb{R}^d$, θ is the unique mean zero solution to

$$\nabla\cdot\left(e^{-\beta V_1(x,y)}(I + \nabla_y\theta(x,y))\right) = 0, \quad y \in \mathbb{T}^d, \tag{7}$$

- We can pass to the mean field limit $N \to +\infty$ using the results from e.g. Dawson (1983), Oelschlager (1984) to obtain a McKean-Vlasov-Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = \nabla \cdot \Big(M(\nabla V_0 p + \nabla \Psi p + (\nabla F * p) p) + \beta^{-1} \nabla \cdot M p + \beta^{-1} \nabla \cdot (M p) \Big). \tag{8}$$

- The mean field $N \to +\infty$ and the homogenization $\epsilon \to 0$ limits commute **over finite time intervals**.
- This is a nonlinear equation and more than one invariant measures can exist, depending on the temperature. Eqn (8) can exhibit **phase transitions**.
- The phase/bifurcation diagrams can be different depending on the order with which we take the limits. For example:

$$V^{\epsilon}(x) = \frac{x^2}{2} + \cos(x/\epsilon).$$

- Consider the case $F(x) = \theta\frac{x^2}{2}$, take $N \to +\infty$ and keep ϵ fixed. The invariant distribution(s) are:

$$
\begin{aligned}
p^{\epsilon}(x; m, \theta, \beta) &= \frac{1}{Z^{\epsilon}} e^{-\beta(V^{\epsilon}(x)+\theta(\frac{1}{2}x^2 - x\,m)}, && (9) \\
Z^{\epsilon} &= \int e^{-\beta(V^{\epsilon}(x)+\theta(\frac{1}{2}x^2 - x\,m)}\,dx, && (10)
\end{aligned}
$$

- where

$$
m = \int x p^{\epsilon}(x; m, \theta, \beta)\,dx. \qquad (11)
$$

- Take first $\epsilon \to 0$ and then $N \to +\infty$. The invariant distribution is

$$
\begin{aligned}
p(x; m, \theta, \beta) &= \frac{1}{Z} e^{-\beta(V_0(x)+\Psi(x)+\theta(\frac{1}{2}x^2 - x\,m)}, && (12) \\
Z &= \int e^{-\beta(V_0(x)+\Psi(x)+\theta(\frac{1}{2}x^2 - x\,m)}\,dy, && (13)
\end{aligned}
$$

- where

$$
m = \int x p(x; m, \theta, \beta)\,dx. \qquad (14)
$$

- The number of invariant measures is given by the number of solutions to the self-consistency equations (11) and (14).

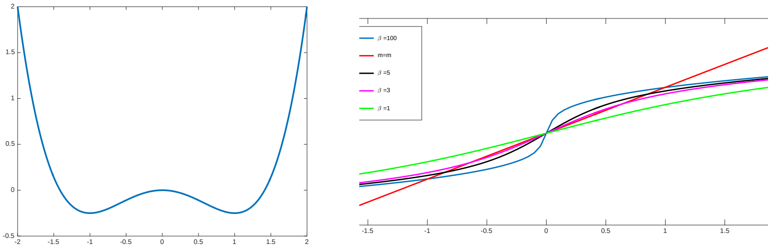


Figure: Self-consistency equation for the bistable potential.

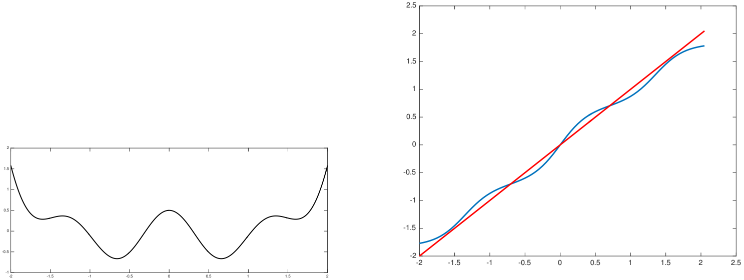


Figure: Self-consistency equation for the bistable potential with additive fluctuations.

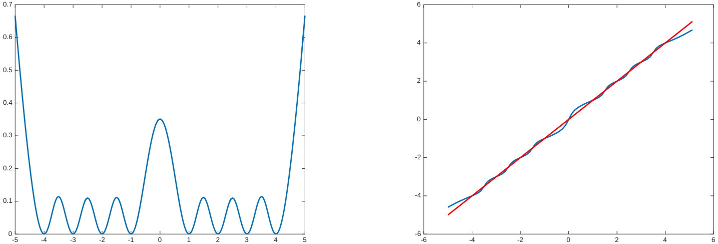


Figure: Potential and solution of self-consistency equation for the potential $V(q) = \frac{1}{\sum_{\ell=-N}^{N} |q-q_\ell|^{-2}}$ (used in the Thesis of Dr Z. Trstanova).

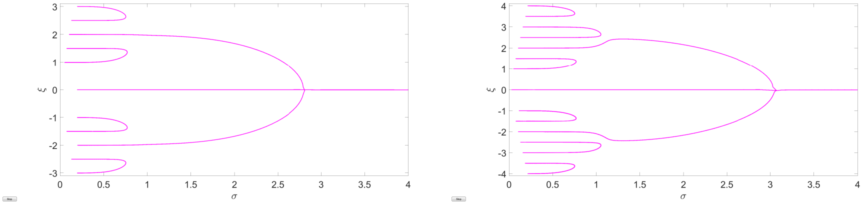


Figure: Bifurcation diagram for for the potential $V(q) = \frac{1}{\sum_{\ell=-N}^{N} |q - q_\ell|^{-2}}$ for the order parameter m as a function of β^{-1} (plots by Dr S. Gomes).