Linear response for macroscopic observables in high-dimensional systems

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Joint work with Georg Gottwald
Consider a mixing chaotic dynamical system $x_n = T(x_{n-1})$, with a physical invariant measure $\mu$. 
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\[
\frac{1}{N} \sum_{n=0}^{N-1} \Phi(x_n) \xrightarrow{N \to \infty} \int \Phi(x) \, d\mu(x) =: \mathbb{E}[\Phi]
\]
Consider a smooth family of mixing chaotic dynamical systems $x_n = T^\varepsilon(x_{n-1})$, with physical invariant measures $\mu^\varepsilon$. The physical measures encode long-term ergodic behaviour of $x_n$. Mathematically, for observables $\Phi$ and Lebesgue-a.e. $x_0$,

$$\frac{1}{N} \sum_{n=0}^{N-1} \Phi(x_n) \xrightarrow{N \to \infty} \int \Phi(x) \, d\mu^\varepsilon(x) =: \mathbb{E}^\varepsilon[\Phi]$$
Linear response theory

\[ \mathbb{E}^\varepsilon[\Phi] := \int \Phi(x) \, d\mu^\varepsilon(x) \]

Linear response theory (LRT) answers: What is \( \frac{d}{d\varepsilon} \mu^\varepsilon \)? (e.g. for Taylor approximations)
Linear response theory

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(e.g. for Taylor approximations)
Linear response theory

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Linear response theory (LRT) answers: What is \( \frac{d}{d\epsilon} \mathbb{E}^\epsilon[\Phi] \)?
(e.g. for Taylor approximations)

\[ \ldots \text{supposing} \mathbb{E}^\epsilon[\Phi] \text{ is differentiable} \]
LRT in practice

The application of linear response theory to climate systems has met with some success:

- Toy models: Majda et al ’07, ’10, Lucarini & Sarno ’11
- Barotropic models: Bell ’80, Gritsun & Dymnikov ’99, Abramov & Majda ’09
- Quasi-geostrophic models: Dymnikov & Gritsun ’01
- Atmospheric models: North et al ’04, Cionni et al ’04, work of Gritsun and others ’02, ’07, ’10, Ring & Plumb ’08
- Coupled climate models: Langen & Alexeev ’05, Kirk & Davidoff ’09, Fuchs et al ’14, Ragone et al ‘15
LRT in practice

However:

- Rough responses are known in atmospheric and ocean dynamics (e.g. Chekroun et al. '14).
- The failure of linear response needs very long time series to be visible (Gottwald, W. & Wouters '16).
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LRT in theory

Analytically, we know LRT works in

- Statistical mechanics: Kubo ’66
- Stochastic dynamical systems: Hänggi ’78, Hairer & Majda ’10
- Axiom A (uniformly hyperbolic dissipative chaos): Ruelle ’97-8
LRT in theory

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- Axiom A (uniformly hyperbolic dissipative chaos): Ruelle ’97-8
- Other dissipative systems . . . ?

Baladi and others (’08, ’10, ’14, ’15) proved there is no \textbf{linear response for quadratic maps}, even Whitney differentiability.

\[
x_{n+1} = ax_n(1 - x_n)
\]
The question

In this talk we will address the following question:

*When and why does linear response occur at macroscopic scales in high-dimensional systems?*
The question

In this talk we will address the following question:

When and why does linear response occur (for all practical purposes) at macroscopic scales in high-dimensional systems?
We study a reasonably simple multiscale system:

\[ M \text{ subsystems } q^{(i)} \ni f(\cdot; \Phi, a^n, c) \]

Parameters \[ a^n = u \]

\[ \Phi(q^{(i)}) \]

\[ \Phi(q_{\text{mean field}}) \]

We will derive reductions for mean-field dynamics \( \Phi \), and discuss (very rich) LRT properties of these systems.
We study a more simple multiscale system:

\[ M \text{ subsystems } q^{(i)} \xrightarrow{} f(\cdot; a^{(i)}, c) \]

Parameters \( a^{(i)} = u \)

Mean field \( \Phi \)
We study a more simple multiscale system:

\[ M \text{ subsystems } q^{(i)} \rightarrow f(\cdot; a^n, e) \]

Parameters \( a^n = v \)

We will derive reductions for mean-field dynamics \( \Phi \), and discuss (very rich) LRT properties of these systems.
The model

We study a more simple multiscale system:

\[ M \text{ subsystems } q \cup f(\cdot; a, c) \]

Parameters \( a, \nu, \ldots \)

\[ E, e, b, e, y, y \]

driver, piggy, ft, 01gal

d, t, t

mean field, My, mean field

\[ E, b, f, dongs, mean field \]

\[ M \text{ subsystems } q \cup Jfc \]

\[ a, \nu \]

<table>
<thead>
<tr>
<th>microscopic subsystem</th>
<th>macroscopic observables</th>
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<tr>
<td></td>
<td>uncoupled</td>
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\[ \triangleleft \text{ mean field } \Phi \]

We will derive reductions for mean-field dynamics \( \Phi \), and discuss (very rich) LRT properties of these systems.
Uncoupled case

System parameters: $a^{(j)}$, $j = 1, \ldots, M$ sampled from measure $\nu$

Microscopic dynamics:

$$q_n^{(j)} = f(q_{n-1}^{(j)}; a^{(j)}, \varepsilon), \ j = 1, \ldots, M$$

Mean-field observable:

$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q_n^{(j)})$$

Each subsystem $q^{(j)}$ evolves independently: suppose they have physical measures $\mu^{a^{(j)}, \varepsilon}$ and are mixing.
Two (nested) ways to take expectations:

- Over dynamics, i.e. initial conditions: \( \mathbb{E}^\varepsilon[\cdots] \)
- Over dynamical systems, i.e. selection of parameters \( a^{(j)} \) (if relevant): \( \langle \mathbb{E}^\varepsilon[\cdots] \rangle \)
Uncoupled case: expectations

Two (nested) ways to take expectations:

- Over dynamics, i.e. initial conditions: $\mathbb{E}^\varepsilon[\cdots]$
- Over dynamical systems, i.e. selection of parameters $a^{(j)}$ (if relevant): $\langle \mathbb{E}^\varepsilon[\cdots] \rangle$
We are interested in behaviour with respect to $\varepsilon$ of

$$E^\varepsilon[\Phi] = \frac{1}{M} \sum_{j=1}^{M} E^\varepsilon[\phi(q^{(j)})]$$

The $q^{(j)}$ will move independently toward statistical equilibrium, so

$$E^\varepsilon[\phi(q^{(j)})] = \int \phi(q) d\mu^{a^{(j)},\varepsilon}(q)$$

function of $\varepsilon$ and $a^{(j)} \sim \nu$
Because the $a^{(j)}$ are randomly selected, a CLT in $\langle \cdot \rangle$ gives

$$
E^\varepsilon[\Phi] = \frac{1}{M} \sum_{j=1}^{M} E^\varepsilon[\phi(q^{(j)})] = \bar{\Phi}^\varepsilon + \frac{1}{\sqrt{M}} \eta^\varepsilon + o(1/\sqrt{M})
$$

where $\eta^\varepsilon$ is a mean-zero Gaussian process in $\varepsilon$, and

$$
\bar{\Phi}^\varepsilon = \langle E^\varepsilon[\phi(q)] \rangle = \int \int \phi(q) d\mu^{a,\varepsilon}(q) d\nu(a)
$$

So response of mean-field $\Phi$ is $\bar{\Phi}^\varepsilon$ plus small correction for finite ensemble size.
LRT of $\Phi^\varepsilon$

$\Phi^\varepsilon = \langle \mathbb{E}^\varepsilon [\phi(q)] \rangle = \int \int \phi(q) \, d\mu^{a,\varepsilon}(q) \, d\nu(a)$

- Clearly if all microscopic subsystems satisfy LRT then so does $\Phi^\varepsilon$. 
LRT of \( \Phi^{\varepsilon} \)

\[
\Phi^{\varepsilon} = \langle \mathbb{E}^{\varepsilon}[\phi(q)] \rangle = \int\int \phi(q) \, d\mu^{a;\varepsilon}(q) \, d\nu(a)
\]

- Clearly if all microscopic subsystems satisfy LRT then so does \( \Phi^{\varepsilon} \).
- On the other hand if the microscopic subsystems violate LRT and \( \nu \) is discrete (e.g. \( \nu = \delta_{a_0} \)), then \( \Phi^{\varepsilon} \) will not have LRT.
LRT of $\Phi^\varepsilon$

If $\nu$ is smooth (e.g. $\frac{d\nu}{da} \in BV$), then averaging over $d\nu(a)$ can give "collective" linear response of microscopic systems that may violate LRT:

- **An easy case:** If $f$ can be written as $f(\cdot; a + K\varepsilon)$:

  $$\frac{d\Phi^\varepsilon}{d\varepsilon} = \int \frac{d}{d\varepsilon} \left( \int \phi(q) d\mu^{a+K\varepsilon}(q) \right) d\nu(a)$$
LRT of $\Phi^\epsilon$

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- **An easy case:** If $f$ can be written as $f(\cdot; a + K\epsilon)$:

\[
\frac{d\Phi^\epsilon}{d\epsilon} = \int \frac{d}{d\epsilon} \left( \int \phi(q) d\mu^{a+K\epsilon}(q) \right) d\nu(a)
= \int K \frac{d}{da} \left( \int \phi(q) d\mu^{a+K\epsilon}(q) \right) d\nu(a)
\]
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\frac{d\Phi^\epsilon}{d\epsilon} = \int \frac{d}{d\epsilon} \left( \int \phi(q) d\mu^{a+K\epsilon}(q) \right) d\nu(a)
\]

\[
= \int K \frac{d}{da} \left( \int \phi(q) d\mu^{a+K\epsilon}(q) \right) d\nu(a)
\]

\[
= -K \int \int \phi(q) d\mu^{a+K\epsilon}(q) \frac{d\nu}{da}
\]

\[
\implies \text{LRT holds}
\]
LRT of $\Phi^\epsilon$

- If $f(\cdot; a, \varepsilon)$ is a family of (analytic) unimodal maps:
LRT of $\Phi^\varepsilon$

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  - These maps obey LRT along topological conjugacy classes (Ruelle ’09);

\[ \varepsilon \]
\[ \varepsilon_0 \]
\[ \nu \]
\[ a \]
• If \( f( \cdot ; a, \varepsilon) \) is a family of (analytic) unimodal maps:
  • These maps obey LRT along topological conjugacy classes (Ruelle ’09);
  • Avila et al. (’03) conjectured that topological conjugacy classes of these maps have a uniformly analytic codimension-one lamination.
LRT of $\Phi^\varepsilon$

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  - These maps obey LRT along topological conjugacy classes (Ruelle '09);
  - Avila et al. ('03) conjectured that topological conjugacy classes of these maps have a uniformly analytic codimension-one lamination.

This may imply $\Phi^\varepsilon$ has linear response.
Smooth family of unimodal maps:

\[ f(q; a, \varepsilon) = (a + 4\varepsilon q(1 - q))q(1 - q), \]
\[ \nu \sim \text{Cosine}(3.75, 0.05) \]
LRT of $\eta^\varepsilon$

$$E^\varepsilon[\Phi] = \tilde{\Phi}^\varepsilon + \frac{1}{\sqrt{M}} \eta^\varepsilon + o(1/\sqrt{M})$$

Finite $M$ correction $\eta^\varepsilon$ is almost surely as rough as the individual $q^{(j)}$ responses.

Thus, for finite $M$, $\Phi$ may only have “approximate” LRT:

![Graph showing LRT for different M values](image-url)
What about the dynamics of $\Phi_n$?
What about the *dynamics* of $\Phi_n$? The $q^{(j)}$s are independent of each other, so for any $n$

$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q^{(j)}_n)$$

is a sum of independent random variables. Thus

$$\Phi_n = \mathbb{E}^\varepsilon[\Phi] + \frac{1}{\sqrt{M}} \zeta_n + o(1/\sqrt{M})$$

where $\zeta_n, n \in \mathbb{N}$ are mean-zero Gaussian random variables.
Macroscopic reduction

When $M \gg 1$, $\zeta$ *appears* to converge to a stationary Gaussian process.
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The autocorrelation function is the average over $\nu$ of the microscopic autocorrelations:

$$\text{Cov}[\zeta_m, \zeta_n] = \langle \text{Cov}[\phi(q_m), \phi(q_n)] \rangle.$$ 

Hence $\zeta$ has decay of correlations and can be approximated by e.g. an AR process.
Mean-field coupled case

System parameters: \( a^{(j)} \), \( j = 1, \ldots, M \) sampled from measure \( \nu \)

Microscopic dynamics:

\[
q^{(j)}_n = f(q^{(j)}_{n-1}; \Phi_{n-1}, a^{(j)}, \varepsilon), \quad j = 1, \ldots, M
\]

Mean-field driver/observable:

\[
\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q^{(j)}_n)
\]
Externally-coupled system

System parameters: $a^{(j)}$, $j = 1, \ldots, M$ sampled from measure $\nu$

External driver: $d_n$

Microscopic dynamics:

$$q_n^{(j)} = f(q_{n-1}^{(j)}; d_{n-1}, a^{(j)}, \varepsilon), \ j = 1, \ldots, M$$

Mean-field observable:

$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q_n^{(j)})$$

Suppose $q^{(j)}$ have time-dependent physical measures $\mu_n^{d, a^{(j)}, \varepsilon}$ with decay of correlations.
Externally-coupled system

We can make the same CLT reduction as before,

$$
\Phi_n = \langle \mathbb{E}^\varepsilon[\Phi_n|d] \rangle + \frac{1}{\sqrt{M}} \eta_n^{d,\varepsilon} + \frac{1}{\sqrt{M}} \zeta_n^d + o(1/\sqrt{M}),
$$

Parameters of this reduction are now time-dependent and depend on past history of $d$. 
**Macroscopic reduction of coupled system**

**Ansatz:** if $M \gg 1$, the coupled system can be approximated by setting $d_n \equiv \Phi_n$. 

![Diagram](image)
Macroscopic reduction of coupled system

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Macroscopic reduction of coupled system

This gives the macroscopic reduction:

\[ \Phi_n = \langle \mathbb{E}^\varepsilon [\Phi_n | \Phi] \rangle + \frac{1}{\sqrt{M}} \eta_{n,\varepsilon}^{\Phi} + \frac{1}{\sqrt{M}} \zeta_n^{\Phi} + o(1/\sqrt{M}) \]
This gives the macroscopic reduction:

\[ \Phi_n = \langle \mathbb{E}^\varepsilon [\Phi_n | \Phi] \rangle + \frac{1}{\sqrt{M}} \eta_n^{\Phi,\varepsilon} + \frac{1}{\sqrt{M}} \zeta_n^{\Phi} + o(1/\sqrt{M}) \]

\[ =: F(\Phi_{n-1}, \Phi_{n-2}, \ldots ; \varepsilon) \]

self-generated noise

usually smaller than \( \tilde{\zeta} \)
LRT of coupled system: finite $M$

\[
\Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon) + \frac{1}{\sqrt{M}} \eta_{n,\varepsilon} + \frac{1}{\sqrt{M}} \tilde{\zeta}_n + o(1/\sqrt{M})
\]

defines a stochastic dynamical system in $\Phi$. 
LRT of coupled system: finite $M$

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\Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon) + \frac{1}{\sqrt{M}} \eta^n \Phi,\varepsilon + \frac{1}{\sqrt{M}} \zeta^n \Phi + o(1/\sqrt{M})
\]

defines a stochastic dynamical system in $\Phi$. Modulo $\eta$’s:
LRT of coupled system: finite $M$

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defines a stochastic dynamical system in $\Phi$.

Modulo $\eta$’s:

- The noise $\tilde{\zeta}_\Phi$ generates (annealed) LRT in the microscopic particles, so this noisy system is $\sim$smooth in $\Phi$ and $\varepsilon$. 
LRT of coupled system: finite $M$

\[ \Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon) + \frac{1}{\sqrt{M}} \eta_{n}^{\Phi,\varepsilon} + \frac{1}{\sqrt{M}} \tilde{\zeta}_{n}^{\Phi} + o(1/\sqrt{M}) \]

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- So $\Phi$ obeys LRT for finite $M$. 
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- The noise $\tilde{\zeta}_n$ generates (annealed) LRT in the microscopic particles, so this noisy system is $\sim$smooth in $\Phi$ and $\varepsilon$.
- So $\Phi$ obeys LRT for finite $M$.
- Thus so does $\Phi$. 
LRT of coupled system: finite $M$

LRT for unimodal microscopic components, $\nu \sim \text{Cosine}$:

![Graph showing the behavior of LRT for different values of $M$](image)

- $M = 300$
- $M = 10,000$
- $M = 300,000$
LRT of coupled system: finite $M$

LRT for unimodal microscopic components, $\nu$ discrete:
Thermodynamic limit

As $M \to \infty$ the CLT reduction reduces to the law of large numbers:

$$\Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon).$$

If we have LRT without coupling, this defines a smooth dynamical system. External forcing washes out over time because of microscopic mixing, so

$$\Phi_n \approx F(\Phi_{n-1}, \Phi_{n-2}, \ldots, \Phi_{n-K}; \varepsilon),$$

i.e. emergent dynamics of $\Phi_n$ are low-dimensional.
Thermodynamic limit

If dynamics converges to equilibrium $\Phi_n \equiv \Phi^\varepsilon$ we have

$$\Phi^\varepsilon = F(\Phi^\varepsilon, \Phi^\varepsilon, \ldots; \varepsilon) := F_0(\Phi^\varepsilon; \varepsilon),$$

which is a smooth function if the microscopic subsystems have “collective” linear response. Then,

$$\frac{d\Phi^\varepsilon}{d\varepsilon} = \left(1 - \frac{\partial F_0}{\partial \Phi^\varepsilon}\right)^{-1} \frac{\partial F_0}{\partial \varepsilon}$$

(+ stability) and hence $\Phi$ has LRT.
Thermodynamic limit

For unimodal microscopic component example, \( \frac{d\nu}{dx} \in C^3 \), we see saddle-node bifurcation:
What about other limiting macroscopic dynamics?

- LRT in thermodynamic limit is difficult to study accurately using naive methods: need both long time series and very large microscopic ensembles.
- However, we can use transfer operator methods to approximate the dynamics of the microscopic distributions $\mu_{\Phi_n}$.
- For uniformly expanding $f$, Chebyshev spectral methods are very good at this (W. '19).
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Macroscopic dynamics in thermo. limit

We choose an $f$ uniformly expanding with perturbation parameter $\varepsilon$ regulating the strength of an appropriate mean-field coupling. For large $\varepsilon$ we see period doubling bifurcation to chaos:
Macroscopic dynamics in thermo. limit

The attracting $\Phi$ dynamics look unimodal:
LRT in thermodynamic limit

We have breakdown of LRT in the thermodynamic limit:
LRT in thermodynamic limit

We have breakdown of LRT in the thermodynamic limit:

We got this from hyperbolic microscopic components!
Macroscopic dynamics in thermo. limit

Side question:
Chaotic hypothesis says macroscopic dynamics should be hyperbolic (i.e. splitting between stable and unstable directions). Is this the case?
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Answer: No. There are homoclinic tangencies.
Macroscopic dynamics in thermo. limit

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Chaotic hypothesis says macroscopic dynamics should be hyperbolic (i.e. splitting between stable and unstable directions). Is this the case?
Answer: No. There are homoclinic tangencies.

How do we know? Continuation with Chebyshev transfer operator methods (Poltergeist.jl).
Macroscopic dynamics in thermo. limit

\[ \varepsilon = 30.061831392 \ldots \]
Macroscopic dynamics in thermo. limit

\[ \varepsilon = 30.061831392 \ldots \]

*zoom in...*
Macroscopic dynamics in thermo. limit

\[ \varepsilon = 30.061831392 \ldots \]
Various mechanisms by which linear response may emerge *and/or break down* in large coupled chaotic systems:

- Macroscopic LRT from inhomogeneous microscopic variables that individually violate LRT
- LRT in large (finite) chaotic systems via feedback of self-generated noise
- In thermodynamic limit LRT may depend on structure of macroscopic dynamics
  - This may be non-hyperbolic chaos, leading to LRT violation

Mostly these depend on the system’s network structure! 
Further directions

• More rigorous study of some of these phenomena (e.g. Sélley and Tanzi ’20)
• Study of chaotic networks beyond global, mean-field couplings
Further details


Aside on periodic windows

Unimodal maps have periodic dynamics on a dense (but not full measure) parameter set—i.e., non-mixing. To keep things simple, we avoid this by adding “hidden” dynamics $r_n^{(j)} \in [0, 1]$: 

$$f(q, r; a, \varepsilon) = \begin{cases} 
(\tilde{f}(q; a, \varepsilon), 2r), & r \leq 1/2 \\
(q, 2r - 1), & r > 1/2.
\end{cases}$$

This makes the unimodal $q^{(j)}$ dynamics mixing while retaining the same invariant measures.

(N.B. at statistical equilibrium, $\{r_n \geq 1/2\}_{n \in \mathbb{N}}$ are i.i.d. Bernoulli.)
If dynamical system $x_n = f(x_{n-1})$ is mixing with respect to measure $\mu$ then for all $w \in L^2(\mu)$ with $\mathbb{E}[w] = 1$,

$$
\mathbb{E}[\phi(x_n)w(x_0)] = \int \phi(x_n)w(x_0) \, d\mu(x_0) \xrightarrow{n \to \infty} \mathbb{E}[\phi]
$$

More generally, are going to assume that if $\tilde{\mu}$ is a “nice” measure,

$$
\int \phi(x_n) \, d\tilde{\mu}(x_0) \xrightarrow{n \to \infty} \mathbb{E}[\phi]
$$