Numerics, a model and statistics for the stochastically forced vorticity equation

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1. Results for PDE/SPDE
2. A reduced model
3. Comparison SDE/SPDE
4. ... propose different timestep approximation.
Vorticity Equations

Consider

\[ \partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \quad \mathbf{u} = \left( \begin{array}{c} \partial_y (-\Delta^{-1}) \\ -\partial_x (-\Delta^{-1}) \end{array} \right) \omega. \]  

(1)

On the asymmetric torus

\[(x, y) \in D_\delta := [0, 2\pi \delta] \times [0, 2\pi] \]

with \( \delta \approx 1 \).

Periodic boundary conditions, and viscosity \( 0 < \nu \ll 1 \).

The relation between \( \mathbf{u} \) and \( \omega \) is known as the Biot-Savart law.
Stochastically Forced Vorticity

\[ \partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega + \frac{\partial \mathcal{W}}{\partial t}, \quad \mathbf{u} = \left( \begin{array}{c} \partial_y (-\Delta^{-1}) \\ -\partial_x (-\Delta^{-1}) \end{array} \right) \omega. \]  

The noise is white in time, colored in space, and takes the form, for \( \mathbf{k} = (k_1, k_2) \neq (0, 0) \),

\[ \mathcal{W}(t, x, y) = \sqrt{2\nu} \sum_{\mathbf{k} \in \mathcal{K} \subset \mathbb{Z}^2 \setminus \{(0,0)\}} \sigma_{\mathbf{k}} e^{i(\frac{k_1}{\delta}x + k_2y)} \beta_{\mathbf{k}}(t). \]

\( \beta(t) = \{ \beta_{\mathbf{k}}(t) \} \) are i.i.d. Wiener processes.
\[ \partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega + \frac{\partial \mathcal{W}}{\partial t}, \quad \mathbf{u} = \begin{pmatrix} \partial_y (-\Delta^{-1}) \\ -\partial_x (-\Delta^{-1}) \end{pmatrix} \omega. \] 

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(3)

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Spatial correlation \( \sigma_k \)

We assume there exist fixed positive constants \( C_0 \) and \( \alpha_0 \) such that

\[ |\sigma_k| \leq C_0 e^{-\alpha_0 |k|^2}. \]

Solutions then analytic in space [Mattingly 2002].
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Solutions then analytic in space [Mattingly 2002].

To insure the vorticity remains real valued for all times we impose

\[\bar{\sigma}_{\mathbf{k}} = \sigma_{-\mathbf{k}} \quad \bar{\beta}_{\mathbf{k}} = \beta_{-\mathbf{k}}.\]

We choose \( \sigma_{(0,0)} = 0 \) so property \( \int_{D_\delta} \omega = 0 \) is preserved.
Deterministic equation:

- Time-asymptotic rest state of zero

\[ \omega_{x\mathrm{bar}}(x, t) = e^{-\nu \delta^2 t} \sin(x/\delta), \quad \omega_{y\mathrm{bar}}(y, t) = e^{-\nu t} \sin y, \]

or similarly with sine replaced by cosine.

Dipoles are given by

\[ \omega_{\text{dipole}}(x, y, t) = e^{-\nu \delta^2 t} \sin(x/\delta) + e^{-\nu t} \sin y, \]

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Deterministic equation:

- Time-asymptotic rest state of zero
- Quasi-stationary states: bars and dipoles

These rapidly attract nearby solutions and correspond to transient structures: a key role in the long-time evolution of solutions

[Beck Cooper Spiliopoulos, Beck & Wayne ’13, Bouchet & Simonnet ’09]
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The $x$- and $y$-bar states

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  The $x$- and $y$-bar states

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\omega_{xbar}(x, t) = e^{-\nu t} \sin(x/\delta), \quad \omega_{ybar}(y, t) = e^{-\nu t} \sin y,
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X-bar, Y-bar and Dipoles: with $\delta = 1$

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X-bar, Y-bar and Dipoles: with $\delta = 1$

x-bar: $\omega_{xbar} = \sin(x)$  
y-bar: $\omega_{ybar} = \sin(y)$

Dipole: $\omega_{dipole} = \sin(x) + \sin(y)$
SPDE in Fourier space:

\[ \dot{\hat{\omega}}_k = -\frac{\nu}{\delta^2} |k|_\delta^2 \hat{\omega}_k - \frac{\delta}{2} \sum_{j+l = k} \langle j^\perp, l \rangle \left( \frac{1}{|l|_\delta^2} - \frac{1}{|j|_\delta^2} \right) \hat{\omega}_j \hat{\omega}_l + \sqrt{2\nu} \sigma_k \dot{\beta}_k, \]

where

\[ |k|_\delta^2 = k_1^2 + \delta^2 k_2^2, \quad k^\perp = (k_2, -k_1). \]  

▶ Low Modes:

**x-bar states**: \( e^{-\frac{\nu}{\delta^2} t} \cos(x/\delta) \) and \( e^{-\frac{\nu}{\delta^2} t} \sin(x/\delta) \)
correspond to solutions with energy only in the \( k = (\pm 1, 0) \) modes.
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y-bar states: \( e^{-\nu t} \cos(y) \) and \( e^{-\nu t} \sin(y) \),
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    \item \textbf{\textit{y-bar states}:} \( e^{-\nu t} \cos(y) \) and \( e^{-\nu t} \sin(y) \), correspond to solutions with energy only in the \( k = (0, \pm 1) \) modes.
    \item \textbf{Dipole states}: energy in both the \( k = (\pm 1, 0) \) and \( k = (0, \pm 1) \).
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**Dipole states:** energy in both the \( k = (\pm 1, 0) \) and \( k = (0, \pm 1) \). These “low modes” have lowest value of \( |k|_\delta \) defined by (4).

▶ High Mode: Any mode \( \hat{\omega}_k \) with \( |k| > \max\{1, \delta^2\} \)
To measure the relative energy in the low modes, define

\[ Z_{\text{vort}}(t) := \frac{|\hat{\omega}_{(1,0)}(t)|^2}{|\hat{\omega}_{(1,0)}(t)|^2 + |\hat{\omega}_{(0,1)}(t)|^2}, \]

\[ Z_{\text{vort}}(t) \in [0, 1] \]
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$\triangleright$ $Z_{vort}(t) \in [0, 1]$

$\triangleright$ If the dynamics drive $Z_{vort}(t)$ to increase to 1, there is more energy in $\hat{\omega}(1,0)$ relative to $\hat{\omega}(0,1)$, indicating the system is in an $x$-bar state.
Stochastic Order Parameter

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- If \( Z_{\text{vort}}(t) \) falls toward 0, the system would be observed to be in a \( y \)-bar state.
- If \( Z_{\text{vort}}(t) \) instead stays near 1/2, the system is in a dipole state with relative energy in the low modes comparable in magnitude.
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- If \( Z_{vort}(t) \) instead stays near 1/2, the system is in a dipole state with relative energy in the low modes comparable in magnitude.
Similar to [Bouchet & Simonnet 09], take \( \sum_{k \in K} e^{-\alpha_0 |K|^2} = 1. \)

Space Discretization: Fourier

Time Discretization: Finely discretized Tamed Euler-Maruyama method (see later).

\[
\bar{Z}_{vort}(t) = \frac{1}{N} \sum_{i=1}^{N} Z_i^{vort}(t),
\]

Time averages of these Monte Carlo averages:

Introduce a "burn-in time", \( t_{burn} \)

Define this time average for any function \( f(t) \) defined on \( t_{burn} \leq t \leq T \) to be

\[
A(f, t_{burn}) := \frac{1}{T - t_{burn}} \int_{t_{burn}}^{T} f(t) \, dt.
\]
\( \delta = 1 \) : dipole solution

\( \delta > 1 \) : x-bar solution

\( \delta < 1 \) : y-bar solution
Vorticity: $\delta > 1$

An individual trajectory transitions among quasi-stationary states.
Vorticity: $\delta > 1$

An individual trajectory transitions among quasi-stationary states.

On average, the system is close to an $x$-bar state.
Vorticity: $\delta = 1.1$

Average contour plot of vorticity.

$\bar{Z}_{vort}(t)$ with 95% confidence interval.
Vorticity: $\delta = 1$

$\bar{Z}_{vort}(t)$ with 95% confidence interval.

Average contour plot of vorticity.
Vorticity: $\delta = 0.9$

$\bar{Z}_{vort}(t)$ with 95% confidence interval.

Average contour plot of vorticity.
Vorticity: $\delta = 1.1$, $\delta = 1.0$, $\delta = 0.9$
Finite Dimensional system

Use lowest eight Fourier modes:

\[ \omega_1 := \hat{\omega}(1,0), \quad \omega_2 := \hat{\omega}(-1,0), \quad \omega_3 := \hat{\omega}(0,1), \quad \omega_4 := \hat{\omega}(0,-1), \]
\[ \omega_5 := \hat{\omega}(1,1), \quad \omega_6 := \hat{\omega}(-1,1), \quad \omega_7 := \hat{\omega}(1,-1), \quad \omega_8 := \hat{\omega}(-1,-1). \]

\(\omega_{1,2,3,4}\) correspond to the low modes
\(\omega_{5,6,7,8}\) represent the role of all the high modes.
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\]

\(\omega_{1,2,3,4}\) correspond to the low modes
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Since the solution \(\omega(x, y)\) is real valued, the following complex conjugacy relationship still hold,

\[
\omega_1 = \bar{\omega}_2, \quad \omega_3 = \bar{\omega}_4, \quad \omega_5 = \bar{\omega}_8, \quad \omega_7 = \bar{\omega}_8.
\]

Based on centre manifold from deterministic case.
SDE Model

Set $\alpha_1 = (1 + \delta^2)$, $\alpha^4 = (1 + 4\delta^2)$, $\alpha_4 = (4 + \delta^2)$ and $\beta = (\delta^2 - 1)$ then

\[
\dot{\omega}_3 = -\nu \omega_5 - \delta \omega_1 \omega_7 - \delta^6 \delta (3 + \delta^2)
\]
SDE Model

Set $\alpha_1 = (1 + \delta^2)$, $\alpha^4 = (1 + 4\delta^2)$, $\alpha_4 = (4 + \delta^2)$ and $\beta = (\delta^2 - 1)$ then

\[
\dot{\omega}_1 = -\frac{\nu}{\delta^2} \omega_1 + \frac{1}{\delta \alpha_1} [\omega_3 \omega_7 - \bar{\omega}_3 \omega_5] + \frac{3 \delta^6}{2 \nu \alpha_4 \alpha_1^2} \omega_1 (|\omega_5|^2 + |\omega_7|^2) + \sqrt{2 \nu} \sigma_1 \dot{W}_1
\]

\[
\dot{\omega}_3 = -\nu \omega_3 + \frac{\delta^3}{\alpha_1} [\bar{\omega}_1 \omega_5 - \omega_1 \bar{\omega}_7] + \frac{3 \delta^2}{2 \nu \alpha_4 \alpha_1^2} \omega_3 (|\omega_5|^2 + |\omega_7|^2) + \sqrt{2 \nu} \sigma_3 \dot{W}_3
\]

\[
\dot{\omega}_5 = -\nu \frac{\alpha_1}{\delta^2} \omega_5 - \frac{\beta}{\delta} \omega_1 \omega_3 - \frac{\delta^6 (3+\delta^2)}{2 \nu \alpha_4 \alpha_1^2} \omega_5 |\omega_1|^2 - \frac{1 + 3 \delta^2}{2 \nu \delta^2 \alpha_4 \alpha_1} \omega_5 |\omega_3|^2 + \sqrt{2 \nu} \sigma_5 \dot{W}_5
\]

\[
\dot{\omega}_7 = -\nu \frac{\alpha_1}{\delta^2} \omega_7 + \frac{\beta}{\delta} \omega_1 \bar{\omega}_3 - \frac{\delta^6 (3+\delta^2)}{2 \nu \alpha_4 \alpha_1^2} \omega_7 |\omega_1|^2 - \frac{1 + 3 \delta^2}{2 \nu \delta^2 \alpha_4 \alpha_1} \omega_7 |\omega_3|^2 + \sqrt{2 \nu} \sigma_7 \dot{W}_7.
\]

Take

$$\sigma_{1,3} = e^{-\alpha_0} \quad \text{and} \quad \sigma_{5,7} = e^{-2\alpha_0}.$$

Order parameter for SDE model:

$$Z_{\text{red}}(t) := \frac{|\omega_1(t)|^2}{|\omega_1(t)|^2 + |\omega_3(t)|^2}.$$
Simulation of $\bar{Z}_{red}(t)$ with noise for $\nu = 0.001$.
For $\delta > 1$, the order parameter increases: x-bar state.
For $\delta < 1$, the order parameter decreases: y-bar state.
When $\delta = 1$, $\bar{Z}_{red}(t)$ remains near 1/2 indicating a dipole state.
Perturbation Analysis

Use backward Kolmogorov equation to derive PDEs for insight on $Z_{red}(t)$ as $\delta \to 1$ ...

0. Set $\delta^2 = 1 \pm \epsilon$
1. Scale – get fast-slow system
2. Obtain backward Kolmogorov equation
3. Average out the fast variables and look at $\mathbb{E}[Z_{red}(t)]$ as $\delta \to 1$

evolves to $y$-bar state $\hat{\epsilon} = 0.1$ evolves to $x$-bar state $\hat{\epsilon} = -0.1$. 
Comparison SPDE & SDE: $\delta = 1.1, \delta = 1.0, \delta = 0.9$
Comparison SPDE & SDE: \(\delta = 1.1, \delta = 1.0, \delta = 0.9\)
A note on numerics...

We used Tamed Euler methods and fixed steps: the S(P)DEs are nonlinear with non-global Lipschitz drifts

\[ du = [Au + f(u)] \, dt + dW. \]

What goes wrong with EM? Consider simple 1D SDE

\[ dX = -X^3 \, dt + \sigma \, dW \]
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The associated Euler map with steps size \( \Delta t \) for deterministic Eq.

\[ Y_{n+1} = Y_n - \Delta t Y_n^3 \]
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▶ stable equilibrium solution at 0
▶ unstable two-cycle at \( \left\{ \pm \sqrt{2/\Delta t} \right\} \).

So the basin of attraction of the zero solution is \( |Y_0| < \sqrt{2/\Delta t} \).
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So the basin of attraction of the zero solution is \( |Y_0| < \sqrt{2/\Delta t}. \)
▶ Outside of the basin of attraction: oscillation and growth!
Tamed Euler-Maruyama methods

[Hutzenthaler, Jentzen, Kloeden], [Hutzenthaler, Jentzen], [Gyongy, Sabanis, Siska], etc

► Idea: introduce higher order perturbation of the flow
Simplest: Drift-tamed Euler-Maruyama

\[ Y_{n+1} = Y_n + \frac{\Delta t}{1 + \Delta t \|f(Y_n)\|} f(Y_n) + g(Y_n) \Delta W_{n+1} \]
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Prove moment bounds

\[ \sup_{n \in \mathbb{N}} \sup_{n \in \{0,1,\ldots,N\}} \mathbb{E}[\| Y_n \|^p] < \infty. \] (5)

Strong convergence

\[ \left( \mathbb{E} \left[ \sup_{t \in [0,T]} \| X(t) - \bar{Y}_t \|^p \right] \right)^{1/p} \leq C_p \Delta t^{1/2} \]

➤ but use a finite $\Delta t$ in computations.
Adaptive Alternative

Rather than adapt the flow – adapt the timestep

\[ Y_{n+1} = Y_n + \Delta t_{n+1} f(Y_n) + g(Y_n) \Delta W_{n+1}. \]
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- \( \Delta t_n \) sequence random timesteps: \( \Delta t_{n+1} \) determined by \( Y_n \).
- Let \( \{t_n := \sum_{i=1}^{n} \Delta t_i\}_{n=1}^{N} \) with \( t_0 = 0 \). \( N \) random.
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- Let \( \{ t_n := \sum_{i=1}^{n} \Delta t_i \}_{n=1}^{N} \) with \( t_0 = 0 \). \( N \) random.
- Let \( \Delta t_n \) satisfy \( \Delta t_{\text{min}} < \Delta t_n < \Delta t_{\text{max}} \) where

\[ \Delta t_{\text{max}} = \rho \Delta t_{\text{min}} \quad 0 < \rho \in \mathbb{R} \]

\( \Delta t_{\text{min}} \) : ensures finite number of time steps over \([0, T]\).
If \( \Delta t < \Delta t_{\text{min}} \) use backstop method.
\( \Delta t_{\text{max}} \) : prevents stepsizes from becoming too large.
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- \( \Delta t_n \) sequence random timesteps: \( \Delta t_{n+1} \) determined by \( Y_n \).
- Let \( \{ t_n := \sum_{i=1}^{n} \Delta t_i \}_{n=1}^{N} \) with \( t_0 = 0 \). \( N \) random.
- Let \( \Delta t_n \) satisfy \( \Delta t_{\text{min}} < \Delta t_n < \Delta t_{\text{max}} \) where

\[ \Delta t_{\text{max}} = \rho \Delta t_{\text{min}} \quad 0 < \rho \in \mathbb{R} \]

\( \Delta t_{\text{min}} \): ensures finite number of time steps over \([0, T]\).
If \( \Delta t < \Delta t_{\text{min}} \) use backstop method.
\( \Delta t_{\text{max}} \): prevents stepsizes from becoming too large.

- Admissible step: Take \( \Delta t_{n+1} \) such that

\[ \|f(Y_n)\|^2 \leq R_1 + R_2 \|Y_n\|^2. \]

For example \( \Delta t_{n+1} \leq \Delta t_{\text{max}} \|f(Y_n)\|^{-1}. \)
- Convergence as \( \Delta t_{\text{max}} \rightarrow 0 \). [C. Kelly & G.L. IMAJNA 2017]
Consider SPDE

\[ du = [Au + f(u)] \, dt + G(u) \, dW. \]

with a mild solution

\[ u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(u(s)) \, ds + \int_0^t e^{(t-s)}G(u(s)) \, ds \]

We take trace class noise \( W \) and write

\[ W(t) = \sum_{j=1}^{\infty} \sqrt{q_j} \phi_j(x) \beta_j(t). \]

For \( f \) we take: one sided Lipschitz and growth condition

\[ \|Df(x)\| \leq c(1 + \|x\|^c) \]

For \( G \) - a global Lipschitz condition.

Under these conditions:

Existence and bounded moments in [Jentzen and Pusnik].
Space discretization- FE/Spectral

\[ dX^h = \left( A_h X^h + P_h F(X^h) \right) dt + P_h B(X^h) dW. \]

Discretize mild solution in time by Exponential method:

\[ X_{n+1}^h = e^{\Delta t_n A_h} X_n^h + A_h^{-1}(e^{\Delta t_n A_h} - I)P_h F(X_n^h) + P_h B(X_n^h) \Delta W_{n+1}. \]

**Theorem** With admissible timestep \( \Delta t_n \) have strong convergence of exponential method & for initial data \( X_0 \in L_2(D(A)^{\gamma/2}) \), \( 0 \leq \gamma \leq 1 \)

\[ \left( \mathbb{E} \left[ \| X(T) - Y_N \|^2 \right] \right)^{1/2} \leq C(\Delta t_{\text{max}}^{\gamma/2} + h^r). \]
Numerical evidence

![Graph showing averaged order parameter Z with parameters \( \delta = 1.1 \), \( \nu = 0.1 \), and NumTrials = 1 over time from 0 to 2000.](image)
Numerical evidence

Averaged Order Parameter $Z$

$\delta = 1.1, \quad \nu = 0.1, \quad \text{NumTrials} = 1$
Numerical evidence
Numerical evidence: timestep distribution
Numerical evidence: timestep distribution
Summary

1. Stochastic vorticity: see different states for $\delta > 1$, $\delta = 1$ and $\delta < 1$.

2. Developed a finite dimensional SDE model

3. Numerical studies show a match on transitions between states

4. Would be interesting to take same noise paths: from SPDE to SDE.

5. Would be interesting to look closer as $\delta \approx 1$.

6. Results with adaptive timestepping...