A detectability criterion for sequential data assimilation

Jason Frank

Mathematical Institute
Utrecht University

Multiscale '20, Geneva
Jan 30, 2020

with Sergiy Zhuk (IBM Dublin)

Data assimilation

- Predictive dynamical model (usually chaotic, imperfect; here: *perfect*):

$$\dot{x} = f(t,x), \quad x(t) \in \mathbf{R}^d, \quad t \in [0,T]$$

- Observational data (e.g. instrument measurements; usually discrete, incomplete, noisy):

$$y(t) = h(\tilde{x}(t)) + \eta(t), \quad h : \mathbf{R}^d \to \mathbf{R}^m, m \leq d, \quad \eta \sim \mathcal{N}(0, R^{-1})$$

- Combine to construct a better estimate of either **the state at a given time** or **the motion on an interval** than could be achieved by either source alone.
- **Example:** used in weather prediction to construct an initial condition from data measured over the recent past.

Motivation

Originally motivated to understand what minimal observations are needed to construct a complete state estimation.

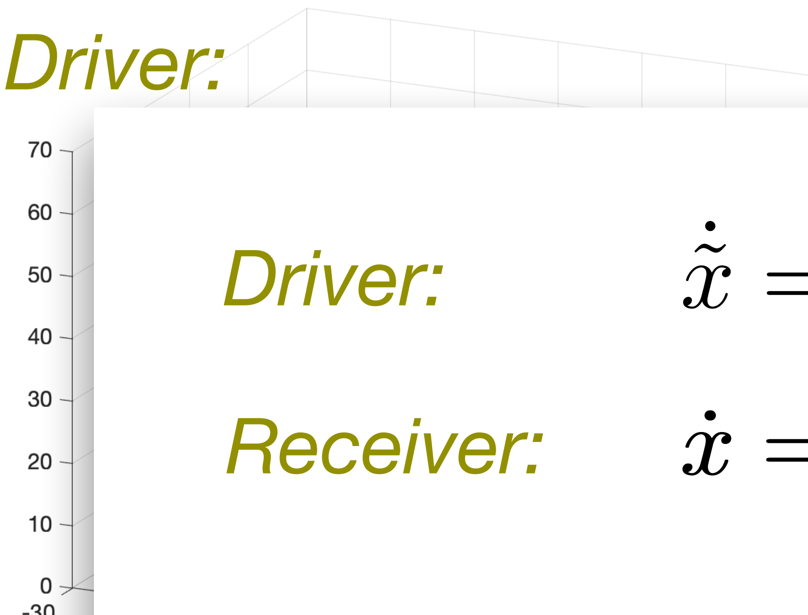
We construct a continuous time synchronous data assimilation method that explicitly uses a decomposition of the tangent space into expanding and decaying subspaces.

Lyapunov vectors and data assimilation in the literature:

- AUS methods (Trevisan, Carrassi, Bocquet, Grudzien, …)
- Mentioned in work by Ghil et al., González-Tokman & Hunt, Gottwald & Reich, …
- *De Leeuw, Dubinkina, Frank, Steyer, Tu, Van Vleck 2018*.

Synchronization of chaotic dynamics

Driver:



Observation time series:

Driver: $\dot{\tilde{x}} = f(t, \tilde{x}) \qquad y(t) = H\tilde{x}(t)$

Receiver: $\dot{x} = f(t, x(t)) + L(t)(y(t) - Hx(t))$

Transverse linear dynamics:

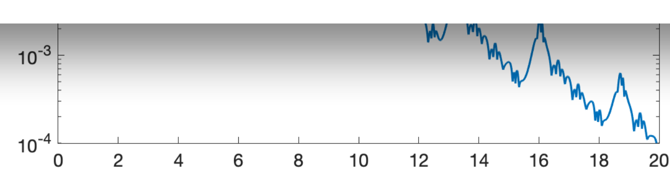
$$w(t) = x(t) - \tilde{x}(t)$$

$$\dot{w} = [A(t) - L(t)H]\, w + N(t, w)$$

$$A(t) = Df(x(t))$$

Sync: $\lim_{t\to\infty} \|w(t)\| \le Ce^{-\mu t}$

$\dot{x} = f(t, x(t)) + g(y(t) - Hx(t))$



$$\lim_{t\to\infty} \|x(t) - \tilde{x}(t)\| \le Ce^{-\mu t}$$

(e.g. Pecora & Carroll 1990)

Lyapunov exponents and stability

Lyapunov developed the stability theory of nonautonomous linear systems in his thesis (1892)

$$\dot{x}(t) = A(t)x(t)$$

Characteristic function measures asymptotic rate of growth/decay

$$\chi(x_0) = \lim_{t\to\infty} \frac{1}{t} \log \|x(t)\|$$

Assumes at most d values on $\boldsymbol{R}^d$, the Lyapunov exponents, with associated Lyapunov vectors

$$\lambda_1 \geq \cdots \geq \lambda_d$$

$$v_1(t), \ldots, v_d(t)$$

(Berreira & Pesin, AMS Lecture Series, 2000)

Lyapunov exponents and stability

(1) Linear systems

x(t)=0 is exponentially stable if

$$\dot{x}(t) = A(t)x(t)$$

$$\lambda_1 < 0$$

2. Quasi-linear systems

x(t)=0 is stable in a *neighborhood of 0* if (1) holds

$$\dot{x} = A(t)x + N(t,x), \quad N(t,0) = 0$$

$$\|N(t,x) - N(t,\tilde{x})\| \leq C\|x - \tilde{x}\|^p, \quad p > 1$$

3. Nonlinear systems

Orbit x*(t) attracts a neighborhood of itself if (1) holds

$$\dot{x} = f(t,x), \quad x(t) = x^*(t) + w(t)$$

$$\begin{aligned} \dot{w} &= Df(x^*)w + [f(x) - f(x^*) - Df(x^*)w] \\ &= A(t)\,w + N(t,w) \end{aligned}$$

(Berreira & Pesin, AMS Lecture Series, 2000)

Computation of Lyapunov Exponents

Procedure to compute the LEs yields an orthogonal basis for the associated Lyapunov vectors.

Continuous QR factorization $\dot{X}(t) = A(t)X(t), \quad X(t) = Q(t)R(t)$

$$\dot{R} = BR \qquad B = Q^T AQ - S$$

$$S = -S^T$$

$$B = \begin{bmatrix} B_{11} & * & * \\ & \ddots & * \\ & & B_{kk} \end{bmatrix}$$

$$\dot{Q} = (I - QQ^T)AQ - QS$$

$$S = \begin{bmatrix} 0 & -\mathrm{tril}(Q^T AQ) \\ \mathrm{tril}(Q^T AQ) & 0 \end{bmatrix}$$

In essence a power iteration.
First k columns of Q span the k fastest growing Lyapunov vectors. Q is d x k.

LEs appear ordered on the diag of B:

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t B_{ii}(s)\, ds$$

Dieci, Jolly & Van Vleck 2001

Sequential DA with tangent splitting

Recei

Transv

$, w)$

Want $\tilde{R} = Q^T L H Q$ upper tri. positive diag.

Choose $L = QUH^T$

then $\tilde{R} = Q^T Q U H^T H Q$

$= U H^T H Q$

Suppose U is invertible: $U^{-1}\tilde{R} = H^T H Q$

A good choice is the QR factorization:

$$\tilde{Q}\tilde{R} = H^T H Q$$

Consequently, $U = \tilde{Q}^T$

$$L = pQ\tilde{Q}^T H^T$$

Note tha

Choose

then the

itself.

$B = Q$

pper

triangular with positive diagonal!

Detectability criterion

Recall $\tilde{Q}\tilde{R} = H^T H Q$

The Lyapunov vector v_j is *detectable* if

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \tilde{R}_{ii}(s)\, ds > 0, \quad i = 1, \ldots, j$$

The *pair* $(A(t), H)$ is detectable if all Lyapunov vectors corresponding to nonnegative Lyapunov exponents are detectable. *(Needs rank $H \geq$ dimension nonstable space).*

Theorem. If (A(t),H) is detectable then there exists $\alpha>0$ such that all Lyapunov exponents of the fundamental matrix equation

$$\dot{W} = (A(t) - \alpha L(t) H) W, \qquad L(t) = Q(t)\tilde{Q}(t)^T H^T$$

are negative.

Observation operator

By construction H^T and Q have full rank. Because they are ‘tall’ matrices, they have no null space.

Consequently, $H^T HQx = 0$ implies $HQx = 0$. But HQ has dim. $m \times k$ and rank $\min\{m, k\}$ unless there is a nontrivial linear combination of the columns of Q in $\ker(H)$, in which case H doesn’t ‘see’ the whole nonstable space.

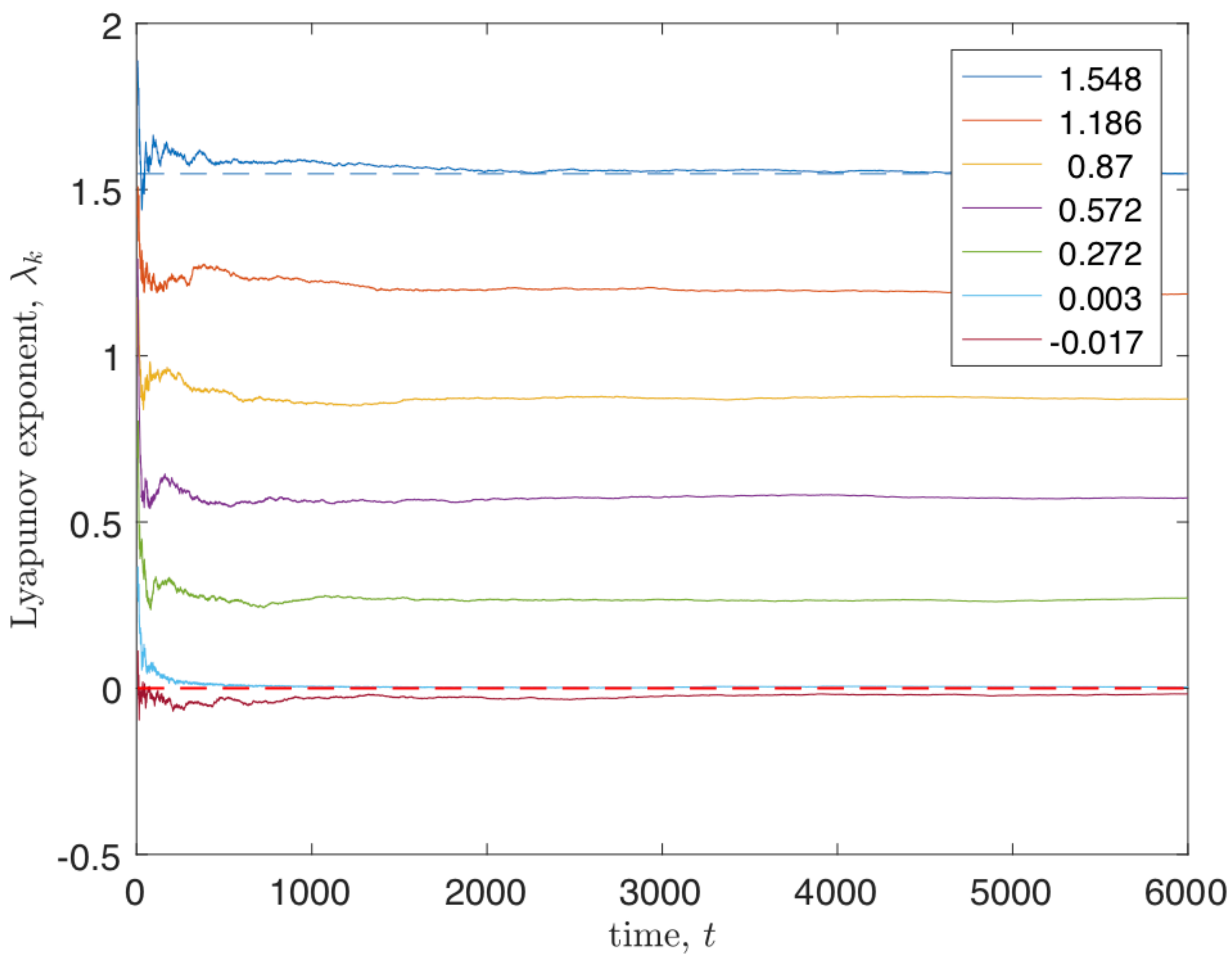
Conclusion: We need $m \geq k$ and HQ full rank for detectability.

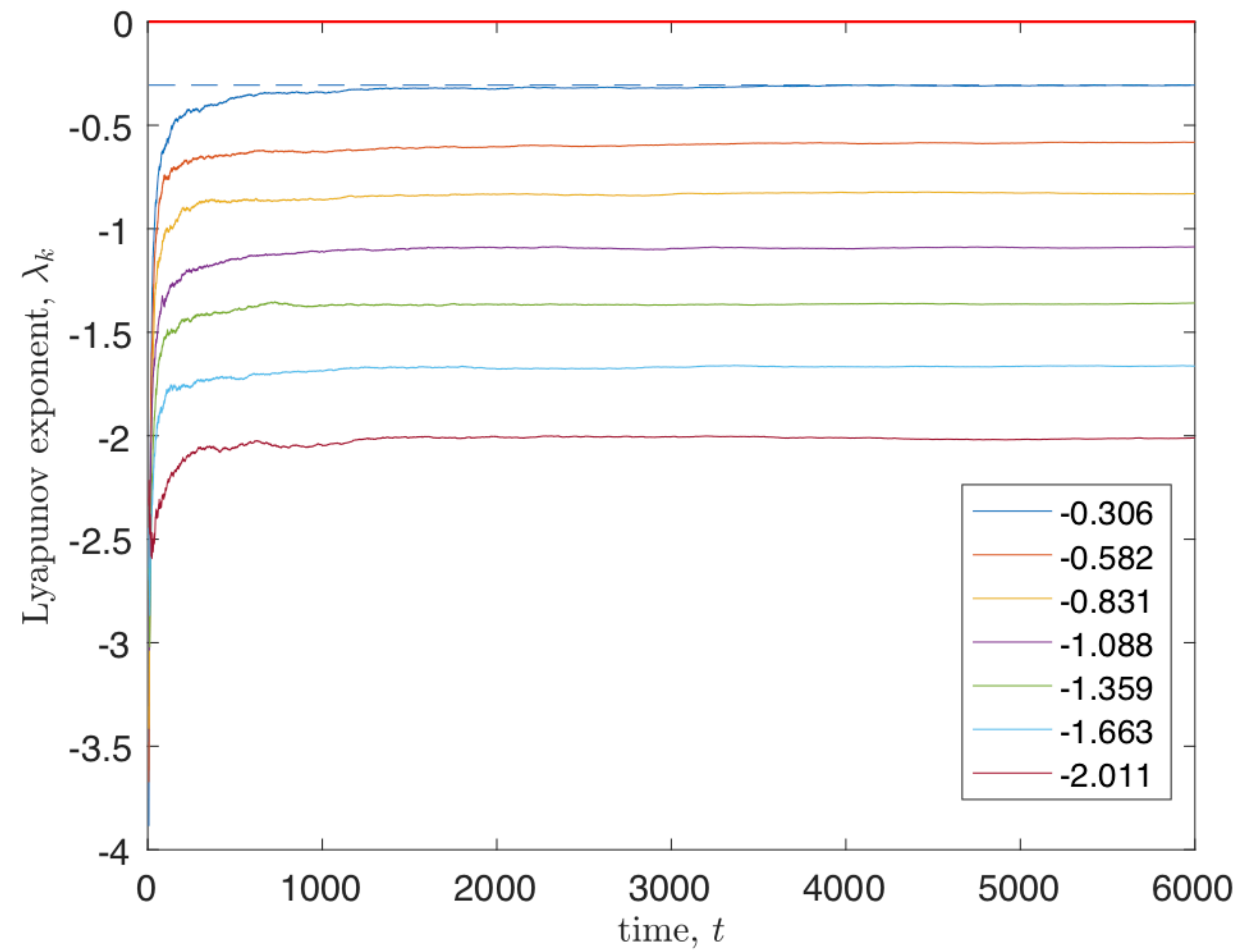
Numerical experiment

- Lorenz 96 model, M=18 lattice points, initial condition

$$x_j(0) = \sin 2\pi j/M + \mathcal{N}(0, \varepsilon^2)$$

H: first 7 eigenmodes of the lattice Laplacian





Numerical experiment

- Lorenz 96 model, M=18 lattice points, initial condition

$$x_j(0) = \sin 2\pi j/M + \mathcal{N}(0, \varepsilon^2)$$

100-member ensemble, H = first 7 eigenmodes

$$\varepsilon = 0.01$$

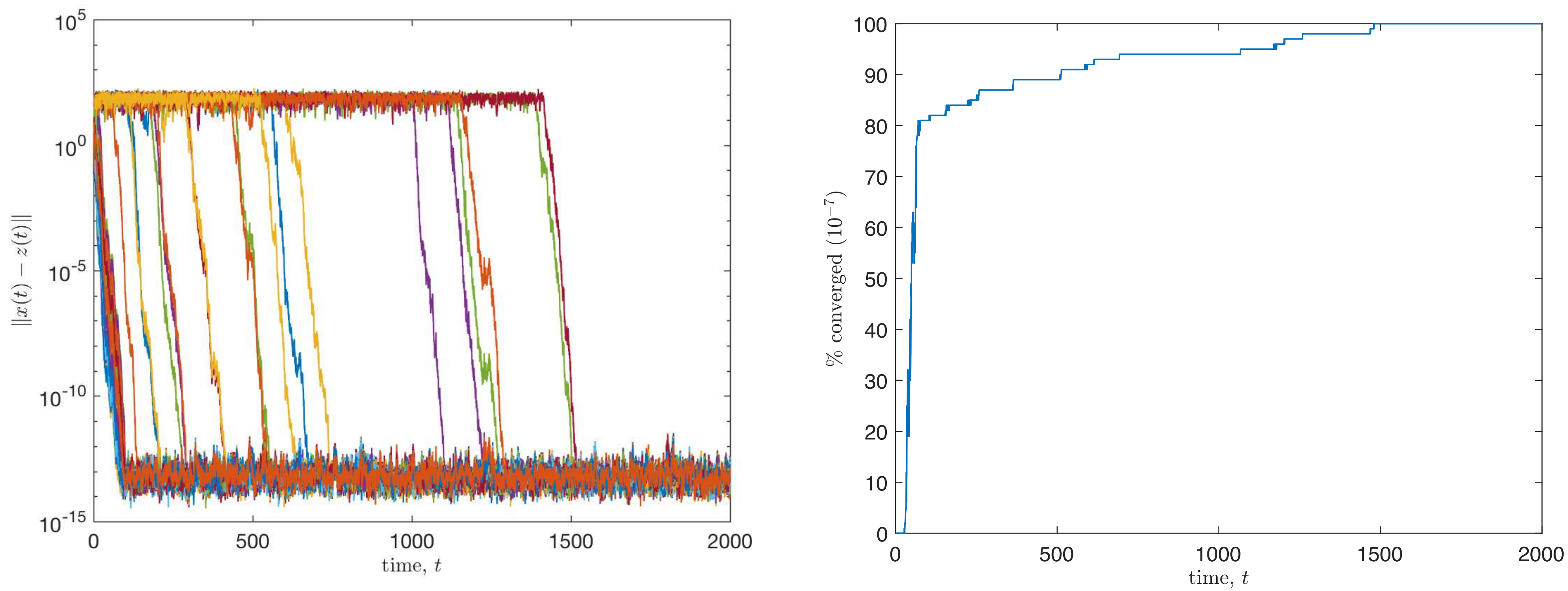


Figure 2: Convergence of filter (29) for the Lorenz '96 model (32) with $k = 7$. Left, the errors $\|\xi(t)\|$ for a 100-member ensemble of perturbed initial conditions. Right, the number of ensemble members converged to tolerance $\|\xi(t)\| < 10^{-7}$ at time t.

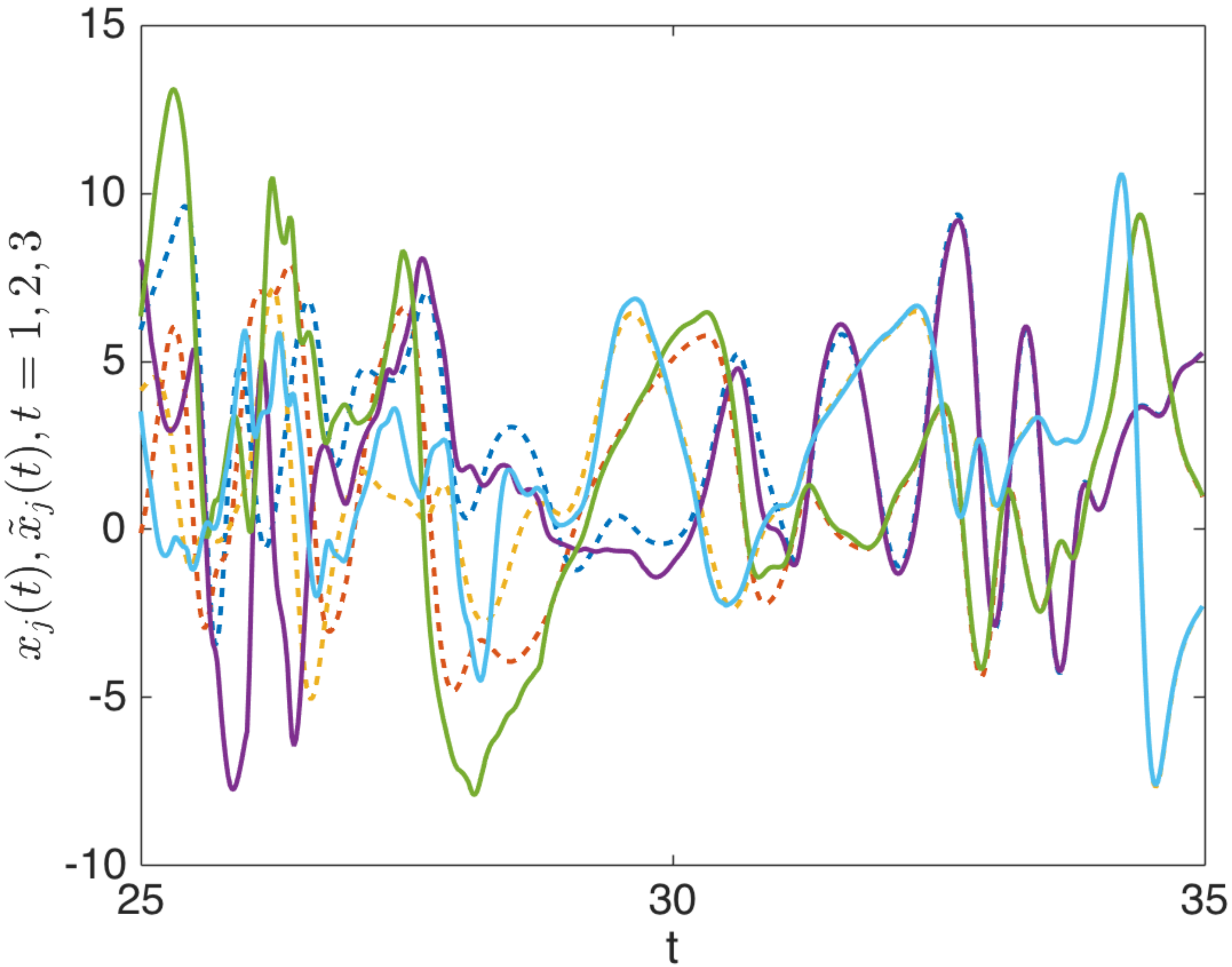
Numerical experiment

- Lorenz 96 model, M=18 lattice points, initial condition

$$x_j(0) = \sin 2\pi j/M + \mathcal{N}(0, \varepsilon^2)$$

100-member ensemble, H = first 7 eigenmodes

$\varepsilon = 0.1$



Numerical experiment

- Lorenz 96 model, M=18 lattice points, initial condition

$$x_j(0) = \sin 2\pi j/M + \mathcal{N}(0, \varepsilon^2)$$

100-member ensemble, H = first 8 eigenmodes

$$\varepsilon = 0.1$$

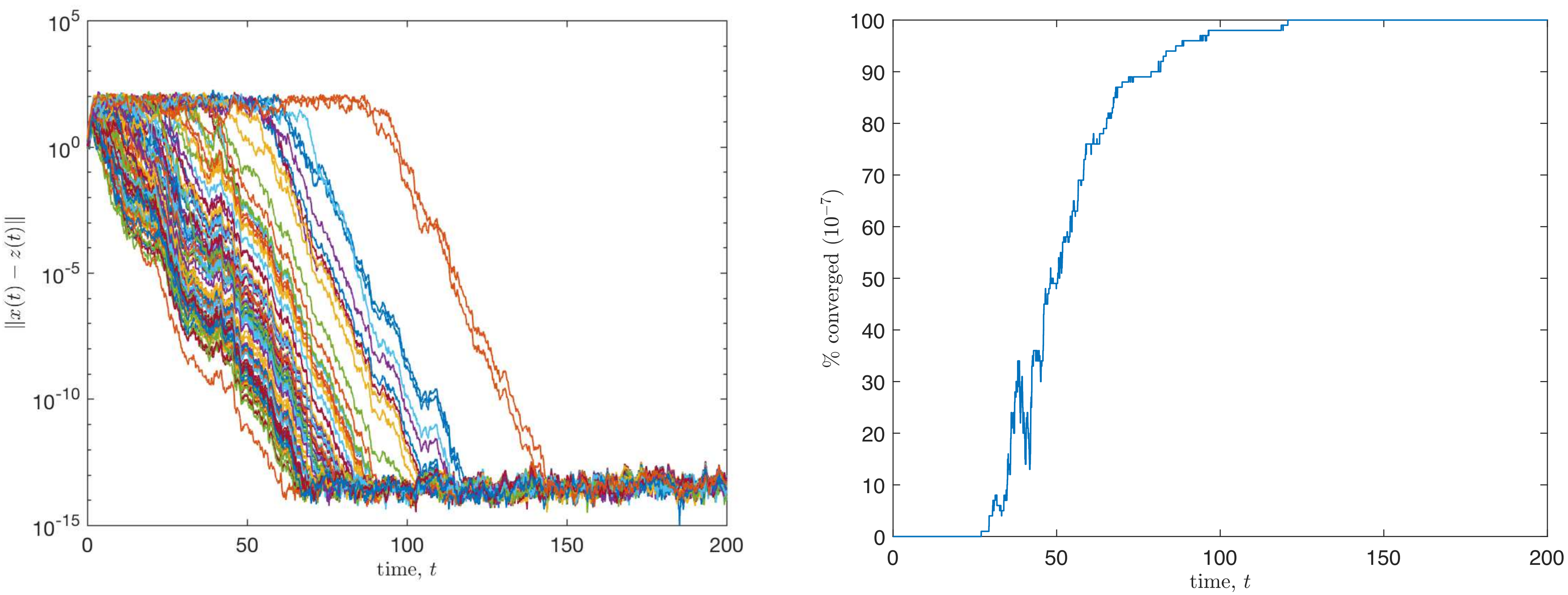


Figure 3: Convergence of filter (29) for the Lorenz '96 model (32) with $k = 8$. Left, the errors $\|\xi(t)\|$ for a 100-member ensemble of perturbed initial conditions. Right, the number of ensemble members converged to tolerance $\|\xi(t)\| < 10^{-7}$ at time t.

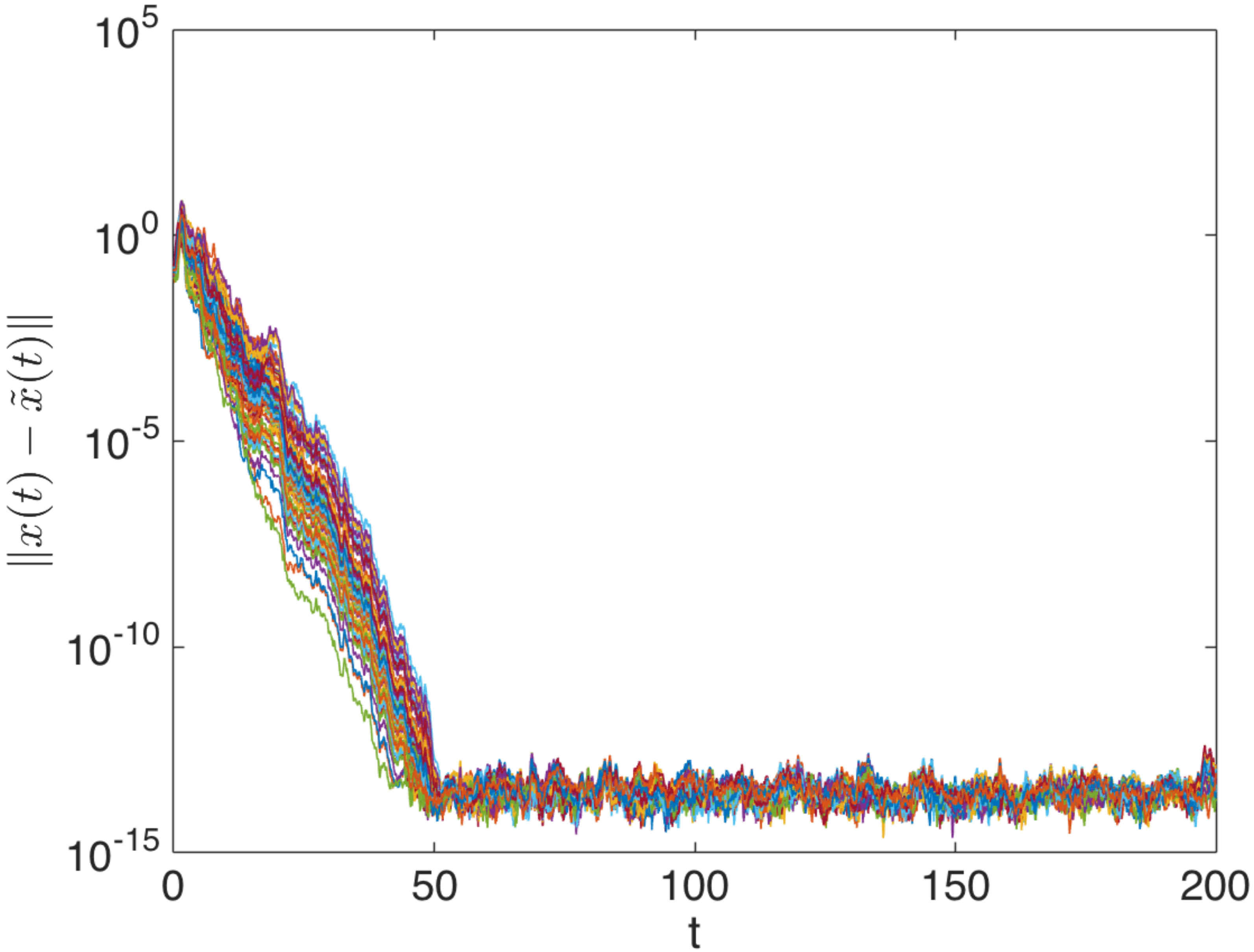
Numerical experiment

- Lorenz 96 model, M=18 lattice points, initial condition

$$x_j(0) = \sin 2\pi j/M + \mathcal{N}(0, \varepsilon^2)$$

100-member ensemble, H = first 8 eigenmodes

$\varepsilon = 0.01$

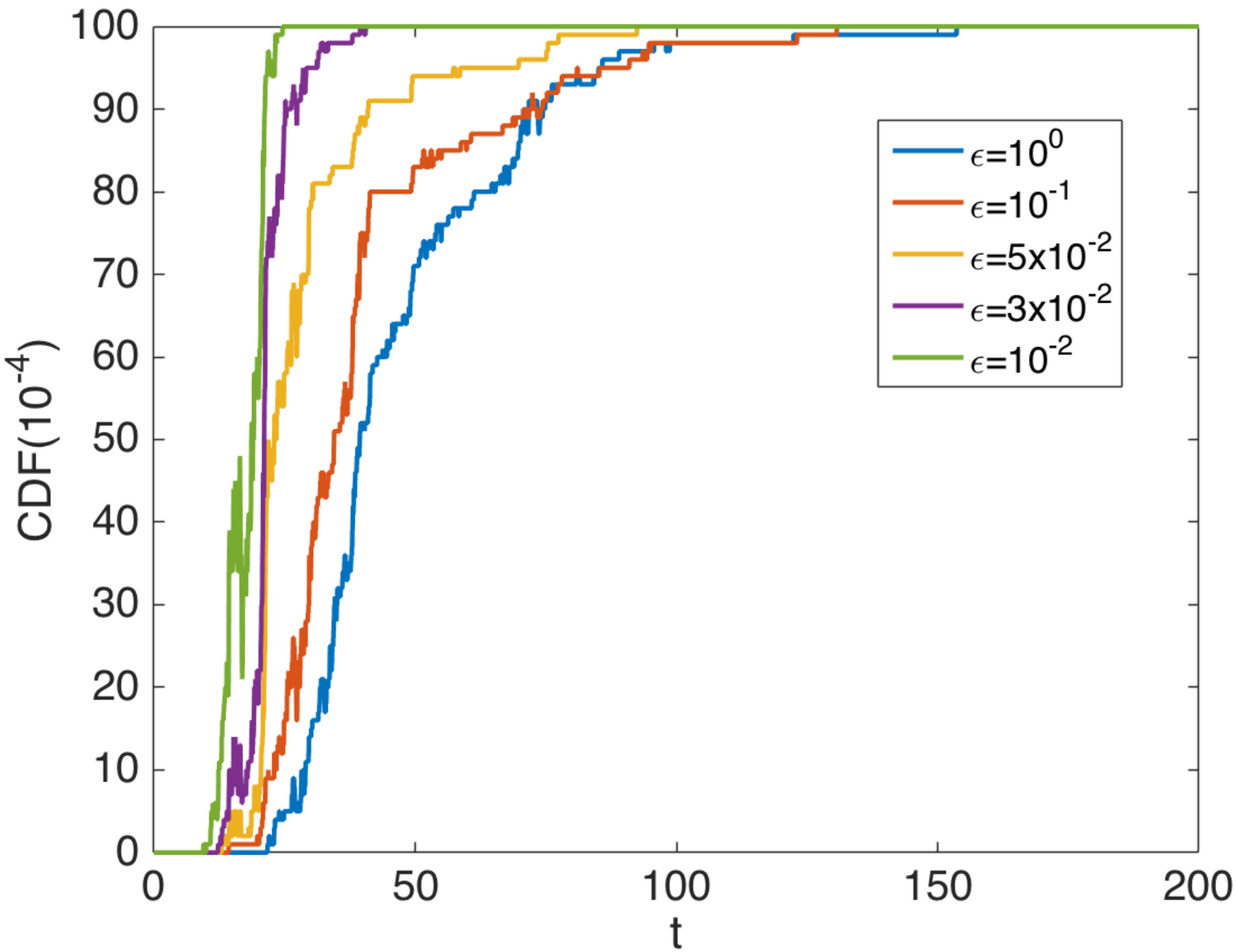


Numerical experiment

- Lorenz 96 model, M=18 lattice points, initial condition

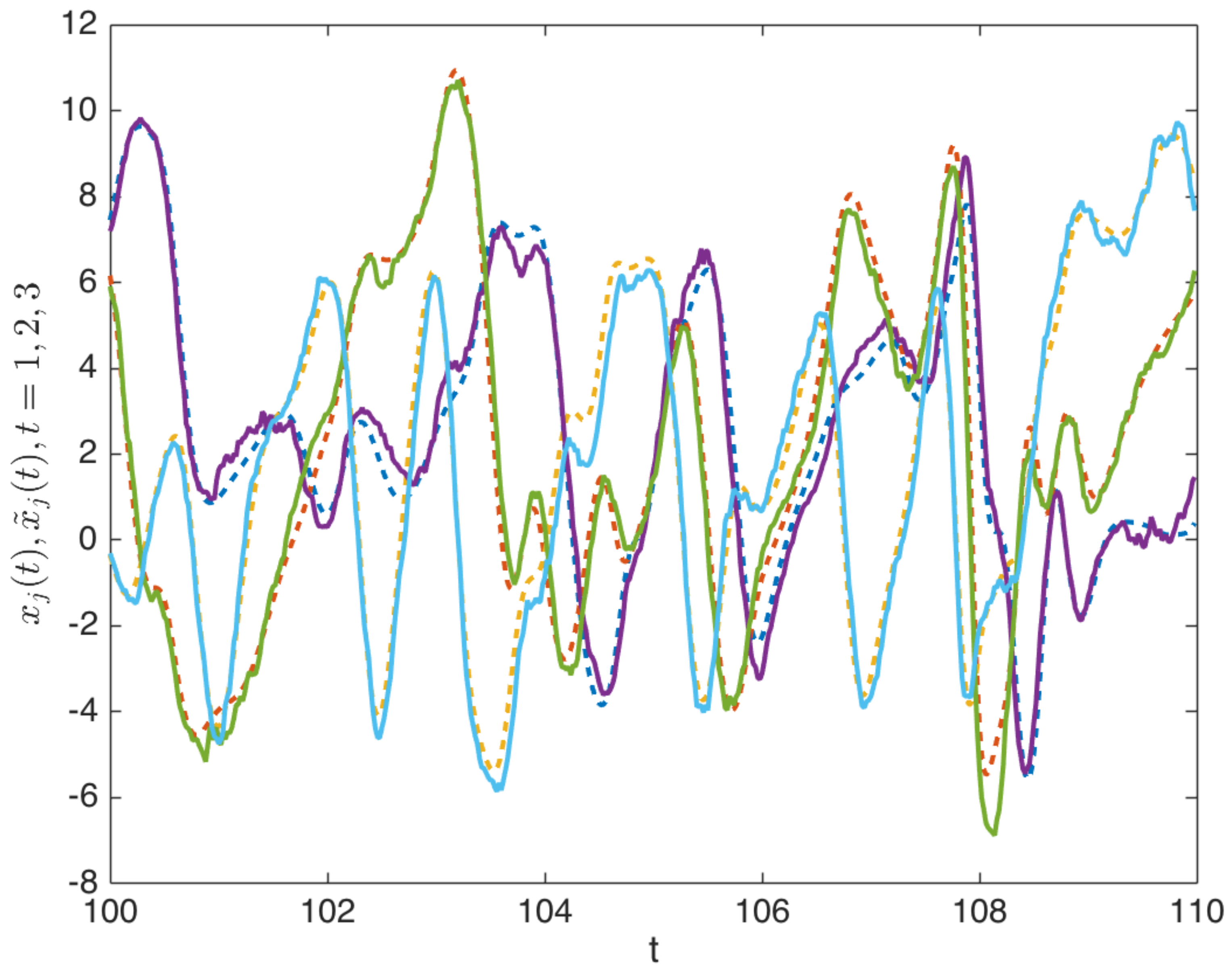
$$x_j(0) = \sin 2\pi j/M + \mathcal{N}(0, \varepsilon^2)$$

100-member ensemble, H = first 8 eigenmodes



Numerical experiment

- Lorenz 96 model, M=18 lattice points, initial condition
- Noisy data: $\eta(t) \sim \mathcal{N}(0, 4)$



Numerical experiment

- Lorenz 96 model, M=18 lattice points, initial condition
- Noisy data: $\eta(t) \sim \mathcal{N}(0, 4)$

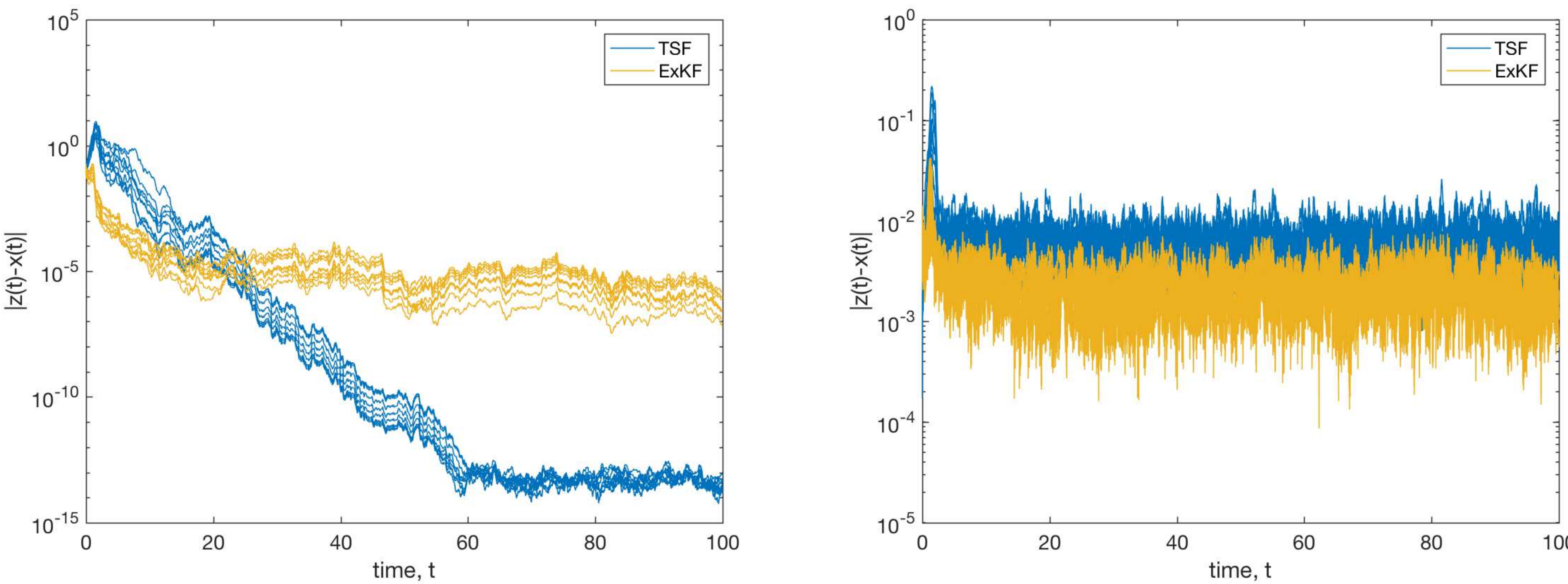


Figure 4: Comparison of the filter (29) and the ExKF (33) for the Lorenz '96 model (32) with $k = 8$. Left, the errors $\|\xi(t)\|$ for a 10-member ensemble of perturbed initial conditions. Right, the errors $\|\xi(t)\|$ for a 10-member ensemble with random observational error.

Summary

- For synchronization of chaos the data signal must be sufficient to control the nonstable directions (put another way, the latter need to be *detectable*).
- In particular, the observation operator should observe the nonstable tangent space most of the time, and its rank should be at least as large as the dimension of the nonstable space.
- An efficient continuous sequential filter can be constructed that explicitly detects the unstable space.
- Numerical experiments confirm the filter also works when the observations are noisy.