Spectral asymptotics for contracted tensor ensembles

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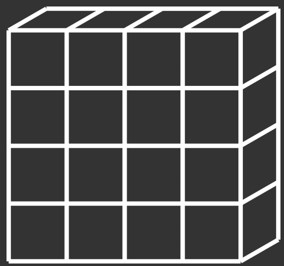
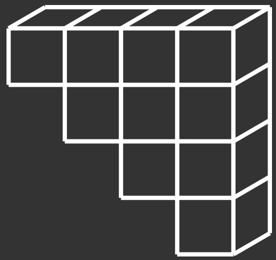
Joint work with Jorge Garza-Vargas

- Setting: d-th order N-dimensional real square symmetric tensors $T_{d,N} = \left(T_{d,N}(k_1, \ldots, k_d)\right)_{(k_1,\ldots,k_d)\in[N]^d} \in \mathcal{S}_{d,N} \subseteq \mathbb{R}^{N^d}$,

$$T_{d,N}(k_1, \ldots, k_d) = T_{d,N}(k_{\sigma(1)}, \ldots, k_{\sigma(d)})$$

- Example: $d = 2$





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- Basic question: how does the randomness of $T_{d,N}$ behave under repeated contractions?

- For $T_{d,N} \in \mathcal{S}_{d,N}$, $p \leq d$, and vectors $v_1, \dots, v_p \in \mathbb{R}^N$,

we define the contracted tensor

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p] \in \mathcal{S}_{d-p,N}$$

by

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p](k_1, \dots, k_{d-p})$$

$$= \sum_{\ell_1, \dots, \ell_p} T_{d,N}(k_1, \dots, k_{d-p}, \ell_1, \dots, \ell_p)\, v_1(\ell_1) \cdots v_p(\ell_p)$$

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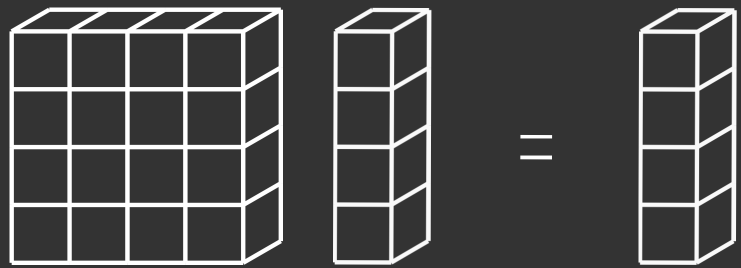

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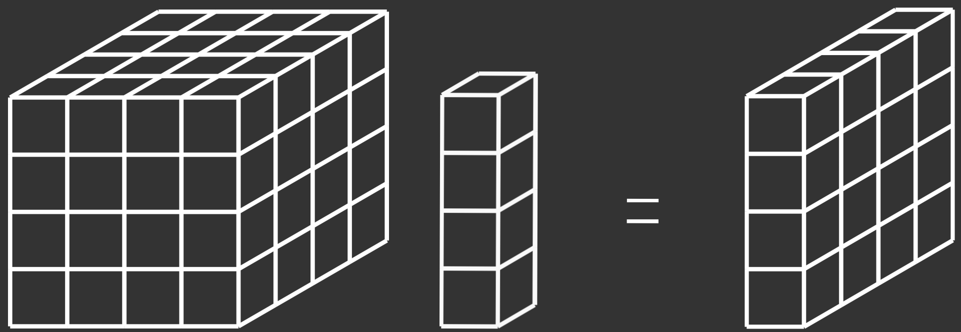

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- If $T_{d,N}$ is a random symmetric tensor, the corresponding contracted tensor ensemble (GCC21) is the family of random matrices $\left\{ T_{d,N}[u^{\otimes d-2}] \right\}_{u \in S^{N-1}}$
- What kind of randomness? A canonical distribution:

(GOTE) $$\frac{1}{Z_{d,N}} e^{-\|H\|_F^2/2} \, dH$$

- (GCC21) For any sequence of unit vectors $u_N \in S^{N-1}$, the empirical spectral distribution of $W_N = \frac{1}{\sqrt{N}} \underset{\text{GOTE}}{T_{3,N}}[u_N]$ converges weakly almost surely to the semicircle distribution with $\sigma^2 = \frac{1}{6}$

- In general, W_N is not a Wigner matrix:

$$T_{3,N}[u_N](j,k) = \sum_{\ell} T_{3,N}(j,k,\ell)\, u_N(\ell)$$

- Proof relies on Stein's method and $d = 3$

- Question 1: what about higher order $d \geq 4$?

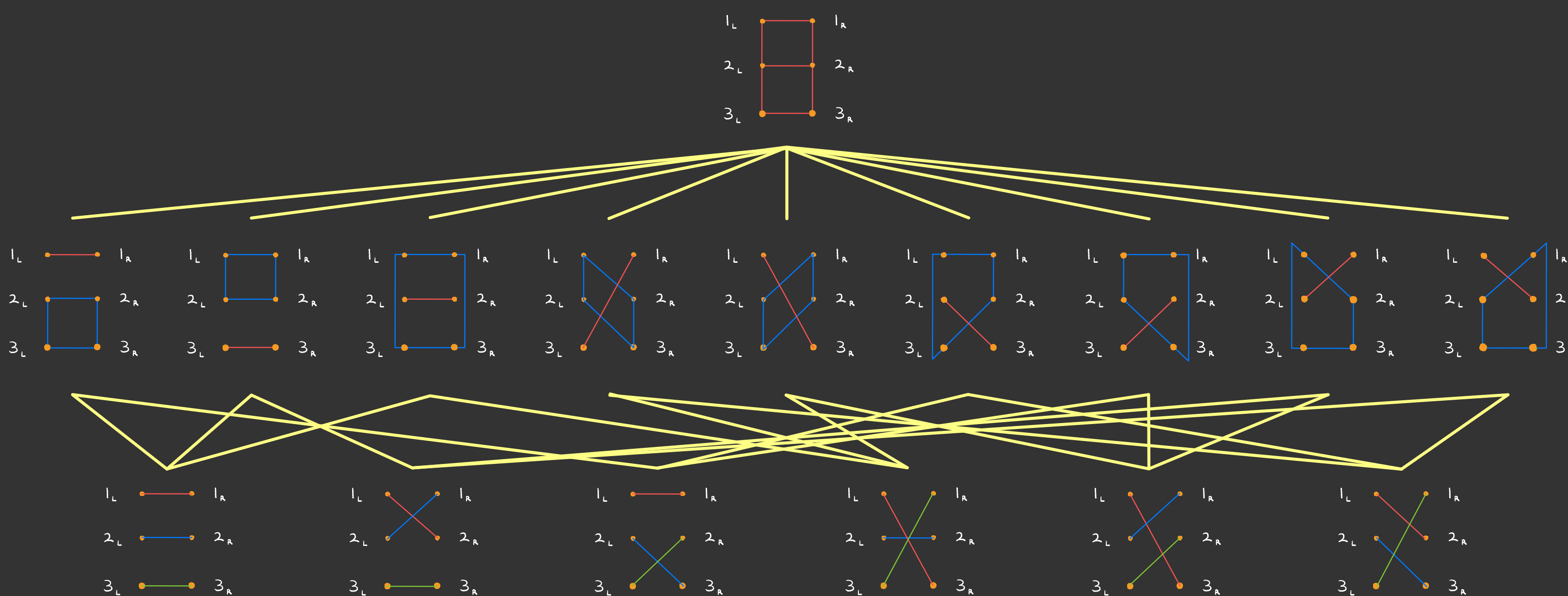
$$T_{3,N}[u_N](j,k) = \sum_{\ell} T_{3,N}(j,k,\ell)\, u_N(\ell)$$

$$T_{4,N}[u_N^{\otimes 2}](j,k) = \sum_{\ell_1,\ell_2} T_{4,N}(j,k,\ell_1,\ell_2)\, u_N(\ell_1)\, u_N(\ell_2)$$

- Question 2: universality for general tensor distributions?

- Question 3: general contractions $u_N^{(1)} \otimes \cdots \otimes u_N^{(d-2)} \neq u_N^{\otimes d-2}$?

- Question 4: joint behavior of $\left\{ T_{d,N}[u_N^{(1)} \otimes \cdots \otimes u_N^{(d-2)}] \right\}_{u_N^{(j)} \in S^{N-1}}$



$(UBP(3), \leq)$