CLT for non-Hermitian random band matrices with variance profiles

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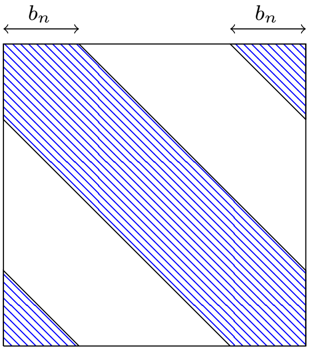
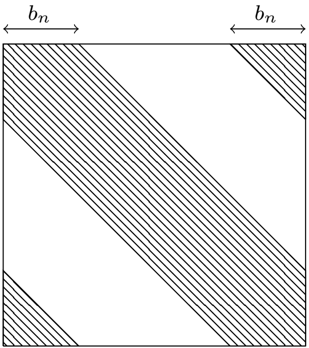
Random Matrices and Random Landscapes



(a) Periodic **non-Hermitian** random band matrix of bandwidth b_n



(b) Weight profile matrix (deterministic) of bandwidth b_n

Figure: Blue/black lines represent the non-zero diagonal vectors.

- How do we analyze fluctuations of the random measure $\mu_M (= \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i})$ via linear eigenvalue statistics?
- Define the centered linear eigenvalue statistics

$$\mathcal{L}_f(M)^\circ = \sum_{i=1}^{n} f(\lambda_i) - nf(0)$$

- Rider, Silverstein (*The Annals of Probability*, 2006) had shown that if $f_1, f_2, \ldots, f_k$ are analytic function supported on $\mathbb{D}_4 := \{z : |z| \leq 4\}$, then [1] $(\mathcal{L}^\circ_{f_1}(M), \mathcal{L}^\circ_{f_2}(M), \ldots, \mathcal{L}^\circ_{f_k}(M)) \xrightarrow{d} \mathcal{N}_k(0, \Sigma)$ as $n \to \infty$, where

$$\Sigma_{ij} = \frac{1}{\pi} \int_{\mathbb{D}_1} f_i'(\eta)\overline{f_j'(\eta)}\, d\Re(\eta) d\Im(\eta).$$

Theorem (J., 2022)

Let $M = \frac{1}{\sqrt{c_n}} X$, where X be an $n \times n$ random band matrix of bandwidth b_n, and $c_n = 2b_n + 1$. Then $\sqrt{c_n/n}(\mathcal{L}^\circ_{f_1}(M), \mathcal{L}^\circ_{f_2}(M), \dots, \mathcal{L}^\circ_{f_k}(M)) \xrightarrow{d} \mathcal{N}_k(0, \Sigma)$ as $n \to \infty$, where

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if $\nu := \lim_{n \to \infty} (c_n/n) \in (0, 1]$ and M is periodic,

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Technical notes:

- $f_1, f_2, \ldots, f_k$ are complex analytic functions supported on $\mathbb{D}_{1+\epsilon}$.
- x_{ij}s are assumed to be complex random variables with $\mathbb{E}[x_{ij}] = 0$, $\text{Var}(x_{ij}) = 1$ and satisfy Poincaré inequality.

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¿If $c_n = n$ i.e., $\nu = 1$, do we recover full matrix's variance?

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Clearly, $\text{sinc}(k\pi) = \mathbf{1}_{\{k=0\}}$. It can be shown that

$$-\frac{1}{4\pi^2} \oint_{\partial \mathbb{D}_1} \oint_{\partial \mathbb{D}_1} \frac{f_i(z)\overline{f_j(w)}}{(z\bar{w} - 1)^2} \, dz \, d\bar{w} = \frac{1}{\pi} \int_{\mathbb{D}_1} f_i'(\eta)\overline{f_j'(\eta)} \, d\Re(\eta) d\Im(\eta).$$

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Let $M = \frac{1}{\sqrt{c_n}} X$, where X be an $n \times n$ random band matrix of bandwidth b_n, and $c_n = 2b_n + 1$. Then $\sqrt{c_n/n}(\mathcal{L}^\circ_{f_1}(M), \mathcal{L}^\circ_{f_2}(M), \ldots, \mathcal{L}^\circ_{f_k}(M)) \xrightarrow{d} \mathcal{N}_k(0, \Sigma)$ as $n \to \infty$, where

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¿But, what if $\nu = 0$?

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$$\Sigma_{ij} = -\frac{1}{4\pi^2} \int_{\mathbb{R}} \oint_{\partial\mathbb{D}_1} \oint_{\partial\mathbb{D}_1} \frac{f_i(z)\overline{f_j(w)}\text{sinc}(\pi t)}{(z\bar{w} - \text{sinc}(\pi t))^2} \, dz \, d\bar{w} \, dt$$

if $\lim_{n\to\infty} (c_n/n) = 0$.

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¿Can we attach weights, $h_{ij} x_{ij}$?

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if $\lim_{n \to \infty} (c_n/n) = 0$.

Particular case, when $c_n = o(n)$ i.e., $\nu = 0$ and $f(z) = z^p$, the variance is

$$\frac{p}{\pi}\int_{\mathbb{R}} \mathrm{sinc}^p(t)\, dt$$

Here are first few values

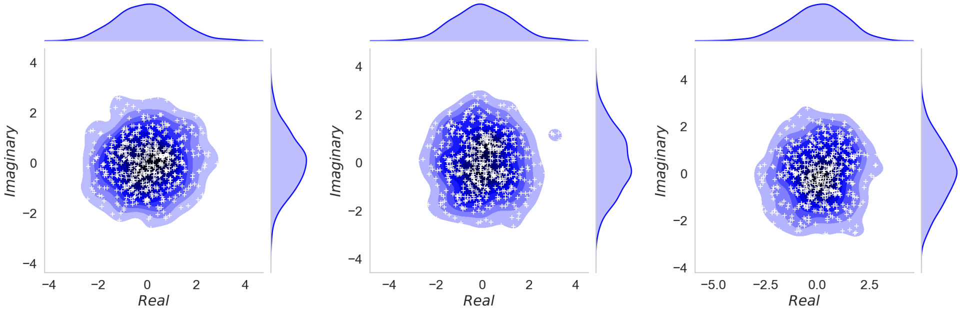
$$\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{sinc}(t)\, dt = 1 \qquad \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{sinc}^2(t)\, dt = 1$$

$$\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{sinc}^3(t)\, dt = \frac{3}{4} \qquad \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{sinc}^4(t)\, dt = \frac{2}{3}$$

$$\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{sinc}^5(t)\, dt = \frac{115}{192} \qquad \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{sinc}^6(t)\, dt = \frac{11}{20}.$$

Limiting variance - Band vs Full matrix

$f(z)$	z	z^2	z^3	z^4	z^5	z^6
Band matrix	1.00	2.00	2.25	2.67	2.9948	3.3
Full matrix	1.00	2.00	3.00	4.00	5.00	6.00



(a) $f(z) = z^4$ (b) $f(z) = z^5$ (c) $f(z) = z^6$

Figure: A heat map of 500 samples of $\sqrt{c_n/n}\mathcal{L}_f^\circ(M)$ and marginal densities of real and imaginary parts are plotted on the complex plane. M is a 4000×4000 random band matrix of bandwidth $b_n = n^{0.3}$ with i.i.d. complex Gaussian entries.

¡Thank you for your attention!