Critical-subcritical moments of moments of random matrices

Mo Dick Wong

Department of Mathematical Sciences, Durham University

Random Matrices and Random Landscapes
Conference in honour of Yan Fyodorov's 60th birthday

Joint work with Jon Keating

RMT: some number-theoretic motivations

U_N: Haar-distributed unitary matrix of size $N \times N$.

$$|\det(I_N - e^{-i\theta} U_N)| \overset{N \sim \log \frac{T}{2\pi}}{\longleftrightarrow} \left|\zeta(\frac{1}{2} + iTU + x)\right|, \qquad U \sim \mathrm{Uniform}[0,1].$$

- Montgomery–Dyson: correlation functions $\leftrightarrow$ sine kernel.
- Keating–Snaith: moments of ζ.
  - RMT result: $\mathbb{E}_{\mathrm{U}(N)}\left[|\det(I_N - e^{-i\theta} U_N)|^{2s}\right] \overset{N\to\infty}{\sim} C_s N^{s^2}$.
  - NT conjecture: $\mathbb{E}\left[\left|\zeta\left(\frac{1}{2} + iTU\right)\right|^{2s}\right] \overset{T\to\infty}{\sim} A_s C_s \left(\log \frac{T}{2\pi}\right)^{s^2}$.

Moments of moments

Recent years: statistical behaviour of ζ on a 'short interval' of the critical line, e.g.

$$\log \left| \zeta \left(\frac{1}{2} + i(TU + t) \right) \right|, \qquad t \in [0, 1].$$

as $T \to \infty$.

- Fyodorov–Keating: a good model could be

$$\log |\det \left(I_N - e^{-i\theta} U_N \right)|, \qquad \theta \in [0, 2\pi], \qquad N \sim \log \frac{T}{2\pi} \to \infty.$$

- Moments of moments: for $k, s \geq 0$ we want to study

$$\mathrm{MoM}_{\mathrm{U}(N)}(k, s) := \mathbb{E}_{\mathrm{U}(N)} \left[\left(\frac{1}{2\pi} \int_0^{2\pi} |\det \left(I_N - e^{-i\theta} U_N \right)|^{2s} d\theta \right)^k \right]$$

as $N \to \infty$.

Fyodorov–Keating conjecture

$$\mathrm{MoM}_{\mathrm{U}(N)}(k,s) \overset{N\to\infty}{\sim} \begin{cases} \left(\dfrac{G(1+s)^2}{G(1+2s)\Gamma(1-s^2)}\right)^k \Gamma(1-ks^2) N^{ks^2}, & k < 1/s^2, \\ c_{k,s} N^{k^2s^2-k+1}, & k > 1/s^2. \end{cases}$$

- $k, s \in \mathbb{N}$: Bailey–Keating (2019), Assiotis–Keating (2020).
- $k \in \mathbb{N}, s \geq 0$: Claeys–Krasovsky (2015), Fahs (2019).
- What about $k = 1/s^2$?

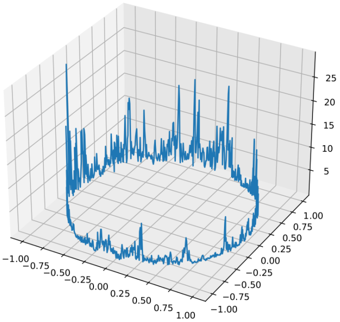

Figure: A realisation of $|\det(I_N - e^{-i\theta} U_N|^{\frac{1}{\sqrt{2}}}$, $N = 1000$

Theorem (Keating–W' 2022)

Let $k \geq 2$ by any integer. Then

$$\mathrm{MoM}_{\mathrm{U}(N)}\left(k, \frac{1}{\sqrt{k}}\right) \overset{N\to\infty}{\sim} \frac{k-1}{\Gamma(1-\frac{1}{k})^k} \left[\frac{G(1+\frac{1}{\sqrt{k}})^2}{G(1+\frac{2}{\sqrt{k}})}\right]^k N \log N.$$

(Conjecture: same expression holds for any $k > 1$.)

Ingredients: a preview

- Log-characteristic polynomial $\leftrightarrow$ (mollified) log-correlated Gaussian field.
- $\mathrm{MoM}_{U(N)}$ $\leftrightarrow$ moments of Gaussian multiplicative chaos (GMCs)
  - Analysis of exponential functional of Brownian motion
  - Fyodorov–Bouchaud formula: exact integrability of 1d GMC.