Invertible Polynomial Ensembles

JPDF on $I \subseteq \mathbb{R}$: $\quad p(x_1,\ldots,x_N) = \frac{1}{Z_N}\, \Delta_N(x_1,\ldots,x_N)\ \det\left[\varphi_i(x_j)\right]_{i,j=1}^{N}$

Applications of averages over polynomial ensembles:

- Elementary Particle Physics - Fermions with finite non-zero temperature.
- Non-Hermitian RMT: Statistics of eigenvectors (Fyodorov, et al 2017).

Invertibility conditions: Assume the functions $\varphi_l(x) = \varphi(a_l, x)$ are analytic for real parameters $a_1,\ldots,a_N$. For a family $\{\pi_k\}_{k=0}^{\infty}$ of monic polynomials:

$$\pi_k(a) = \int_I dx\ x^k\ \varphi(a,x), \quad k = 0,1,\ldots \tag{1}$$

In addition, we assume that Eq. (1) is invertible, i.e. we write

$$z^k = \int_{I'} ds\ F(s,z)\ \pi_k(s), \quad k = 0,1,\ldots, \tag{2}$$

where I' is a certain contour in the complex plane and $F : I' \times \mathbb{C} \to \mathbb{C}$ is analytic.

Averages of characteristic polynomials

The correlation kernel of a polynomial ensemble can be obtained via

$$K_N(x,y) = \frac{1}{x-y} \operatorname*{Res}_{z=y} \left(\mathbb{E}_p \left[\frac{D_N(x)}{D_N(z)} \right] \right), \quad \text{where} \quad D_N(z) = \prod_{n=1}^{N} (z - x_n)$$

denotes the characteristic polynomial associated with $x_1, \ldots, x_N$.
For any invertible polynomial ensemble:

$$\mathbb{E}_p \left[\frac{\prod_{m=1}^{M} D_N(z_m)}{\prod_{l=1}^{L} D_N(y_l)} \right] \quad \Rightarrow \quad \begin{array}{l} \text{depends only on } M \text{ and } L \text{ as well as} \\ \\ \text{the quadrupel } (\varphi(a,x), I, F(s,z), I') \ . \end{array} \tag{3}$$

Eq. (3) allows simple large N analysis of averages of characteristic polynomials.