Multiplicative chaos of the Brownian loop soup

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Happy birthday, Yan!

Outline

1) Reminders about the *Brownian Loop Soup, GFF, Liouville measure*
2) Statements:
    - convergence of uniform measures on thick points
    - identification of the limit at the critical intensity parameter $\theta = 1/2$
    - Law of the loop soup near a thick point
3) Integrability

1) Reminders

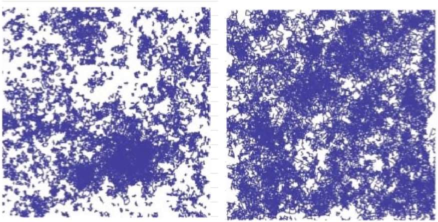
Brownian loop soup

Let $p_t(z,z) = 1/(4\pi t)$,
$\mathbb{P}_{z\to z,t}$ = law of Brownian bridge (speed 2) from z to z of duration $t > 0$.
Let

$$\mu^{\text{loop}}(d\ell) = \textit{loop measure} = \int_{\mathbb{C}} dz \int_0^\infty \frac{dt}{t} p_t(z,z)\mathbb{P}_{z\to z,t}(d\ell).$$

Definition

The Brownian loop soup with intensity $\theta > 0$ is the Poisson point process with intensity measure $\theta\mu^{\text{loop}}$.



Two simulations (by Camia–Gandolfi–Kleban) at $\theta = 0.5$ and $\theta = 2$.

Some important facts

Percolation transition (Sheffield–Werner)

$\theta \leq 1/2$: clusters of loops are bounded
$\theta > 1/2$: unbounded

At $\theta = 1/2$ the outer boundaries of clusters are CLE_4.
$\theta = 1/2$ also special for another reason:

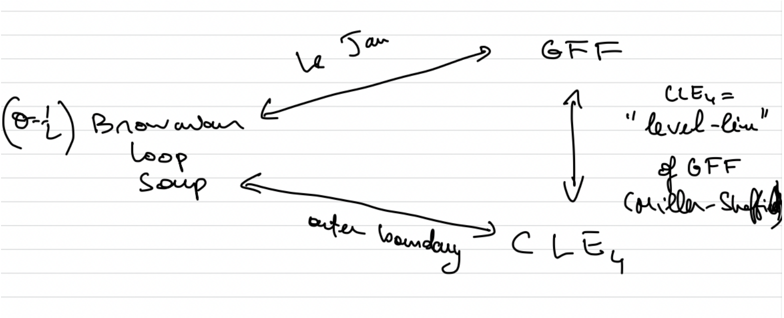
Le Jan's isomorphism

$$: L(\mathcal{L}_D^{\theta=1/2}) :=: \frac{1}{2} h_D^2 :$$

where:

- $L =$ local time,
- $\mathcal{L}_D^{\theta=1/2}$ = loop soup restricted to D,
- $h_D =$ Dirichlet GFF in D.
- $: \cdot :$ "Wick" regularisation

Canonical coupling (Qian–Werner)



All these couplings can be made to hold simultaneously!

GFF and Liouville measure

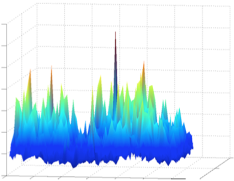
Let $h_D = GFF$ (= Gaussian field with covariance: = Green function).
For $\gamma \in (0,2)$ let μ_γ denote the *Liouville measure* of the GFF ($\approx$ *Gaussian multiplicative chaos* associated to h_D):

$$\mu_\gamma(dx) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \exp(\gamma h_\varepsilon(x)) dx.$$

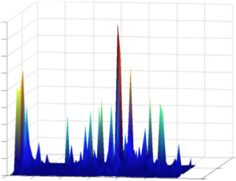
Theorem (Kahane, Robert–Vargas, Duplantier–Sheffield, B., Shamov).

The above limit exists in probability w.r.t. weak convergence, independent of choice of regularisation.
For $\gamma \in [0,2)$, nontrivial: $\mu_\gamma(D) \in (0,\infty)$, a.s.

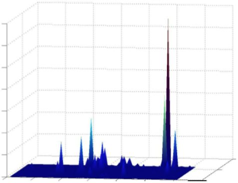
Simulations of approximate density:



$\gamma = 0.2$



$\gamma = 1$



$\gamma = 1.8$

Thick points

Definition

Let $\alpha > 0$. Call a point x thick if

$$\lim_{\varepsilon \to 0} \frac{h_\varepsilon(x)}{\log(1/\varepsilon)} = \alpha.$$

The set $\mathcal{T}_\alpha$ of α-thick points is exceptional:

$$\dim(\mathcal{T}_\alpha) = (2 - \alpha^2/2)_+$$

and is empty if $\alpha > 2$.
However it is *typical* for Liouville measure, e.g.:

$$\mu_\gamma(D \setminus \mathcal{T}_\gamma) = 0, a.s.$$

This talk

Consider Le Jan's coupling between GFF and Brownian loop soup.

Motivating question:

What connections between Liouville measure and the Brownian loop soup?

The field at a Liouville-typical point is exceptionally high.

Ex:

What is the geometry of Brownian loops at a Liouville-typical point?
How much does one loop contribute to Liouville measure?
What measure do the loops describe when the intensity $\theta \neq 1/2$?

2) Main results

Multiplicative chaos of the Brownian loop soup

For any $\theta > 0$ and $\gamma \in (0,2)$, we construct a measure which can be thought of as $\exp(\gamma\sqrt{L(\mathcal{L}_D^\theta)})$.
The *square root* is needed for log correlations.

Let $D_N = D \cap (1/N)\mathbb{Z}^2$. Let $\mathcal{L}_{D_N}^\theta$ denote *Random Walk Loop Soup*, with intensity $\theta\mu_{\text{loop}}^N$

$$\mu_{\text{loop}}^N = \frac{1}{N^2}\sum_{z\in D_N}\int_0^\infty \mathbb{P}_{D_N}^{z\to z;t}(d\ell)p_{D_N}(t,z,z)\frac{dt}{t}$$

where $p_{D_N}(t,x,y)$ is the transition probability kernel of random walk (cts.-time, rate N^2) killed upon leaving D_N.

Local time and thick points

Let $L_x(\ell) = N^2 \int_0^{T(\ell)} 1_{\{\ell(t)=x\}} dt$, and set

$$\mathcal{T}_N(\gamma) = \left\{ x \in D_N : \sum_{\ell \in \mathcal{L}^\theta_{D_N}} L_x(\ell) \geq \frac{1}{2\pi} a (\log N)^2 \right\}$$

where $a = \gamma^2/2$ *is the thickness parameter* in local time units.

Remark: By Dembo–Peres–Rosen–Zeitouni (2001, Acta) this is the correct scaling for a single Brownian trajectory. In particular they computed the "dimension" of $\mathcal{T}_N(\gamma)$...

Uniform measure on thick points

Let

$$\mathcal{M}^N(dx) = \frac{(\log N)^{1-\theta}}{N^{2-\gamma^2/2}} \sum_{x \in D_N} 1_{\{x \in \mathcal{T}_N(\gamma)\}}(dx).$$

Theorem 1. (Aïdékon, B., Jégo, Lupu)

We have the convergence in distribution:

$$\mathcal{M}_N \to \mathcal{M}$$

as $N \to \infty$. The measure $\mathcal{M}$ is measurable w.r.t. the limiting Brownian loop soup $\mathcal{L}_D^\theta$: it is the "*multiplicative chaos of Brownian loop soup*".

Compared to DPRZ (2001): gives dimension, but also log scaling and geometric structure of $\mathcal{T}_N(\gamma)$.
See *Jégo (2020)* for single Brownian trajectory (formally $\theta \to 0^+$).

Hyperbolic cosine of GFF

Theorem 2. (Aïdékon, B., Jégo, Lupu)

Take $\theta = 1/2$ (critical intensity of Loop Soup). Then

$$\mathcal{M}/c_0 = 2\cosh(\gamma h) = e^{\gamma h} + e^{-\gamma h},$$

$c_0 = 2^{1/2}(2\sqrt{2}e^{c_{EM}})^{\gamma^2/2}$. The multiplicative chaos of the loop soup is the cosh of GFF.

Informally, up to constant, $\mathcal{M} = \exp(\gamma\sqrt{L})$. By Le Jan's isomorphism, $\sqrt{L} = |h|$, and

$$e^{\gamma|h|} = 2\cosh(\gamma h)$$

since $|h| \gg 1$ is very large.

Q: link with sinh-Gordon QFT?
Q: law of mass à la Fyodorov–Bouchaud?

Girsanov

Sample $z \sim \mathcal{M}$. What is the conditional law of the loop soup?
How many loops? How thick?

Theorem 3 (Aïdékon, B., Jégo, Lupu).

Let $\theta > 0$. The marginal law of z is $z \sim CR(z,D)^{\gamma^2/2}$. Furthermore, conditional on z, law of $\mathcal{L}$: =

$$\mathcal{L}_D^\theta + \text{loops}_z$$

where loops_z: let $(a_1, a_2, \dots) \sim aPD(0,\theta)$ where $a = \gamma^2/2$,

$$\ell_i \sim PPP(a_i \mu_{z;D}^{\text{excursion}})$$

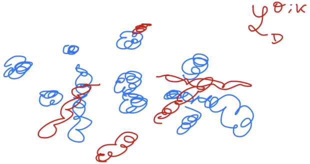
each loop is the concatenation of Itô distributed excursions.

In particular, a.s. z has infinite loop multiplicity, each loop has thickness a_i.

3) Integrability

As in B. (2017), we need a continuum construction.

Start with continuous Brownian loop soup $\mathcal{L}_D^\theta$.
Integrable discretisation: massive loop soup!
Keep each loop ℓ with probability $1 - \exp(-KT(\ell))$, then $K \to \infty$.



so

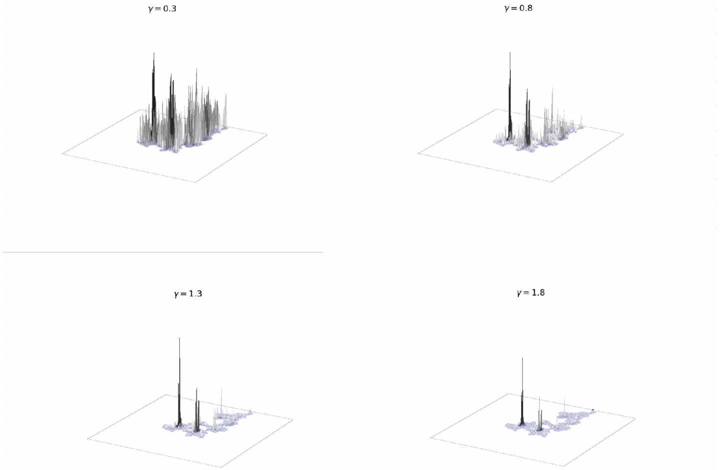
$$\mathcal{L}_D^\theta = \bigcup_{K>0} \mathcal{L}_D^{\theta;K}$$

Let

$$\mathcal{M}_K(dx) = \lim_{\varepsilon \to 0} \sqrt{\log(1/\varepsilon)}\varepsilon^{\gamma^2/2} e^{\gamma \sqrt{L_\varepsilon(x)}} dx.$$

Brownian multiplicative chaos. Exists by prior work of *Jégo (2020)*.

Brownian multiplicative chaos (Jégo 2020)



See also Aïdékon–Hu–Shi (2020).

Integrability of Massive Loop Soup Chaos

Claim:

$$\mathcal{M}(dx) = \lim_{K\to\infty} (\log K)^{-\theta} \mathcal{M}_K(dx)$$

Reveals integrable structure.

Proposition

$$\mathbb{E}[(\log K)^{-\theta} \mathcal{M}_K(dx)] = \frac{1}{a} CR(z, D)^a F(aC(x))$$

where $a = \gamma^2/2$,

$$F(u) = \theta \int_0^u e^{-t} {}_1F_1(\theta, 1, t) dt$$

and

$$C(x) = \int_0^\infty (1 - e^{-Kt}) p_t^D(x, x) dt.$$

When $\theta = 1$, $F(u) = u$ ...!

Q: $\theta = 1$ more integrable. Why?