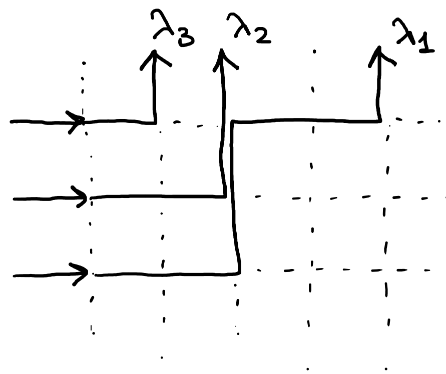
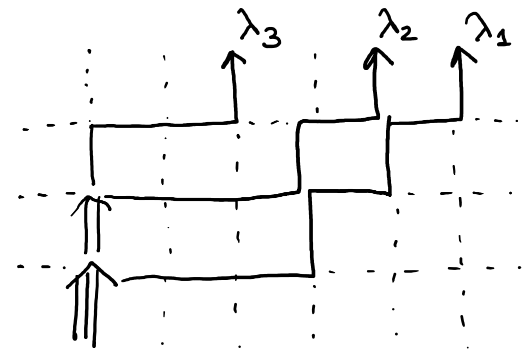
$\mathfrak{sl}(1|1)$ - vertex models: Boson-Fermion correspondence and determinantal point processes

after arXiv: 2109.06718 by Aggarwal-Borodin-Petrov-Wheeler

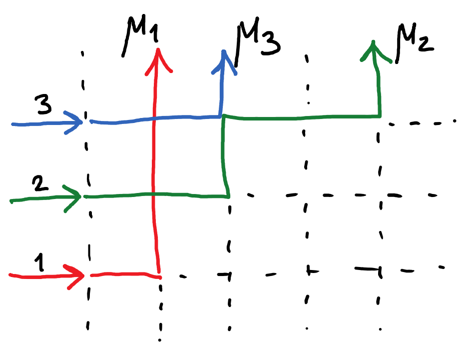


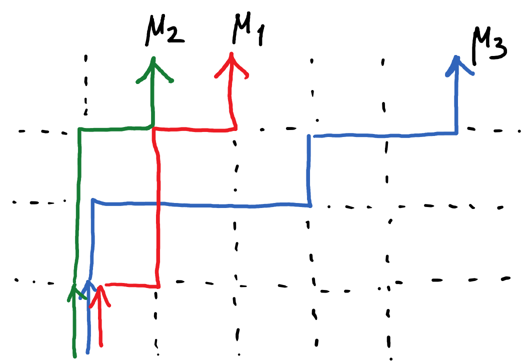


Fused $U_q(\widehat{sl(2)})$ weights

Symmetric Hall-Littlewood / q-Whittaker polys

ASEP, q-TASEP, directed polymers in (1+1)d

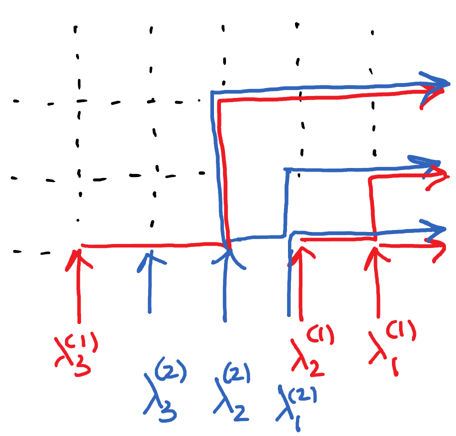


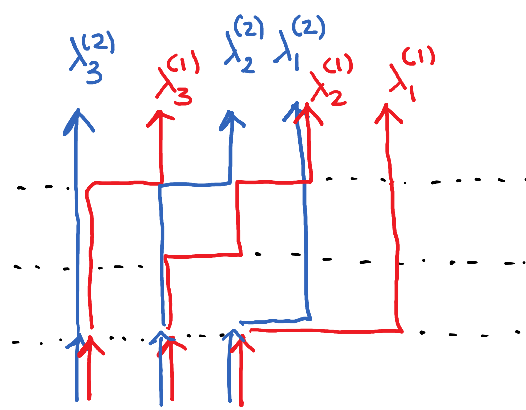


Fused $U_q(\widehat{sl(n)})$ weights

Nonsymmetric Hall-Littlewood & Macdonald polys

Colored (multi-species) exclusion processes





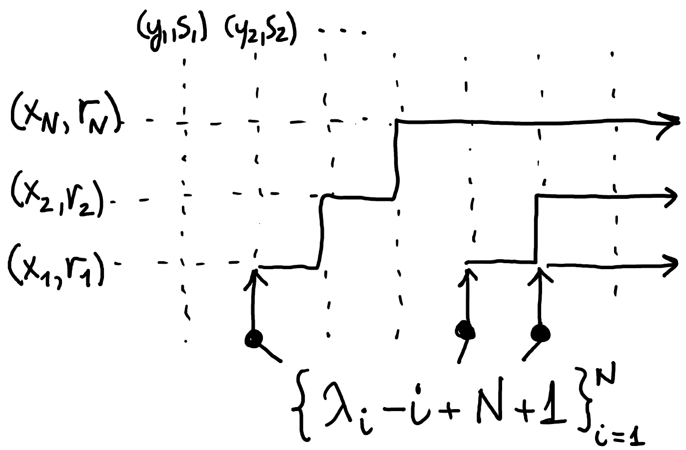
Fused $U_q(\widehat{sl(1|n)})$ weights

(Non)symmetric Macdonald & LLT polynomials

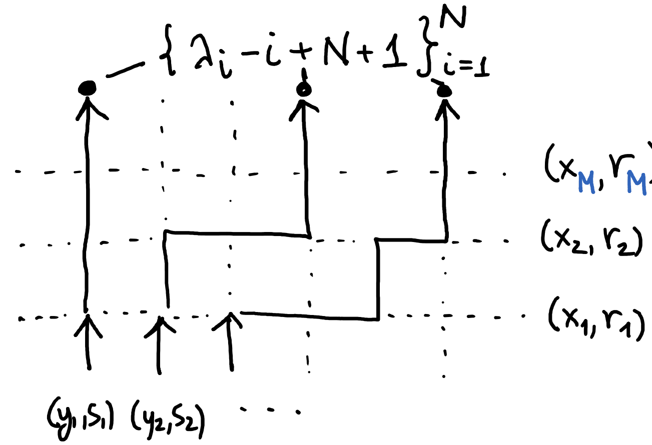
$n=1$: Schur measures, TASEP, last passage percolation

$n>1$: ???

The $\mathfrak{sl}(1|1)$ case. Symmetric functions

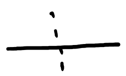


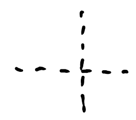
$F_\lambda(\vec{x};\vec{y};\vec{r};\vec{s})$

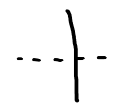


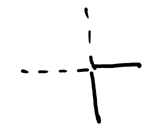
$G_\lambda(\vec{x},\vec{y};\vec{r};\vec{s})$

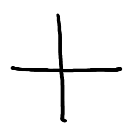
$\vec{x}$: row rapidities
$\vec{y}$: column rapidities
$\vec{r}$: row spin parameters
$\vec{s}$: column spin parameters

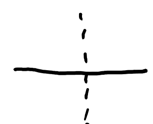
Weights are normalized so that  $= 1$

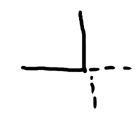
 $= 1$

 $= \dfrac{s^{-2}y - r^{-2}x}{s^{-2}y - x}$

 $= \dfrac{x(r^{-2}-1)}{s^{-2}y - x}$

 $= \dfrac{r^{-2}x - y}{s^{-2}y - x}$

 $= \dfrac{y - x}{s^{-2}y - x}$

 $= \dfrac{y(s^{-2}-1)}{s^{-2}y - x}$

Rem. 1 $s^2 = r^2 = q^{-1}$ is the un-fused $U_q(\widehat{\mathfrak{sl}(1,1)})$ case.
(L,M)-fusion gives $s^2 = q^{-L}$, $r^2 = q^{-M}$.
[Bazhanov - Shadrikov, 1987]

Rem. 2 The free-fermionic condition $a_1a_2 + b_1b_2 = c_1c_2$ holds.

Relation to Schur polynomials

Assume that the weights are invariant under horizontal shifts (column parameters are equal).

$$F_\lambda(\vec{x};y;\vec{r};s) = \prod_{i=1}^{N} \frac{(r_i^{-2}-1)x_i}{1-x_i} \cdot \prod_{1\le i<j\le N} \frac{r_i^{-2}x_i - x_j}{x_i - x_j} \; * \; \det\left[\left(\frac{1-s^2x_i}{s^2(1-x_i)}\right)^{\lambda_j+N-j}\right]_{i,j=1}^{N}$$

$$\frac{F_\lambda(\vec{x};y;\vec{r};s)}{F_{(0^N)}(\vec{x};y;\vec{r};s)} = s_\lambda\left(\left\{\frac{1-s^2x_i}{s^2(1-x_i)}\right\}_{i=1}^{N}\right)$$

Schur polynomial

[Brubaker-Bump-Friedberg, 2011]

$$G_\lambda(\vec{x};y;\vec{r};s) = \prod_{i=1}^{M}\left(\frac{1-s^2r_i^{-2}x_i}{1-s^2x_i}\right)^{N} \; * \; S_\lambda\left(\left\{\frac{s^2(1-x_j)}{1-s^2x_j}\right\}_{j=1}^{M} \;\middle|\; \left\{\frac{s^2(x_jr_j^{-2}-1)}{1-s^2r_j^{-2}x_j}\right\}_{j=1}^{M}\right)$$

supersymmetric Schur polynomial

$$\sum_\lambda S_\lambda(a|b)\,S_\lambda(t) = \prod_{i,j}\frac{1+t_ib_j}{1-t_ia_j}$$

Inhomogeneous analogs of the Cauchy summation$^{F\&G}$, alternant^{F}, Jacobi-Trudi^{G}, Sergeev-Pragacz^{G}, and torus orthogonality^{F} formulas are available.

Schur measures

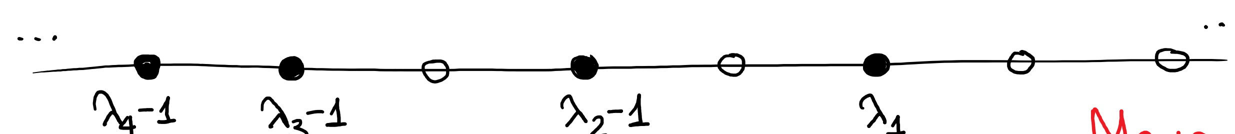
These are probability measures on partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq 0)$ given by

$$\mathrm{Prob}\{\lambda\} = \frac{s_\lambda(x)\, s_\lambda(y)}{Z}, \qquad Z = \sum_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

They define determinantal point processes:

$$\mathrm{Prob}\left\{ \{a_1, \ldots, a_n\} \subset \mathcal{J}(\lambda) \right\} = \det \left[K(a_i, a_j) \right]_{i,j=1}^{n}, \qquad n = 1, 2, \ldots$$

where $\mathcal{J}(\lambda) = \{\lambda_i - i + 1\}_{i \geq 1}$:



Maya diagram

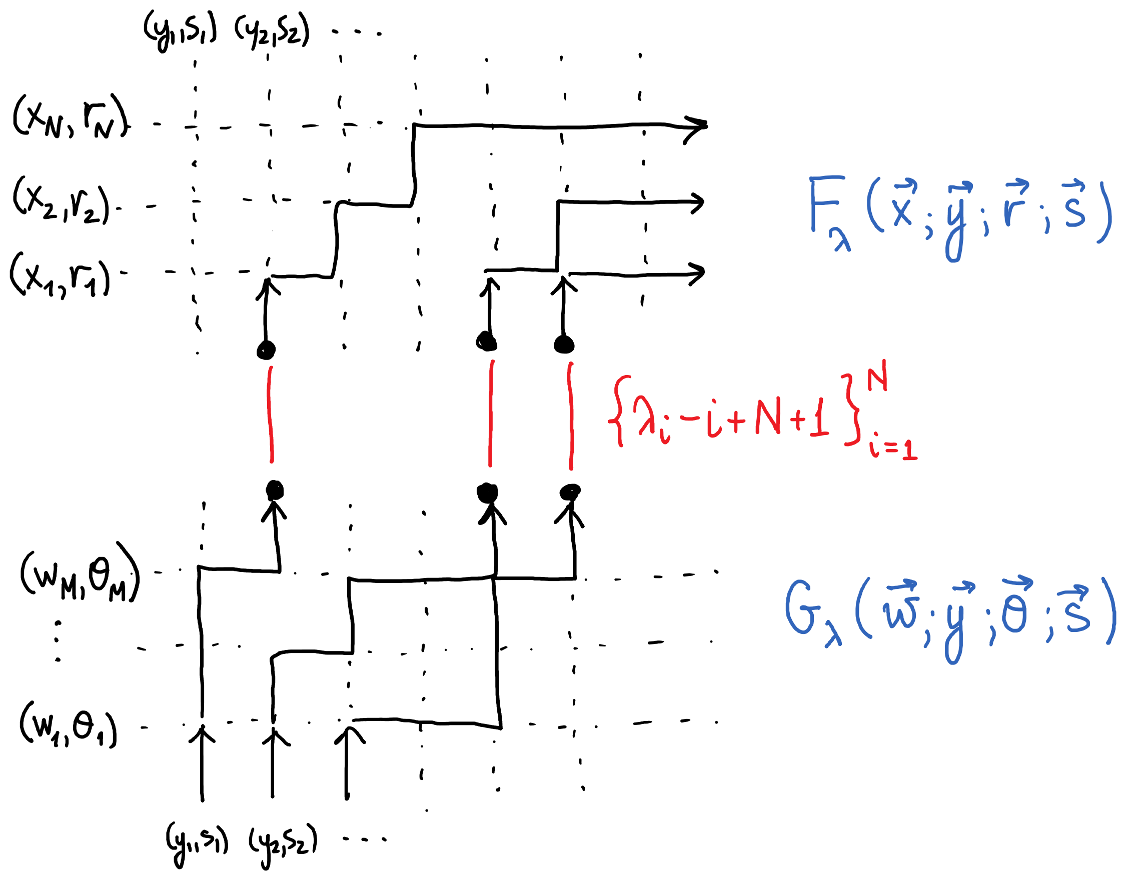
$$K(a,b) = \frac{1}{(2\pi i)^2} \oint\oint_{|u|>|v|} \frac{du\, dv}{u - v} \frac{v^{b-1}}{u^a} \frac{\Phi(u)}{\Phi(v)} \qquad \text{with} \qquad \Phi(z) = \prod_i \frac{1 - z^{-1} y_i}{1 - z x_i}$$

separating x/y poles

[Okounkov, 2000]

Many probabilistic applications.

Fully inhomogeneous Schur-like measures



$$\mathrm{Prob}\{\lambda\} = \frac{F_\lambda(\vec{x};\vec{y};\vec{r};\vec{s})\,G_\lambda(\vec{w};\vec{y};\vec{\theta};\vec{s})}{Z}$$

$$Z = \frac{\prod_{1\le i<j\le N}\left(\frac{x_i}{r_i^2}-x_j\right)\left(\frac{y_i}{s_i^2}-y_j\right)}{\prod_{i,j=1}^{N}(y_i-x_j)} \cdot \prod_{i,j}\frac{\left(x_i-\frac{w_j}{\theta_j^2}\right)}{(x_i-w_j)}$$

Cauchy-like identity

Determinantal correlation functions?

Two approaches: random matrix techniques and Fermionic Fock space.

Fermionic Fock space

Maya diagrams  $= e_J$, $\quad J \subset \mathbb{Z}$, $\quad |J \cap \mathbb{Z}_{\geq 0}| + |\mathbb{Z}_{<0} \setminus J| < \infty$

$$\mathcal{F} := \mathrm{Span}(e_J);$$

e_J's form an orthonormal basis of $\mathcal{F}$

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n \quad \text{with} \quad \mathcal{F}_n = \mathrm{Span}\big(e_J : \underbrace{|J \cap \mathbb{Z}_{\geq 0}| - |\mathbb{Z}_{<0} \setminus J|}_{\mathrm{Charge}(J)} = n\big)$$

$$\{J : \mathrm{Charge}(J) = 0\} \xleftrightarrow{\text{1-to-1}} \{\text{partitions } \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq 0)\}$$

$$J(\lambda) := \{\lambda_i - i + 1\}_{i \geq 1} \longleftrightarrow \lambda$$

Fermionic creation/annihilation operators ψ_j, ψ_j^* are defined by

$$\psi_j e_J = \begin{cases} (-1)^{|J \cap \mathbb{Z}_{>j}|} e_{J \cup \{j\}}, & j \notin J; \\ 0, & j \in J; \end{cases} \qquad \psi_j^* e_J = \begin{cases} (-1)^{|J \cap \mathbb{Z}_{>j}|} e_{J \setminus \{j\}}, & j \in J; \\ 0, & j \notin J. \end{cases}$$

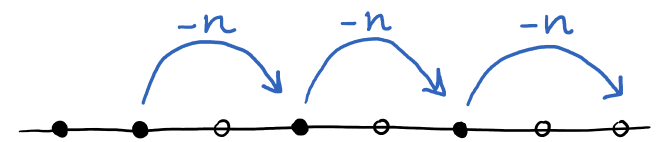
$$[\psi_k, \psi_k^*]_+ = \mathbb{1} \qquad [\psi_k, \psi_\ell]_+ = [\psi_k, \psi_\ell^*]_+ = [\psi_k^*, \psi_\ell^*]_+ = 0 \qquad \psi_j \psi_j^* e_J = \mathbb{1}_{j \in J} e_J.$$

$$\psi_k^2 = (\psi^*)^2 = 0 \qquad\qquad k \neq \ell$$

Boson-Fermion correspondence

Bosonic operators

$$\alpha_n = \sum_{k\in\mathbb{Z}} \psi_{k-n}\psi_k^*, \quad n=\pm1,\pm2,\dots,$$



tries to move each particle by $-n$.

satisfy $[\alpha_n, \alpha_m] = n\cdot\delta_{m+n,0}$. Define

$$\Gamma_\pm(\vec{s}) := \exp\left(\sum_{n=1}^{\infty} s_n\,\alpha_{\pm n}\right).$$

Then

$$\Gamma_-(\vec{t})\, e_{J(\mu)} = \sum_\lambda s_{\lambda/\mu}(\vec{x})\, e_{J(\lambda)} \quad \text{with} \quad t_k = \frac{1}{k}\sum_i x_i^k$$

$\Gamma_-(\{z\})$ "adds variable z" to Schur poly's

$\{z\} = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\right).$

$$\left.\begin{aligned}\sum_{n\in\mathbb{Z}} \psi_n z^n &= z^{\text{Charge}}\, R\, \Gamma_-(\{z\})\,\Gamma_+(-\{z^{-1}\})\\ \sum_{n\in\mathbb{Z}} \psi_n^* z^n &= R^{-1} z^{-\text{Charge}}\, \Gamma_-(-\{z\})\,\Gamma_+(\{z^{-1}\})\end{aligned}\right\}$$

Expression of fermionic operator through bosonic ones

with R being the shift by 1 operator.

[Date-Jimbo-Kashiwara-Miwa, 1981].

Determinantal structure of Schur measures

$$\frac{1}{Z}\sum_{\lambda:\{a_1,\dots,a_n\}\subset \mathfrak{J}(\lambda)} s_\lambda(x)\, s_\lambda(y) = \frac{1}{Z}\left(e_{\mathbb{Z}_{\le 0}},\ \Gamma_+^{(y)}\cdot \psi_{a_1}\psi_{a_1}^* \cdots \psi_{a_n}\psi_{a_n}^* \cdot \Gamma_-^{(x)} e_{\mathbb{Z}_{\le 0}}\right) =$$

$$\sum_\lambda s_\lambda(x)\, e_{\mathfrak{J}(\lambda)} = \Gamma_-^{(x)} e_{\mathbb{Z}_{\le 0}}$$

$$\psi_j \psi_j^* e_{\mathfrak{J}} = \mathbb{1}_{j\in \mathfrak{J}}\, e_{\mathfrak{J}}\,.$$

Wick's lemma

$$= \det\Big[\underbrace{\left(e_{\mathbb{Z}_{\le 0}},\ \Gamma_+^{(y)}\left(\Gamma_-^{(x)}\right)^{-1} \psi_{a_i}\psi_{a_j}^*\ \Gamma_-^{(x)}\left(\Gamma_+^{(y)}\right)^{-1} e_{\mathbb{Z}_{\le 0}}\right)}_{K(a_i,a_j)}\Big]_{i,j=1}^n$$

$\sum_{a,b\in\mathbb{Z}} K(a,b)\, z^a w^b$ is not hard to compute using the commutation relations

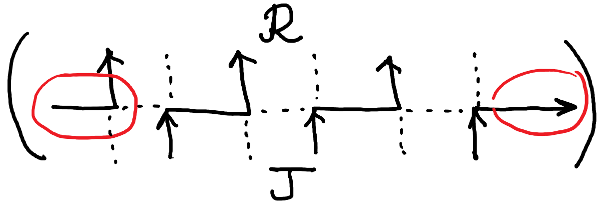
$$\Gamma_+(\vec{s})\,\Gamma_-(\vec{t}) = \underbrace{e^{\sum_{n\ge 1} n s_n t_n}}_{Z} \cdot \Gamma_-(\vec{t})\,\Gamma_+(\vec{s})$$

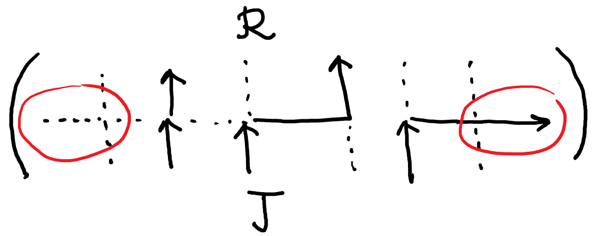
$$\Gamma_\pm^{(x)} \cdot \sum_{n\in\mathbb{Z}} \psi_n z^n = \prod_i \frac{1}{1 - x_i\cdot z^{\pm 1}} \sum_{n\in\mathbb{Z}} \psi_n z^n \cdot \Gamma_\pm^{(x)}$$

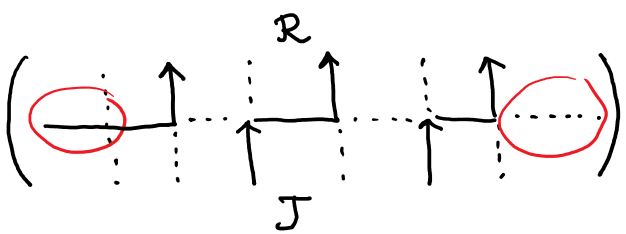
$$\Gamma_\pm^{(x)} \cdot \sum_{n\in\mathbb{Z}} \psi_n^* z^n = \prod_i \left(1 - x_i z^{\pm 1}\right) \cdot \sum_{n\in\mathbb{Z}} \psi_n^* z^n \cdot \Gamma_\pm^{(x)}$$

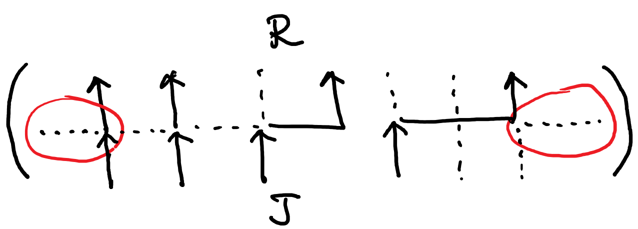
[Okounkov, 2000]

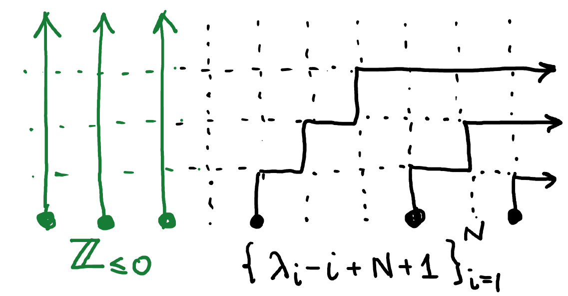
Transfer matrices (ABA operators)

$Ae_J = \sum_R \mathrm{Weight}\Big($  $\Big)\, e_R$

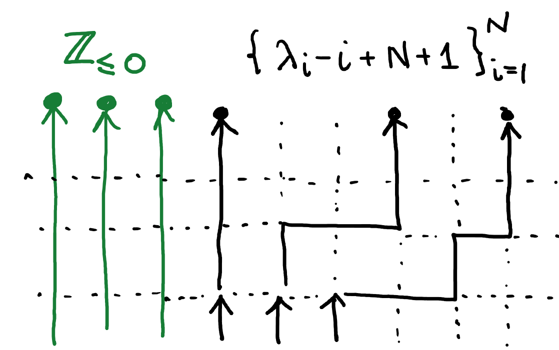
$Be_J = \sum_R \mathrm{Weight}\Big($  $\Big)\, e_R$

$Ce_J = \sum_R \mathrm{Weight}\Big($  $\Big)\, e_R$

$De_J = \sum_R \mathrm{Weight}\Big($  $\Big)\, e_R$



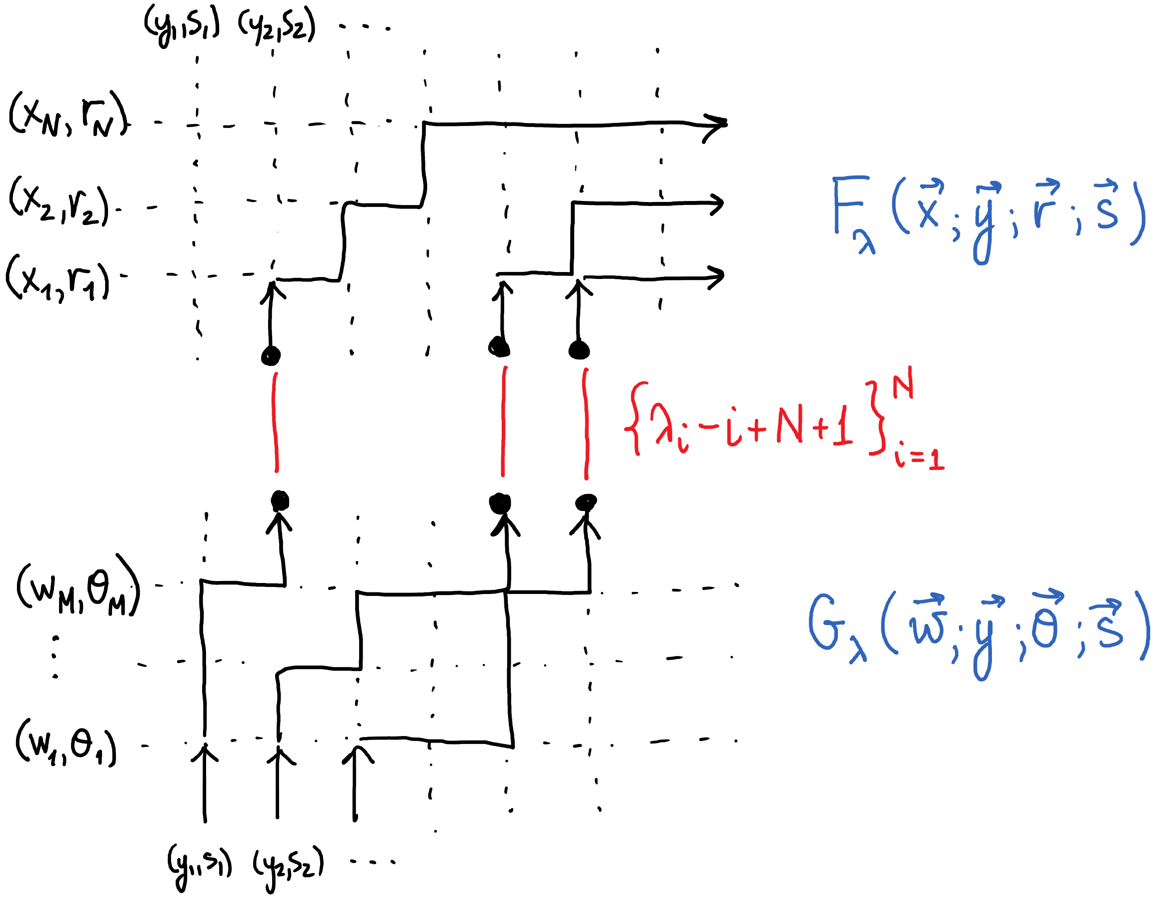
$$F_\lambda \sim \left(e_{\mathbb{Z}_{\le 0}},\ B\cdot B\cdot B\cdot e_{J(\lambda)+N}\right)$$



$$G_\lambda \sim \left(e_{J(\lambda)+N},\ D\cdot D\cdot D\cdot e_{\mathbb{Z}_{\ge N}}\right)$$

For infinite domains the operators need to be properly normalized.
The operators A, B, C, D depend on the row rapidity and spin parameter (and ∞-many column par's)

Correlation functions from transfer matrices



$$\mathrm{Prob}\left\{ \{a_1,\dots,a_n\} \subset \{\lambda_i - i + N + 1\}_{i=1}^{N} \right\} =$$

$$= \frac{1}{Z}\Big(e_{\mathbb{Z}_{\leq 0}},\ \underbrace{B\cdot B\cdot B}_{\substack{\text{Total of } N,\\ \text{arguments } (x_1,r_1),\dots,(x_N,r_N)}}\ \psi_{a_1}\psi^*_{a_1}\cdots\psi_{a_n}\psi^*_{a_n}\ \underbrace{D\cdot D\cdot D}_{\substack{\text{Total of } M,\\ \text{arguments } (w_1,\theta_1),\dots,(w_M,\theta_M).}}\ e_{\mathbb{Z}_{\leq N}} \Big)$$

How to evaluate?

Two simplifications of Yang-Baxter commutation relations

On the whole of $\mathbb{Z}$

(under convergence conditions)

$$B(x_1,r_1)D(x_2,r_2) = \frac{r_2^{-2}x_2 - x_1}{x_2 - x_1} D(x_2,r_2)B(x_1,r_1)$$

$$C(x_1,r_1)D(x_2,r_2) = \frac{r_1^{-2}x_1 - x_2}{r_1^{-2}x_1 - r_2^{-2}x_2} D(x_2,r_2)C(x_1,r_1)$$

$$B(x_1,r_1)C(x_2,r_2) = \frac{r_2^{-2}x_2 - r_1^{-2}x_1}{x_2 - x_1} C(x_2,r_2)B(x_1,r_1)$$

For special spin parameters

$$B(x,t)B\left(z,\sqrt{\tfrac{z}{x}}\right) = 0$$

$$C\left(x,\sqrt{\tfrac{x}{z}}\right)C(z,t) = 0$$

$$B\left(z,\sqrt{\tfrac{z}{x}}\right)D\left(x,\sqrt{\tfrac{x}{z}}\right) = -D\left(z,\sqrt{\tfrac{z}{x}}\right)B\left(x,\sqrt{\tfrac{x}{z}}\right)$$

$$D\left(z,\sqrt{\tfrac{z}{x}}\right)C\left(x,\sqrt{\tfrac{x}{z}}\right) = C\left(z,\sqrt{\tfrac{z}{x}}\right)D\left(x,\sqrt{\tfrac{x}{z}}\right)$$

$$D\left(x,\sqrt{\tfrac{x}{z}}\right)A\left(z,\sqrt{\tfrac{z}{x}}\right) - C\left(x,\sqrt{\tfrac{x}{z}}\right)B\left(z,\sqrt{\tfrac{z}{x}}\right) =$$

$$= A\left(z,\sqrt{\tfrac{z}{x}}\right)D\left(x,\sqrt{\tfrac{x}{z}}\right) + C\left(z,\sqrt{\tfrac{z}{x}}\right)B\left(x,\sqrt{\tfrac{x}{z}}\right)$$

To compare: On a finite segment with generic parameters the YBE implies

$$D(x_2,r_2)B(x_1,r_1) = \frac{r_1^{-2}x_1 - x_2}{r_1^{-2}x_1 - r_2^{-2}x_2}B(x_1,r_1)D(x_2,r_2) + \frac{(1-r_1^{-2})x_1}{r_1^{-2}x_1 - r_2^{-2}x_2}B(x_2,r_2)D(x_1,r_1);$$

$$\frac{x_1(r_1^{-2}-1)}{r_2^{-2}x_2 - x_1}D(x_2,r_2)A(x_1,r_1) + \frac{r_2^{-2}x_2 - r_1^{-2}x_1}{r_2^{-2}x_2 - x_1}C(x_2,r_2)B(x_1,r_1) = \frac{x_1(r_1^{-2}-1)}{r_2^{-2}x_2 - x_1}D(x_1,r_1)A(x_2,r_2) + \frac{x_2 - x_1}{r_2^{-2}x_2 - x_1}B(x_1,r_1)C(x_2,r_2);$$

Corollaries of commutation relations. "Bosons" and Fermions

Define

$$\Psi(x,z) := D\left(x,\sqrt{\tfrac{x}{z}}\right) C\left(z,\sqrt{\tfrac{z}{x}}\right)(-1)^{\text{charge}} \qquad \Psi^*(x,z) := D\left(x,\sqrt{\tfrac{x}{z}}\right) B\left(z,\sqrt{\tfrac{z}{x}}\right).$$

Theorem

$$\boxed{\Psi(x,z) = \sum_{j\in\mathbb{Z}} \varphi_j(x,z)\,\underbrace{\psi_j}_{\text{creation}}, \qquad \Psi^*(x,z) = \sum_{j\in\mathbb{Z}} \varphi^*_j(x,z)\,\underbrace{\psi^*_j}_{\text{annihilation}}}$$

with

$$\varphi_j(x,z) = \begin{cases} \dfrac{y_j(1-s_j^{-2})}{x-s_j^{-2}y_j} \cdot \displaystyle\prod_{k=1}^{j-1} \dfrac{x-y_k}{x-s_k^{-2}y_k}, & j>0; \\[2ex] \dfrac{y_j(1-s_j^{-2})}{x-s_j^{-2}y_j} \displaystyle\prod_{k=j}^{0} \dfrac{x-s_k^{-2}y_k}{x-y_k}, & j\le 0 \end{cases}$$

$$\varphi^*_j(x,z) = \begin{cases} \dfrac{z-x}{z-y_j} \displaystyle\prod_{k=1}^{j-1} \dfrac{z-s_k^{-2}y_k}{z-y_k}, & j>0; \\[2ex] \dfrac{z-x}{z-y_j} \displaystyle\prod_{k=j}^{0} \dfrac{z-y_k}{z-s_k^{-2}y_k}, & j\le 0. \end{cases}$$

inhomogeneous analogs of powers

Compare in the Schur case: $\displaystyle\sum_{n\in\mathbb{Z}} \psi_n z^n = z^{\text{Charge}} R\, \Gamma_-(\{z\})\Gamma_+(-\{z^{-1}\})$, $\displaystyle\sum_{n\in\mathbb{Z}} \psi^*_n z^n = R^{-1} z^{-\text{Charge}}\, \Gamma_-(-\{z\})\Gamma_+(\{z^{-1}\})$.

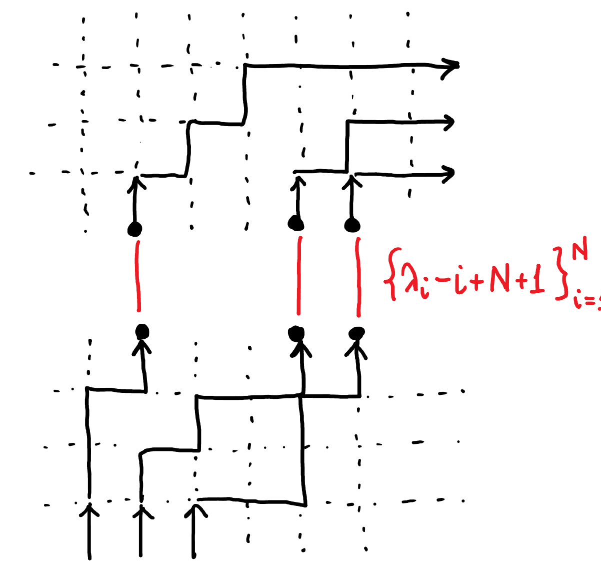
Also,

$$B(x,r)\Psi(u,\xi) = -\frac{u-r^{-2}x}{u-x}\Psi(u,\xi)B(x,r) \qquad \Psi(u,\xi)D(w,\theta) = \frac{u-w}{u-\theta^{-2}w} D(w,\theta)\Psi(u,\xi)$$

$$B(x,r)\Psi^*(\varkappa,v) = \frac{v-x}{u-\theta^{-2}v}\Psi^*(\varkappa,v)B(x,r) \qquad \Psi^*(\varkappa,v)D(w,\theta) = \frac{\theta^{-2}w-v}{w-v} D(w,\theta)\Psi^*(\varkappa,v)$$

$$\Psi^*(\varkappa,v)\Psi(u,\xi) = -\Psi(u,\xi)\Psi^*(\varkappa,v)$$

Schematic computation of the correlation functions



$$\left(e_{\mathbb{Z}_{\le 0}},\, B\cdot B\cdot B\, \psi_{a_1}\psi^*_{a_1}\cdots\psi_{a_n}\psi^*_{a_n}\, D\cdot D\cdot D\, e_{\mathbb{Z}_{\le N}}\right) =$$

coefficient of $\left(e_{\mathbb{Z}_{\le 0}},\, B\cdot B\cdot B\cdot \Psi_1\Psi_1^*\cdots\Psi_n\Psi_n^*\cdot D\cdot D\cdot D\, e_{\mathbb{Z}_{\le N}}\right) \sim$

$$\sim \left(e_{\mathbb{Z}_{\le 0}},\, DDD\, \Psi_1\Psi_1^*\cdots\Psi_n\Psi_n^*\, BBB\, e_{\mathbb{Z}_{\le N}}\right) \sim$$

$$\sim \left(e_{\mathbb{Z}_{\le 0}},\, \hat{D}\hat{D}\hat{D}\, \Psi_1\Psi_1^*\cdots\Psi_n\Psi_n^*\, BBB\, e_{\mathbb{Z}_{\le N}}\right) \sim$$

$$\sim \left(e_{\mathbb{Z}_{\le 0}},\, \Psi_1\Psi_1^*\cdots\Psi_n\Psi_n^*\, (\hat{D}B)(\hat{D}B)(\hat{D}B)\, e_{\mathbb{Z}_{\le N}}\right) \sim$$

$$\sim \left(e_{\mathbb{Z}_{\le 0}},\, \Psi_1\Psi_1^*\cdots\Psi_n\Psi_n^*\, \tilde{\Psi}_1^*\cdots\tilde{\Psi}_N^*\, \overbrace{\hat{\Psi}_1\cdots\hat{\Psi}_N\, e_{\mathbb{Z}_{\le 0}}}\right)$$

and $\left(e_{\mathbb{Z}_{\le 0}},\, \Psi(u_1,\zeta_1)\Psi^*(\varkappa_1,v_1)\cdots\Psi(u_m,\zeta_m)\Psi^*(\varkappa_m,v_m)\, e_{\mathbb{Z}_{\le 0}}\right) \stackrel{\text{Wick}}{=} \det\left[\dfrac{v_j-\varkappa_j}{u_i-v_j}\right]_{i,j=1}^{m}$

Here "$\sim$" means removing factorized expressions.

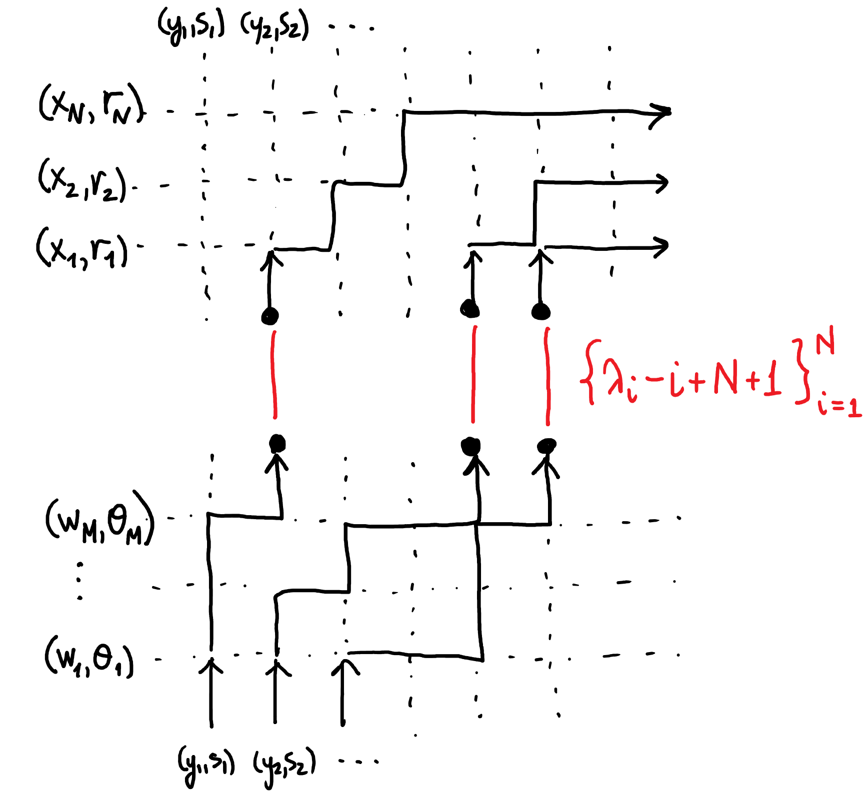
Final determinantal formula

$$\mathrm{Prob}\left\{ \{a_1,\dots,a_n\} \subset \{\lambda_i - i + N + 1\}_{i=1}^{N} \right\} = \det\left[K(a_i,a_j)\right]_{i,j=1}^{n}$$

$$K(a,b) = \frac{1}{(2\pi i)^2} \underset{\substack{\text{around}\\ y,\ \theta^{-2}w}}{\oint} du \underset{\substack{\text{around}\\ y,\ w,\ u}}{\oint} dv \cdot \prod_{k=1}^{N} \frac{(u-y_k)(v-x_k)}{(u-x_k)(v-y_k)} \cdot \prod_{j=1}^{M} \frac{(u-w_j)(v-\theta_j^{-2}w_j)}{(v-w_j)(u-\theta_j^{-2}w_j)} \ *$$

$$* \underbrace{\frac{1}{u-v}}_{\text{interaction between } u \text{ and } v} \cdot \frac{y_a(1-s_a^{-2})}{v-s_a^{-2}y_a} \cdot \frac{1}{u-y_b} \cdot \prod_{i=1}^{a-1} \frac{v-y_i}{v-s_i^{-2}y_i} \cdot \prod_{j=1}^{b-1} \frac{u-s_j^{-2}y_j}{u-y_j}$$

become powers in the homogeneous situation.



Summary

- Fused $U_q(\widehat{sl(1|1)})$ vertex models allow to construct inhomogeneous Schur-like functions and Schur-like measures/processes.

- Algebraic Bethe Ansatz operators with special spin parameters provide an analog of Boson-Fermion correspondence.

- Wick's theorem leads to determinantal correlation functions.

- Bulk limit has been considered, more probabilistic applications hopefully to come.