SUPERSYMMETRIC CLUSTER EXPANSIONS IN RANDOM MATRIX MODELS FOR QUANTUM DIFFUSION

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Cluster expansion

→ powerful tool in statistical mechanics to study thermodynamic limit

→ many versions   here : expansions around a gaussian measure

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Plan

- a cluster expansion in stat. mechanics
- SUSY cluster expansion
- application to RM in quantum diff (in dimension $d>1$)

standard example

$\Lambda = [-L, L]^d \cap \mathbb{Z}^d$ cube

$\varphi_\Lambda : \Lambda \longrightarrow \mathbb{C}$ spin config (or $\varphi_j \in \mathbb{R}^2$)

$\qquad j \longrightarrow \varphi_j$

unnormalized Gibbs measure $\langle \mathcal{O} \rangle_{\Lambda, C} := \int_{(\mathbb{R}^2)^\Lambda} d\mu_C(\varphi, \bar\varphi)\, e^{-V(\bar\varphi\varphi)}\, \mathcal{O}$

- $d\mu_C(\varphi, \bar\varphi) = \dfrac{e^{-\varphi^* C^{-1} \varphi}}{\det C} \prod_{j \in \Lambda} \dfrac{d\bar\varphi_j\, d\varphi_j}{2\pi}$ normalized gauss measure with average 0 and covariance C

- $e^{-V(\bar\varphi\varphi)} = \prod_{j \in \Lambda} e^{-V(\bar\varphi_j \varphi_j)}$ local interaction (ex $V(x) = \lambda x^2$)

- $\mathcal{O}$ local observable (ex $\bar\varphi_{j_0}\, \varphi_{k_0}$)

- in general $C \in \mathbb{C}^{\Lambda \times \Lambda}$ $\quad d\nu_C(\varphi, \bar\varphi)$ well def if $\mathrm{Re}\, C^{-1} > 0$

  here set $C^{-1} = -\Delta + m^2$ $\quad \Delta$ lattice Laplacian, $m > 0$

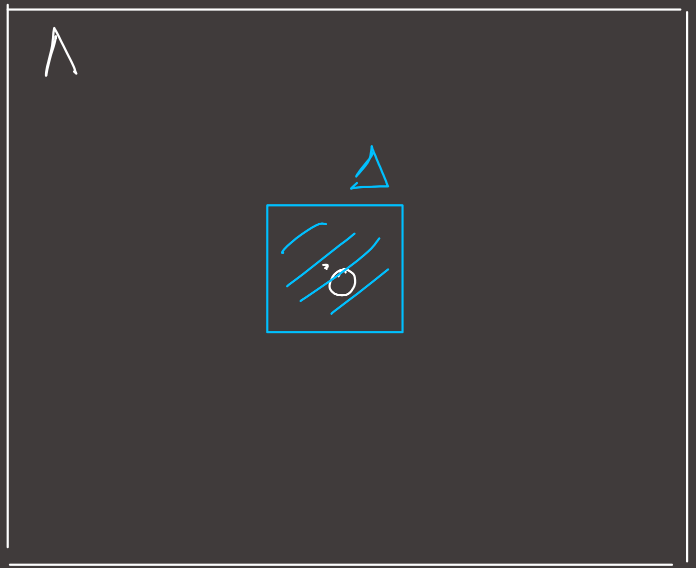
- study $\lim_{\Lambda \nearrow \mathbb{Z}^d} \dfrac{\langle \sigma \rangle_{\Lambda, C}}{\langle 1 \rangle_{\Lambda, C}}$ $\leftarrow$ normalization $\quad$ ex: set $\sigma = \bar\varphi_0 \varphi_0$

cluster exp $\quad \dfrac{\langle \bar\varphi_0 \varphi_0 \rangle_{\Lambda, C}}{\langle 1 \rangle_{\Lambda, C}} = \displaystyle\sum_{\substack{X \subset \Lambda \\ 0 \subset X}} I_X$ $\leftarrow$ abs. convergent sum unif in Λ

(remains valid as $\Lambda \nearrow \mathbb{Z}^d$)

how to construct the cluster exp?

heuristics: $C_{ij} = (-\Delta + m^2)^{-1}_{ij} \simeq e^{-|i-j|m} \implies d\nu_C$ approx factored at distance $> \frac{1}{m}$

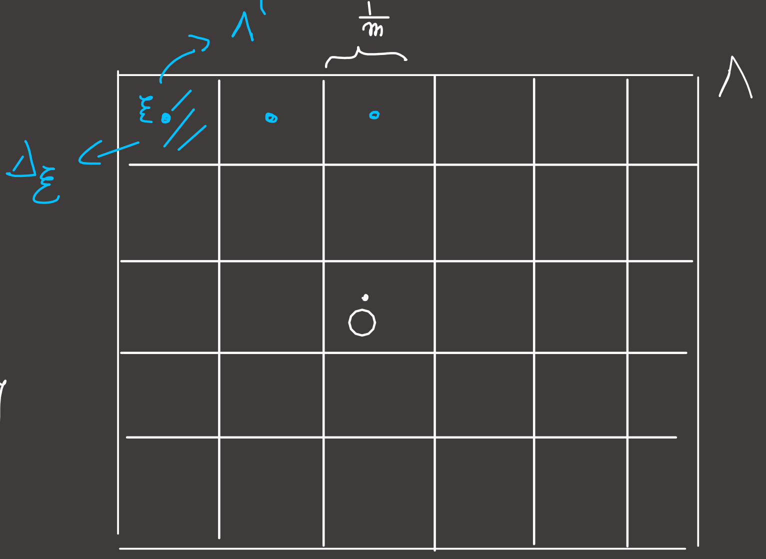


$$\implies \text{expect} \quad \frac{\langle \bar{\varphi}_0 \varphi_0 \rangle_{\Lambda, C}}{\langle 1 \rangle_{\Lambda, C}} \simeq \frac{\langle \bar{\varphi}_0 \varphi_0 \rangle_{\Delta, C_{\Delta\Delta}}}{\langle 1 \rangle_{\Delta, C_{\Delta\Delta}}}$$

Δ = cube of side $\frac{1}{m}$ containing 0

make this rigorous

→ partion Λ into cubes of side $\frac{1}{m}$

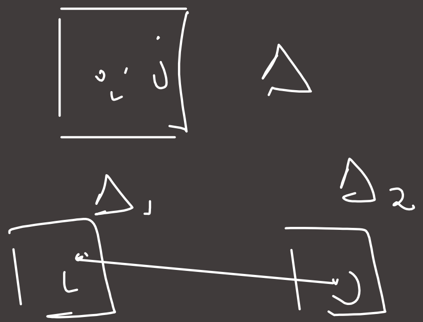


Λ' = lattice of centers

$$\Lambda = \bigcup_{\xi \in \Lambda'} \Delta_\xi \qquad B = \{ \Delta_\xi \mid \xi \in \Lambda' \}$$

→ interpolate the covariance: set $s \in [0,1]$

$$C(s)_{ij} := \begin{cases} C_{ij} & \text{if } ij \in \Delta \text{ for some } \Delta \in B \\ s\, C_{ij} & \text{if } i \in \Delta_1,\ j \in \Delta_2 \quad \Delta_1 \neq \Delta_2 \end{cases}$$



• limit cases

$$C(1) = C \longrightarrow \text{initial integral}$$

$$C(0) = \sum_{\Delta \in \mathcal{B}} C_{\Delta\Delta} \longleftarrow \text{block diag}$$

$$C_{ij}(s) := \begin{cases} C_{ij} & \text{if } i,j \in \Delta \quad \text{for some } \Delta \in \mathcal{B} \\ s\, C_{ij} & \text{if } \begin{array}{l} i \in \Delta_1 \\ j \in \Delta_2 \end{array} \quad \Delta_1 \neq \Delta_2 \end{cases}$$

$$\Rightarrow \frac{\int d\mu_{C(0)}\, e^{-V} \bar{\varphi}_0 \varphi_0}{\int d\mu_{C(0)}\, e^{-V}} = \frac{\int d\mu_{C_{\Delta_0\Delta_0}}\, e^{-V_{\Delta_0}} \bar{\varphi}_0 \varphi_0 \cdot \cancel{\prod_{\Delta \neq \Delta_0} \int d\mu_{C_{\Delta\Delta}}\, e^{-V_\Delta}}}{\int d\mu_{C_{\Delta_0\Delta_0}}\, e^{-V_{\Delta_0}} \cdot \cancel{\prod_{\Delta \neq \Delta_0} \int d\mu_{C_{\Delta\Delta}}\, e^{-V_\Delta}}} = \frac{\langle \mathcal{O} \rangle_{\Delta_0}}{\langle 1 \rangle_{\Delta_0}}$$

• interpolated covariance

$$C(s) = \underbrace{s}_{>0}\, C + (1-s) \overbrace{\sum_{\Delta \in \mathcal{B}} \underbrace{C_{\Delta\Delta}}_{>0}}^{C(0)} \longrightarrow C(s)^{-1} > 0 \Rightarrow d\mu_{C(s)} \text{ well defined}$$

set $I(s) := \int d\mu_{C(s)}\, e^{-V}$ (same arg with $e^{-V}\sigma$)

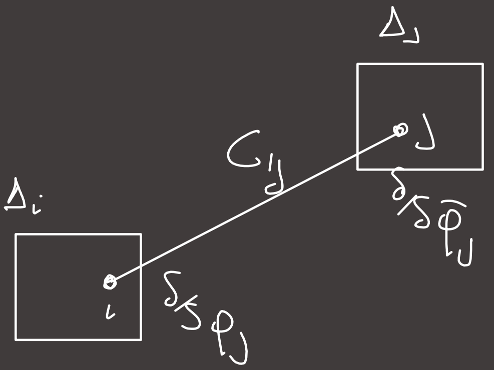
first order Taylor exp

$$I(1) = I(0) + \int_0^1 ds\, \frac{d}{ds} I(s)$$

↑ init. integral ↑ factored int.

Key formula (integration by parts)

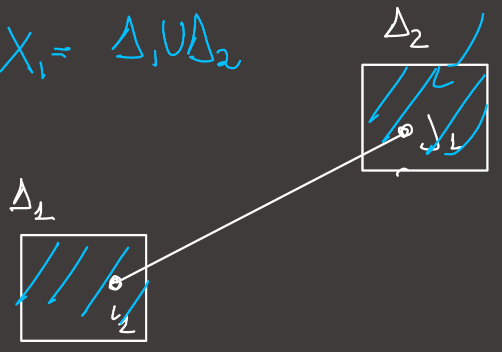
$$\boxed{\frac{d}{ds}\int d\mu_{C(s)}\, e^{-V} = \sum_{i,j\in\Lambda} \frac{d}{ds} C(s)_{ij} \int d\mu_{C(s)}\, \frac{\delta}{\delta\varphi_i}\frac{\delta}{\delta\varphi_j}\, e^{-V}}$$



next step

$X_1 = \Delta_1 \cup \Delta_2$

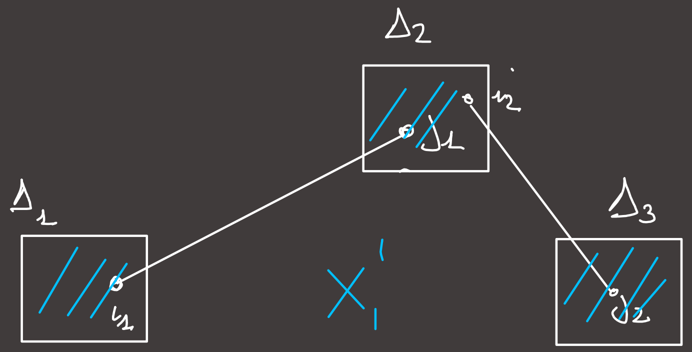
connected region

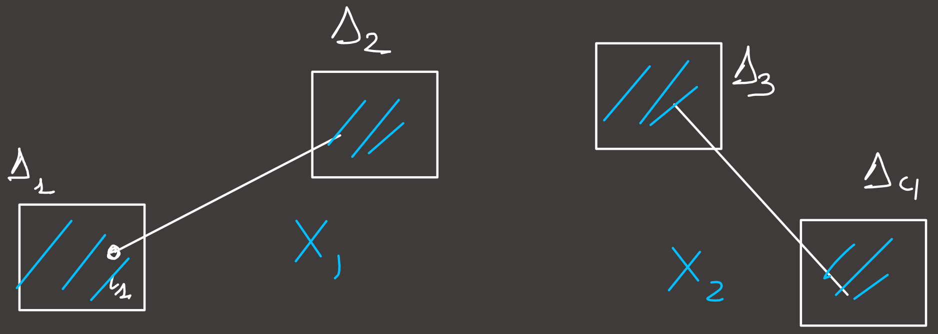


$$C(s_2, s_1)_{ij} = \begin{cases} C(s_1)_{ij} & \text{if } \; i,j \in X_1 \; \text{ or } \; i,j \in X_1^c \cap \Delta \\ s_2\, C(s_1)_{ij} & \text{if } \; i \in X_1, \; j \in X_1^c \\ & \text{or } \; i,j \in X_1^c \quad i \in \Delta \quad j \in \Delta' \\ & \qquad \Delta \neq \Delta' \end{cases}$$

$\longrightarrow$ first order Taylor exp in s_2

$\Longrightarrow$





repeat until all cubes are used. → (Brydges - Kennedy - Taylor, Abdesselam - Rivasseau)

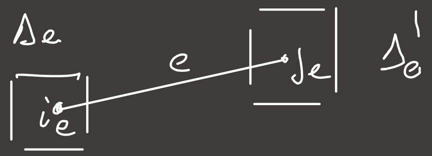
$$\boxed{\int_{(\mathbb{R}^2)^\Lambda} d\mu_C \, e^{-V} = \sum_{n \ge 1} \frac{1}{n!} \sum_{\substack{Y_1 \dots Y_n \subset \Lambda \\ Y_i \cap Y_j = \emptyset \\ \cup_j Y_j = \Lambda}} \prod_{j=1}^{n} A(Y_j)}$$

→ Y_j = set of cubes $\forall j$

$$A(Y) = \sum_{T \in \mathcal{T}(Y')} \sum_{\{i_e \in \Delta'_e,\ j_e \in \Delta_e\}_{e \in T}} \prod_{e \in T} C_{i_e j_e} \prod_{e \in T} \int_0^1 ds_e \; P_T(s) \int_{(\mathbb{R}^2)^{Y}} d\mu_{C(s)} \left(\prod_{e \in T} \frac{\delta}{\delta \varphi_{i_e}} \frac{\delta}{\delta \varphi_{j_e}} \right) e^{-V}$$

→ $\mathcal{T}(Y') = \{$ spanning trees on $Y' \}$

→ $e \in T$: $e = (\Delta_e, \Delta'_e)$



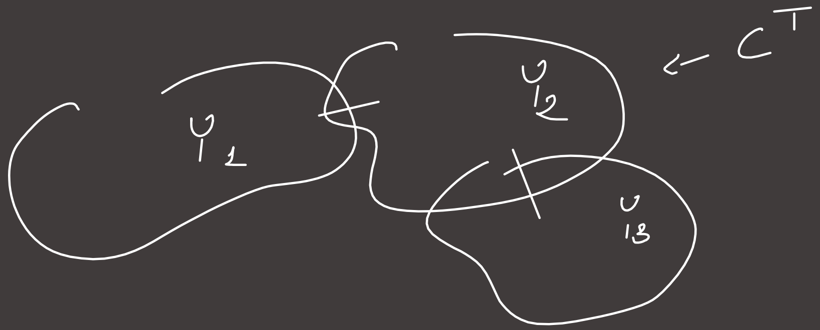
→ remove the constraint $\gamma_i \cap \gamma_j = \emptyset$ additional exp (Mayer exp)

→ final result

$$\frac{\langle \bar{\varphi}_0 \varphi_0 \rangle_\Lambda}{\langle 1 \rangle_\Lambda} = \sum_{\substack{X \subset \Lambda \\ 0 \in \Lambda}} \mathcal{I}_X$$

connected comp cont the obs.

$$\mathcal{I}_X = \sum_{n \geq 1} \frac{1}{n!} \sum_{\gamma_1, \dots, \gamma_n \subset \Lambda} A_0(\gamma_1) \prod_{j=2}^{n} A(\gamma_j) \, C^T(\gamma_1, \dots, \gamma_n)$$



A susy cluster expansion

same setting as above

$$\varphi_j, \varphi_j^* \longrightarrow \Phi_j = \begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix} \qquad \Phi_j^* = (\bar\varphi_j, \bar\chi_j) \qquad \varphi_j \in \mathbb{C} \qquad \chi_j, \bar\chi_j \text{ Grassman}$$

$$d\mu_C(\varphi,\bar\varphi) \longrightarrow d\mu_C(\Phi,\Phi^*) = e^{-\Phi^* C^{-1}\Phi} \prod_j \frac{d\bar\varphi_j \, d\varphi_j}{2\pi} d\bar\chi_j \, d\chi_j$$

normalized susy gauss measure

$$V(\bar\varphi_j\varphi_j) \longrightarrow V(\Phi_j^*\Phi_j)$$

$$\int_{(\mathbb{R}^2)^\Lambda} d\mu_C(\varphi,\bar\varphi)\, e^{-V(\bar\varphi\varphi)}\, \sigma \longrightarrow \int_{(\mathbb{R}^{2|2})^\Lambda} d\mu_C(\Phi,\Phi^*)\, e^{-V(\Phi^*\Phi)}\, \sigma$$

disadvantage: we must integrate out the Grassm var before doing bounds
→ algebra may be complicated

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advantage: no normalization factor $\langle 1 \rangle_{\Lambda,C} = 1$ (by susy)

$e^{-\bar{\Phi}^* C^{-1} \bar{\Phi}} \, d\bar{\Phi}^* d\bar{\Phi}$ automatically normalized

no Maier expansion

cluster exp is much easier!

$$\langle \bar{\varphi}_0 \varphi_0 \rangle = \sum_{\substack{X \subset \Lambda \\ 0 \in X}} A(X)$$

Application to RM models for quantum diffusion

Example 1 random Schrödinger

$H \in \mathbb{R}^{\Lambda \times \Lambda}$ $\quad H := -\Delta_\Lambda + \lambda V$

$$(-\Delta_\Lambda)_{ij} = \begin{cases} -1 & \text{if } |i-j|=1 \\ \sum_{k \,|\, |k-j|=1} 1 & \text{if } i=j \\ 0 & \text{oth} \end{cases}$$

Lattice Laplace op

$\lambda > 0$ param

$V = \operatorname{diag}(V_x)_{x\in\Lambda}$ $\quad V_x$ iid rv with $\mathbb{E}[V_x] = 0$

random potential $\quad \operatorname{Var}[V_x] = 1$

density of states $\longrightarrow$ set $z = E + i\varepsilon$ $\quad E \in \mathbb{R},\ \varepsilon > 0$

study $\quad \lim_{\varepsilon \downarrow 0} \frac{1}{|\Lambda|} \operatorname{Im} \mathbb{E}_\Lambda \left[\operatorname{tr} (z-H)^{-1} \right]$

• SUSY repr: assume $d\rho(V_x)$ admits some finite moment

$$\frac{1}{|\Lambda|} \operatorname{tr} \mathbb{E}_\Lambda\left[(z-H)^{-1} \right] = \int_{(\mathbb{R}^{2|2})^\Lambda} \prod_{x\in\Lambda} d\Phi_x^* d\Phi_x \; e^{iz\sum_x \Phi_x^* \Phi_x} \underbrace{\mathbb{E}\left[e^{-i\sum_{xy} H_{xy} \Phi_x^* \Phi_y} \right]}_{\text{Fourier transf.}} \frac{\sum_x \bar{S}_x S_x}{|\Lambda|}$$

$$= \int_{(\mathbb{R}^{2|2})^\Lambda} \prod_{x\in\Lambda} d\Phi_x^* d\Phi_x \; e^{-i\Phi^*(-\Delta-z)\Phi} \prod_x \hat{\rho}(\lambda \Phi_x^* \Phi_x) \; \frac{\sum_x \bar{S}_x S_x}{|\Lambda|}$$

$$= \int_{(\mathbb{R}^{2|2})^\Lambda} d\mu_C(\Phi, \Phi^*) \; e^{V(\Phi^*\Phi)} \frac{\sum_x \bar{S}_x S_x}{|\Lambda|}$$

• $d\mu_C(\Phi, \Phi^*)$ normalized gauss measure with covariance $C = \dfrac{1}{i(-\Delta - z)}$

• V local potential

BIG PROBLEM ! $\quad C$ imaginary $\Rightarrow$ need to use strong oscillations $\quad$ hard

recent progress on DOS :
- Antinucci, Fresta, Porta AHP 2020 $\rightsquigarrow$ hyerarchical approx via RG
- Fresta MPAG 2021 $\rightsquigarrow$ RS at large disorder via cluster expansion

key tool $\to$ SUSY Fourier transform

caution $\to$ large disorder $\Rightarrow$ expand around e^{-V} instead of d_{μ_c}

Example 2   random band matrix

$H \in \mathbb{C}^{\Lambda \times \Lambda}$ $H^* = H$ $i=j$ $H_{ii} \sim N_{\mathbb{R}}(0, J_{ii})$

$i<j$ $H_{ij} \sim N_{\mathbb{C}}(0, J_{ij})$

$$J_{ij} = (-W^2 \Delta + 1)^{-1}_{ij} \approx e^{-\frac{|i-j|}{W}}$$

$0 < W :=$ band width

$$H = \begin{pmatrix} & H \sim \mathrm{iid} & \sim 0 \\ & & W \\ \sim 0 & & \end{pmatrix} \quad \leftarrow d = 1$$

- **SUSY repr**   set $z = E + i\varepsilon$ $E \in \mathbb{R}$, $\varepsilon > 0$

$$\frac{1}{|\Lambda|} \mathrm{tr}\, \mathbb{E}\left[(z-H)^{-1} \right] = ? \quad \longrightarrow \quad \text{similar as in RS}$$

Rand Schr. $\quad H = -\Delta + \lambda V$

$$\frac{\mathrm{tr}\ \mathbb{E}\left[(z-H)^{-1}\right]}{|\Lambda|} = \int_{(\mathbb{R}^{2|2})^{\Lambda}} d\mu_C(\bar\Phi, \bar\Phi^*)\ e^{V(\bar\Phi^*\bar\Phi)}\ \frac{\sum_x \bar S_x S_x}{|\Lambda|}$$

spin $\quad \bar\Phi_x = \begin{pmatrix} S_x \\ \chi_x \end{pmatrix}$ supervector

$$d\mu_C = \prod_x d\bar\Phi_x\, d\bar\Phi_x^*\ e^{-\sum_{xy} \bar\Phi_x^* C^{-1}_{xy} \bar\Phi_y}$$

$$e^{V} = \prod_x \hat{g}(\bar\Phi_x^* \bar\Phi_x)$$

band matrix

$$\frac{\mathrm{tr}\ \mathbb{E}\left[(z-H)^{-1}\right]}{|\Lambda|} = \int_{(\mathbb{R}^{2|2})^{\Lambda}} d\mu_J(M)\ e^{V(\bar\Phi^*\bar\Phi)}\ \frac{\sum_x a_x}{|\Lambda|}$$

$\to \quad M_x = \begin{pmatrix} a_x & \bar\rho_x \\ \rho_x & i b_x \end{pmatrix}$ supermatrix $\quad a, b \in \mathbb{R}$, $\ \rho, \bar\rho$ Grassm

$$\to \quad d\mu_J = \prod_x dM_x\ e^{-\sum_{xy} \mathrm{Str}\, M_x M_y J^{-1}_{xy}}$$

$$\to \quad e^{V} = \prod_x \frac{1}{\mathrm{Sdet}(z - M_x)}$$

advantage $i(-\Delta - z) \longrightarrow J^{-1} = -W^2\Delta + \mathrm{Id}$ real symm

price to pay → work with supermatrices

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cluster exp? → heuristics for $W \gg 1$

• $$d\nu_J(M) = \underbrace{e^{-W^2 \sum_{|x-y|=1} \mathrm{Str}(M_x - M_y)^2}}_{\text{exp small unless } M_x \sim M_y \ \forall x,y} e^{-\sum_x \mathrm{Str}\, M_x^2} \prod_x dM_x$$

⟹ expect $M_x \simeq M \ \forall x$ ⟹ $$\int \left(e^{-\mathrm{St}M^2} \frac{1}{\mathrm{Sdet}\, z - M} \right)^{|\Lambda|} a \, dM$$

- <u>saddle approx</u>

$$\int e^{-|\Lambda|\left[\mathrm{Str}\, M^2 - \ln \mathrm{Sdet}\, z-M\right]} dM \approx a_s \int e^{-|\Lambda|(m_1+im_2)\mathrm{Str}\, M^2} dM$$

$m_1 > 0$, $m_2 \in \mathbb{R}$

- <u>renormalized covariance</u>

$$J^{-1} \to B^{-1} = -W^2\Delta + m_1 + i m_2 = C^{-1} + i m_2$$

→ partly complex

$$\int d\nu_J(M)\, e^{-V(M)} = \underbrace{\int d\nu_B(M)\, e^{-V_r(M)}}_{\text{starting point for cluster exp.}}$$

$|B_{xy}| \approx e^{-\frac{|x-y|}{W}} \Rightarrow d\nu_{B^{-1}}$ approx factored at distance $W \Rightarrow \int_{(\mathbb{R}^{2|2})^\Lambda} d\nu_B\, e^{-V_r^a} \approx \int_{(\mathbb{R}^{2|2})^\Delta} d\nu_{B_{\Delta\Delta}}\, e^{-V_{r\Delta}}$

Δ = cube of side W cont. O

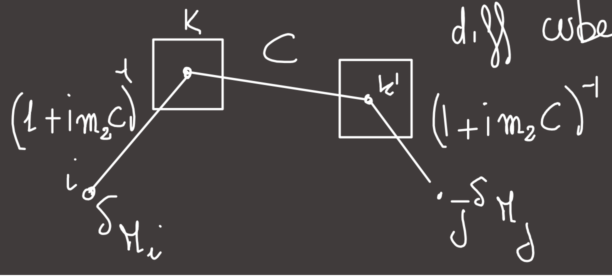
problem $\quad B(s) = s\,B + (1-s)\, \mathcal{I}_\Delta B_{\Delta\Delta}$

does not have the right properties $\longrightarrow$ need $\begin{cases} \mathrm{Re}\, B(s)^{-1} > 0 \\ B(s) \text{ same exp decay as } B \end{cases}$

solution $\quad$ interpolate only the real part $\quad B(s) := \left[C^{-1}(s) + i m_2 \right]^{-1}$

(+) we can iterate the interp keeping all required properties

(−) $\quad \dfrac{d}{ds} B(s) = \left(\mathrm{Id} + i m_2 C(s) \right)^{-1} \dfrac{d}{ds} C(s) \left(\mathrm{Id} + i m_2 C(s) \right)^{-1}$



diff cubes

$\longrightarrow$ the tree construction is more involved

→ can we prove something with this ?

regularity of DOS at large W

$\begin{cases} d=3 \quad \text{D., Pinson, Spencer} \qquad \text{CMP} \quad 2002 \\ d=2 \quad \text{D., Lager} \qquad \text{AHP} \quad 2017 \end{cases}$

→ open problems

- better cluster exp?
- use it to study $\mathbb{E}\left[\left| (z-H)^{-1}_{xy} \right|^2 \right]$?
- applicable to other matrix models

HAPPY BIRTHDAY YAN!