# The rough-smooth boundary in dimer models

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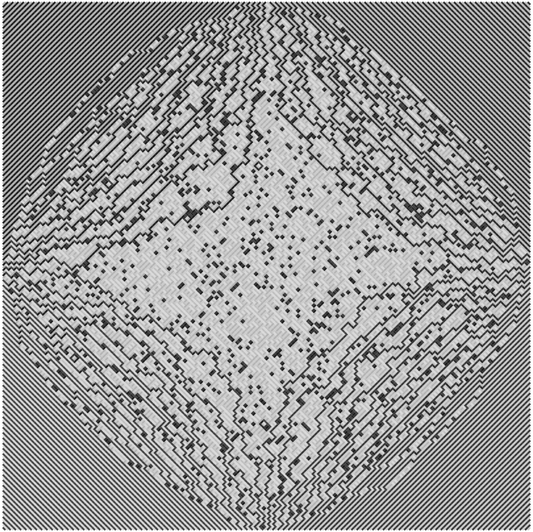
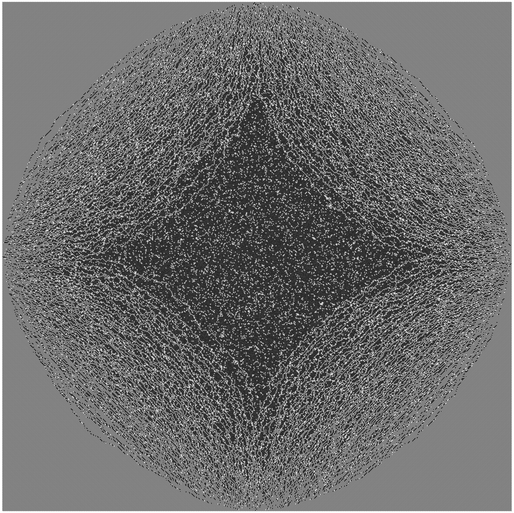
# Two-periodic Aztec diamond

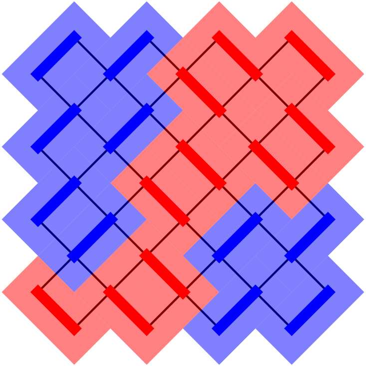


# Two-periodic Aztec diamond



# Aztec diamond

A **domino tiling** of an Aztec diamond shape corresponds to a **dimer configuration** on the Aztec graph.



# Probability measure

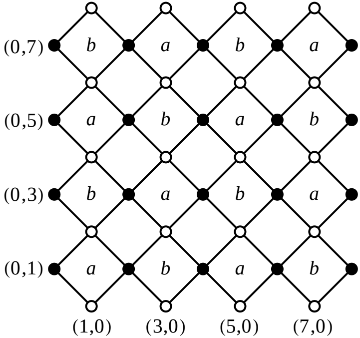
Let $\nu(e) > 0$ be the **weight** of the edge $e$ in the graph $\mathcal{G}$. The probability of a certain **dimer cover** $C$, i.e. each vertex is covered exactly once, is

$$\frac{1}{Z} \prod_{e \in C} \nu(e).$$

$Z$ is the **partition function**.

# Two Periodic Weighting

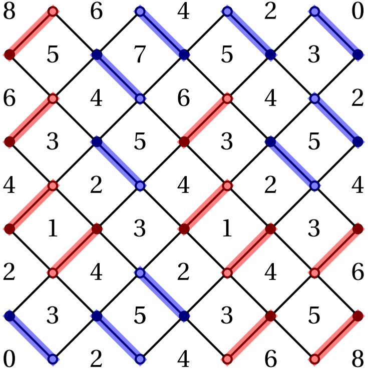
The **two-periodic weighting** of the Aztec diamond is defined in the following way. For a two-colouring of the faces, the edge weights around a particular coloured face alternate between $a$ and $b$, we have **a-edges** and **b-edges**. E.g. for a size 4 Aztec diamond



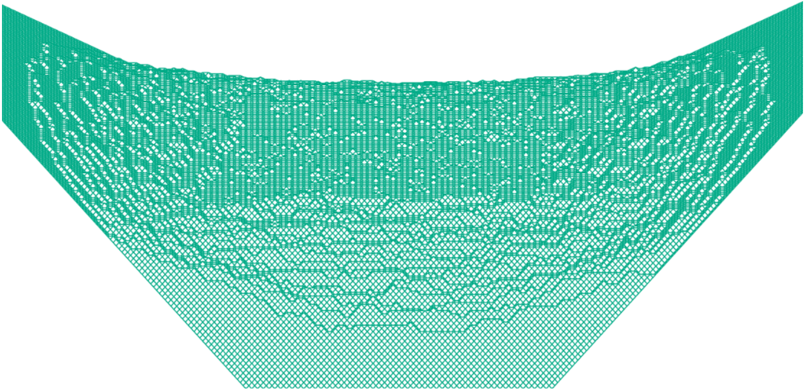
# Aztec diamond height function

To each tiling of an Aztec diamond we can associate a **height function**. The heights sit on the faces of the Aztec graph. The height differences between two faces are given by

- $+3$ if we cross a dimer with a white vertex to the right
- $-1$ if we do not cross a dimer and have a white vertex to the right



# Two-periodic Aztec diamond height function



# Variational principle for the limit height shape

The limiting height function solves

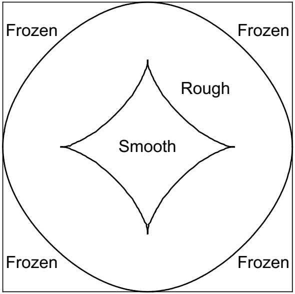
$$\inf_h \int_\Omega \sigma(\nabla h)\, dx,$$

where $\sigma$ is the surface tension. (Cohn-Kenyon-Propp).

Properties investigated by Kenyon and Okounkov.

Recent breakthrough work by Astala-Duse-Prause-Zhong, *Dimer models and Conformal structures*. Investigate possible geometries and prove regularity results. **Pokrovsky-Talapov law**: height function $\sim d^{3/2}$ at typical boundary point of the rough region.

# Phases




The curve in the picture is a degree 8 curve with two real components. We get three regions which are called **frozen, rough and smooth**.

# Phases

Kenyon, Okounkov and Sheffield have characterized the different **limiting translation invariant Gibbs measures** that are possible for bipartite dimer models on the plane.

There are three classes of Gibbs measures, **frozen**, **rough** and **smooth**.

*Correlations* between dominos decay polynomially with distance in the rough region, and exponentially in the smooth region.

# The Kasteleyn method

For the Aztec diamond graph we define the **Kasteleyn matrix** by

$$\mathbb{K}(b,w) = \begin{cases} \nu(b,w) & \text{if } e=(b,w) \text{ is horizontal} \\ \mathrm{i}\nu(b,w) & \text{if } e=(b,w) \text{ is vertical} \\ 0 & \text{otherwise (i.e. no edge between } b \text{ and } w) \end{cases}$$

Theorem (Montroll-Potts-Ward, Kenyon)

*If $e_i = (b_i, w_i)$, then the probability that $e_1, \dots, e_m$ belong to a dimer cover is*

$$\mathbb{P}(e_1,\dots,e_m) = \det\left(\mathbb{K}(b_i,w_i)\mathbb{K}^{-1}(w_i,b_j)\right)_{1\le i,j\le m}$$

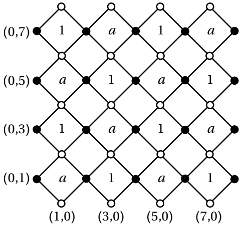
This means that the dimers form a **determinantal point process** with correlation kernel $K(e_i,e_j) = \mathbb{K}(b_i,w_i)\mathbb{K}^{-1}(w_i,b_j)$, $e_i = (b_i, w_i)$.

# Dimer-dimer correlations at the rough-smooth boundary

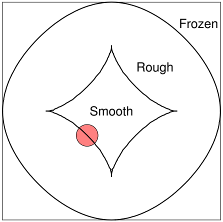
(Based on joint work with S. Mason, *Dimer-dimer correlations at the rough-smooth boundary*, arXiv:2110.14505; formula for the inverse Kasteleyn matrix from J., Chhita.)

Consider a size $n$ two-periodic Aztec diamond with $n$ very large. Formula for the inverse Kasteleyn matrix

$$K_{a,1}^{-1}(x,y) = \mathbb{K}_{1,1}^{-1}(x,y) - C_{\omega_c}(x,y) + R(x,y) + B^*(x,y).$$



# Dimer-dimer correlations at the rough-smooth boundary




$B^*$ is exponentially small in $n$. Write

$x = n(1+\xi)(1,1)+(2a_1-1,2a_2), y = n(1+\xi)(1,1)+(2b_1,2b_2-1)$.

$\xi = \xi_c = -\frac{1}{2}\sqrt{1-2c}$, $c = \frac{a}{1+a^2}$, gives asymptotic boundary.

# Dimer-dimer correlations at the rough-smooth boundary

We are close to the boundary: $\xi_c - \xi \to 0$ as $n \to \infty$.
Let $G(w) = \frac{1}{\sqrt{2c}}(w - \sqrt{w^2 + 2c})$ and

$$g_\xi(w) = \log w - \xi \log G(w) + \xi \log G(w^{-1}).$$

Then $g'_\xi(w)$ has roots at $\pm\omega_c$, $\pm\bar{\omega}_c$, where $\omega_c$ is in the first quadrant. $\omega_c = \mathrm{i}$ iff $\xi = \xi_c$. $R$ is an error term for our purposes.

$$|R(x,y)| \leq C|G(\omega_c)|^{b_1 - b_2 + a_2 - a_1} \min\left(\frac{1}{n^{1/3}}, \frac{1}{\sqrt{n\sqrt{\xi_c - \xi}}}\right),$$

for $|a_i|, |b_i| \leq \max(n^{1/3}, \sqrt{n\sqrt{\xi_c - \xi}})$.

# Dimer-dimer correlations at the rough-smooth boundary

In the region we are investigating,

$$K_{a,1}^{-1}(x,y) = \mathbb{K}_{1,1}^{-1}(x,y) - C_{\omega_c}(x,y) + \text{negligible}$$
$$= \mathbb{K}_{s_1,s_2}^{-1}(x,y) + \text{negligible},$$

where $\mathbb{K}_{s_1,s_2}^{-1}$ gives a rough Gibbs measure in the whole plane. $\mathbb{K}_{1,1}^{-1}$ can be expressed in terms of the integrals

$$E_{k,\ell} = \frac{\mathrm{i}^{-k-\ell}}{4(1+a^2)\pi\mathrm{i}} \int_{|w|=1} \frac{G(w)^{\ell} G(1/w)^{k}}{\sqrt{w^2+2c}\sqrt{1/w^2+2c}} \frac{dw}{w}.$$

$C_{\omega_c}(x,y)$ can be expressed in terms of the integral the same integral but integrated over $\Gamma_{\omega_c}$ which consists of two short arcs on the unit circle around $\mathrm{i}$ and $-\mathrm{i}$ of length $c\sqrt{\xi_c - \xi}$.
$k \approx (x_2 - y_2)/2$ and $\ell \approx (x_1 - y_1)/2$.

# The rough-smooth boundary. Correlation asymptotics

Consider two dimers along the main diagonal oriented orthogonally to the diagonal. Think of $n$ as very large but fixed and consider growing $r$.

Assume $n^{-2/3} << \xi_c - \xi$ (*not right at the boundary*) and $\xi_c - \xi < \delta_n \to 0$ (*not fully in the rough region*).

- $r_{\min} < r < c_1 \log \frac{1}{\sqrt{\xi_c - \xi}}$: corr $\sim c e^{-r/\alpha}$ (**exponential decay**)
- $c_1 \log \frac{1}{\sqrt{\xi_c - \xi}} < r < c_2 \frac{1}{\sqrt{\xi_c - \xi}}$: corr $\sim c(\xi_c - \xi)$ (**constant**)
- $c_2 \frac{1}{\sqrt{\xi_c - \xi}} < r << \sqrt{n\sqrt{\xi_c - \xi}}$ and $r \sim \frac{d}{\sqrt{\xi_c - \xi}}$:
  corr $\sim (\xi_c - \xi)\frac{\sin^2 d}{d^2}$ (**power law and oscillatory**)

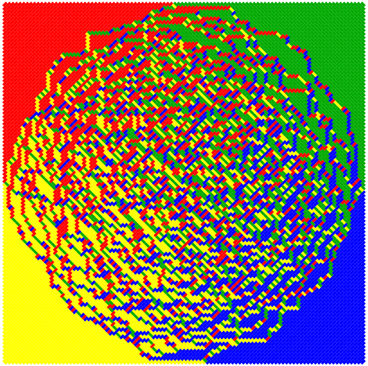
# The rough-smooth boundary. Correlation asymptotics

**Two length scales**

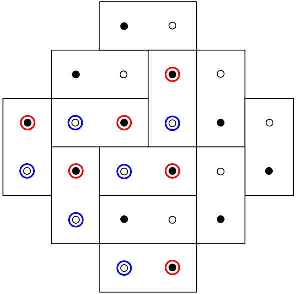
1) the lattice spacing
2) the distance $\frac{1}{\sqrt{\xi_c - \xi}}$

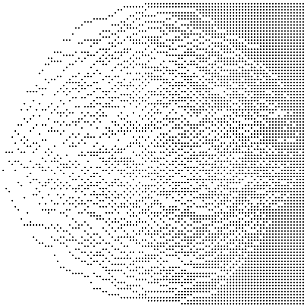
The results can be thought of as the decay of correlations for infinite volume Gibbs measures in the rough phase close to the smooth phase.

# Frozen-Rough boundary

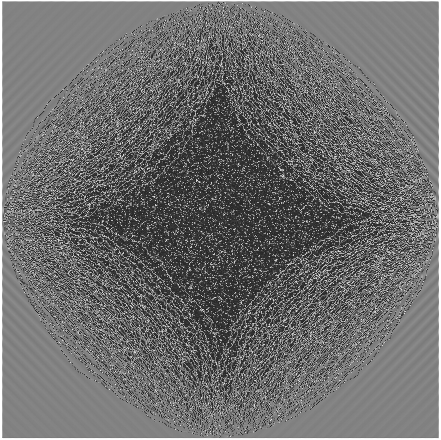


# Frozen-Rough boundary



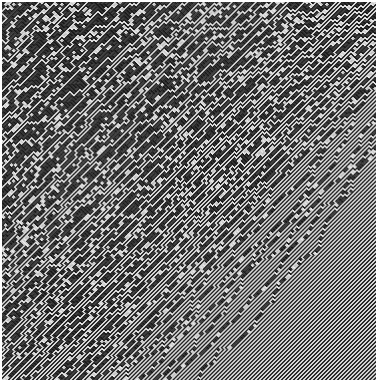


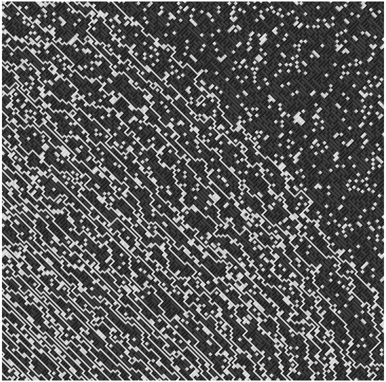
# The rough-smooth boundary



# The rough-smooth boundary

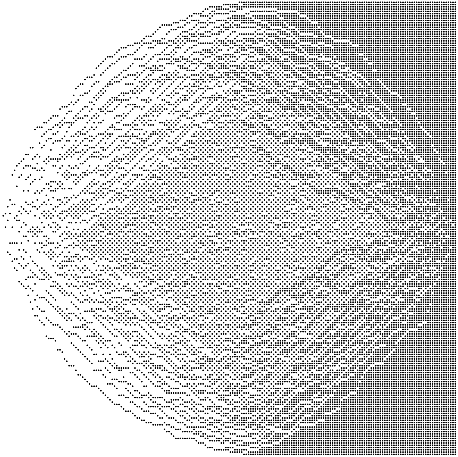
To the left part of frozen-rough boundary, to the right part of rough-smooth boundary.



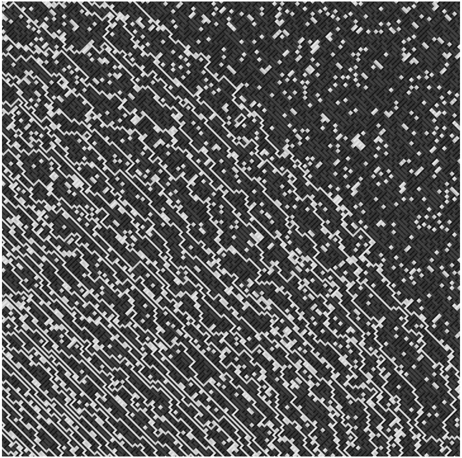


# The rough-smooth boundary

Particles in the two-periodic Aztec diamond.



# The rough-smooth boundary



What are the "long paths" that we see in the picture? Can we define a boundary that converges to the Airy process?

# Squishing

(Based on joint work with Beffara and Chhita.)

An *a*-**dimer** is a dimer that covers an *a*-edge. They are **oriented** from white to black.

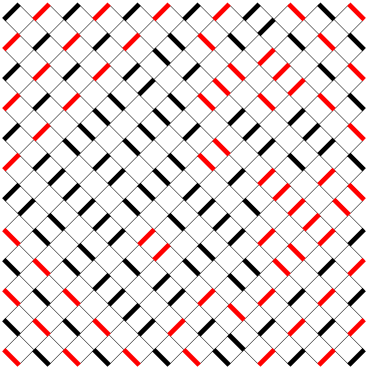
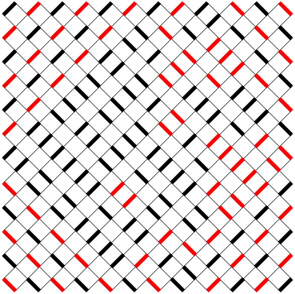
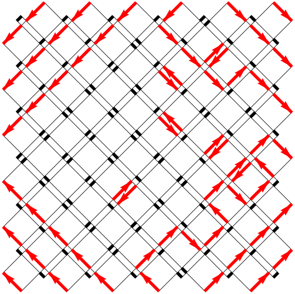


Figure: The red dimers are *a*-dimers, and the black *b*-dimers.

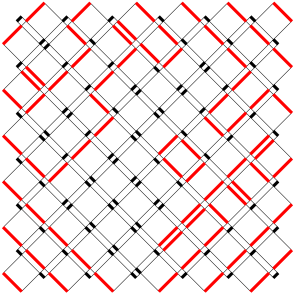
# Squishing

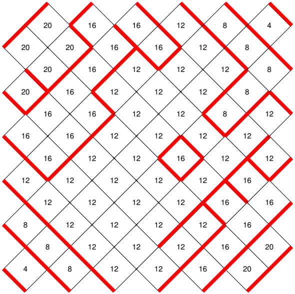
We let the *b*-faces become smaller, go to zero in size.

# Squishing

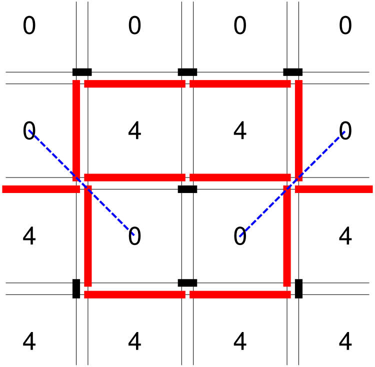
We get **double edges**, **loops** and **paths**.





# Paths and Loops

To get a unique split between paths and loops and get well-defined loops we need a convention. We use **mirrors**.



# Paths

The paths go between the boundaries.

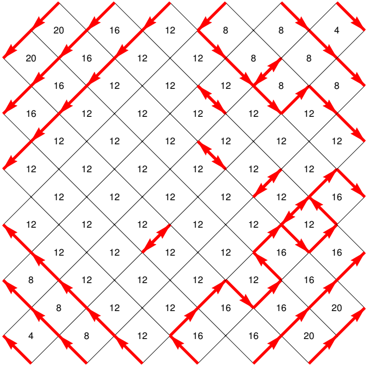



Figure: After squishing.

# Paths

The paths go between the boundaries.

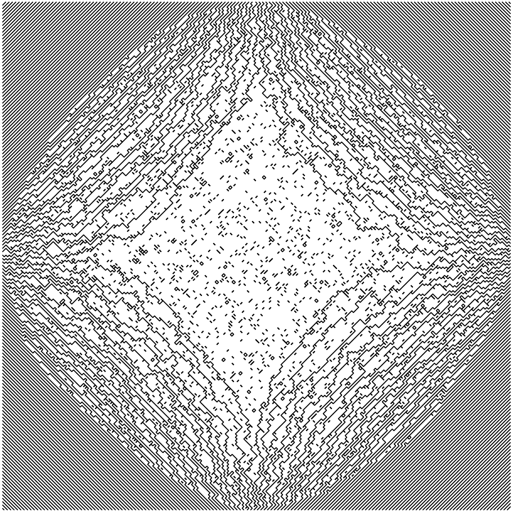
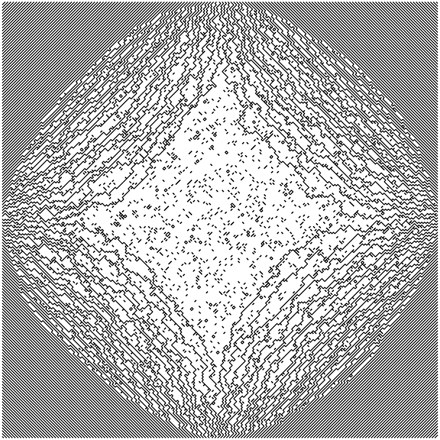


Figure: After squishing, $n = 300, a = 0.5$.

# What we would like to prove

With high probability, if we go along the main diagonal there is a last path in the third quadrant close to the asymptotic rough-smooth boundary and this path converges to the Airy process.

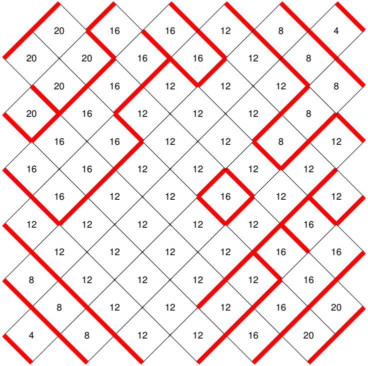


# What we can prove

Let $h(f)$ be the height at the face $f$ in the Aztec diamond. Then we can split it into two parts:

$$h(f) = h_\ell(f) + h_c(f)$$

where $h_\ell(f)$ is the **loop height** and $h_c(f)$ is the **corridor height**.



# What we can prove

Assume that $a < 1/3$. Imbed the interval $A$ as a discrete interval of length $\sim m^{1/3}$ in the Aztec diamond at the rough-smooth boundary. Define the **random signed measure**

$$\kappa_m(\{\beta\} \times A) = \frac{1}{4}(h_c(F_+) - h_c(F_-)),$$

where $F_+$ and $F_-$ are the end-faces of the discrete imbedded interval. Then $\kappa_m(\{\beta\} \times A)$ converges in terms of Laplace transforms to $\mu_{\mathrm{Ai}}(\{\beta\} \times A)$ as $m \to \infty$, where $\mu_{\mathrm{Ai}}$ is the Airy kernel point process.

We expect that with high probability $\kappa_m$ is actually a positive measure. We should think of $\kappa_m$ as counting the number of paths between the two faces.

Happy birthday Yan!