

Counting equilibria of complicated dynamics

Random Matrices and Random Landscape, CSF, Ascona

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General introduction

Equilibria of complex dynamics

Characterising the **number** and **nature** of equilibria for dynamics involving many interacting agents/particles is a central question in many fields

- Ecology
- Economics
- Physics
- Computer science algorithms

Studied by a large community in Physics and Maths:

Altieri, Auffinger, Belius, Ben-Arous, Biroli, Bourgade, Bray, Cammarota, Dean, Fyodorov, Khoruzhenko, Le Doussal, McKenna, Ros, Subag, Zeitouni...

Equilibria of complex dynamics

The dynamics of N particles x_i 's can quite generally be described by

$$\partial_t x_i = -\partial_{x_i} U_i(x_i) + F_i(\mathbf{x}) \quad \mathbf{x} = \{x_1, \dots, x_N\} \quad i = 1, \dots, N$$

$U_i(x)$: single-body potential

$F_i(\mathbf{x})$: stems from the interactions

An equilibrium is characterised by the simultaneous vanishing of all the terms in the right-hand side

Counting equilibria and the Kac-Rice formula

Using the Kac-Rice formula, one can obtain

$\mathcal{N}_{\text{tot}}$: total number of equilibria

$$\mathcal{N}_{\text{tot}} = \int d\mathbf{x} \prod_{i=1}^N \delta \left(F_i - \partial_{x_i} U_i(x_i) \right) |\det(J(\mathbf{x}))|$$

$$J_{ij}(\mathbf{x}) = \partial_{x_j} F_i - \delta_{ij} \partial_{x_i}^2 U_i(x_i)$$

$\mathcal{N}_{\text{st}}$: number of stable equilibria

$$\mathcal{N}_{\text{st}} = \int d\mathbf{x} \prod_{i=1}^N \delta \left(F_i - \partial_{x_i} U_i(x_i) \right) |\det(J(\mathbf{x}))| \Theta(J(\mathbf{x}))$$

Where $\Theta(J(\mathbf{x}))$ ensures that the Jacobian matrix $J(\mathbf{x})$ has all its eigenvalues with positive real part

From complex to random

- To characterise essential features of highly complicated dynamics, one can replace the exact interactions by random interactions with the correct symmetries
- Constitutes one of the main features of May's model and many theoretical ecology paper since
- This is at the heart of the ubiquitous use of random matrix theory in many different contexts

Random dynamics with
homogeneous relaxation rates

Random dynamics with homogeneous relaxation rates

Let us first focus our attention to the case

(Fyodorov, Khoruzhenko '16; Ben Arous, Fyodorov, Khoruzhenko '21; Arnould de Pirey, van Wijland '22)

$$\partial_t x_i = -\mu x_i - \partial_{x_i} V(\mathbf{x}) + \frac{1}{\sqrt{N}} \sum_{j=1}^N \partial_{x_j} A_{ij}(\mathbf{x})$$

$$\langle V(\mathbf{x}_1) V(\mathbf{x}_2) \rangle = N \Gamma_V \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{2N} \right) \quad \langle A_{ij}(\mathbf{x}_1) A_{kl}(\mathbf{x}_2) \rangle = N \Gamma_A \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{2N} \right) (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \quad \langle A_{ij}(\mathbf{x}_1) V(\mathbf{x}_2) \rangle = 0$$

$V(\mathbf{x})$ corresponds to a random gradient term

$A_{ij}(\mathbf{x})$ corresponds to a random solenoidal term

The case of $A_{ij}(\mathbf{x}) = 0$ has been treated extensively over the past ~20 years since (Fyodorov '04)

Annealed complexity

The main quantities of interest in the large N limit are the so-called complexities:

The total annealed complexity is defined as

$$\Xi_{\text{tot}} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle \mathcal{N}_{\text{tot}} \rangle, \quad \langle \mathcal{N}_{\text{tot}} \rangle = \int d\mathbf{x} \langle \prod_{i=1}^N \delta(F_i - \mu x_i) | \det(\mu \mathbb{I} + J(\mathbf{x})) | \rangle$$

The annealed complexity of stable equilibria reads

$$\Xi_{\text{st}} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle \mathcal{N}_{\text{st}} \rangle, \quad \langle \mathcal{N}_{\text{st}} \rangle = \int d\mathbf{x} \langle \prod_{i=1}^N \delta(F_i - \mu x_i) | \det(\mu \mathbb{I} + J(\mathbf{x})) | \Theta(\mu \mathbb{I} + J(\mathbf{x})) \rangle$$

For a gradient descent dynamics these complexities were shown to display a phase transition: (Fyodorov '04)

- In the “topologically trivial” phase $\Xi_{\text{tot}} = \Xi_{\text{st}} = 0$
- In the “topologically complex” phase $\Xi_{\text{tot}} > \Xi_{\text{st}} > 0$

Covariance structure

Before applying the Kac-Rice formula, let us analyse the covariances of the force and Jacobian matrix for a fixed position $\mathbf{x}$:

$F_i(\mathbf{x}) = -\partial_{x_i} V(\mathbf{x}) + \frac{1}{\sqrt{N}} \sum_{j=1}^N \partial_{x_j} A_{ij}(\mathbf{x})$ is a mean zero Gaussian random variable with

$$\langle F_i(\mathbf{x}) F_j(\mathbf{x}) \rangle = \sigma^2 \delta_{ij} \quad \langle F_i(\mathbf{x}) J_{jk}(\mathbf{x}) \rangle = 0$$

$$\langle J_{ij}(\mathbf{x}) J_{kl}(\mathbf{x}) \rangle = \frac{c}{N} (\delta_{ik} \delta_{jl} + \tau (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl}))$$

Covariance structure

$$\langle F_i(\mathbf{x}) F_j(\mathbf{x}) \rangle = \sigma^2 \delta_{ij} \quad \langle F_i(\mathbf{x}) J_{jk}(\mathbf{x}) \rangle = 0$$

c is the magnitude of the randomness

$$\langle J_{ij}(\mathbf{x}) J_{kl}(\mathbf{x}) \rangle = \frac{c}{N} (\delta_{ik} \delta_{jl} + \tau (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl}))$$

τ is the fraction of gradient dynamics

The forces $F_i(\mathbf{x})$ are i.i.d. Gaussian random variables

Forces $F_i(\mathbf{x})$ and Jacobian matrix $J(\mathbf{x})$ are independent

The Jacobian matrix $J(\mathbf{x})$ belongs to the real Elliptic Gaussian Ensemble (rEGE) + an independent random Gaussian variable

This ensemble is non-Hermitian and the eigenvalues of $J(\mathbf{x})$ are complex

(in contrast with the gradient case $\tau = 1$)

The statistics of all these objects is independent of the position $\mathbf{x}$

Applying Kac-Rice

Using this simple covariance structure:

The average total number of equilibria reads

$$\langle \mathcal{N}_{\text{tot}} \rangle = \langle | \det(\mu \mathbb{I} + J(\mathbf{0})) | \rangle \int d\mathbf{x} \prod_{i=1}^N \langle \delta(F_i - \mu x_i) \rangle = \frac{\langle | \det(\mu \mathbb{I} + J(\mathbf{0})) | \rangle}{\mu^N}$$

The average number of stable equilibria reads

$$\langle \mathcal{N}_{\text{st}} \rangle = \frac{\langle | \det(\mu \mathbb{I} + J(\mathbf{0})) | \Theta(\mu \mathbb{I} + J(\mathbf{0})) \rangle}{\mu^N}$$

Laplace method

In the large N limit, it is convenient to re-write

$$\langle \mathcal{N}_{\text{tot}} \rangle = \frac{\langle |\det(\mu \mathbb{I} + J(\mathbf{0}))| \rangle}{\mu^N} = \int \frac{d\xi}{\sqrt{2\pi\tau c/N}} e^{-N \left[\frac{\xi^2}{2c\tau} - \frac{1}{N} \ln \langle |\det((\mu + \xi)\mathbb{I} + X)| \rangle_{X \in rEGE} + \ln \mu \right]}$$

Using now the strong self-averaging property for the real Elliptic Gaussian Ensemble (rEGE)

(Fyodorov, Khoruzhenko '16, Ben Arous, Fyodorov, Khoruzhenko '21)

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle |\det((\mu + \xi)\mathbb{I} + X)| \rangle_{X \in rEGE} = \lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{Tr} \ln |(\mu + \xi)\mathbb{I} + X| \rangle_{X \in rEGE} = \int d^2z \rho(z) \ln |\mu + \xi - z|$$

One readily obtains $\Xi_{\text{tot}} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle \mathcal{N}_{\text{tot}} \rangle = \max_{\xi} \left[-\frac{\xi^2}{2c\tau} + \int d^2z \rho(z) \ln |\mu + \xi - z| \right] - \ln \mu$

And similarly $\Xi_{\text{st}} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle \mathcal{N}_{\text{st}} \rangle = \max_{\xi \geq \min_k \Re(z_k) - \mu} \left[-\frac{\xi^2}{2c\tau} + \int d^2z \rho(z) \ln |\mu + \xi - z| \right] - \ln \mu$

Total annealed complexity

- It undergoes the same “complexity trivialisation transition”

(Fyodorov, Khoruzhenko '16)

$$\mathbb{E}_{\text{tot}}(c) = \begin{cases} 0, & c \leq c_t = \mu^2 \\ \frac{1}{2} \left(\frac{\mu^2}{c} - 1 - \ln \frac{\mu^2}{c} \right), & c > c_t = \mu^2 \end{cases}$$

- Independent of τ , i.e. independent of the nature of the dynamics
- Vanishes quadratically at the transition:

$$\mathbb{E}_{\text{tot}} = \frac{(c - \mu^2)^2}{4\mu^2} + O(c - \mu^2)^3$$

Reminder:

c is the magnitude of the randomness

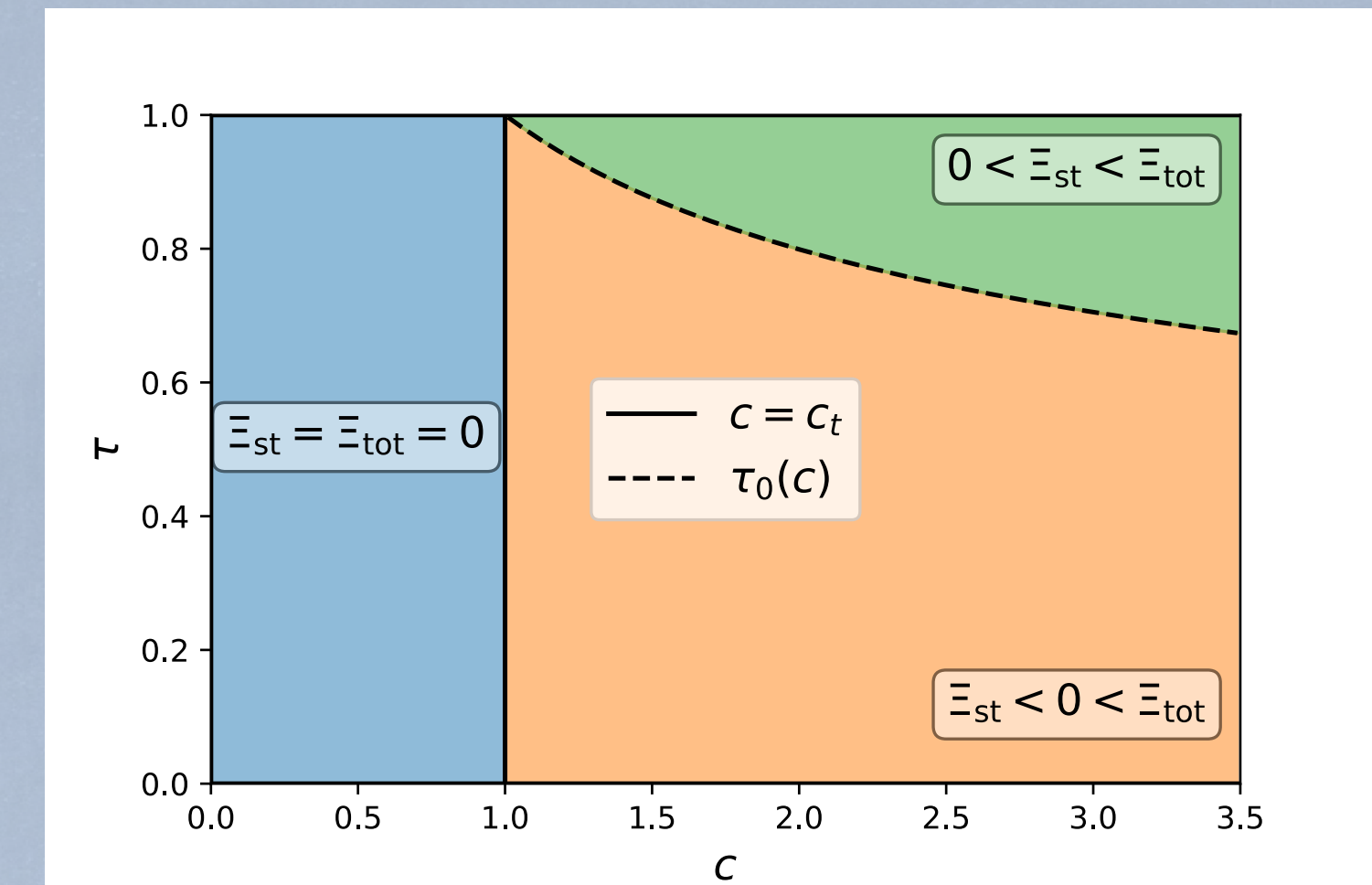
τ is the fraction of gradient dynamics

Annealed complexity of stable equilibria

- It undergoes two transitions: “complexity trivialisation transition”

(Ben Arous, Fyodorov, Khoruzhenko '21)

$$\Xi_{\text{st}}(c, \tau) = \begin{cases} 0, & c \leq c_t = \mu^2 \\ \Xi_{\text{st}}(c, 1) - \frac{1-\tau}{2c\tau} \left(\sqrt{c} - \mu \right)^2, & c > c_t = \mu^2 \end{cases}$$



- For $c > c_t$, $\Xi_{\text{st}}(c, \tau)$ positive for $\tau > \tau_0(c)$: exponential number of stable equilibria
- For $c > c_t$, $\Xi_{\text{st}}(c, \tau)$ negative for $\tau < \tau_0(c)$: exponentially small probability to have a stable equilibrium
- Vanishes quadratically $\tau < 1$ or cubically $\tau = 1$ at the transition:

$$\Xi_{\text{st}} = -\frac{(1-\tau)}{8\tau\mu^4}(c - \mu^2)^2 + \frac{(c - \mu^2)^3}{24\mu^6} + O(c - \mu^2)^4$$

Reminder:

c is the magnitude of the randomness

τ is the fraction of gradient dynamics

Results for heterogeneous relaxation rates

Random dynamics with uniform relaxation rates

We now turn to **inhomogeneous relaxation rates** (LACT, Fyodorov '22)

$$\partial_t x_i = -\mu_i x_i - \partial_{x_i} V(\mathbf{x}) + \frac{1}{\sqrt{N}} \sum_{j=1}^N \partial_{x_j} A_{ij}(\mathbf{x})$$

$$\langle V(\mathbf{x}_1) V(\mathbf{x}_2) \rangle = N \Gamma_V \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{2N} \right) \quad \langle A_{ij}(\mathbf{x}_1) A_{kl}(\mathbf{x}_2) \rangle = N \Gamma_A \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{2N} \right) (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

$$\langle A_{ij}(\mathbf{x}_1) V(\mathbf{x}_2) \rangle = 0$$

$V(\mathbf{x})$ corresponds to a random gradient term

$A_{ij}(\mathbf{x})$ corresponds to a random solenoidal term

Only gradient case studied in depths by Ben Arous, Bourgade, McKenna '21

Gradient case: Connection to elastic manifold

The gradient case of this problem is related to the elastic manifold in d-dimensions

$$\mathcal{H} = \frac{1}{2} \sum_{\vec{x}, \vec{y} \in L^d} \vec{u}(\vec{x}) (-t\Delta + \mu \mathbb{I})_{\vec{x}, \vec{y}} \vec{u}(\vec{y}) + \sum_{\vec{x}} V(\vec{u}(\vec{x}), \vec{x})$$

$$\langle V(\vec{u}_1, \vec{x}_1) V(\vec{u}_2, \vec{x}_2) \rangle = N \Gamma \left(\frac{|\vec{u}_1 - \vec{u}_2|^2}{2N} \right) \delta_{\vec{x}_1, \vec{x}_2}$$

Complexity derived in (Fyodorov, Le Doussal '20)

Shown to correspond to a simpler model and re-derive rigorously

(Ben Arous, Bourgade, McKenna '21)

$$H = \frac{1}{2} \sum_i \mu_i x_i^2 + \tilde{V}(\mathbf{x})$$

Applying Kac-Rice

The covariance structure is identical and the derivation follows through

The average total number of equilibria reads

$$\langle \mathcal{N}_{\text{tot}} \rangle = \langle | \det(D^\mu + J(\mathbf{0})) | \rangle \int d\mathbf{x} \prod_{i=1}^N \langle \delta(F_i - \mu_i x_i) \rangle = \frac{\langle | \det(D^\mu + J(\mathbf{0})) | \rangle}{\det(D^\mu)}$$

The average number of stable equilibria reads

$$\langle \mathcal{N}_{\text{st}} \rangle = \frac{\langle | \det(D^\mu + J(\mathbf{0})) | \Theta(D^\mu + J(\mathbf{0})) \rangle}{\det(D^\mu)}$$

$$D_{ij}^\mu = \mu_i \delta_{ij}$$

$J(\mathbf{0})$: matrix in REGE
+ random Gaussian

Laplace method

In the large N limit, it is convenient to re-write

$$\langle \mathcal{N}_{\text{tot}} \rangle = \frac{\langle |\det(D^\mu + J(\mathbf{0}))| \rangle}{\det(D^\mu)} = \int \frac{d\xi}{\sqrt{2\pi\tau c/N}} e^{-N \left[\frac{\xi^2}{2c\tau} - \frac{1}{N} \ln \langle |\det(\xi \mathbb{I} + D^\mu + X)| \rangle_{X \in rEGE} + \int d\lambda n_\mu(\lambda) \ln \lambda \right]} \quad n_\mu(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \mu_i)$$

Supposing the strong self-averaging property

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle |\det(\xi \mathbb{I} + X + D_\mu)| \rangle_{X \in rEGE} = \lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{Tr} \ln |\xi \mathbb{I} + X + D_\mu| \rangle_{X \in rEGE} = \int d^2z \rho(z) \ln |\xi - z|$$

One readily obtains $\Xi_{\text{tot}} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle \mathcal{N}_{\text{tot}} \rangle = \max_{\xi} \left[-\frac{\xi^2}{2c\tau} + \int d^2z \rho(z) \ln |\xi - z| \right] - \int d\lambda n_\mu(\lambda) \ln \lambda$

And similarly $\Xi_{\text{st}} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle \mathcal{N}_{\text{st}} \rangle = \max_{\xi \geq \min_k \Re(z_k)} \left[-\frac{\xi^2}{2c\tau} + \int d^2z \rho(z) \ln |\xi - z| \right] - \int d\lambda n_\mu(\lambda) \ln \lambda$

Results:

Total annealed complexity

- It generically undergoes a “complexity trivialisation transition”

(LACT, Fyodorov '22)

$$\bar{E}_{\text{tot}}(c) = \begin{cases} 0, & c \leq c_t \\ \int_{c_t}^c \frac{d\omega}{2\omega^2} \Upsilon(\omega)^2, & c > c_t \end{cases}$$

$$\frac{1}{c} = \int d\lambda \frac{n_\mu(\lambda)}{\lambda^2 + \Upsilon^2(c)}$$

$$c_t = I_2(\mu)^{-1}$$

$$I_p(\mu) = \int \frac{d\lambda}{\lambda^p} n_\mu(\lambda)$$

Reminder:

$$n_\mu(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \mu_i)$$

c is the magnitude of the randomness

τ is the fraction of gradient dynamics

- Independent of τ , i.e. independent of the nature of the dynamics

Results:

Total annealed complexity

- The behaviour at the transition depends on the small λ behaviour:

$$n_\mu(\lambda) \sim \lambda^{\eta-1}$$

- If $\eta > 4$: $c_t > 0, I_4(\mu) < \infty$ and $\Xi_{\text{tot}} = \frac{I_2(\mu)^2}{4I_4(\mu)}(c - c_t)^2 + O(c - c_t)^3$

$$c_t = I_2(\mu)^{-1}$$

Behaviour previously derived for gradient dynamics $\tau = 1$

$$I_p(\mu) = \int \frac{d\lambda}{\lambda^p} n_\mu(\lambda)$$

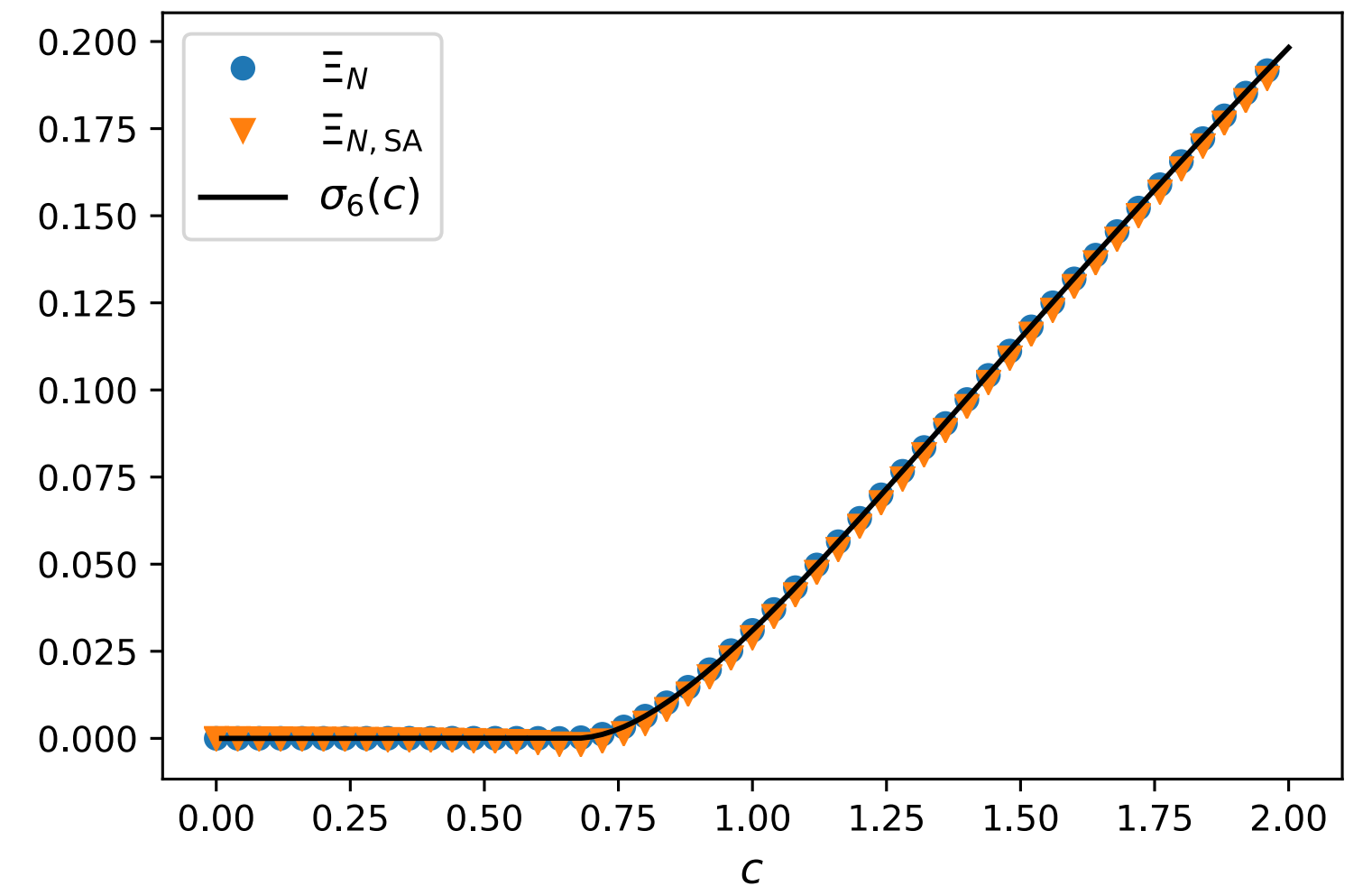
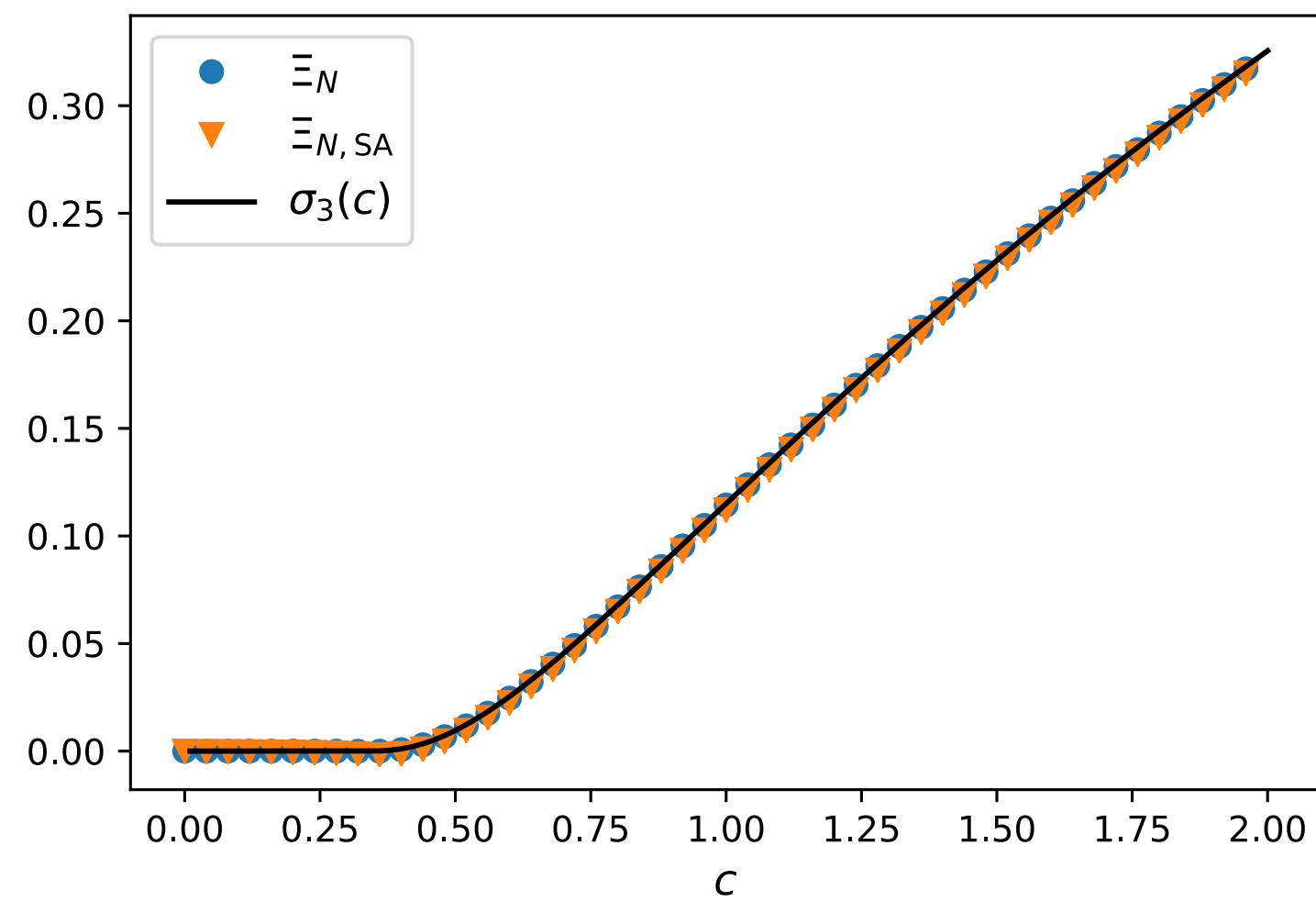
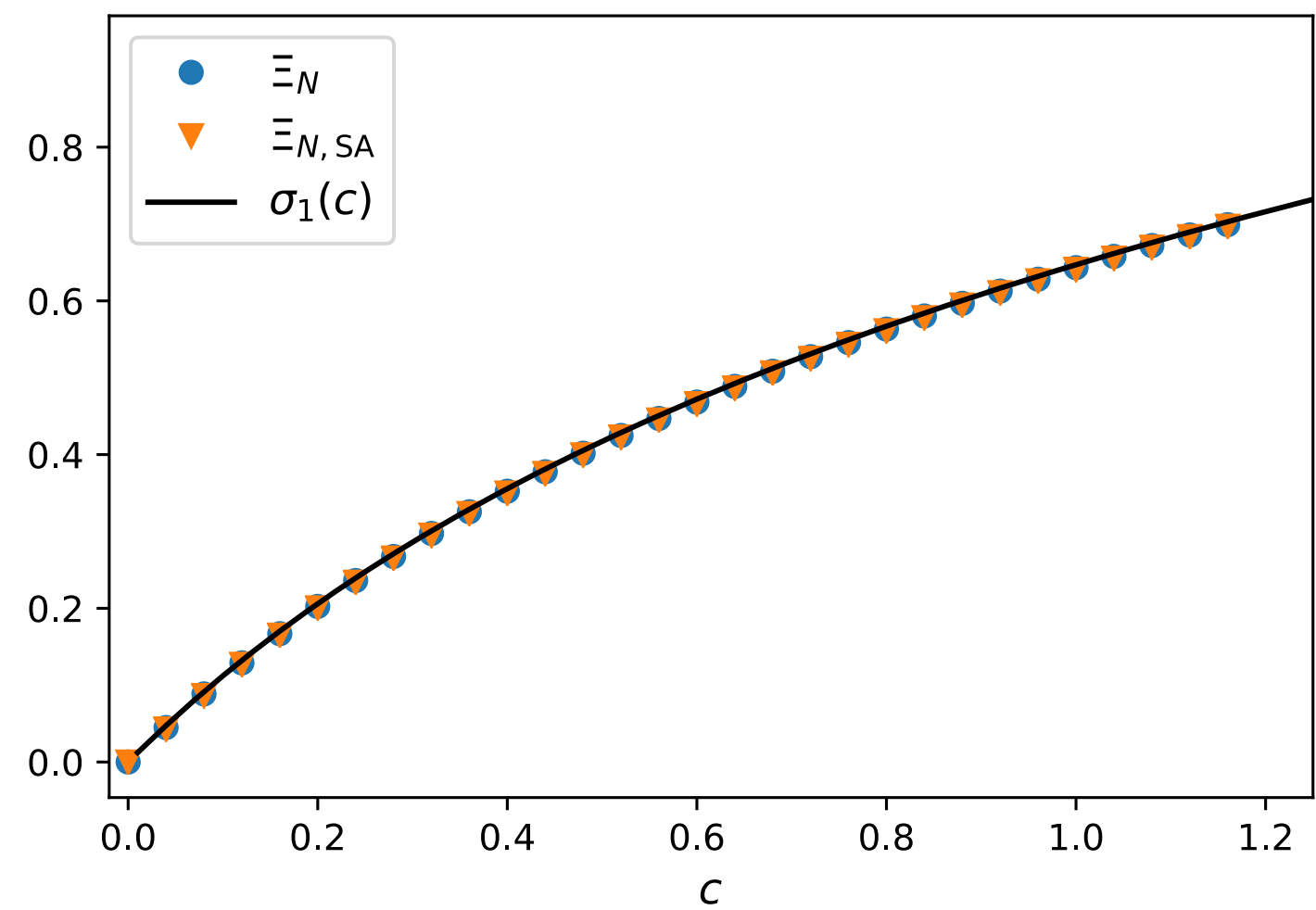
(Ben Arous, Bourgade, McKenna '21)

- If $\eta < 4$: $c_t > 0, I_4(\mu) = \infty$ and $\Xi_{\text{tot}} = C_1(\eta)(c - c_t)^{\frac{\eta}{\eta-2}} + o(c - c_t)^{\frac{\eta}{\eta-2}}$

- If $\eta < 2$: $c_t = 0, I_4(\mu) = \infty$ and $\Xi_{\text{tot}} = C_2(\eta)(c - c_t)^{\frac{\eta}{2-\eta}} + o(c - c_t)^{\frac{\eta}{2-\eta}}$

Results:

Total annealed complexity



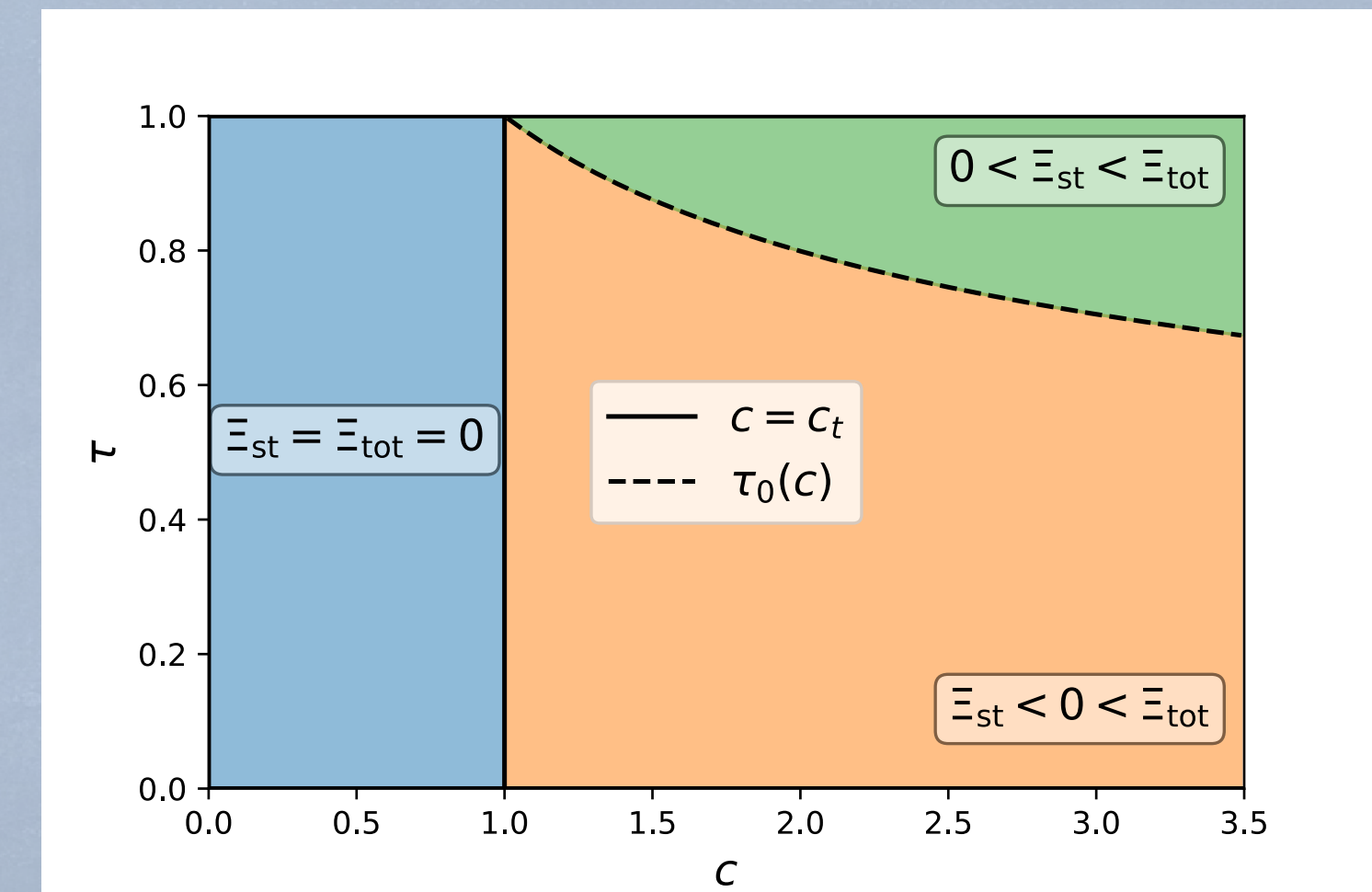
Results:

Annealed complexity of stable equilibria

- It generically undergoes a “complexity trivialisation transition”

(LACT, Fyodorov '22)

$$\Xi_{\text{st}}(c, \tau) = \begin{cases} 0, & c \leq c_t \\ \int_{c_t}^c \frac{d\omega}{2\omega^2} \nu(\omega)^2 - \frac{1-\tau}{2c\tau} \nu(c)^2, & c > c_t \end{cases} \quad \frac{1}{c} = \int d\lambda \frac{n_\mu(\lambda)}{(\lambda + \nu(c))^2}$$



- For $c > c_t$, $\Xi_{\text{st}}(c, \tau)$ positive for $\tau > \tau_0(c)$: exponential number of stable equilibria
- For $c > c_t$, $\Xi_{\text{st}}(c, \tau)$ negative for $\tau < \tau_0(c)$: exponentially small probability to have a stable equilibrium

Results:

Annealed complexity of stable equilibria

- The behaviour at the transition depends on the small λ behaviour:

$$n_\mu(\lambda) \sim \lambda^{\eta-1}$$

- If $\eta > 3$: $c_t > 0, I_3(\mu) < \infty$ and $\Xi_{\text{st}} = -\frac{1-\tau}{8\tau} \frac{I_2(\mu)^5}{I_3(\mu)^2} (c - c_t)^2 + \frac{I_2(\mu)^6}{24I_3(\mu)^2} (c - c_t)^3 + O(c - c_t)^4$

Behaviour previously derived for gradient dynamics $\tau = 1$

(Ben Arous, Bourgade, McKenna '21)

$$c_t = I_2(\mu)^{-1}$$

- If $\eta < 3$: $c_t = I_2(\mu)^{-1} > 0, I_3(\mu) = \infty$

$$I_p(\mu) = \int \frac{d\lambda}{\lambda^p} n_\mu(\lambda)$$

- If $\eta < 2$: $c_t = I_2(\mu)^{-1} = 0, I_3(\mu) = \infty$

Derivation

Optimisation problem

- The complexities are solution of optimisation problems involving the spectral density $\rho(z)$ of $X + D_\mu$ with $X \in rEGE$

$$\Xi_{\text{tot}} = \max_{\xi} \left[-\frac{\xi^2}{2c\tau} + \frac{1}{2} \int d^2z \rho(z) \ln |(\xi - z)(\xi - \bar{z})| \right] - \int d\lambda n_\mu(\lambda) \ln \lambda$$

- The saddle-point equation reads

$$\xi_* = \frac{c\tau}{2} \partial_\xi \int d^2z \rho(z) \ln |(\xi - z)(\xi - \bar{z})| \Big|_{\xi_*}$$

Spectrum for deformation of rEGE

- One of the main difficulty of the computation is to derive:

$$\Phi_{c,\tau}(z, \bar{z}) = \lim_{N \rightarrow \infty} \frac{1}{2N} \langle \text{Tr} \ln | (z\mathbb{I} + X + D_\mu)(\bar{z}\mathbb{I} + X^\dagger + D_\mu) | \rangle_{X \in rEGE} = \frac{1}{2} \int d^2w \rho(w) \ln | (z - w)(\bar{z} - \bar{w}) |$$

$$\rho(z, \bar{z}) = \frac{1}{2\pi} \partial_{z, \bar{z}}^2 \Phi_{c,\tau}(z, \bar{z})$$

- Method 1: introducing Grassman variables
- Method 2: adapting a method by Khorenzenko ('96) to compute

$$G = \left((z\mathbb{I} - X - D^\mu)(\bar{z}\mathbb{I} - X^\dagger - D^\mu) + \kappa^2\mathbb{I} \right)^{-1}$$

$$\partial_z \Phi_{c,\tau}(z, \bar{z}) = \lim_{\kappa \rightarrow 0} \frac{1}{2N} \langle \text{Tr}[G(\bar{z}\mathbb{I} + X^\dagger + D_\mu)] \rangle_{X \in rEGE}$$

Self-consistent equations

• Defining

$$\gamma_{c,\tau}(z) = \lim_{\kappa \rightarrow 0} \int d^2w \frac{c \kappa \rho(w)}{|z - w| + \kappa^2}$$

$\gamma_{c,\tau}(z) = 0$ outside the special density
 $\gamma_{c,\tau}(z) > 0$ inside the special density

$$\Gamma_{c,\tau}(x) = c\tau \partial_x \Phi_{c,\tau}(x + iy, x - iy) = \lim_{\kappa \rightarrow 0} \int d^2w \frac{c \tau \kappa \rho(w)(x - \Re(w))}{|x + iy - w|^2 + \kappa^2}$$

Saddle-point equation:
 $\xi_* = \Gamma_{c,\tau}(\xi_*)$

These function satisfy the self-consistent equations

$$\gamma_{c,\tau}(z) = \int d\lambda \frac{c \gamma_{c,\tau}(z) n_\mu(\lambda)}{(x - \Gamma_{c,\tau}(x) - \lambda)^2 + \gamma_{c,\tau}(z)^2 + \frac{y^2}{(1 - \tau)^2}} \quad \Gamma_{c,\tau}(x) = \int d\lambda \frac{(x - \Gamma_{c,\tau}(x) - \lambda) c n_\mu(\lambda)}{(x - \Gamma_{c,\tau}(x) - \lambda)^2 + \gamma_{c,\tau}(z)^2 + \frac{y^2}{(1 - \tau)^2}}$$

$$z = x + iy$$

Expression for total complexity

- The total complexity satisfies

$$\mathbb{E}_{\text{tot}} = -\frac{\xi_*^2}{2c\tau} + \Phi_{c,\tau}(\xi_*, \xi_*) - \int d\lambda n_\mu(\lambda) \ln \lambda$$

Useful identities:

$$\partial_c \Phi_{c,\tau}(z, \bar{z}) = \frac{\gamma_{c,\tau}(z)^2}{2c^2} - \frac{\Gamma_{c,\tau}(x)^2}{2c^2\tau}$$

$$\partial_\tau \Phi_{c,\tau}(z, \bar{z}) = -\frac{\Gamma_{c,\tau}(x)^2}{2c\tau^2}$$

Self-consistent equations at the saddle-point $\xi_* = \Gamma_{c,\tau}(\xi_*)$

$$\gamma_{c,\tau}(\xi_*) = \Upsilon(c) = \int d\lambda \frac{c \Upsilon(c) n_\mu(\lambda)}{\lambda^2 + \Upsilon(c)^2} \quad \xi_* = - \int d\lambda \frac{\lambda c n_\mu(\lambda)}{\lambda^2 + \Upsilon(c)^2}$$

Expression for total complexity

Useful identities:

$$\partial_c \Phi_{c,\tau}(z, \bar{z}) = \frac{\gamma_{c,\tau}(z)^2}{2c^2} - \frac{\Gamma_{c,\tau}(x)^2}{2c^2\tau}$$

One can then compute

$$\partial_\tau \Xi_{\text{tot}} = (\Gamma_{c,\tau}(\xi_*) - \xi_*) \frac{\partial_\tau \xi_*}{c\tau} + \frac{\xi_*^2 - \Gamma_{c,\tau}(\xi_*)^2}{2c\tau^2} = 0$$

$$\partial_\tau \Phi_{c,\tau}(z, \bar{z}) = -\frac{\Gamma_{c,\tau}(x)^2}{2c\tau^2}$$

$$\partial_c \Xi_{\text{tot}} = (\Gamma_{c,\tau}(\xi_*) - \xi_*) \frac{\partial_c \xi_*}{c\tau} + \frac{\xi_*^2}{2c^2\tau} + \frac{1}{2c^2} \left(\gamma_{c,\tau}(\xi_*)^2 - \frac{\Gamma_{c,\tau}(\xi_*)^2}{\tau} \right) = \frac{\Upsilon(c)^2}{2c^2}$$

$$\Rightarrow \Xi_{\text{tot}}(c) = \int_{c_t}^c \frac{d\omega}{2\omega^2} \Upsilon(\omega)^2$$

Expression for complexity of stable equilibria

- The complexity of stable equilibria satisfies instead

$$\Xi_{\text{st}} = -\frac{\lambda_-^2}{2c\tau} + \Phi_{c,\tau}(\lambda_-, \lambda_-) - \int d\lambda n_\mu(\lambda) \ln \lambda, \quad \lambda_- = \min_k \Re(z_k)$$

Self-consistent equations at the leftmost edge $z = \lambda_-$

$$1 = \int d\lambda \frac{c n_\mu(\lambda)}{(\lambda + \nu(c))^2} \quad \nu(c) = \Gamma_{c,\tau}(\lambda_-) - \lambda_- = - \int d\lambda \frac{c \tau n_\mu(\lambda)}{\lambda + \nu(c)} - \lambda_-$$

Useful identities:

$$\partial_c \Phi_{c,\tau}(z, \bar{z}) = \frac{\gamma_{c,\tau}(z)^2}{2c^2} - \frac{\Gamma_{c,\tau}(x)^2}{2c^2\tau} \quad \partial_\tau \Phi_{c,\tau}(z, \bar{z}) = -\frac{\Gamma_{c,\tau}(x)^2}{2c\tau^2} \quad \partial_\tau \lambda_- = \partial_\tau [\Gamma_{c,\tau}(\lambda_-)] = \frac{\Gamma_{c,\tau}(\lambda_-)}{\tau} \quad \partial_c \lambda_- \Big|_{\tau=1} = \frac{\Gamma_{c,1}(\lambda_-)}{c}$$

Expression for complexity of stable equilibria

Which yields

$$\partial_\tau \Xi_{\text{st}} = (\Gamma_{c,\tau}(\lambda_-) - \lambda_-) \partial_\tau \lambda_- + \frac{\lambda_-^2}{2c\tau^2} - \frac{\Gamma_{c,\tau}(\lambda_-)^2}{2c\tau^2} = \frac{\nu(c)^2}{2c\tau^2}$$

Useful identities:

$$\Rightarrow \Xi_{\text{st}}(c, \tau) = \Xi_{\text{st}}(c, 1) - \frac{1 - \tau}{2c\tau} \nu(c)^2$$

$$\partial_c \Phi_{c,\tau}(z, \bar{z}) = \frac{\gamma_{c,\tau}(z)^2}{2c^2} - \frac{\Gamma_{c,\tau}(x)^2}{2c^2\tau}$$

$$\partial_c \Xi_{\text{st}}(c, 1) = (\Gamma_{c,1}(\lambda_-) - \lambda_-) \frac{\partial_c \lambda_- \big|_{\tau=1}}{c} + \frac{\lambda_-^2}{2c^2} - \frac{\Gamma_{c,\tau}(\lambda_-)^2}{2c^2} = \frac{\nu(c)^2}{2c^2}$$

$$\partial_\tau \Phi_{c,\tau}(z, \bar{z}) = -\frac{\Gamma_{c,\tau}(x)^2}{2c\tau^2}$$

$$\Rightarrow \Xi_{\text{st}}(c, 1) = \int_{c_t}^c \frac{d\omega}{2\omega^2} \nu(\omega)^2$$

$$\partial_\tau \lambda_- = \partial_\tau [\Gamma_{c,\tau}(\lambda_-)] = \frac{\Gamma_{c,\tau}(\lambda_-)}{\tau}$$

$$\partial_c \lambda_- \big|_{\tau=1} = \frac{\Gamma_{c,1}(\lambda_-)}{c}$$

Conclusion

Conclusion

- Extension of previous results
(only gradient descent dynamics or only homogeneous rates)
- Independence of the total complexity on the fraction of gradient term τ
- Additional transition for the complexity of minima by varying τ
- This transition should occur for the complexity of any fixed extensive index

Future directions

- Study quenched complexity
- Extension to correlated Gaussian fields:

Application to wave propagation in non-linear disordered medium (Spivak, Zyuzin '04)

$$\left(-\frac{\hbar^2}{2m} \Delta + u(\vec{r}) + \beta |\phi(\vec{r})|^2 - \epsilon \right) \phi(\vec{r}) = 0$$

$u(\vec{r})$: disordered potential ϵ : energy

Future directions

For a given realisation of $u(\vec{r})$:

$$-\frac{\hbar^2}{2m}\Delta n_i(\vec{r}) = E_i n_i(\vec{r}) \quad \int d\vec{r} n_i(\vec{r}) n_j(\vec{r}) = \delta_{ij} \quad |\phi(\vec{r})|^2 = \sum_i u_i \sqrt{E_i} n_i(\vec{r})$$

$$\left(-\frac{\hbar^2}{2m}\Delta + u(\vec{r}) + \beta |\phi(\vec{r})|^2 - \epsilon \right) \phi(\vec{r}) = 0 \Rightarrow -E_i u_i + f_i(\vec{u}) = 0$$

Through scattering computation:

$$\langle f_i(\vec{u}) f_j(\vec{u}) \rangle = a \delta_{ij} \quad \langle f_i(\vec{u}) \partial_{u_j} f_k(\vec{u}) \rangle = 0 \quad \langle \partial_{u_i} f_j(\vec{u}) \partial_{u_k} f_l(\vec{u}) \rangle \propto \frac{(j/l)^{1/3} + (l/j)^{1/3}}{(|i-k| + |j-l|)^{2/3}}$$

Happy birthday Yaw!