Universality for free fermions point processes

Joint work with A. Deleporte. arXiv: 2109.02121

1. Probabilistic model of free fermions on $\mathbb{R}^d$.
2. Relation to random matrix theory.
3. Limits of the correlation functions.
4. Central limit theorems

1 Model

Quantum state of a particle is described by a wave function $\psi \in L^2(\mathbb{R}^3)$.

$|\psi(x)|^2 =$ p.d.f. for the position x

$|\hat{\psi}(\xi)|^2 =$ p.d.f. for the momentum ξ $(m=1)$

Many-body system $\psi \in L^2(\mathbb{R}^{3N})$ N particles.

$|\psi(x_1,\dots,x_N)|^2$ is a probability measure invariant under S_N.

$\psi \in L^2_A(\mathbb{R}^{3N})$ "Antisymmetric" $\boxed{\text{Fermions}}$

Quantum evolution is given by the Schrödinger eq.

$$i\hbar \frac{d}{dt}\psi = H_N \psi$$

Planck cst $\hbar \simeq 10^{-34}$ J·s

Hamiltonian - operator on $L^2_A(\mathbb{R}^{3N})$ $\boxed{H \geq 0}$

Stationary states: $H_N \Psi = \Lambda \Psi$

energy of the state

$$H_N = \sum_{j=1}^{N} \left(-\hbar^2 \Delta_{x_j} + V(x_j) \right) + \frac{\beta}{N} \sum_{i<j} W(x_i, x_j)$$

kinetic energy   trap (external potential $V : \mathbb{R}^d \to \mathbb{R}_+$ continuous growing)

$\beta = 0$ — non-interacting or free fermions.

H_N has pure point spectrum $\{\Lambda_j\}_{j \in \mathbb{N}_0}$; $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots$

$\hbar \propto N^{-1/d}$; $\Lambda_1 \simeq N \lambda_1$.

Ground state: $\langle \Psi | H_N \Psi \rangle = \Lambda_1$

Slater det: $\Psi(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det_{N \times N} \left(\phi_i(x_j) \right)$

$\{\phi_i\}$ orthonormal family in $L^2(\mathbb{R}^3)$

$$\underbrace{\left(-\hbar^2 \Delta + V \right)}_{H} \phi_j = \varepsilon_j \phi_j \quad ; \quad \varepsilon_1(\hbar) \leq \varepsilon_2(\hbar) \leq \cdots$$

Fix $\mu > 0$ and let $N = \# \{ i : \varepsilon_i(\hbar) \leq \mu \}$.

Fermi energy

Limit as $\hbar \to 0$.

$\hbar$ = "microscopic scale".

$$\boxed{N \simeq \left(\frac{\omega_d}{2\pi\hbar} \right)^d Z_\mu}$$

Probability on $\mathbb{R}^{3N}$: $\mathbb{P}_N(x_1, \dots, x_N) = \frac{1}{N!} \left| \det_{N\times N} \left(\phi_i(x_j) \right) \right|^2$

$$= \frac{1}{N!} \det_{N\times N} \left[\Pi_N (x_i, x_j) \right]$$

$$\boxed{\Pi_N(x,y) = \sum_{i=1}^{N} \phi_i(x)\, \phi_j(y)}$$

$\mathbb{P}_N$ is a det. point process on $\mathbb{R}^d$ with kernel Π_N

1) The correlation fcts of the measure $\mathbb{P}_N$ are

$$\rho_k(x_1, \dots, x_k) = \det_{k\times k} \left[\Pi_N(x_i, x_j) \right] \quad \text{for } k \in \mathbb{N}.$$

2) For $g : \mathbb{R}^d \to \mathbb{R}_+$

$$\mathbb{E}_N \left[\prod_{j=1}^{N} g(x_j) \right] = \det_{L^2(\mathbb{R}^d)} \left[1 + (g-1) \widehat{\Pi}_N \right].$$

3) $$\boxed{\widehat{\Pi}_N = \mathbb{1}_{\{H \leq \mu\}}}$$ orthogonal proj.

- Like an orthogonal polynomial ensemble
  but no recurrence relations for $(\phi_i)_{i \in \mathbb{N}}$
  no Christoffel - Darboux formula
  no Riemann - Hilbert problem

(2) Random matrices ($d=1$)

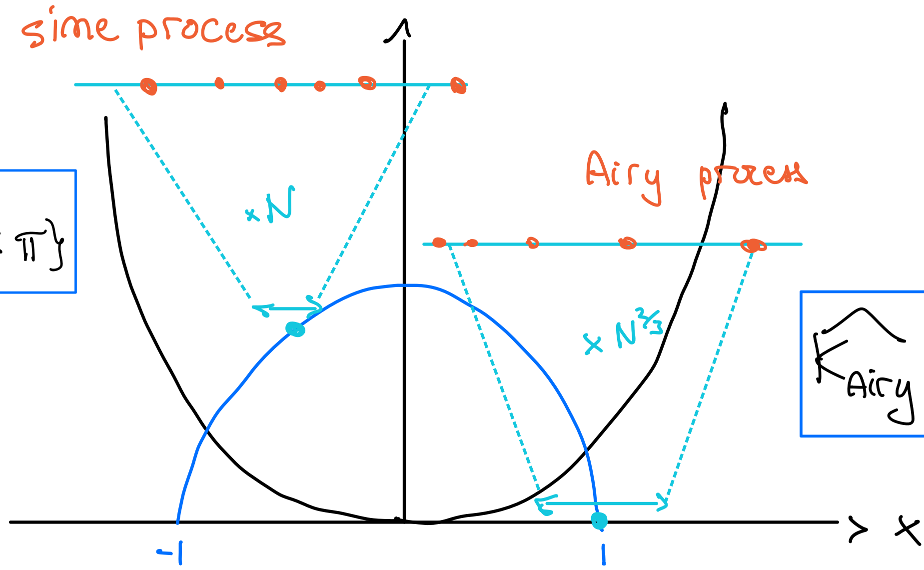
Proposition: The GUE is the ground state of a Free Fermi gas confined by $V(x) = x^2$ on $\mathbb{R}$ with $\begin{cases} \hbar = \frac{1}{2N} \\ \mu = 1 \end{cases}$.

Idea: Hermite functions satisfy the ODE

$$\left(-\frac{1}{4N^2}\Delta + x^2\right)\phi_j = \frac{j-\frac{1}{2}}{N}\phi_j \quad ; \quad j \in \mathbb{N}$$

Thm: For GUE, as $\hbar \to 0$ (or $N \to \infty$)

- $\mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j} \longrightarrow \frac{2}{\pi}\sqrt{1-x^2}\, dx$.
- For $x \in (-1,1)$; $\frac{1}{\rho(x)N} K_N\left(x + \frac{u}{\rho(x)N}, x + \frac{v}{\rho(x)N}\right) \longrightarrow K_{\text{sine}}(u,v)$.
- For $x \in \{\pm 1\}$; $\frac{1}{\rho(x)N} K_N\left(x \pm \frac{u}{(2N)^{2/3}}, x \pm \frac{v}{(2N)^{2/3}}\right) \longrightarrow K_{\text{Airy}}(u,v)$.



$\boxed{\widehat{K_{\text{sine}}} = \mathbb{1}_{\{-\Delta \leq \pi\}}}$

$\boxed{\widehat{K_{\text{Airy}}} = \mathbb{1}_{\{-\Delta + x \leq 0\}}}$

Analogous results for free fermions in a general trap? Universality?

V. Eisler. Universality in the full counting statistics of trapped fermions. *Physical review letters*, 111(8):080402, 2013.

D. S. Dean, P. Le Doussal, S. N. Majumdar, and G. Schehr. Universal ground-state properties of free fermions in a d-dimensional trap. *EPL (Europhysics Letters)*, 112(6):60001, 2015.

Weyl's law

D. Robert and B. Helffer. Comportement semi-classique du spectre des hamiltoniens quantiques elliptiques. In *Annales de l'institut Fourier*, volume 31, pages 169–223, 1981.

$$\begin{cases} \text{Number of particles} \quad N = \mathrm{Tr}(\Pi_N) \sim \left(\dfrac{\omega_d}{2\pi\hbar}\right)^d Z_\mu \\ N^{-1}\,\Pi_N(x,x) \longrightarrow \rho(x) = \dfrac{1}{Z_\mu}\big(\mu - V(x)\big)_+^{d/2} \quad \text{as } \hbar \to 0. \end{cases}$$

Assumptions: $\mu > 0$ is fixed,

- $V \in L^1_{loc}(\mathbb{R}^d \to \mathbb{R}_+)$.
- $\{V \leq \mu + \delta\}$ is compact & $V \in C^\infty$ on this set.

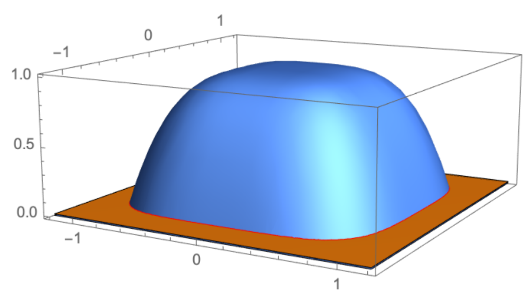
Thm 1 (Law of large number – Density of states)

For any $\varepsilon > 0$, $\quad \mathbb{P}_N\left[\mathrm{dist}\left(\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}, \rho\right) \geq \varepsilon\right] \leq C_\varepsilon\, e^{-cN\varepsilon^2}.$

Ex: 1) $V(x) = x^2$ on $\mathbb{R}$ and $\mu = 1$ (GUE)

$$\rho(x) = \frac{2}{\pi}\sqrt{1-x^2}.$$

2) $V(x) = |x|^4$ on $\mathbb{R}^2$ and $\mu = 1$



③ Limits of correlation functions

Thm 2 (scaling limits)

- For $z \in \{V < \mu\}$, as $\hbar \to 0$,

$$\left\{(x_j - z)(N\rho(z))^{\frac{1}{d}}\right\}_{j=1}^{N} \xrightarrow{d} \text{determinantal point process on } \mathbb{R}^d \text{ with op. } \hat{K}_d = \mathbb{1}_{\{-\Delta \leq \frac{2\pi}{\omega_d}\}}.$$

- For $z \in \{V = \mu, \nabla V \neq 0\}$, as $\hbar \to 0$,

$$\left\{(x_j - z)\,\hbar^{2/3}\right\}_{j=1}^{N} \xrightarrow{d} \text{determinantal point process on } \mathbb{R}^d \text{ with op. } \hat{K}_{Ai,d} = \mathbb{1}_{\{-\Delta + \nabla V(z)\cdot x \leq 0\}}$$

Bulk kernel:

$$K_d(u,v) = \int_{\mathbb{R}^d} \mathbb{1}_{\{|\xi| \leq \frac{1}{\omega_d}\}} e^{2\pi i \xi\cdot(u-v)} d\xi = \frac{J_{d/2}\left(\frac{2\pi|u-v|}{\omega_d}\right)}{(\omega_d|u-v|)^{d/2}}$$

In 1d, $J_{1/2}(r) = \sqrt{\frac{2}{\pi r}}\sin(r)$.

$$|K_d(u,v)|^2 \sim \frac{1}{\omega_d|u-v|^{d+1}} \cdot \frac{\sin^2(\cdot)}{\pi^2} \quad \text{as} \quad |u-v| \to \infty.$$

S. Torquato, A. Scardicchio, and C. E. Zachary. Point processes in arbitrary dimension from fermionic gases, random matrix theory, and number theory. *Journal of Statistical Mechanics: Theory and Experiment*, 2008(11):P11019, 2008.

D. S. Dean, P. Le Doussal, S. N. Majumdar, and G. Schehr. Universal ground-state properties of free fermions in a d-dimensional trap. *EPL (Europhysics Letters)*, 112(6):60001, 2015.

(4) Fluctuations of linear statistics

Thm 3 (Concentration bounds)

Let $f \in C_c^\infty(\{V < \mu\})$. For any $t \geq 0$,

$$\mathbb{P}_N\left[\left|\sum_{j=1}^N f(x_j) - \mathbb{E}\sum_{j=1}^N f(x_j)\right| \geq \sqrt{N\hbar}\, t\right] \leq 2e^{-ct^2}.$$

Thm 4 (Central limit theorem)

For $d \geq 2$, if $0 < \int_{\mathbb{R}^d} |\nabla f|^2 < \infty$, then as $N \to \infty$,

$$\frac{\sum_{j=1}^N f(x_j) - \mathbb{E}\sum_{j=1}^N f(x_j)}{\sqrt{\mathrm{Var}\left(\sum_{j=1}^N f(x_j)\right)}} \xrightarrow{d} \mathcal{N}_{0,1}.$$

Idea: $\mathrm{Var}\left(\sum_{j=1}^N f(x_j)\right) \longrightarrow \infty$ as $N \to \infty$.

Q: Obtain asymptotics of $\mathrm{Var}\left(\sum_{j=1}^N f(x_j)\right) \sim N\hbar\, \Sigma_V(f)$

What is $\Sigma_V(f)$?

Thm 5 (counting statistics)

Let $\Omega_1, \dots, \Omega_k$ be smooth subsets of $\{V < \mu\}$

Let

$$X(\Omega) = \frac{\sum_{j=1}^{N} \mathbb{1}(x_j \in \Omega) - N \int_{\Omega} \rho_V}{\sqrt{N \frac{2\pi \cancel{\hbar}}{\omega_d Z_\mu} \log N \frac{\omega_{d-1}}{\pi^2} \int_{\partial\Omega} (\mu - V(x))^{\frac{d-1}{2}} \, dx}}$$

If $\operatorname{meas}_{d-1}(\partial\Omega_i \cap \partial\Omega_j) = 0$, then

$$\big(X(\Omega_1), \dots, X(\Omega_k)\big) \xrightarrow{d} \mathcal{N}_{0, I_k}.$$

Happy Birthday Yan

Thank you!

Semiclassical analysis

$$H = -\hbar^2\Delta + V = \mathrm{Op}_\hbar\left((x,\xi)\to|\xi|^2+V(x)\right)$$

$$\mathrm{Op}_\hbar(a)(x,y)\mapsto \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\frac{(x-y)\cdot\xi}{\hbar}}\, a\left(\tfrac{x+y}{2},\xi\right)d\xi$$

- If $a\in C_c^\infty(\mathbb{R}^{2d})$, $\mathrm{Op}_\hbar(a)$ is trace-class.
- In general a is a symbol:

$$a(x,\xi;\hbar) = \underbrace{a_0(x,\xi)+\hbar\, a_1(x,\xi)+\hbar^2 a_2(x,\xi)+\dots}_{a^2(x,\xi;\hbar)}$$

For any $k\in\mathbb{N}_0$,

$$\mathrm{Op}_\hbar(a)(x,y)\mapsto \frac{1}{(2\pi\hbar)^d}\left\{\int_{\mathbb{R}^d} e^{i\frac{(x-y)\cdot\xi}{\hbar}}\, a^k\left(\tfrac{x+y}{2},\xi,\hbar\right)+\underbrace{\mathcal{O}(\hbar^{k+1})}_{\text{in trace norm}}\right\}$$

<u>Functional calculus:</u> For any $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}\to\mathbb{R}_+)$

$$\varphi(-\hbar^2\Delta+V) = \mathrm{Op}_\hbar(a)$$

where $a_0(x,\xi)=\varphi(|\xi|^2+V(x))$ & a_k are supported on a small neighborhood of $\{(x,\xi):\varphi(|\xi|^2+V(x))>0\}$.

Approximation of free fermions kernel:

$$\hat{\Pi}_N = \mathbb{1}_{\{H\leq\mu\}}\ (x,y)\mapsto \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\frac{(x-y)\cdot\xi}{\hbar}}\,\mathbb{1}_{\{|\xi|^2+V(\frac{x+y}{2})\leq\mu\}}\,d\xi$$

__Sketch:__ The kernel of the rescaled point process $\left\{\frac{x_j - z}{\varepsilon}\right\}_{j=1}^{N}$ is

$$\varepsilon^d \, \Pi_N\left(z+\varepsilon u \, , \, z+\varepsilon v\right) \simeq \frac{\varepsilon^d}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\varepsilon\frac{(u-v)\cdot\xi}{\hbar}} \, \mathbb{1}_{\left\{|\xi|^2 + V\left(z + \frac{u+v}{2}\varepsilon\right) \leq \mu\right\}} \, d\xi$$

$\boxed{\varepsilon = \hbar/a}$

$$\simeq \frac{1}{(2\pi a)^d} \int_{\mathbb{R}^d} e^{i\frac{(u-v)\cdot\xi}{a}} \, \mathbb{1}_{\left\{|\xi|^2 \leq \underbrace{\mu - V(z)}_{>0 \text{ if } z \in \text{bulk}} + O(\hbar)\right\}} \, d\xi$$

$\boxed{2\pi a = \omega_d \sqrt{\mu - V(z)}}$

$$\simeq \underbrace{\int_{\mathbb{R}^d} e^{i 2\pi (u-v)\cdot\xi} \, \mathbb{1}_{\left\{|\xi| \leq \frac{1}{\omega_d}\right\}} \, d\xi}_{K_d(u,v) \text{ — density } 1 \text{ on } \mathbb{R}^d}$$

*

At the edge, $\boxed{V(z) = \mu}$, the operator associated with the rescaled point process $\left\{\frac{x_j - z}{\varepsilon}\right\}_{j=1}^{N}$ is

$$\mathbb{1}_{\left\{-\hbar^2\varepsilon^{-2}\Delta + V(z+\varepsilon\,\cdot) \leq \mu\right\}} \overset{\boxed{\varepsilon = \hbar^{2/3}}}{\simeq} \underbrace{\mathbb{1}_{\left\{-\Delta + \nabla V(z)\cdot x \leq 0\right\}}}_{\equiv \mathbb{1}_{\{-\Delta + x_1 \leq 0\}} \text{ if } \boxed{\nabla V(z) \neq 0}}$$

__Edge kernel:__

$$K_{Ai,d}(x,y) = \int_0^\infty Ai(x_1+s)\, Ai(x_2+s) \underbrace{\frac{J_{\frac{d-1}{2}}\left(\sqrt{s}\,|x^\perp - y^\perp|\right)}{\left(2\pi |x^\perp - y^\perp|\right)^{\frac{d-1}{2}}}}_{=1 \text{ if } d=1} s^{\frac{d-1}{2}} \, ds$$

$x = (x_1, x^\perp)$ & $y = (y_1, y^\perp)$