Generalized multifractality
at Anderson-localization critical points

Alexander D. Mirlin

Karlsruhe Institute of Technology

$\approx$ 30 years ago

J.F. Karcher, N. Charles, I.A. Gruzberg, A.D.M., Ann.Phys. 435, 168584 (2021)

J.F. Karcher, I.A. Gruzberg, A.D.M., PRB 105, 184205 (2022)

J.F. Karcher, I.A. Gruzberg, A.D.M., arXiv:2206.12226 (2022)

J.F. Karcher, I.A. Gruzberg, A.D.M., in preparation

I.A. Gruzberg, A.D.M., M.R. Zirnbauer, PRB 87, 125144 (2013)

J. Karcher, Karlsruhe

I. Gruzberg, Ohio

N. Charles, Ohio

M. Zirnbauer, Köln

Anderson localization

Philip W. Anderson

1958 “Absence of diffusion in certain random lattices”

Quantum particle moving in a random potential

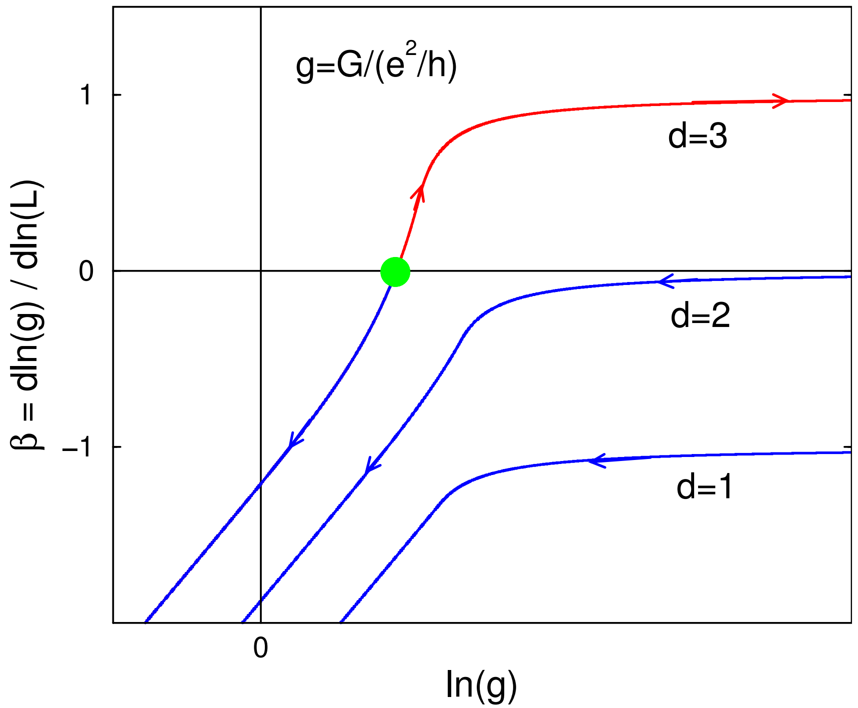
sufficiently strong disorder $\longrightarrow$ quantum localization

$\longrightarrow$ **eigenstates exponentially localized, no diffusion**

$\longrightarrow$ **Anderson insulator**

Nobel Prize 1977

Anderson Metal-Insulator Transitions



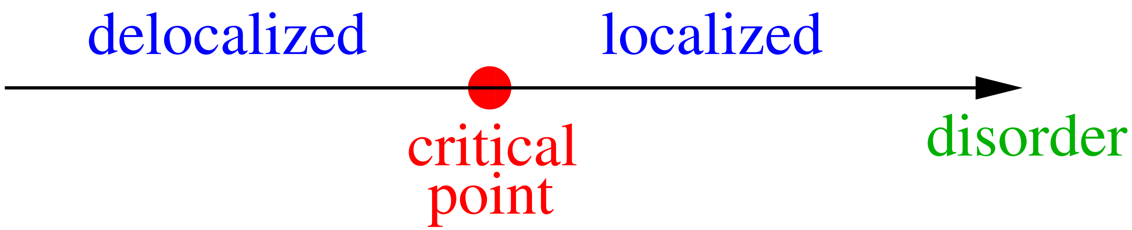
Connection with scaling theory of critical phenomena: Thouless '74; Wegner '76

Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

scaling variable:
dimensionless conductance $g = G/(e^2/h)$

RG for field theory: non-linear σ-model
Wegner '79

$d > 2$: **Anderson metal-insulator transition**



in some symmetry classes there is an Anderson MIT also in 2D

review: Evers, ADM, Rev. Mod. Phys. 80, 1355 (2008)

Field theory: non-linear σ-model

action:

$$S[Q] = \frac{\pi\nu}{4}\int d^d r \, \mathrm{Tr}\,[-D(\nabla Q)^2 - 2i\omega\Lambda Q], \qquad Q^2(r) = 1$$

Wegner'79

σ**-model manifold:** symmetric space

e.g., unitary Wigner-Dyson symmetry class:

- bosonic replicas: $\mathcal{M}_B = \mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n)$, $\quad n \to 0$

non-compact ("hyperboloid")

- fermionic replicas: $\mathcal{M}_F = \mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$, $\quad n \to 0$

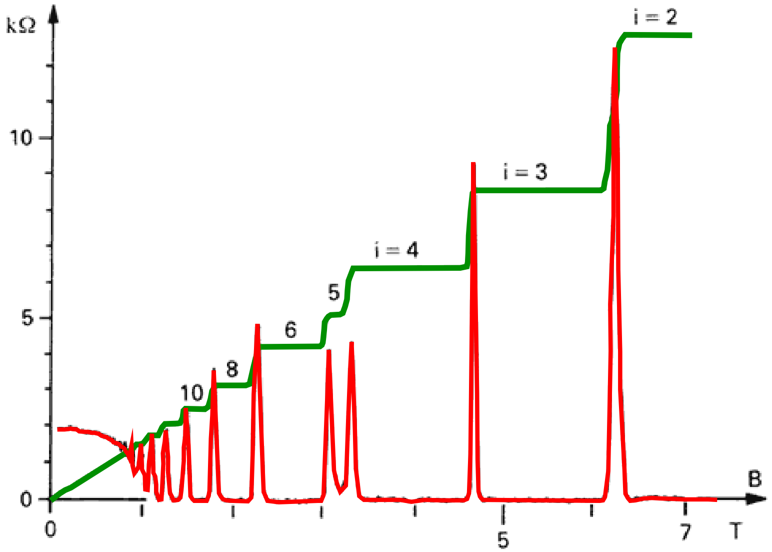
compact ("sphere")

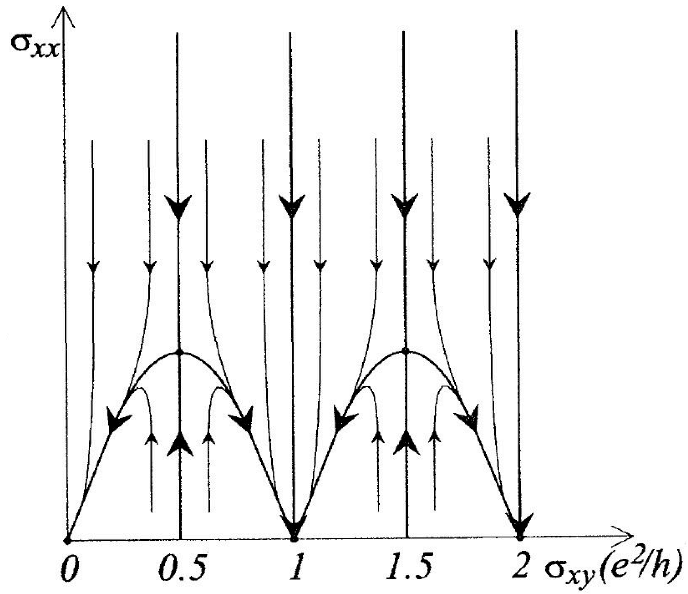
- supersymmetry (Efetov'83): $\mathrm{U}(1,1|2)/\mathrm{U}(1|1)\times\mathrm{U}(1|1)$

$\mathcal{M} = \{\mathcal{M}_B \times \mathcal{M}_F\}$ "dressed" by anticommuting variables

Anderson localization and topology

Paradigmatic example: Integer Quantum Hall Effect





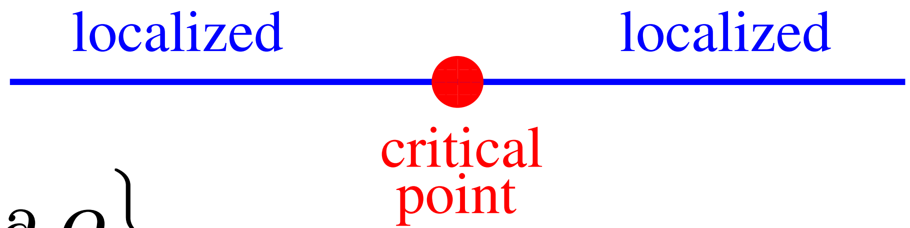
von Klitzing '80 ; Nobel Prize '85

IQHE flow diagram

Khmelnitskii' 83, Pruisken' 84

Field theory:

σ-model with topological term



$$S = \int d^2r \left\{ -\frac{\sigma_{xx}}{8} \mathrm{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \mathrm{Tr} \epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

Anderson-localization transitions between topologically distinct insulating phases

Also: Anderson-localization critical theories emerge on surfaces of disordered topological insulators and superconductors.

Multifractality at Anderson transitions

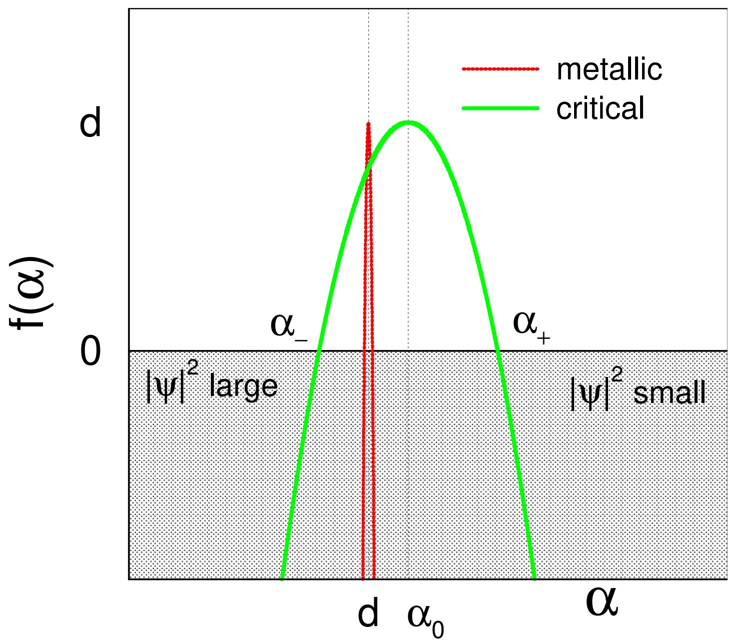
$P_q = \int d^d r |\psi(\mathrm{r})|^{2\mathrm{q}}$ inverse participation ratio

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

$\tau_q = \underbrace{d(q-1)}_{\text{normal}} + \underbrace{\Delta_q}_{\text{anomalous}}$ multifractality

$\tau_q \longrightarrow$ Legendre transformation

$\longrightarrow$ singularity spectrum $f(\alpha)$



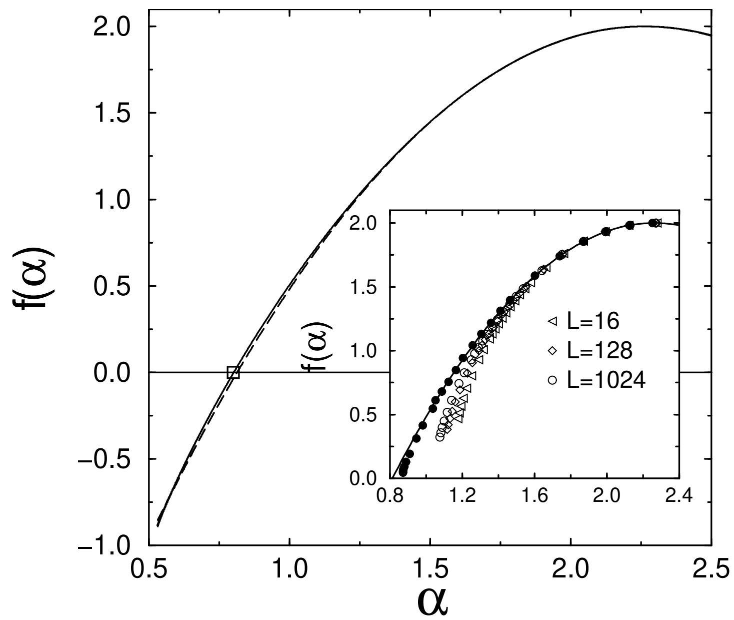
wave function statistics:

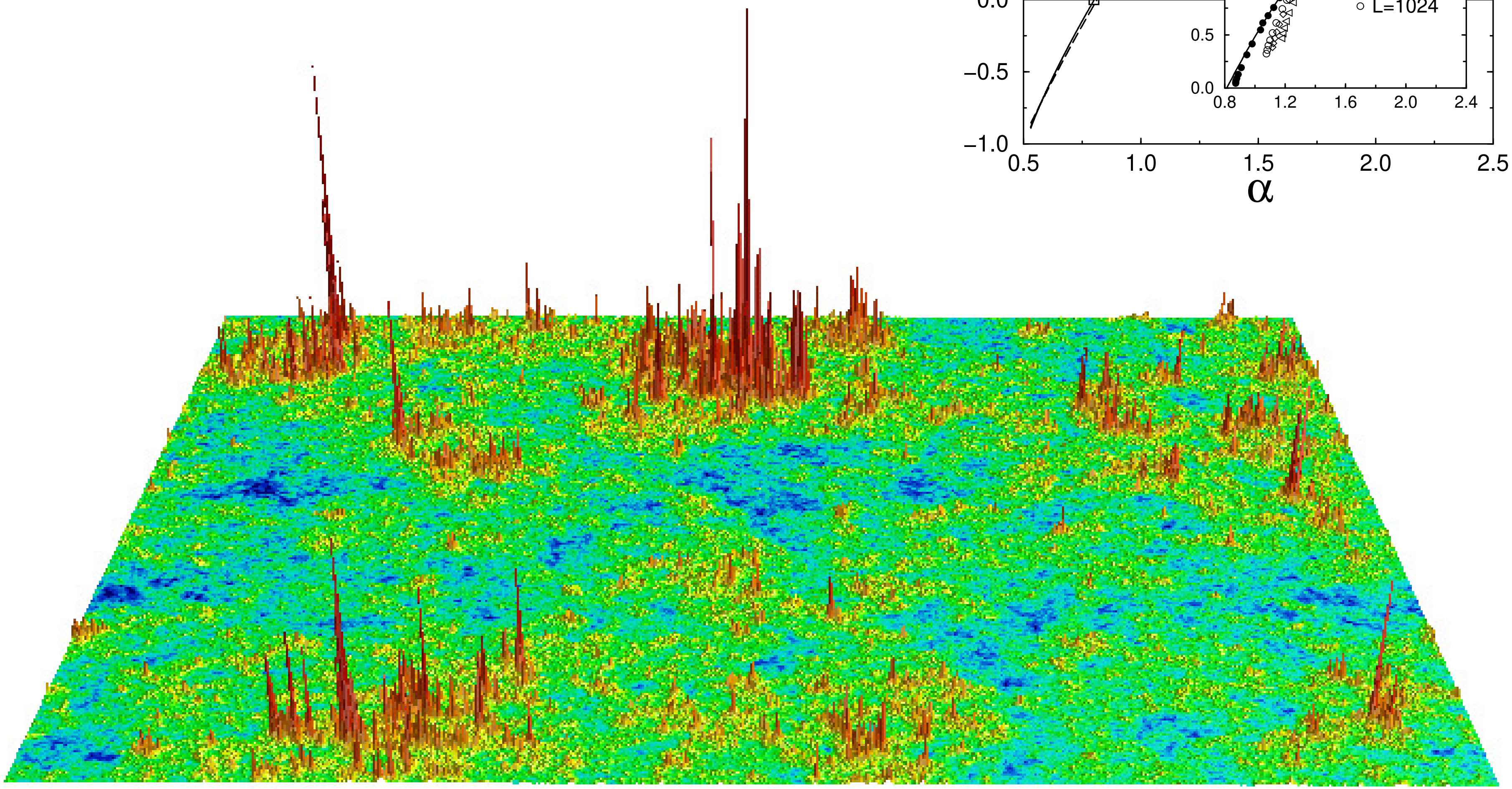
$$\mathcal{P}(\ln |\psi^2|) \sim L^{-d+f(\ln|\psi^2|/\ln L)}$$

$L^{f(\alpha)}$ – measure of the set of points where $|\psi|^2 \sim L^{-\alpha}$

Example: Multifractality at the Quantum Hall transition

Evers, Mildenberger, ADM '01





Multifractality in terms of local DOS

LDOS moments at criticality $\langle \rho^q \rangle \sim L^{-x_q}$

- **Wigner-Dyson classes:**

  average LDOS $\langle \rho \rangle$ is not critical

$$x_1 \equiv 0 \qquad\qquad x_q \equiv \Delta_q$$

- **Unconventional classes (chiral or particle-hole symmetry):**

  average LDOS in general critical $\langle \rho \rangle \sim L^{-x_1}$

$$\Delta_q = x_q - q x_1$$

Symmetry of distribution of local DOS (Wigner-Dyson classes)

Non-linear σ-model

$\longrightarrow$ **distribution of local Green function** $\quad G_R(r,r)/\pi\langle\rho\rangle = u - i\tilde{\rho}$

$$P(u,\tilde{\rho}) = \frac{1}{2\pi\tilde{\rho}} P_0\left((u^2+\tilde{\rho}^2+1)/2\tilde{\rho}\right)$$

$\longrightarrow$ **symmetry of LDOS distribution:** $\quad \mathcal{P}_\rho(\tilde{\rho}) = \tilde{\rho}^{-3}\mathcal{P}_\rho(\tilde{\rho}^{-1})$

ADM, Fyodorov '94 $\quad (\beta = 2)$

Alternative derivation (for $\beta = 1, 2, 4$) via relation to a scattering problem: system with a channel attached at a point r

S-matrix $\quad S = (1 - iK)/(1 + iK) \qquad K = \frac{1}{2}V^\dagger G_R(r,r)V = u - i\tilde{\rho}$

distribution $\;P(S)\;$ **invariant with respect to the phase θ of $S = \sqrt{r}e^{i\theta}$.**

Fyodorov, Savin, Sommers '04-05

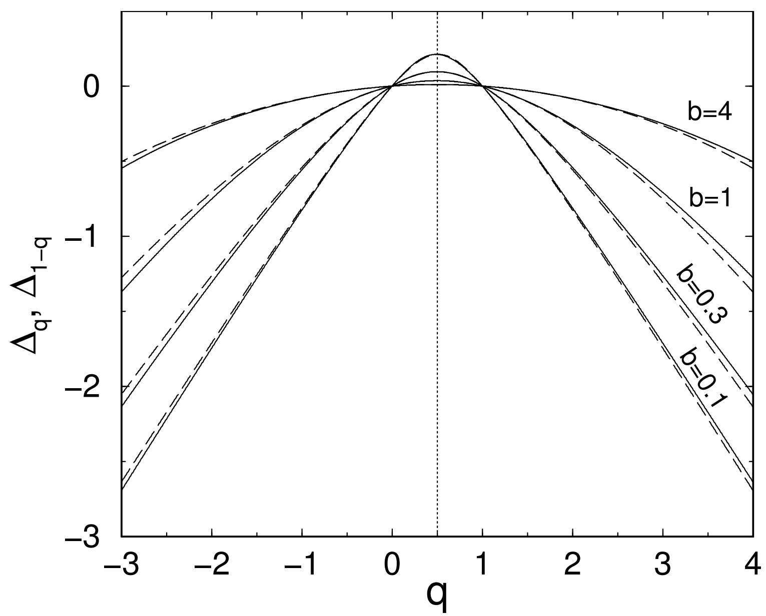
Symmetry of multifractal spectra (Wigner-Dyson classes)

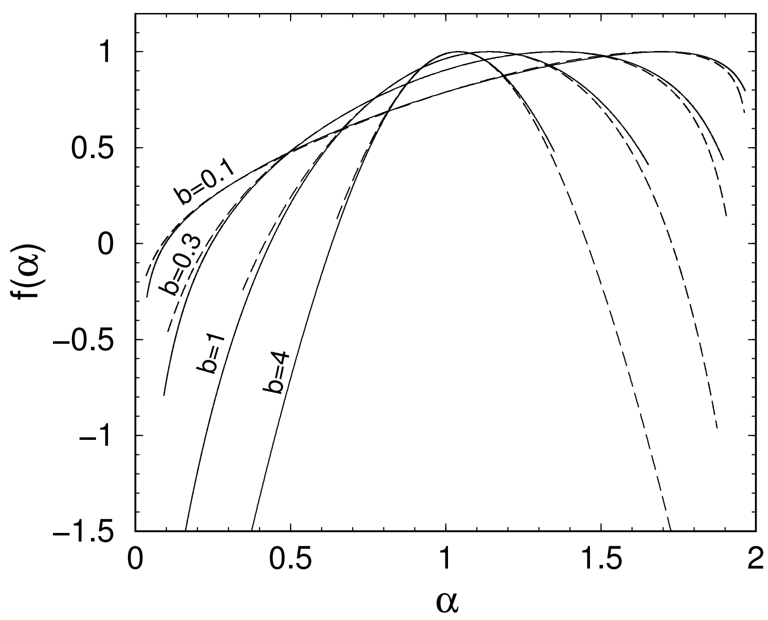
ADM, Fyodorov, Mildenberger, Evers '06

LDOS distribution in σ-model $+$ **universality**

$\longrightarrow$ **exact symmetry of the multifractal spectrum:**

$$\Delta_q = \Delta_{1-q} \qquad\qquad f(2d-\alpha) = f(\alpha) + d - \alpha$$





$\longrightarrow$ **probabilities of unusually large**

and unusually small $|\psi^2(r)|$ are related !

Disordered electronic systems: 10 symmetry classes

Altland, Zirnbauer '97

Wigner-Dyson classes

	$\hat{T}$	$\hat{P}$	$\hat{C}$	symbol
unitary				A
orthogonal	1			AI
symplectic	−1			AII

$\hat{T}$ – antiunitary, commutes with H,
$\hat{T}^2 = \pm 1$

$\hat{C}$ – unitary, anticommutes with H,
$\hat{C}^2 = 1$

$\hat{P}$ – antiunitary, anticommutes with H,
$\hat{P}^2 = \pm 1$

Chiral classes

	$\hat{T}$	$\hat{P}$	$\hat{C}$	symbol
unitary			1	AIII
orthogonal	1	1	1	BDI
symplectic	−1	−1	1	CII

$$H = \begin{pmatrix} 0 & \mathrm{t} \\ \mathrm{t}^\dagger & 0 \end{pmatrix}$$

Bogoliubov-de Gennes classes

$\hat{T}$	$\hat{P}$	$\hat{C}$	symbol
	−1		C
1	−1	1	CI
	1		D
−1	1	1	DIII

$$H = \begin{pmatrix} \mathrm{h} & \Delta \\ -\Delta^* & -\mathrm{h}^T \end{pmatrix}$$

Disordered electronic systems: Sigma-model target spaces

Symmetry Class	NLσM (n-c|c)	Compact (fermionic) space	Non-compact (bosonic) space
A	AIII|AIII	$U(2n)/U(n)\times U(n)$	$U(n,n)/U(n)\times U(n)$
AI	BDI|CII	$Sp(4n)/Sp(2n)\times Sp(2n)$	$SO(n,n)/SO(n)\times SO(n)$
AII	CII|BDI	$SO(2n)/SO(n)\times SO(n)$	$Sp(2n,2n)/Sp(2n)\times Sp(2n)$
AIII	A|A	$U(n)$	$GL(n,\mathbb{C})/U(n)$
BDI	AI|AII	$U(2n)/Sp(2n)$	$GL(n,\mathbb{R})/O(n)$
CII	AII|AI	$U(n)/O(n)$	$GL(n,\mathbb{H})/Sp(2n)$ $\equiv U^*(2n)/Sp(2n)$
C	DIII|CI	$Sp(2n)/U(n)$	$SO^*(2n)/U(n)$
CI	D|C	$Sp(2n)$	$SO(n,\mathbb{C})/SO(n)$
BD	CI|DIII	$O(2n)/U(n)$	$Sp(2n,\mathbb{R})/U(n)$
DIII	C|D	$O(n)$	$Sp(2n,\mathbb{C})/Sp(2n)$

Generalized multifractality:
Eigenfunction pure-scaling observables

$\boldsymbol{\lambda} = (q_1, \ldots, q_n)$ **labels representations.**

Here q_j can be in general arbitrary complex numbers.

Building blocks: $\boldsymbol{\lambda} = \underbrace{(1, \ldots, 1)}_{m \text{ times}} \equiv (1^m)$

"Spinless" symmetry classes: neither $\hat{T}^2 = -1$ nor $\hat{P}^2 = -1$

$\longrightarrow$ classes **A, AI, AIII, BDI, D**

$$P_{(1^m)}[\psi] = |\det(\psi_i(\mathbf{r}_j))_{m\times m}|^2 \equiv \left| \det \begin{pmatrix} \psi_1(\mathbf{r}_1) & \psi_2(\mathbf{r}_1) & \ldots & \psi_m(\mathbf{r}_1) \\ \psi_1(\mathbf{r}_2) & \psi_2(\mathbf{r}_2) & \ldots & \psi_m(\mathbf{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(\mathbf{r}_m) & \psi_2(\mathbf{r}_m) & \ldots & \psi_m(\mathbf{r}_m) \end{pmatrix} \right|^2$$

$\mathbf{r}_1, \ldots \mathbf{r}_\mathrm{m}$ — **close spatial points,** $\psi_1, \ldots \psi_m$ — **close-in-energy eigenfunctions**

For arbitrary $\boldsymbol{\lambda} = (q_1, \ldots, q_n)$:

$$P_\lambda[\psi] = (P_{(1^1)}[\psi])^{q_1-q_2}(P_{(1^2)}[\psi])^{q_2-q_3} \cdots (P_{(1^{n-1})}[\psi])^{q_{n-1}-q_n}(P_{(1^n)}[\psi])^{q_n}$$

$\langle P_\lambda[\psi] \rangle$ **are pure-scaling eigenfunction observables.**

The proof goes via a mapping to the sigma model.

Generalized multifractality:
Eigenfunction pure-scaling observables. Spinful case

"Spinful" symmetry classes: either $\hat{T}^2 = -1$ or $\hat{P}^2 = -1$ (or both)

$\longrightarrow$ Kramers-type (near-)degeneracy, classes AII, CII, C, CI, DIII

Building blocks: $\boldsymbol{\lambda} = (1^m)$

$$P_{(1^m)}[\psi] = \det \left(\begin{array}{c|c} (\psi_{i,\uparrow}(\mathbf{r}_j))_{m\times m} & (\psi_{\bar{\imath},\uparrow}(\mathbf{r}_j))_{m\times m} \\ \hline (\psi_{i,\downarrow}(\mathbf{r}_j))_{m\times m} & (\psi_{\bar{\imath},\downarrow}(\mathbf{r}_j))_{m\times m} \end{array} \right)$$

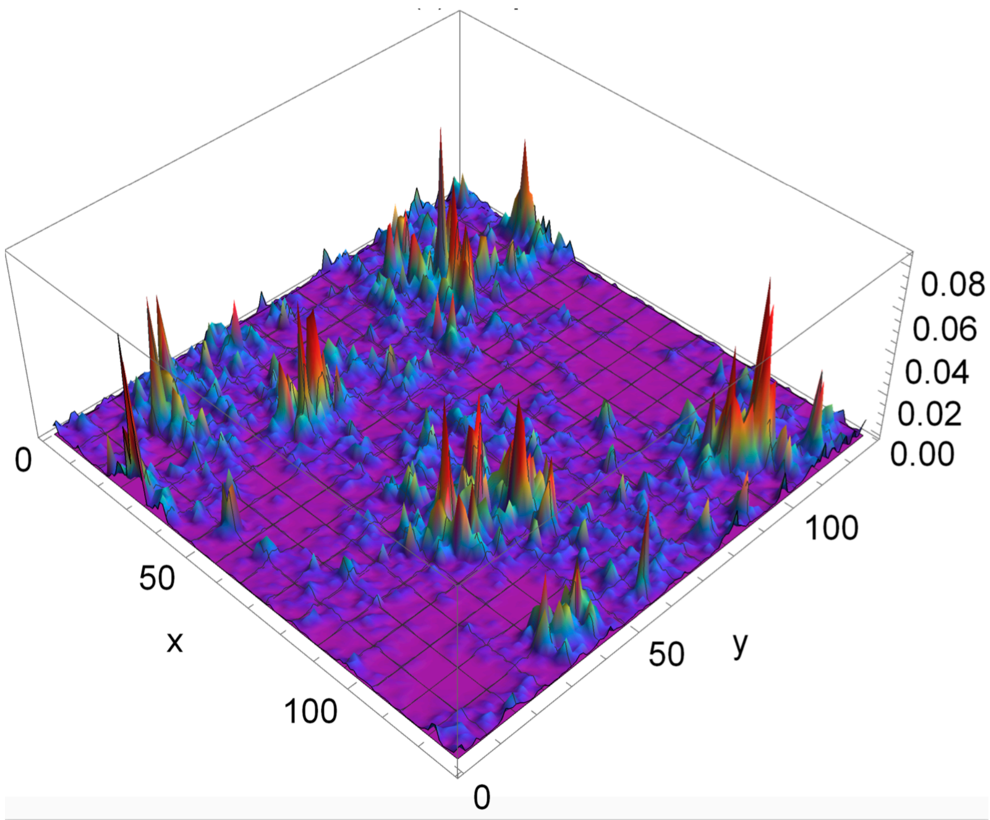
$$= \det \left(\begin{array}{ccc|ccc} \psi_{1,\uparrow}(\mathbf{r}_1) & \dots & \psi_{m,\uparrow}(\mathbf{r}_1) & \psi_{\bar{1},\uparrow}(\mathbf{r}_1) & \dots & \psi_{\bar{m},\uparrow}(\mathbf{r}_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \psi_{1,\uparrow}(\mathbf{r}_m) & \dots & \psi_{m,\uparrow}(\mathbf{r}_m) & \psi_{\bar{1},\uparrow}(\mathbf{r}_m) & \dots & \psi_{\bar{m},\uparrow}(\mathbf{r}_m) \\ \hline \psi_{1,\downarrow}(\mathbf{r}_1) & \dots & \psi_{m,\downarrow}(\mathbf{r}_1) & \psi_{\bar{1},\downarrow}(\mathbf{r}_1) & \dots & \psi_{\bar{m},\downarrow}(\mathbf{r}_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \psi_{1,\downarrow}(\mathbf{r}_m) & \dots & \psi_{m,\downarrow}(\mathbf{r}_m) & \psi_{\bar{1},\downarrow}(\mathbf{r}_m) & \dots & \psi_{\bar{m},\downarrow}(\mathbf{r}_m) \end{array} \right)$$

For arbitrary $\boldsymbol{\lambda} = (q_1, \dots, q_n)$:

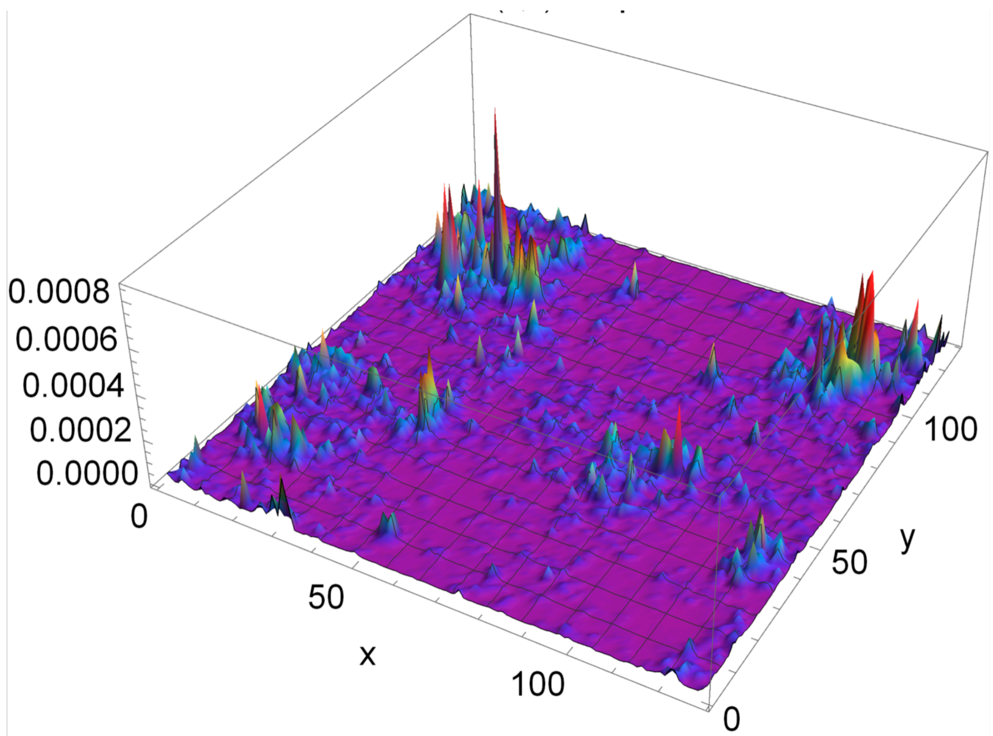
$$P_\lambda[\psi] = (P_{(1^1)}[\psi])^{q_1 - q_2} (P_{(1^2)}[\psi])^{q_2 - q_3} \cdots (P_{(1^{n-1})}[\psi])^{q_{n-1} - q_n} (P_{(1^n)}[\psi])^{q_n}$$

$\langle P_\lambda[\psi] \rangle$ are pure-scaling eigenfunction observables.

The proof goes via a mapping to the sigma model.



$\boldsymbol{P}_{(1)}[\psi]$



$\boldsymbol{P}_{(1,1)}[\psi]$

Scaling operators in sigma-model formalism

- Spinless classes: σ-model composite operators corresponding to wave function correlators $P_{(q_1,\ldots,q_n)}$ are

$$\mathcal{P}_{(q_1,\ldots,q_n)}(Q) = d_1^{q_1-q_2} d_2^{q_2-q_3} \ldots d_n^{q_n},$$

where d_j is the principal minor (determinant of a sub-block) of size $j \times j$ of the matrix

$$\frac{1}{2}(Q_{RR} - Q_{AA} + Q_{RA} - Q_{AR})_{bb}.$$

- Spinful classes: Determinants $\longrightarrow$ Pfaffians of $2j \times 2j$ sub-blocks

These are pure scaling operators.

A proof goes via Iwasawa decomposition $\quad G = NAK$.

Functions $\mathcal{P}_{(q_1,\ldots,q_n)}(Q)$ are N-invariant spherical functions on G/K and have a form of "plane waves" on A.

Iwasawa decomposition and spherical functions

σ-model space: G/K $\quad$ K — maximal compact subgroup

- Iwasawa decomposition: $G = NAK$ $\quad$ $g = nak$

A — maximal abelian in G/K

N — nilpotent $\quad$ ($\longleftrightarrow$ triangular matrices with 1 on the diagonal)

Generalization of Gram-Schmidt (QR) decomposition:

matrix = triangular × unitary

- Eigenfunctions of all G-invariant (Casimir) operators

(in particular, RG transformation) are spherical functions on G/K.

- N-invariant spherical functions on G/K are "plane waves"

$$\varphi_{q,p} = \exp\Big(-2\sum_{j=1}^{n} q_j x_j - 2i\sum_{l=1}^{n} p_l y_l \Big)$$

$x_1, \ldots, x_n;\ y_1, \ldots, y_n$ — natural coordinates on A.

- $p_j = 0 \longrightarrow \phi_q$ is exactly $\mathcal{P}_{(q_1,\ldots,q_n)}(Q)$ introduced above

Weyl symmetries of scaling exponents

Weyl group acts in the space of weights λ
(dual to Lie algebra of A) $\longrightarrow$ invariance of eigenvalues
of any G invariant operator with respect to

(i) reflections: $q_j \to -c_j - q_j$

(ii) permutations: $q_i \to q_j + \dfrac{c_j - c_i}{2}$; $q_j \to q_i + \dfrac{c_i - c_j}{2}$

$c_j = 1 - 2j$,	class A	$c_j = -2j$,	class CI
$c_j = -j$,	class AI	$c_j = 2 - 2j$,	class DIII
$c_j = 3 - 4j$,	class AII	$c_j = 1 - 2j$,	class AIII
$c_j = 1 - 4j$,	class C	$c_j = 1/2 - j$,	class BDI
$c_j = 1 - j$,	class D	$c_j = 2 - 4j$,	class CII

$\longrightarrow$ symmetries of eigenvalues of RG, i.e. scaling exponents x_λ
$\longrightarrow$ earlier found symmetry $x_q = x_{1-q}$ for Wigner-Dyson classes
and many more symmetry relations for all classes

Criticality in 2D

- **Broken spin-rotation invariance: Classes AII, D, DIII**

  Metallic phase (weak antilocalization) and metal-insulator transition

- **Sublattice symmetry: Chiral classes AIII, BDI, CII**

  Critical-metal phase and metal-insulator transition

- **Broken time-reversal invariance:**
  Topological θ term and quantum Hall criticality

  class A — Quantum Hall effect / transition
  class C — Spin Quantum Hall effect / transition
  class D — Thermal Quantum Hall effect / transition

- **Topologically protected criticality on surfaces of topological insulators and superconductors or in models of disordered Dirac fermions**

  classes AII, CII — $\mathbb{Z}_2$ topological term
  classes AIII, CI, DIII — Wess-Zumino term

Generalized multifractality — "fingerprint of a critical point"

Criticality in 2D

- **Broken spin-rotation invariance: Classes $\boxed{\text{AII}}$, D, DIII**

  $\boxed{\text{Metallic phase}}$ (weak antilocalization) and $\boxed{\text{metal-insulator transition}}$

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Generalized multifractality — “fingerprint of a critical point”

Generalized multifractality and conformal invariance in 2D

- Usually (but not always) criticality implies conformal invariance
$\longrightarrow$ conformal field theory (CFT)

- 2D: global conformal invariance is usually promoted to
local conformal invariance (infinite-dimensional Virasoro algebra)

- Most of generalized multifractality correlation functions
depend on system size L in a power-law way, at variance with CFT.

Only those satisfying "neutrality" condition $\sum_i \boldsymbol{\lambda}_i = (-c_1, \ldots, -c_n) \equiv -\boldsymbol{\rho}_b$
have a chance to be described by CFT — but this is not guaranteed.

- Assuming local conformal invariance for such correlation functions
and extending the argument by Bondesan, Wieczorek, Zirnbauer, 2017 $\longrightarrow$

$$x_{\lambda+\lambda'+\lambda''} - x_{\lambda+\lambda'} - x_{\lambda+\lambda''} - x_{\lambda'+\lambda''} + x_\lambda + x_{\lambda'} + x_{\lambda''} = 0$$

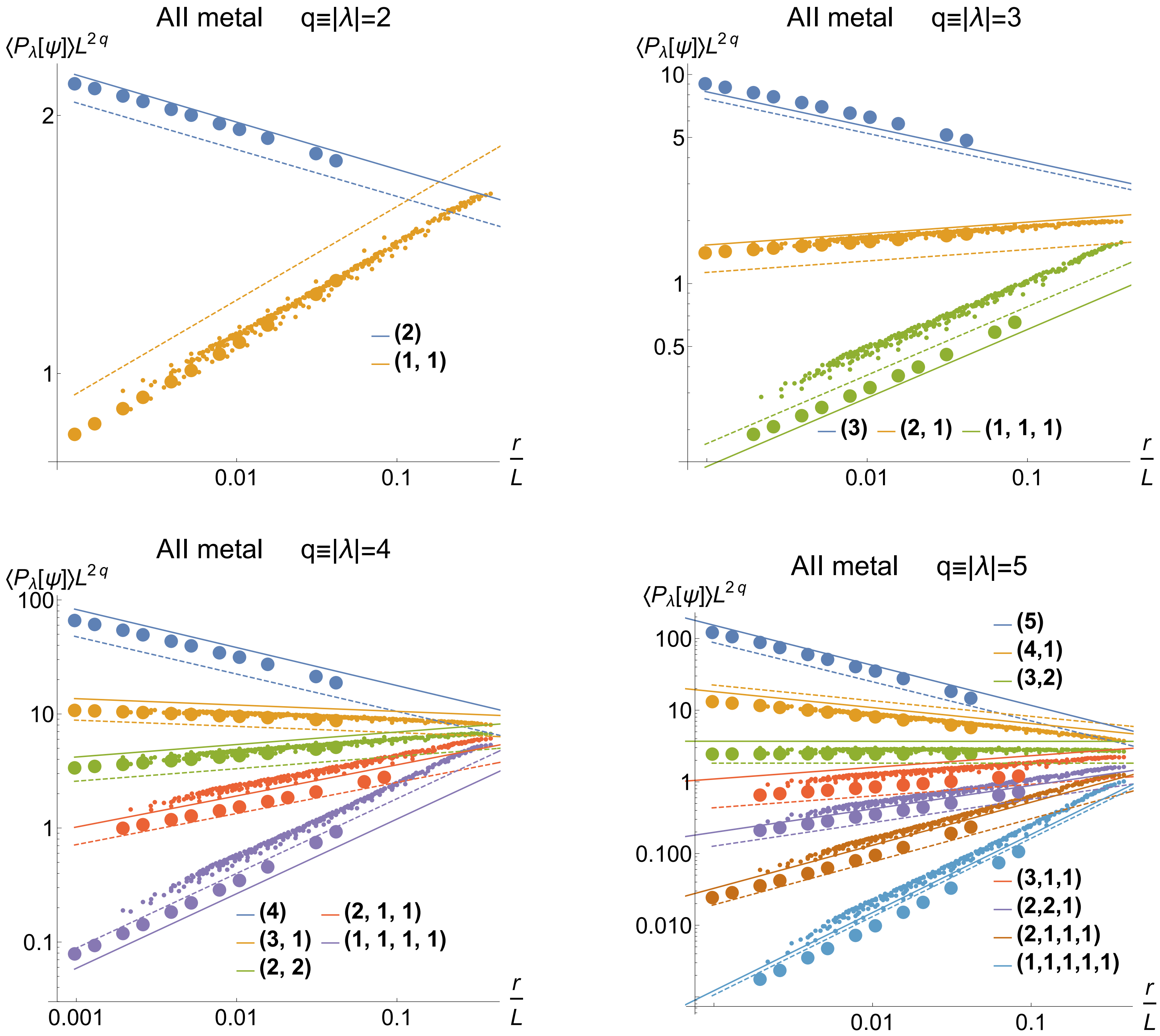
$\longrightarrow$ x_λ – quadratic function of q_j. In combination with Weyl symmetry,
this implies the single-parameter "generalized parabolicity"

$$x_\lambda^{\text{para}} \equiv x^{\text{para}}_{(q_1,q_2,\ldots)} = -b \sum_i q_i(q_i + c_i) \equiv -b\boldsymbol{\lambda}\cdot(\boldsymbol{\lambda} + \boldsymbol{\rho}_b) \equiv -b z_\lambda$$

z_λ– eigenvalues of the Laplace-Beltrami operator on the σ-model target space.

Generalized parabolicity — stringent test of local conformal invariance!

Class AII. Metallic phase: Pure-scaling observables



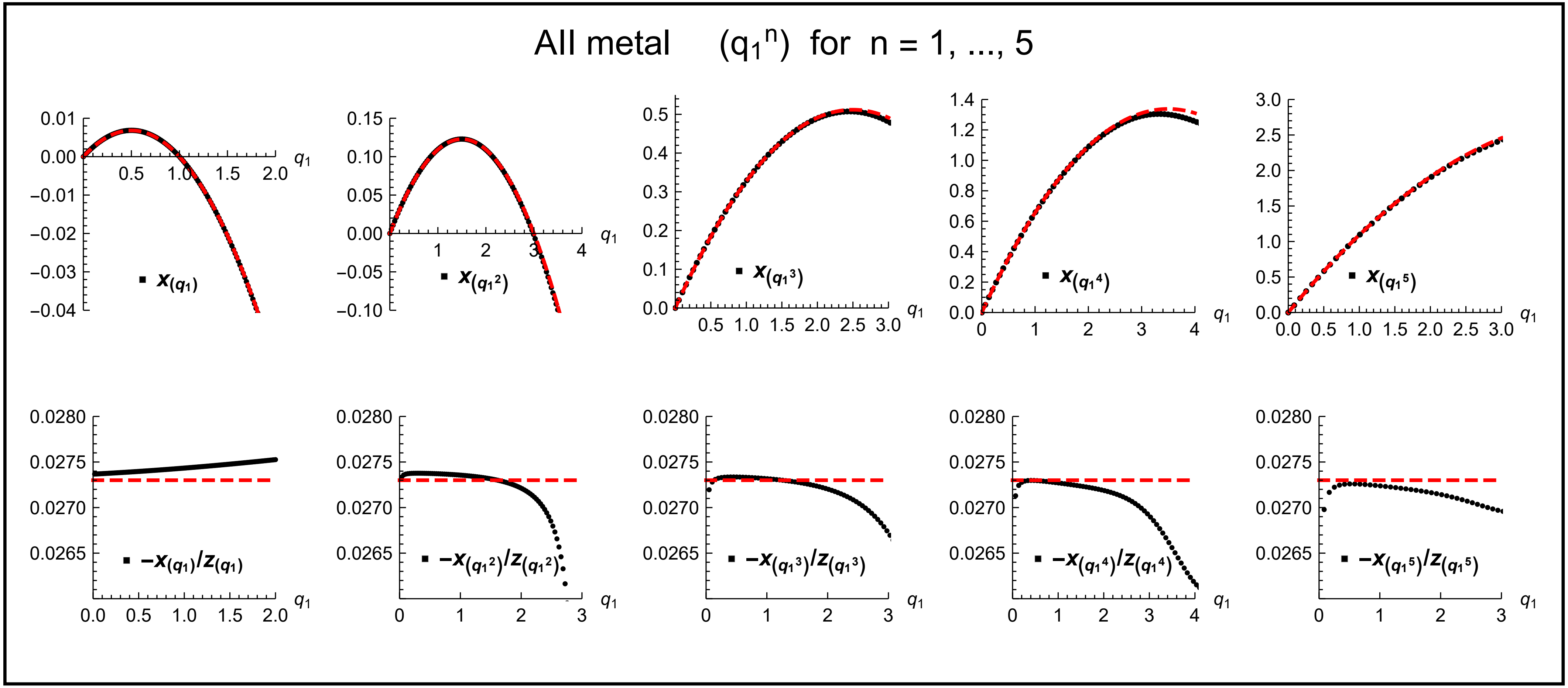
dashed lines:

generalized parabolicity

$x_\lambda^{\text{para}} = -bz_\lambda$

with $b = 0.0273$

Class AII. Metallic phase



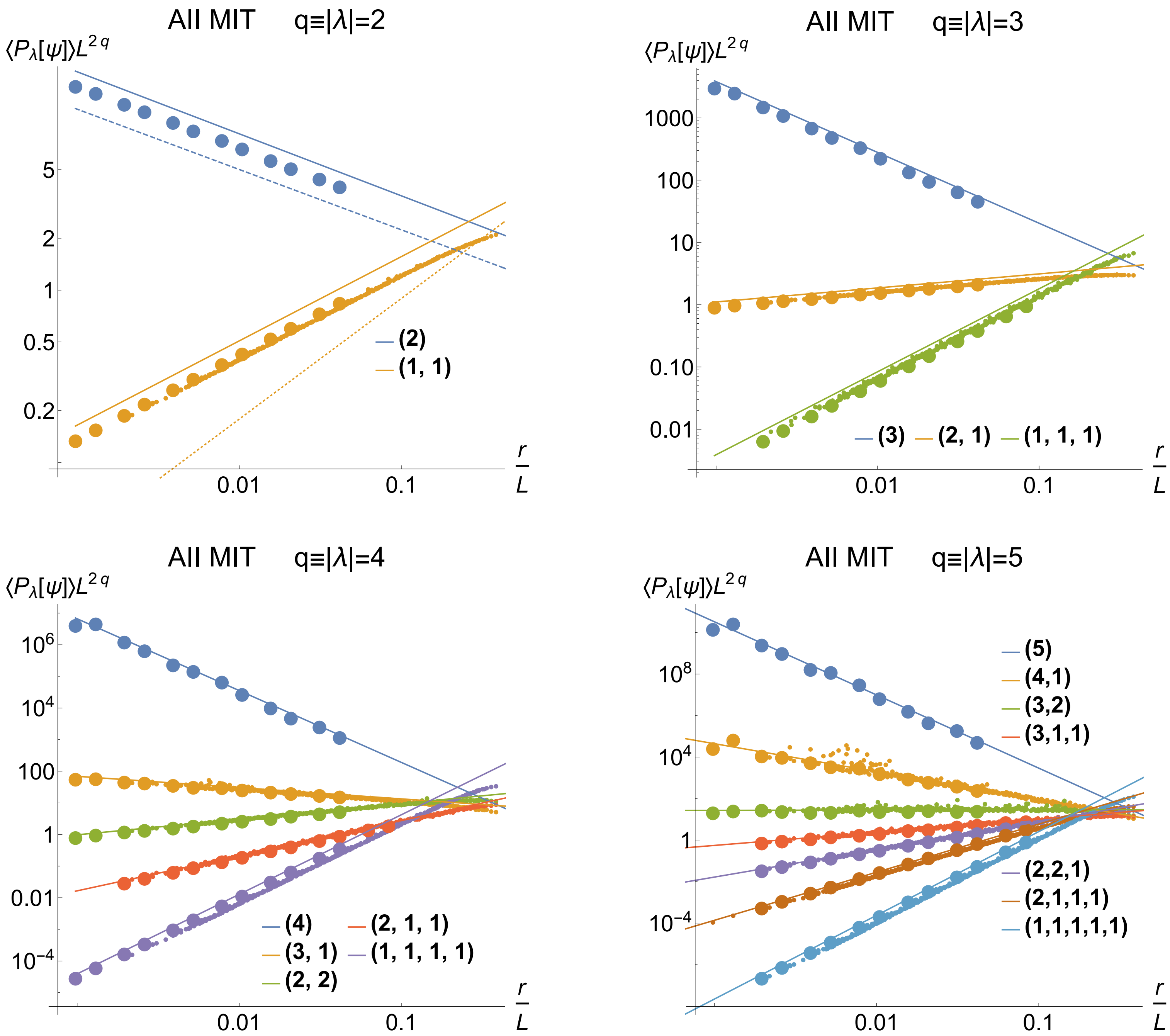
Red dashed lines: generalized parabolicity $x_\lambda^{\rm para} = -b z_\lambda$ with $b = 0.0273$.

Generalized parabolicity holds with an excellent accuracy, in consistency with analytical expectations. It is exact in one-loop order, and there is no two-loop and three-loop corrections in class AII.

Class AII. Scaling exponents x_λ

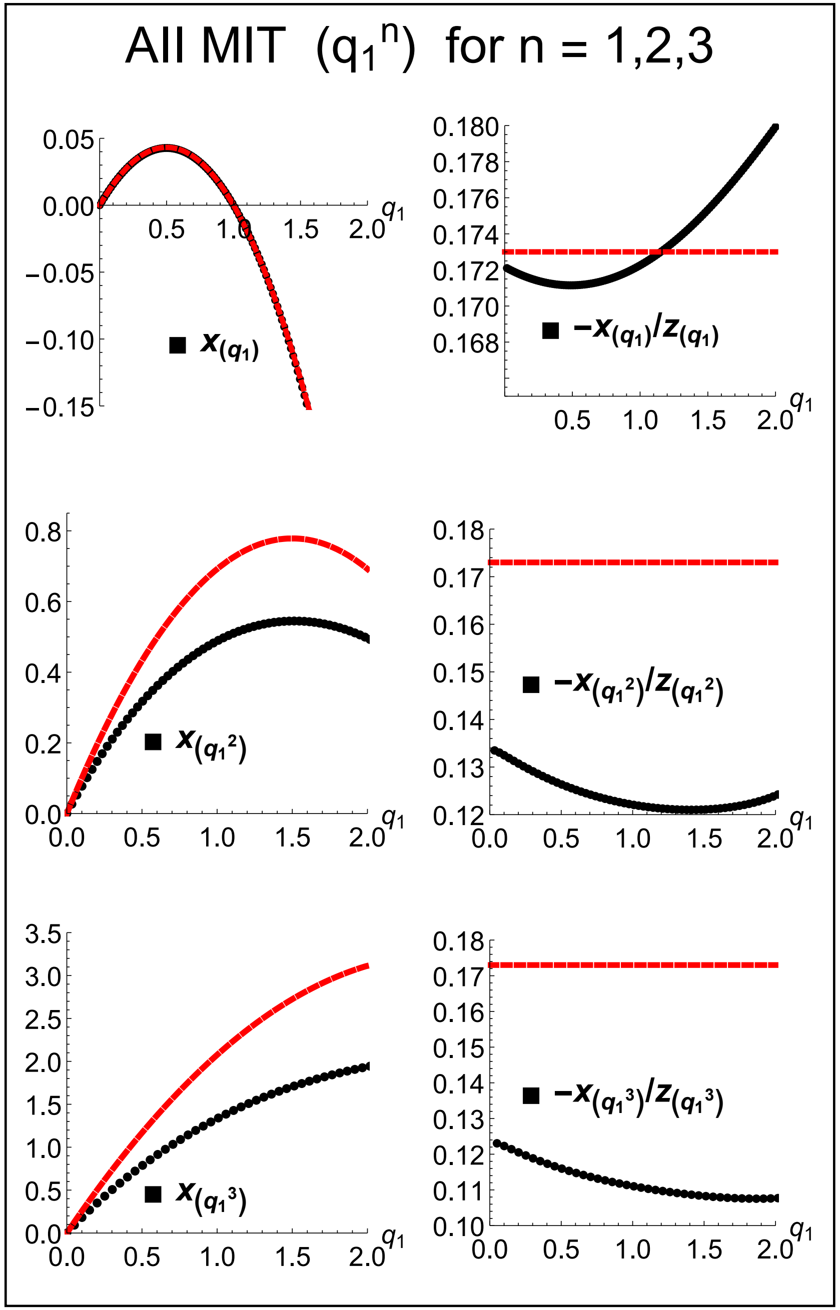
	rep. λ	x_λ^{MIT}	$x_\lambda^{\mathrm{MIT}}/b$	$\boxed{x_\lambda^{\mathrm{metal}}}$	$\boxed{x_\lambda^{\mathrm{metal}}/b}$	$x_\lambda^{\mathrm{para}}$
$q=2$	(2)	-0.361 ± 0.001	-2.08 ± 0.01	-0.0551 ± 0.0001	-2.017 ± 0.005	$-2b$
	(1,1)	0.489 ± 0.001	2.83 ± 0.01	0.1095 ± 0.0001	4.012 ± 0.005	$4b$
$q=3$	(3)	-1.14 ± 0.01	-6.57 ± 0.06	-0.1659 ± 0.0004	-6.08 ± 0.02	$-6b$
	(2,1)	0.225 ± 0.001	1.30 ± 0.01	0.0547 ± 0.0002	2.04 ± 0.01	$2b$
	(1,1,1)	1.333 ± 0.001	7.70 ± 0.01	0.3278 ± 0.0003	12.01 ± 0.01	$12b$
$q=4$	(4)	-2.27 ± 0.05	-13.13 ± 0.29	-0.334 ± 0.001	-12.21 ± 0.04	$-12b$
	(3,1)	-0.36 ± 0.01	-2.06 ± 0.06	-0.0557 ± 0.0005	-2.04 ± 0.02	$-2b$
	(2,2)	0.493 ± 0.005	2.85 ± 0.03	0.1095 ± 0.0005	4.01 ± 0.02	$4b$
	(2,1,1)	1.111 ± 0.003	6.42 ± 0.02	0.2728 ± 0.0005	9.99 ± 0.02	$10b$
	(1,1,1,1)	2.515 ± 0.002	14.54 ± 0.01	0.6545 ± 0.0003	23.97 ± 0.01	$24b$
$q=5$	(5)	-3.52 ± 0.09	-20.37 ± 0.17	-0.559 ± 0.003	-20.48 ± 0.52	$-20b$
	(4,1)	-1.35 ± 0.07	-7.82 ± 0.40	-0.223 ± 0.001	-8.16 ± 0.04	$-8b$
	(3,2)	0.02 ± 0.02	0.08 ± 0.12	-0.0006 ± 0.0009	0.02 ± 0.03	0
	(3,1,1)	0.64 ± 0.01	3.67 ± 0.06	0.1623 ± 0.0008	5.95 ± 0.03	$6b$
	(2,2,1)	1.333 ± 0.005	7.70 ± 0.03	0.327 ± 0.0008	11.97 ± 0.03	$12b$
	(2,1,1,1)	2.316 ± 0.004	13.39 ± 0.02	0.5997 ± 0.0005	21.99 ± 0.02	$22b$
	(1,1,1,1,1)	4.031 ± 0.004	23.30 ± 0.02	1.0895 ± 0.0004	39.91 ± 0.02	$40b$

Class AII. Anderson transition: Pure-scaling observables



generalized parabolicity strongly violated

Class AII. Anderson-transition critical point



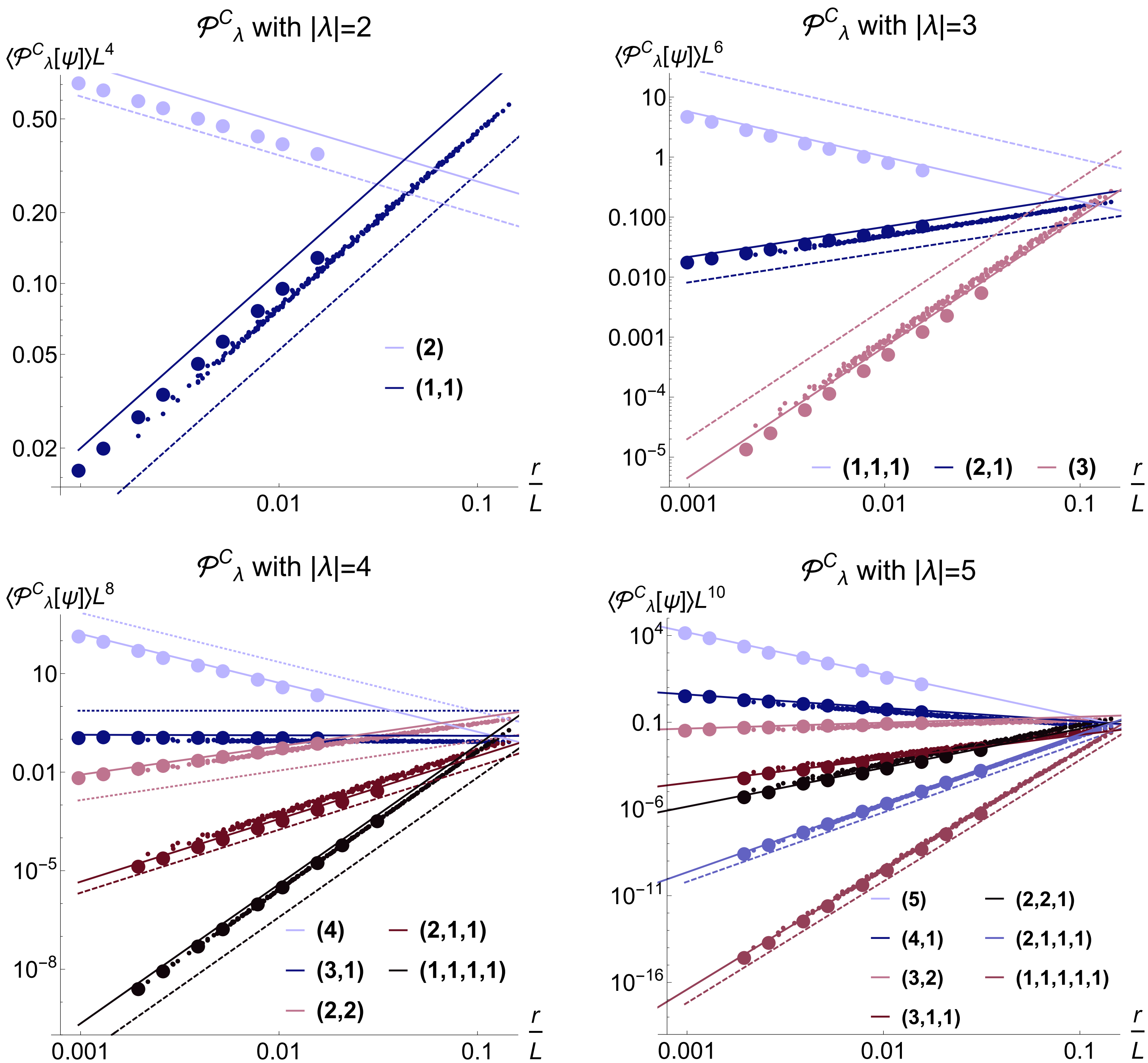
- Weyl symmetry holds nicely (see also the table on the next slide)
- Generalized parabolicity (red lines) strongly violated

$\longrightarrow$ violation of local conformal invariance

Class AII. Scaling exponents x_λ

	rep. λ	$\boxed{x_\lambda^{\mathrm{MIT}}}$	$\boxed{x_\lambda^{\mathrm{MIT}}/b}$	$x_\lambda^{\mathrm{metal}}$	$x_\lambda^{\mathrm{metal}}/b$	$x_\lambda^{\mathrm{para}}$
$q = 2$	(2)	$\boxed{-0.361 \pm 0.001}$	-2.08 ± 0.01	-0.0551 ± 0.0001	-2.017 ± 0.005	$-2b$
	(1,1)	$\boxed{0.489 \pm 0.001}$	2.83 ± 0.01	0.1095 ± 0.0001	4.012 ± 0.005	$4b$
$q = 3$	(3)	-1.14 ± 0.01	-6.57 ± 0.06	-0.1659 ± 0.0004	-6.08 ± 0.02	$-6b$
	(2,1)	0.225 ± 0.001	1.30 ± 0.01	0.0547 ± 0.0002	2.04 ± 0.01	$2b$
	(1,1,1)	$\boxed{1.333 \pm 0.001}$	7.70 ± 0.01	0.3278 ± 0.0003	12.01 ± 0.01	$12b$
$q = 4$	(4)	-2.27 ± 0.05	-13.13 ± 0.29	-0.334 ± 0.001	-12.21 ± 0.04	$-12b$
	(3,1)	$\boxed{-0.36 \pm 0.01}$	-2.06 ± 0.06	-0.0557 ± 0.0005	-2.04 ± 0.02	$-2b$
	(2,2)	$\boxed{0.493 \pm 0.005}$	2.85 ± 0.03	0.1095 ± 0.0005	4.01 ± 0.02	$4b$
	(2,1,1)	1.111 ± 0.003	6.42 ± 0.02	0.2728 ± 0.0005	9.99 ± 0.02	$10b$
	(1,1,1,1)	2.515 ± 0.002	14.54 ± 0.01	0.6545 ± 0.0003	23.97 ± 0.01	$24b$
$q = 5$	(5)	-3.52 ± 0.09	-20.37 ± 0.17	-0.559 ± 0.003	-20.48 ± 0.52	$-20b$
	(4,1)	-1.35 ± 0.07	-7.82 ± 0.40	-0.223 ± 0.001	-8.16 ± 0.04	$-8b$
	(3,2)	$\boxed{0.02 \pm 0.02}$	0.08 ± 0.12	-0.0006 ± 0.0009	0.02 ± 0.03	0
	(3,1,1)	0.64 ± 0.01	3.67 ± 0.06	0.1623 ± 0.0008	5.95 ± 0.03	$6b$
	(2,2,1)	$\boxed{1.333 \pm 0.005}$	7.70 ± 0.03	0.327 ± 0.0008	11.97 ± 0.03	$12b$
	(2,1,1,1)	2.316 ± 0.004	13.39 ± 0.02	0.5997 ± 0.0005	21.99 ± 0.02	$22b$
	(1,1,1,1,1)	4.031 ± 0.004	23.30 ± 0.02	1.0895 ± 0.0004	39.91 ± 0.02	$40b$

Class C. SQH transition: Pure-scaling observables



dashed lines:

analytical results from percolation mapping

SQH transition and classical percolation

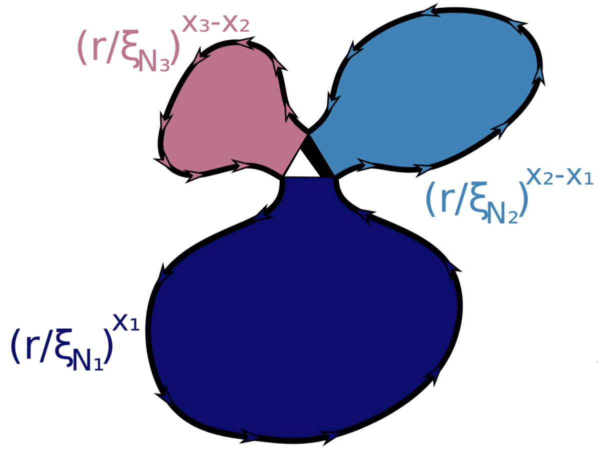
Classical percolation: Probability that n hull segments come close at two points separated by a distance r:

$$P_n(r) \sim r^{-x_n^{\mathrm{h}}}(p-p_c)^{\nu x_n^{\mathrm{h}}}, \qquad r \lesssim \xi = (p-p_c)^{-\nu}$$ Saleur, Duplantier, 1987

n-hull exponents $$x_n^{\mathrm{h}} = \frac{4n^2-1}{12}, \qquad n = 1, 2, \ldots$$

In particular, for $n = 1$, 2, and 3: $$x_1^{\mathrm{h}} = \frac{1}{4}, \qquad x_2^{\mathrm{h}} = \frac{5}{4}, \qquad x_3^{\mathrm{h}} = \frac{35}{12}$$

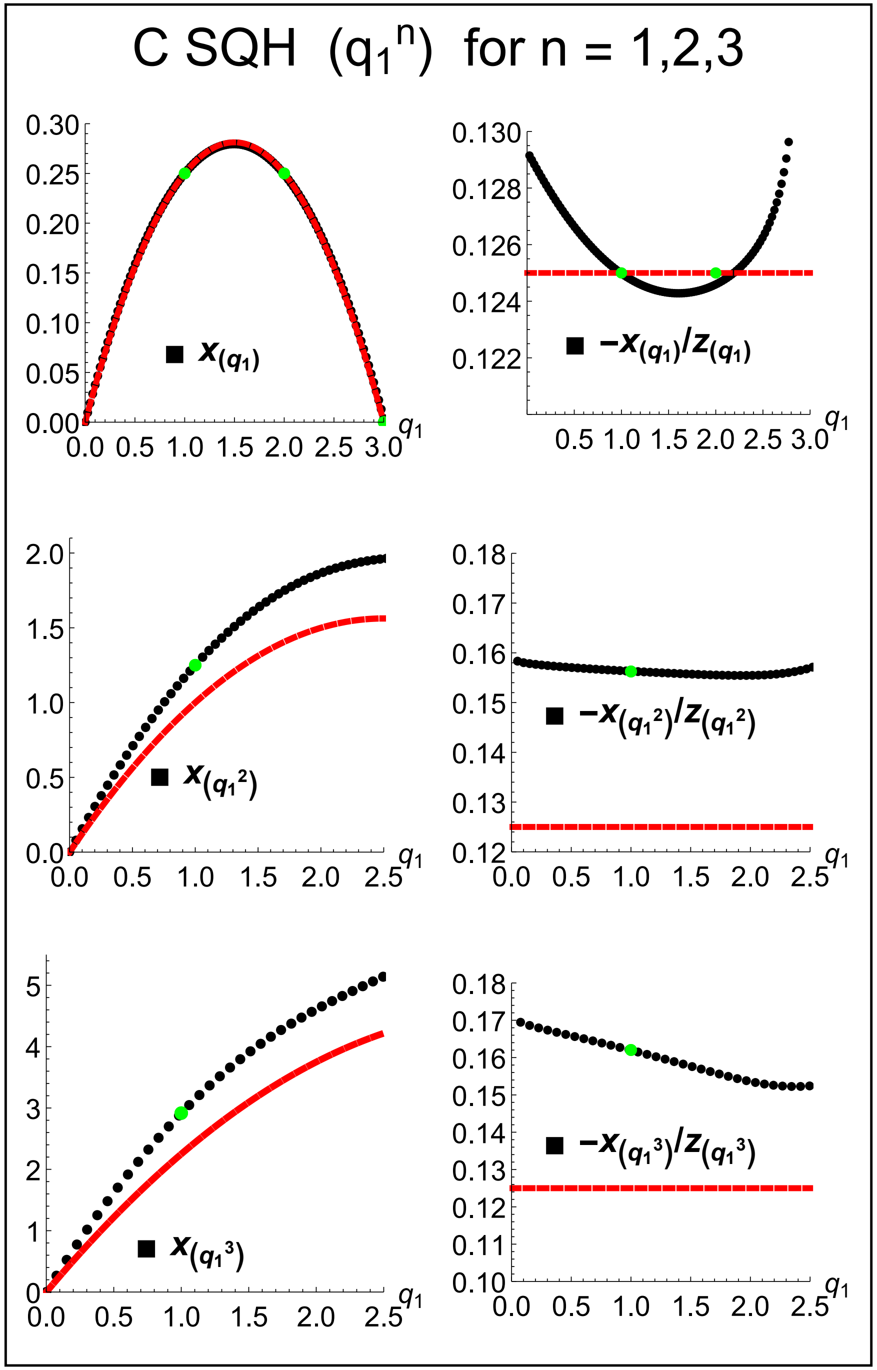
Mapping of SQH transition to percolation for a certain class of correlation functions



Exact results for a subset of SQH exponents x_λ: $\boxed{x_{(1^n)} = x_n^{\mathrm{h}}}$

$$x_{(1)} = x_{(2)} = \frac{1}{4}, \qquad x_{(1,1)} = x_{(2,1)} = \frac{5}{4}, \qquad x_{(1,1,1)} = x_{(2,1,1)} = \frac{35}{12}, \quad \text{etc}$$

Class C. SQH transition.



- **Excellent agreement of numerical values with analytical results (from mapping to percolation; green symbols)**
- **Weyl symmetry holds nicely**
- **Generalized parabolicity (red lines) strongly violated**

$\longrightarrow$ violation of local conformal invariance

SQH transition (class C). Scaling exponents x_λ

	λ	$x_\lambda^{\rm perc}$	$x_\lambda^{\rm qn}$	$x_\lambda^{\rm para}$
$q=1$	(1)	$x_1^{\rm h} = 1/4 = 0.25$	—	$1/4$
$q=2$	(2)	$x_1^{\rm h} = 1/4 = 0.25$	0.249 ± 0.001	$1/4$
	(1,1)	$x_2^{\rm h} = 5/4 = 1.25$	1.251 ± 0.001	1
$q=3$	(3)	0	0.004 ± 0.004	0
	(2,1)	$x_2^{\rm h} = 5/4 = 1.25$	1.249 ± 0.002	1
	(1^3)	$x_3^{\rm h} = 35/12 \simeq 2.917$	2.915 ± 0.002	$9/4$
$q=4$	(4)	—	-0.49 ± 0.02	$-1/2$
	(3,1)	—	0.985 ± 0.007	$3/4$
	(2,2)	—	1.865 ± 0.006	$3/2$
	(2,1,1)	$x_3^{\rm h} = 35/12 \simeq 2.917$	2.911 ± 0.005	$9/4$
	(1^4)	$x_4^{\rm h} = 21/4 = 5.25$	5.242 ± 0.004	4
$q=5$	(5)	—	-1.19 ± 0.06	$-5/4$
	(4,1)	—	0.48 ± 0.03	$1/4$
	(3,2)	—	1.59 ± 0.02	$5/4$
	(3,1,1)	—	2.64 ± 0.02	2
	(2,2,1)	—	3.50 ± 0.02	$11/4$
	$(2,1^3)$	$x_4^{\rm h} = 21/4 = 5.25$	5.23 ± 0.01	4
	(1^5)	$x_5^{\rm h} = 33/4 = 8.25$	8.16 ± 0.01	$25/4$

- Excellent agreement of numerical values $x_\lambda^{\rm qn}$ with analytical results $x_\lambda^{\rm perc}$ (from mapping to percolation)
- Weyl symmetry holds nicely
- Generalized parabolicity ($x_\lambda^{\rm para}$, last column) strongly violated

$\longrightarrow$ violation of local conformal invariance

Extension to other 2D critical points / symmetry classes

- quantum Hall transition (class A)
- metallic phases and MIT in classes D and DIII

Peculiarity: two disjoint components of the σ-model manifold

$\longrightarrow$ domain walls $\longrightarrow$ violation of Weyl symmetry at the MIT

- chiral classes AIII, BDI, CII (models with sublattice symmetry):

critical-metal phases and metal-insulator transitions

Peculiarities:

(i) Observables labeled by a pair of multi-indices:

$\lambda = (q_1, \ldots, q_n)$ and $\bar{\lambda} = (\bar{q}_1, \ldots, \bar{q}_{\bar{n}})$ corresponding to two sublattices

(ii) additional U(1) degree of freedom in the σ model

$\longrightarrow$ affects / restricts Weyl symmetry

- Classes CI, DIII, AIII:

topologically protected critical points at surfaces of topological superconductors or in models of Dirac fermions

$\longrightarrow$ WZNW theories, expect generalized parabolicity,

remains to be verified numerically

Summary

- **Generalized multifractality of wave functions at Anderson transitions**
- **Classification of pure-scaling composite operators in the $\boldsymbol{\sigma}$-model formalism**
- **Construction of pure-scaling eigenfunction observables for all symmetry classes and its numerical verification**
- **Symmetries of scaling exponents: Weyl-group invariance**
- **Numerical evaluation of generalized-multifractality exponents for 2D critical points of various symmetry classes**
- **Analytical evaluation of a certain subset of generalized-multifractality exponents for SQH transition via mapping to percolation**
- **Violation of generalized parabolicity—and thus of local conformal invariance—at SQH transition and at several other 2D Anderson-localization critical points**

Outlook & open questions:

- **models with interaction**
- **experiment**
- **other realizations of generalized multifractality?**

Many happy returns, Yan !