Noninteracting fermions in a 2d rotating trap and random matrix theory

Grégory Schehr, LPTHE
CNRS/Sorbonne Université, Paris

Random matrices and random landscapes,
24-29 July 2022

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Lacroix-A-Chez-Toine, Majumdar, G. S., PRA **99**, 021602 (R), (2019)
Kulkarni, Majumdar, G. S., PRA **103**, 033321 (2021)
Smith, Le Doussal, Majumdar, G. S. , PRA **105**, 043315 (2022)

Ultra-cold atoms in confining potentials

- Recent progress in the experimental manipulation of cold atoms

⇒ to investigate the interplay between quantum and thermal behaviors in many-body systems at low temperature

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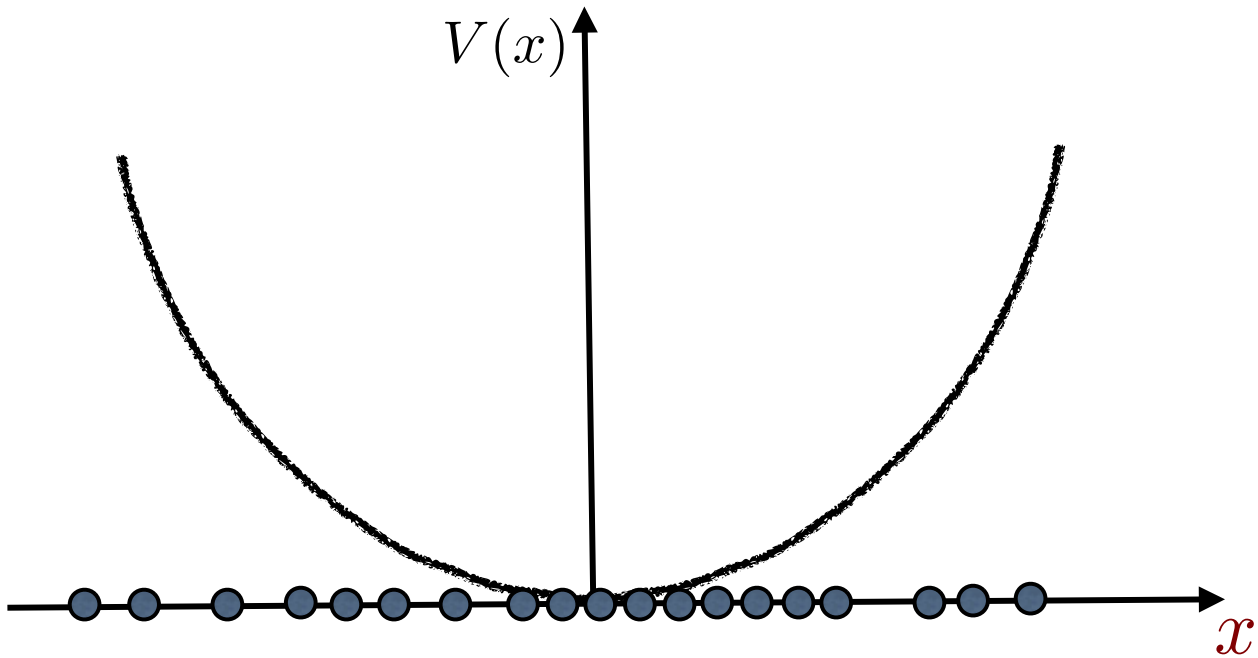
- A common feature of these experiments: presence of a confining potential that traps the atoms within a limited spatial region



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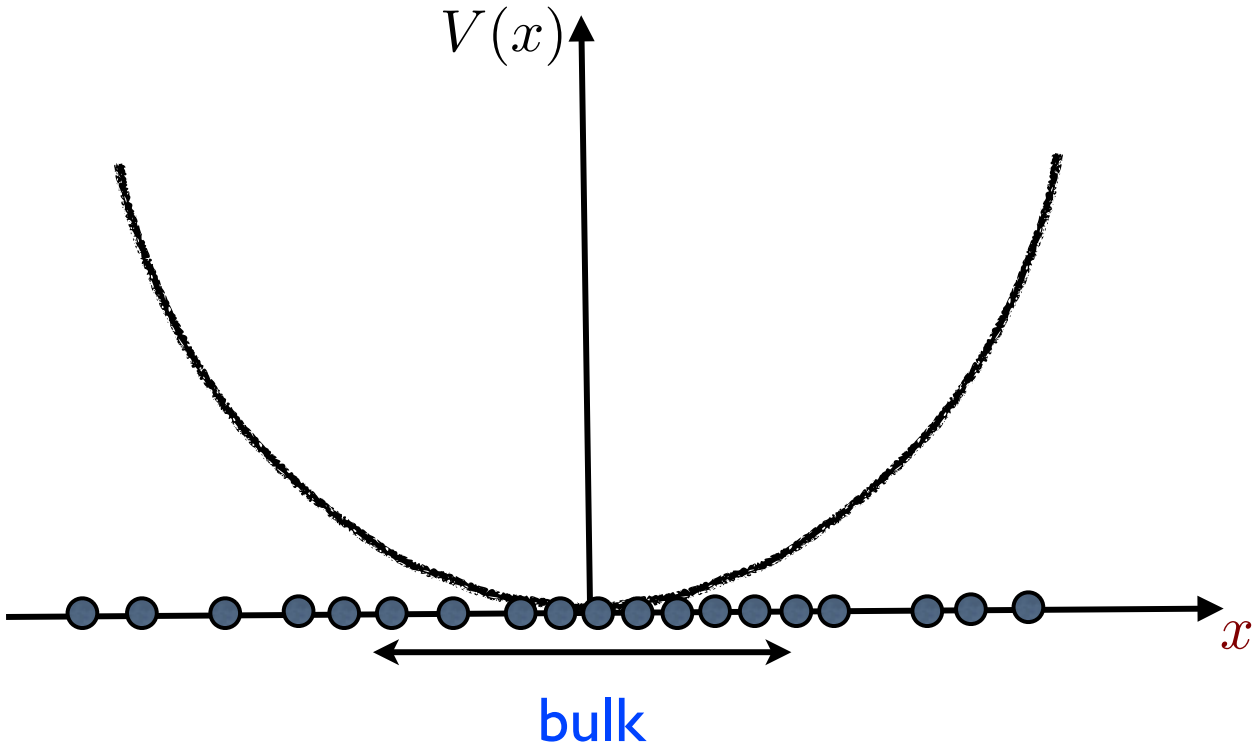

- bulk: traditional many-body physics (translationally invariant system)

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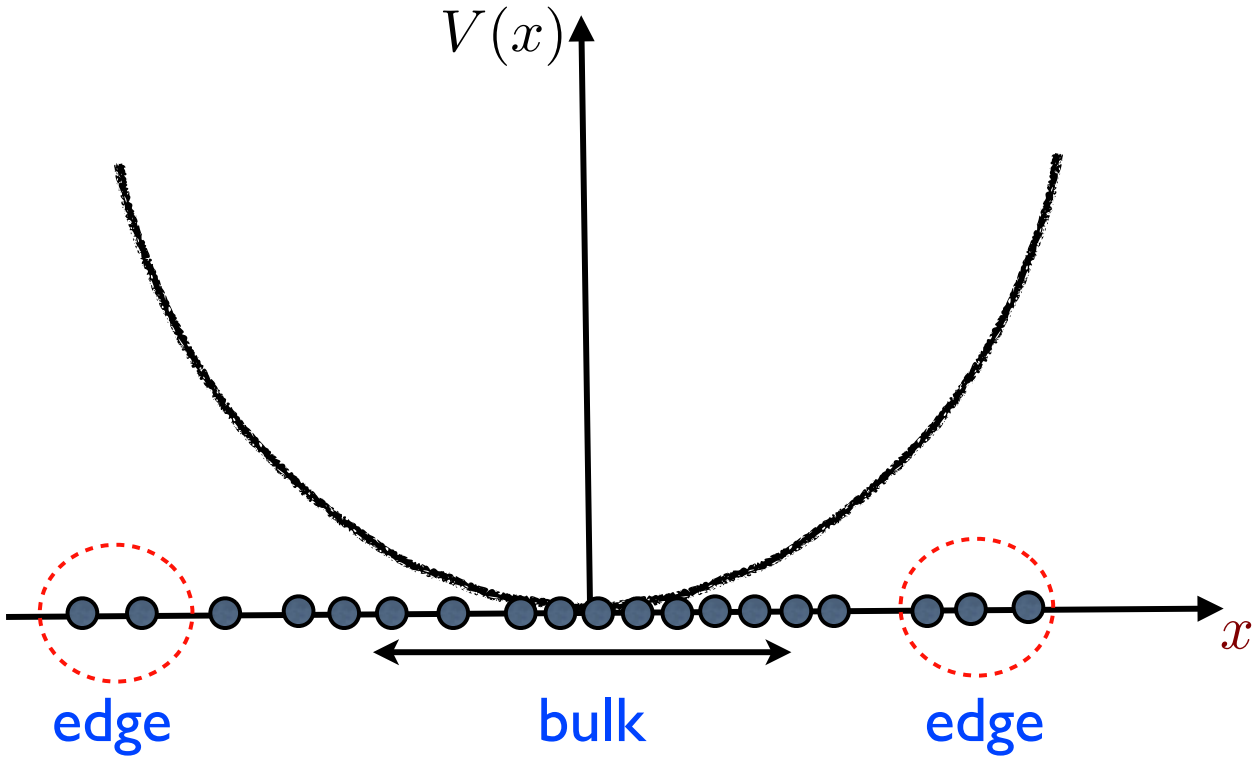

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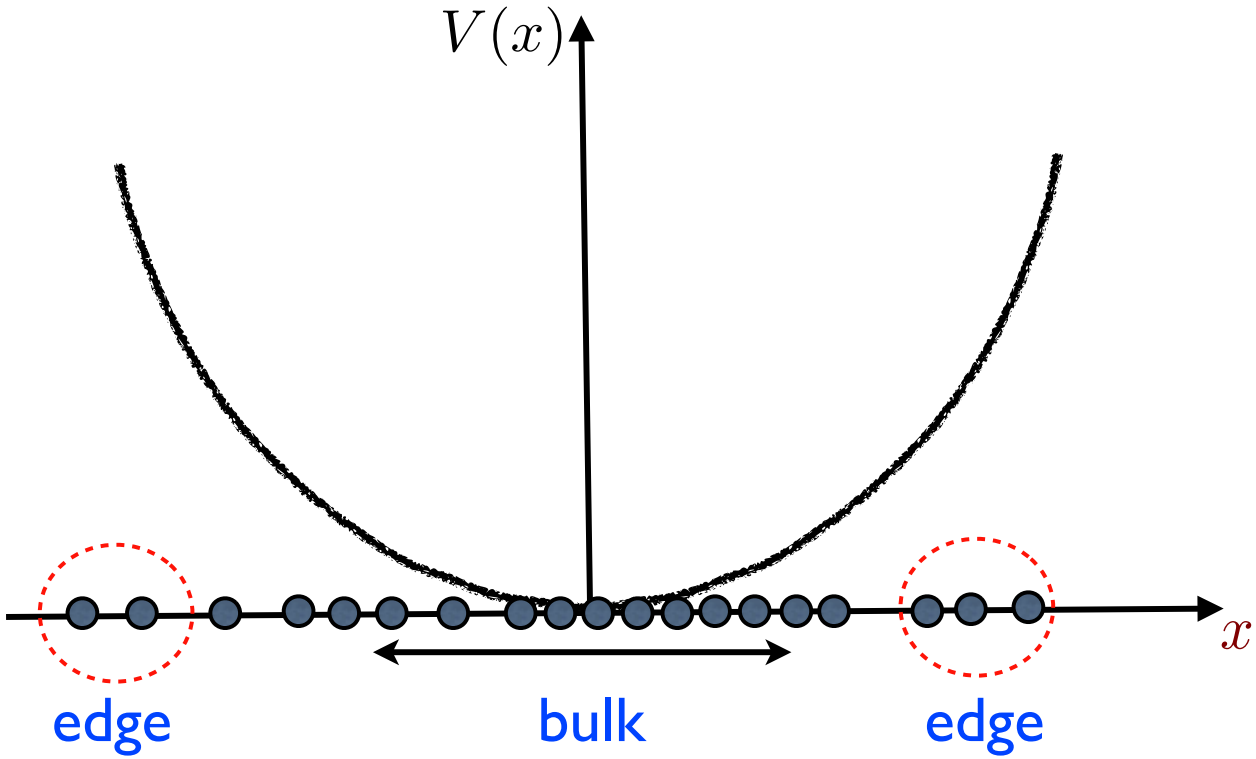
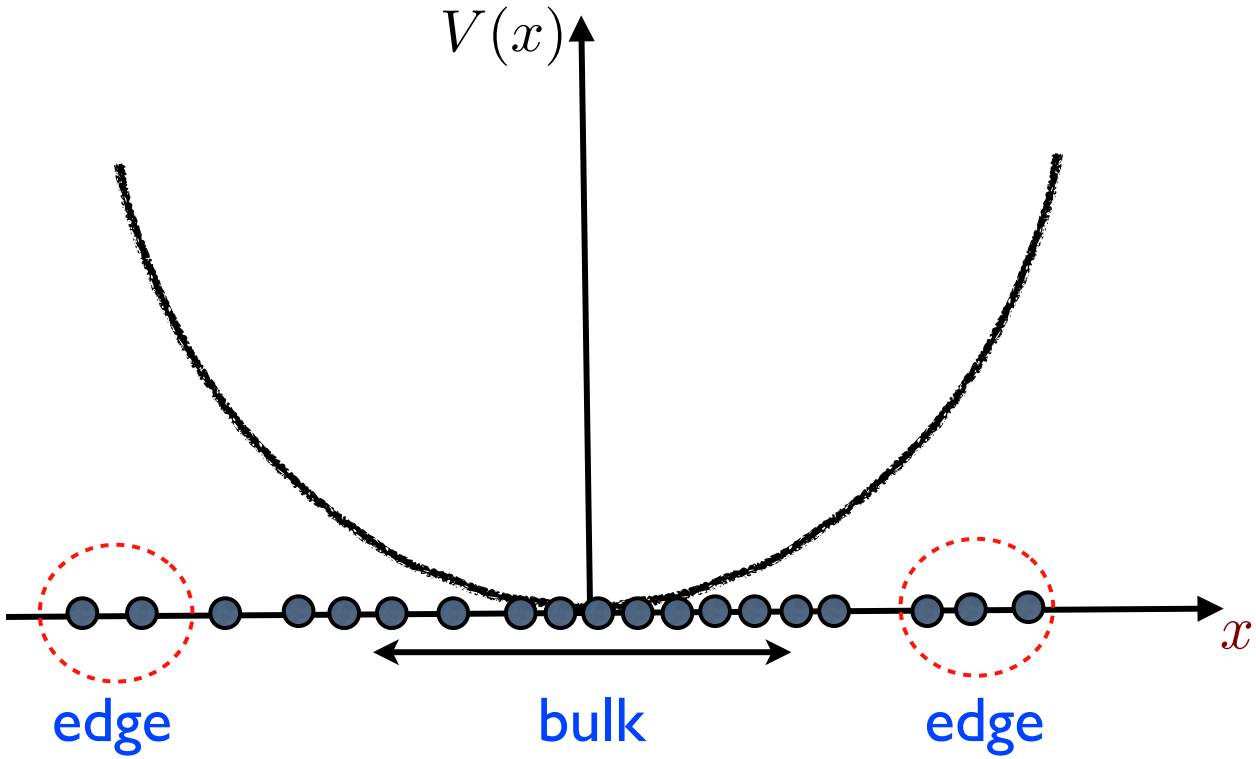


- bulk: traditional many-body physics (translationally invariant system)

  «The uniform electron gas, the traditional starting point for density-based many-body theories of inhomogeneous systems, is inappropriate near electronic edges.»

  W. Kohn, A. E. Mattsson, PRL 1998

Ultra-cold atoms in confining potentials



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This talk: random matrix theory is the ideal tool to study these edge properties

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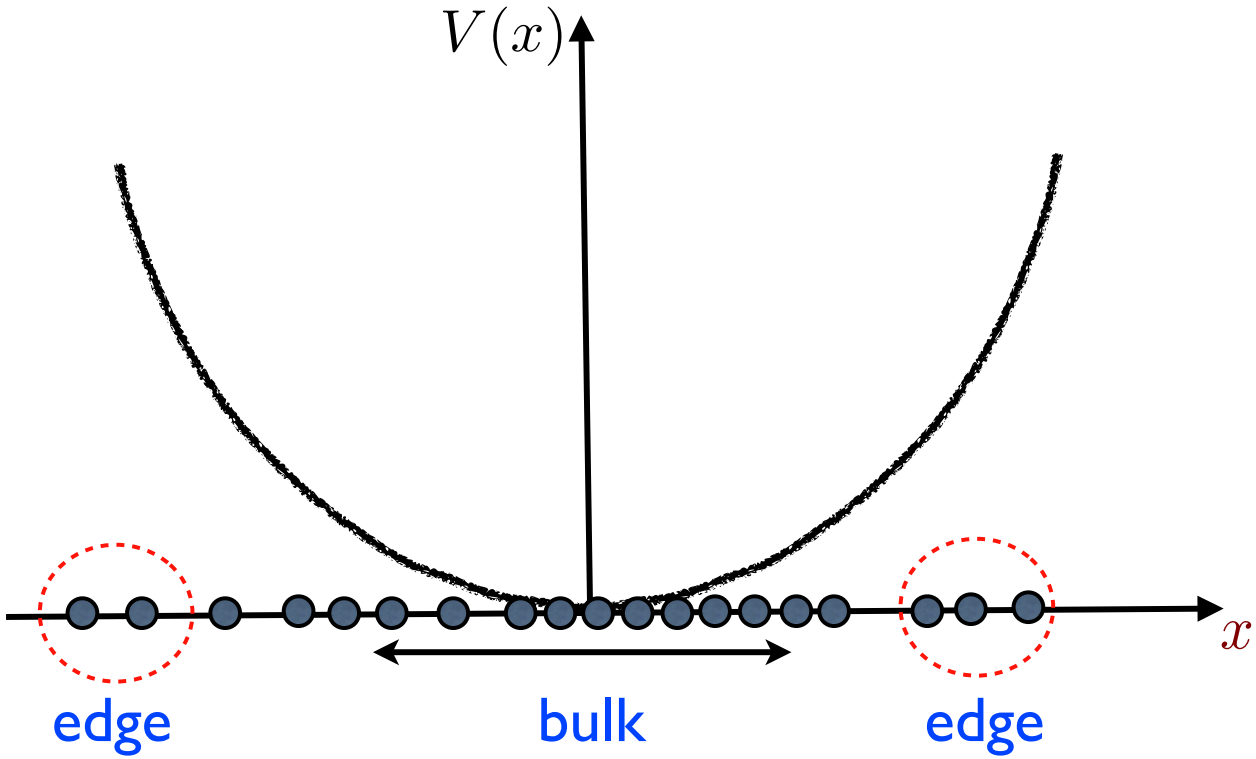
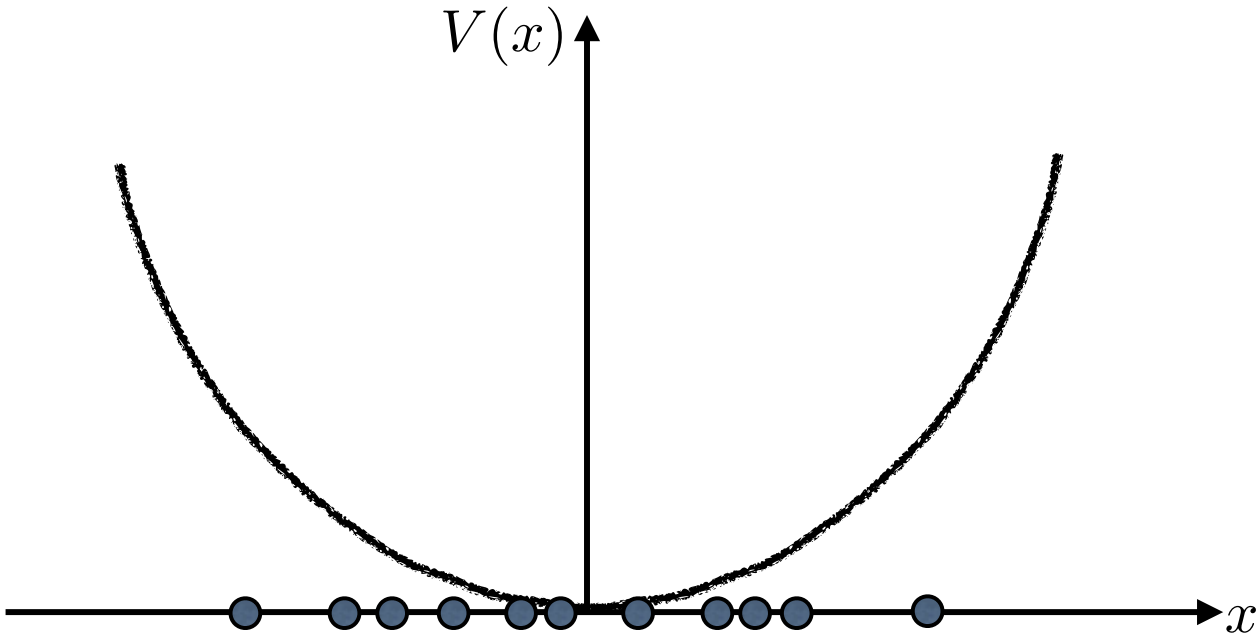


- bulk: traditional many-body physics (translationally invariant system)
- edge: new physics induced by confinement ⇒ universal edge properties

This talk: random matrix theory is the ideal tool to study these edge properties

Spinless free fermions in a 1d harmonic potential



$$V(x) = \frac{1}{2} m \omega^2 x^2$$

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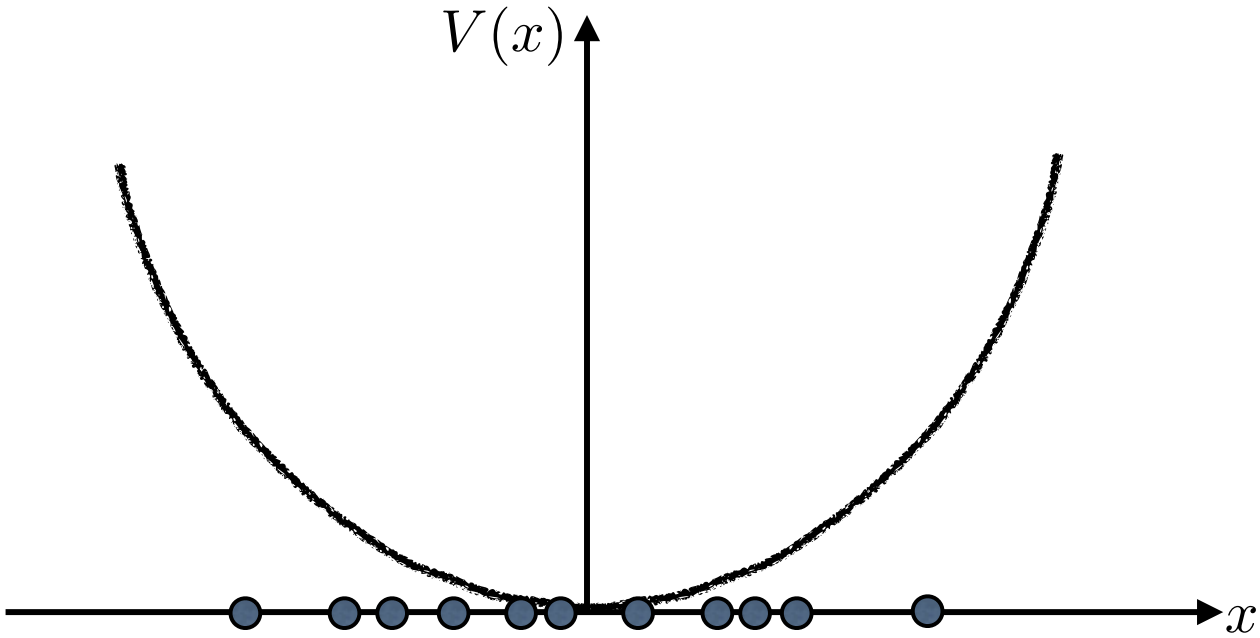

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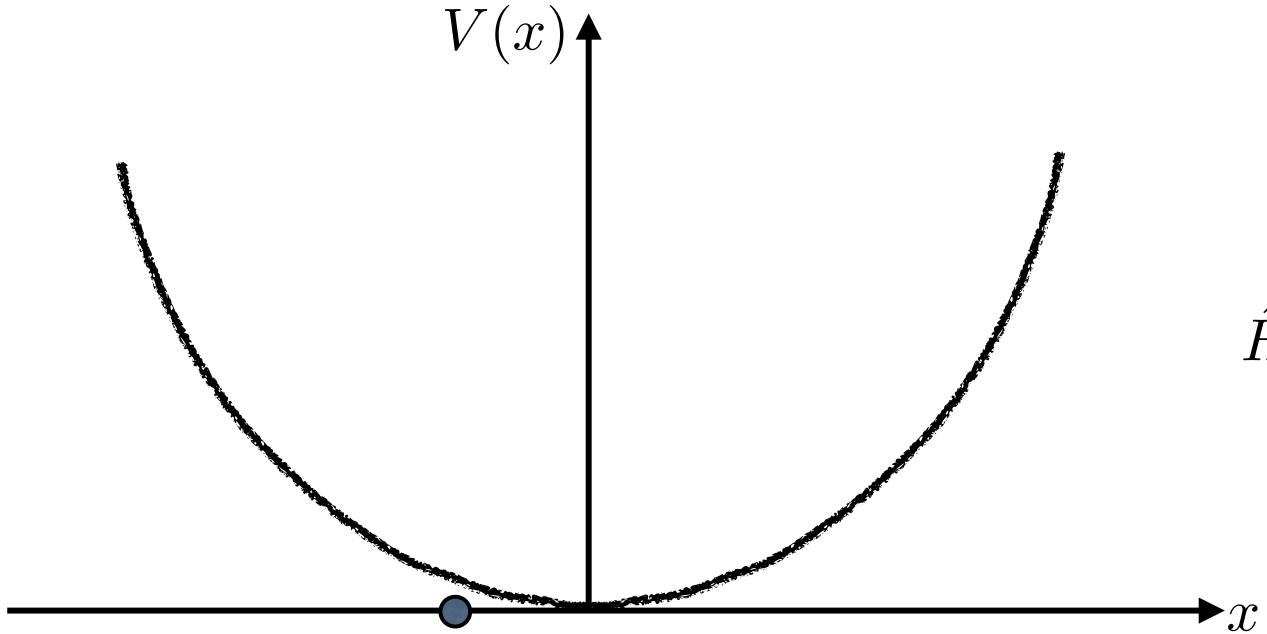
At zero temperature: connection between spinless free fermions in a harmonic trap and Random Matrix Theory (GUE)

Connection between free fermions at T=0 and RMT

- A single quantum particle in a harmonic potential



$$V(x) = \frac{1}{2} m\omega^2 x^2$$

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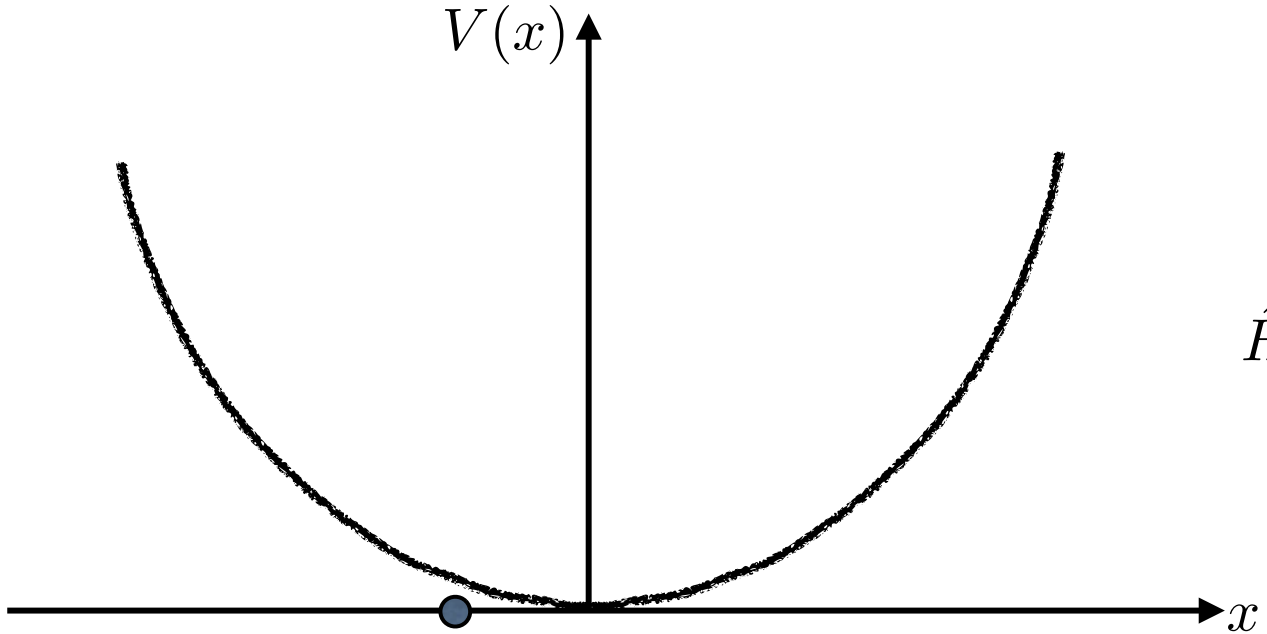

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- Single particle eigenfunctions

$$\hat{H}\,\varphi_E(x) = E\,\varphi_E(x)$$

with $\varphi_E(x \to \pm\infty) = 0$

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi}2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

$$\epsilon_k = \hbar\omega(k + 1/2) \quad , \quad \alpha = \sqrt{m\omega/\hbar}$$

$$k \in \mathbb{N}$$

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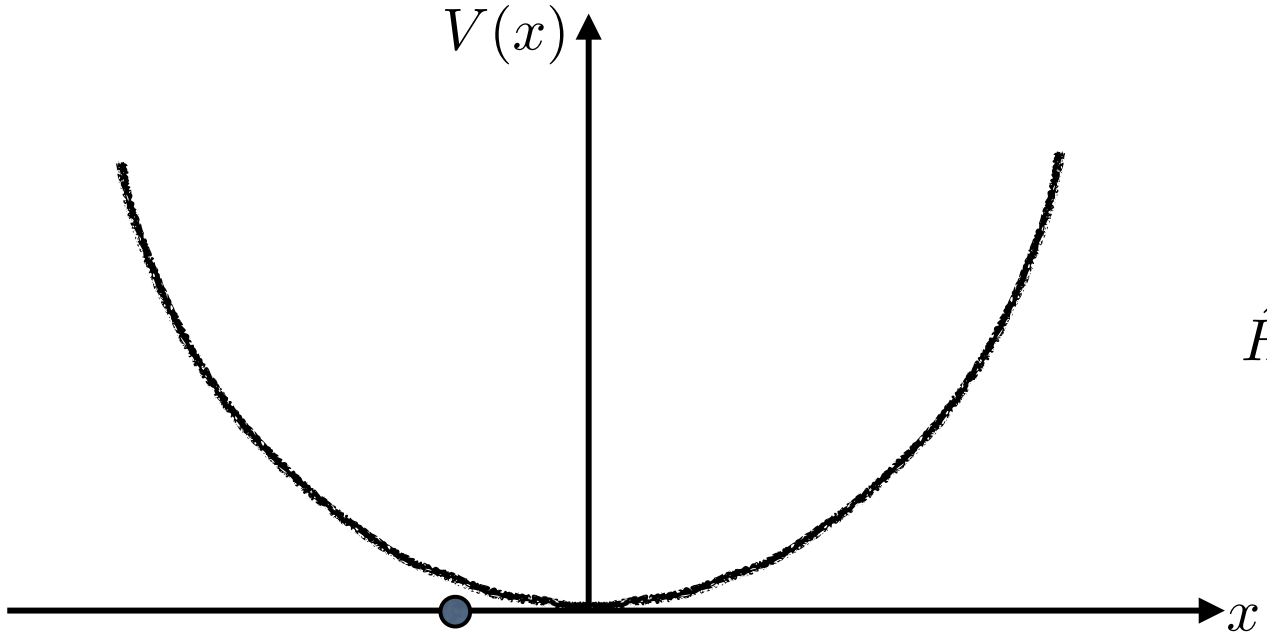

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Hermite polynomial

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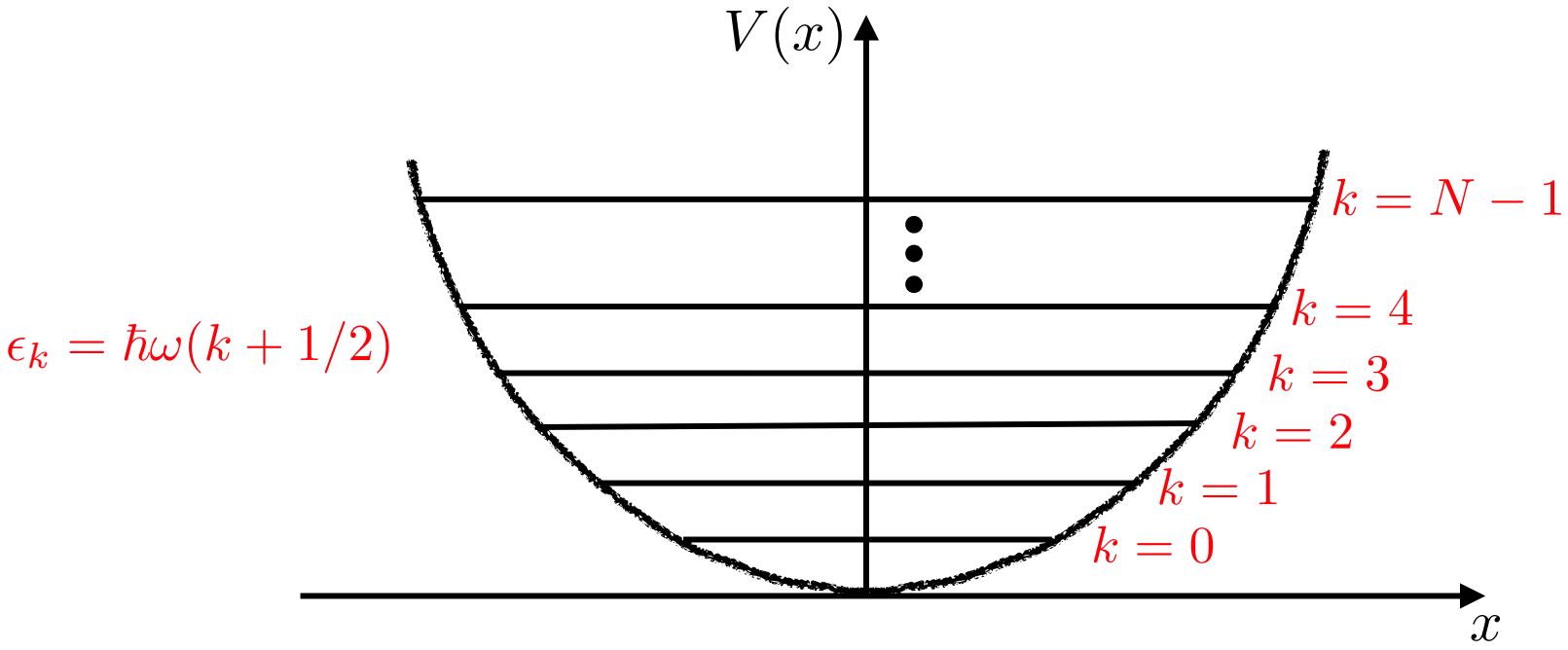

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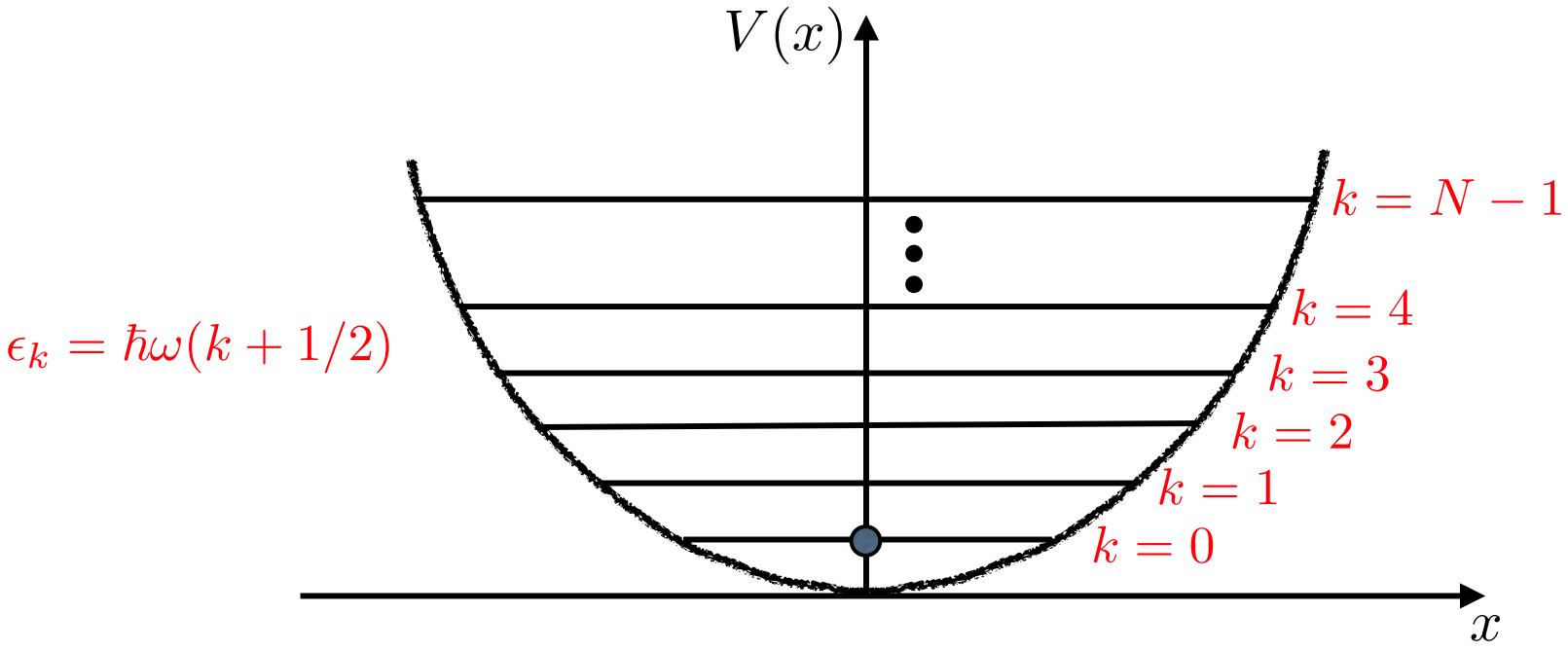

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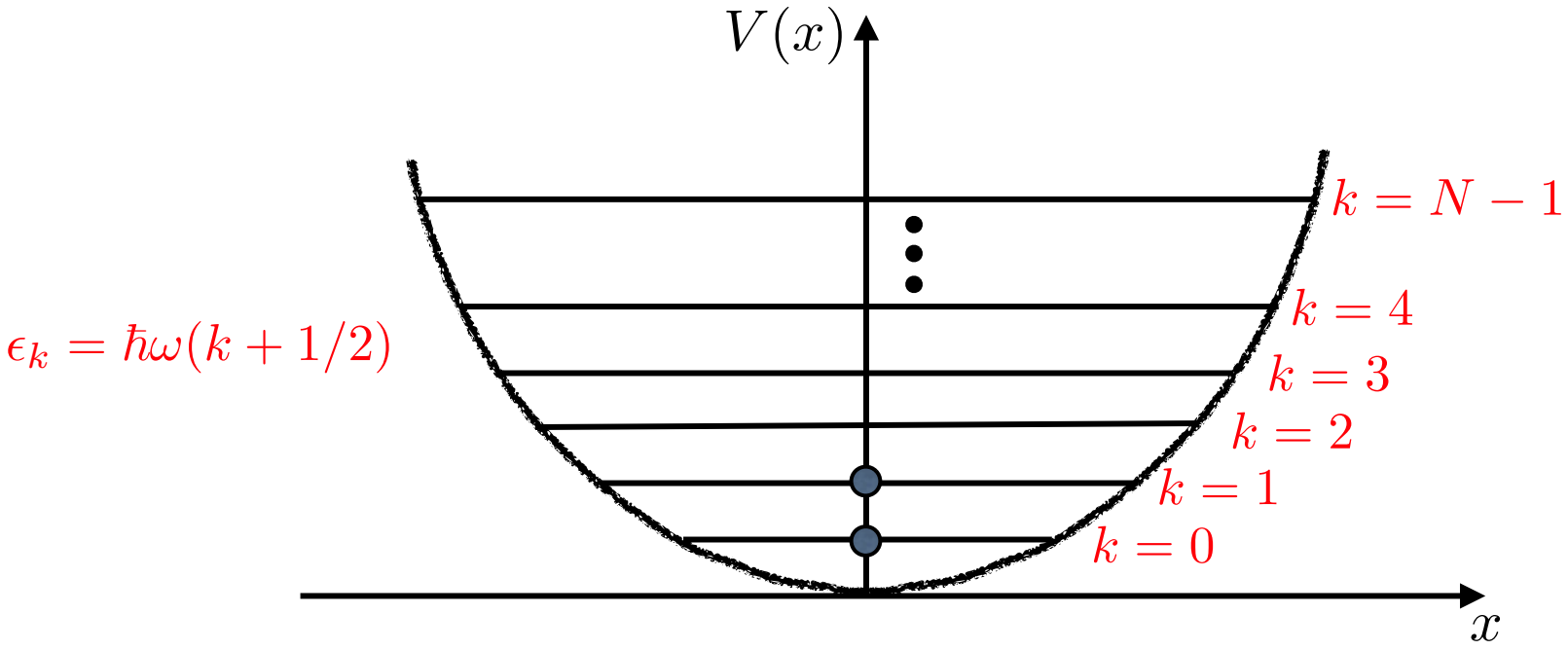

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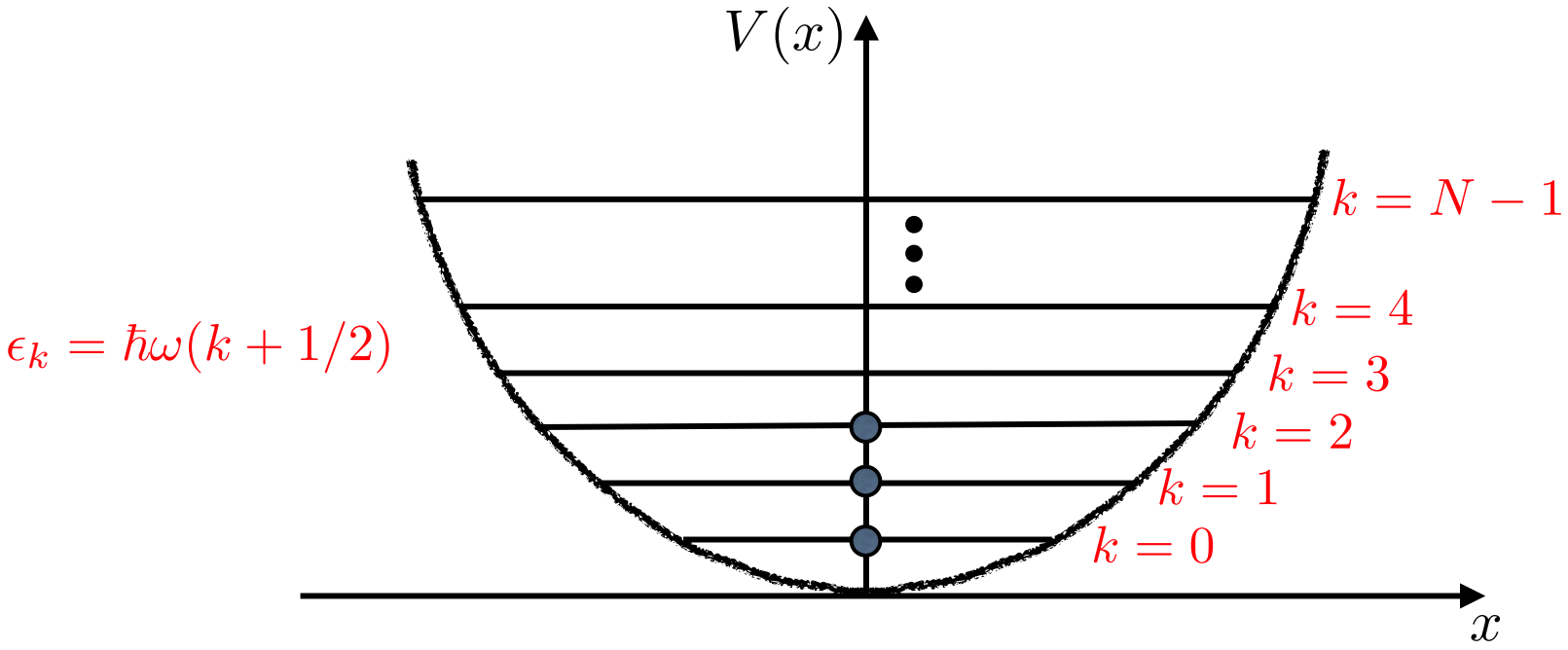

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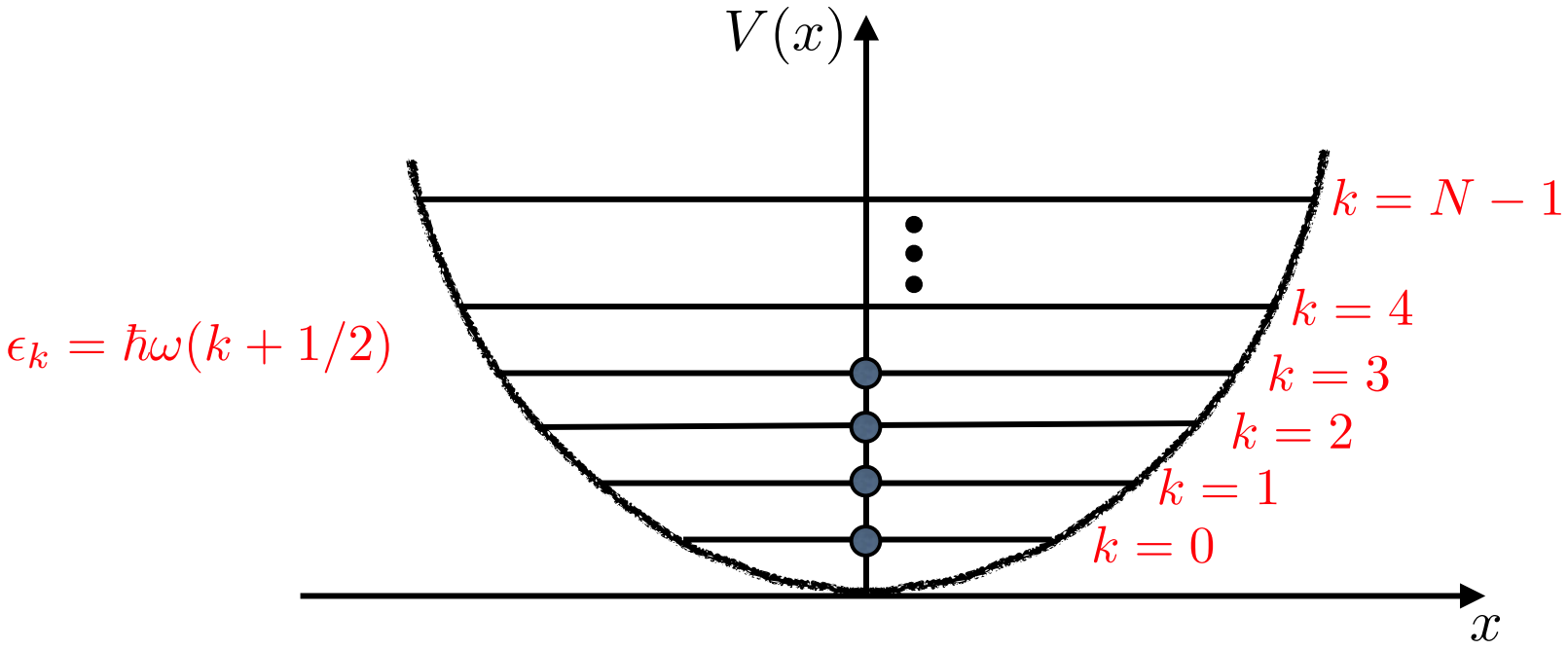

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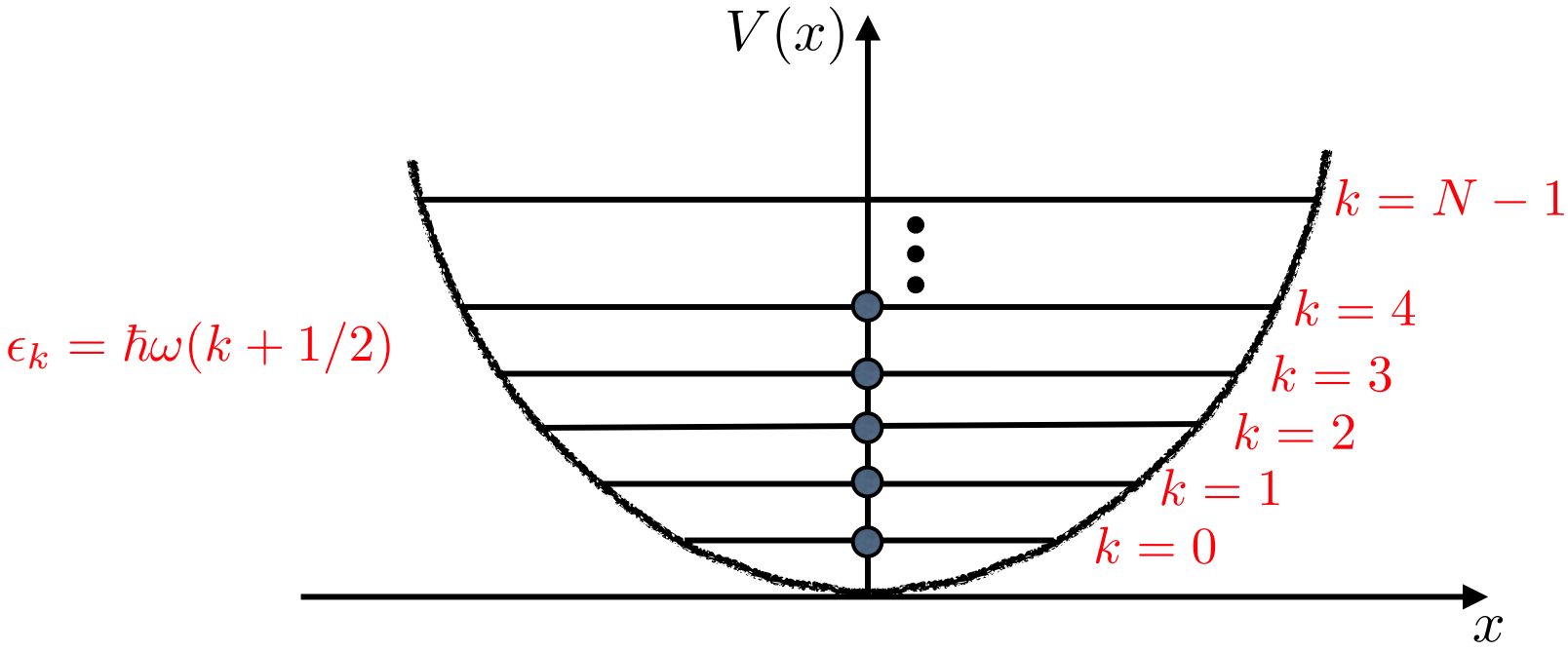

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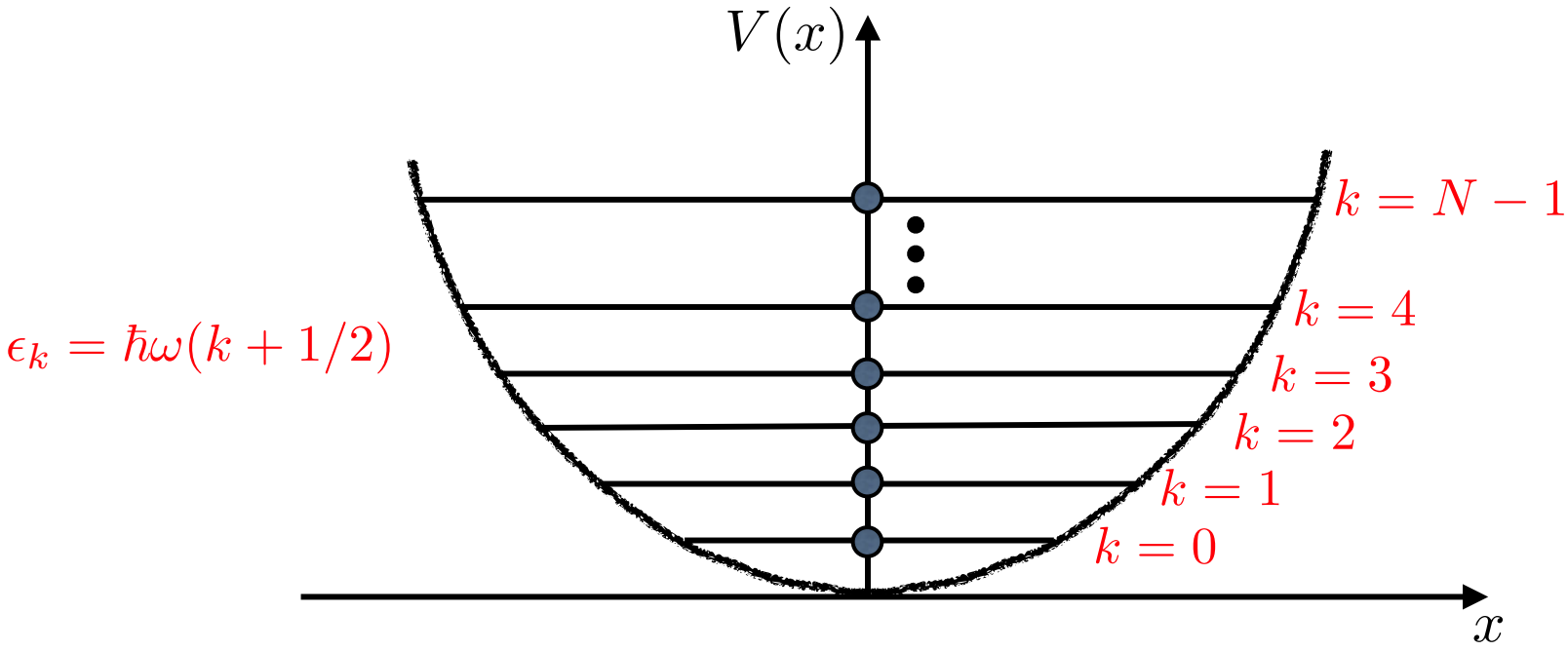

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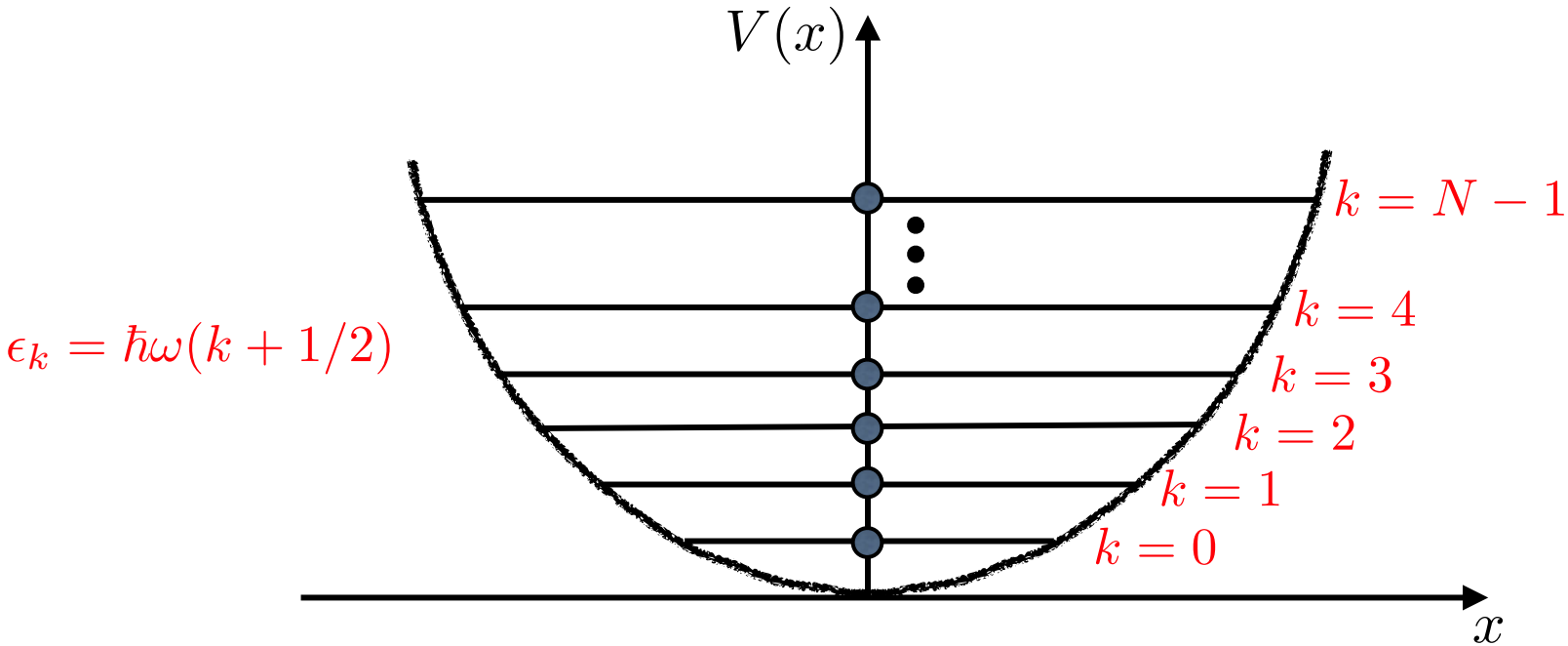

- The N-particle wave function is given by a $N \times N$ Slater determinant

$$\Psi_0(x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)] \qquad 0 \le i \le N-1$$

$$1 \le j \le N$$

$$\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi} 2^k k!}\right]^{1/2} e^{-\frac{\alpha^2 x^2}{2}} H_k(\alpha x)$$

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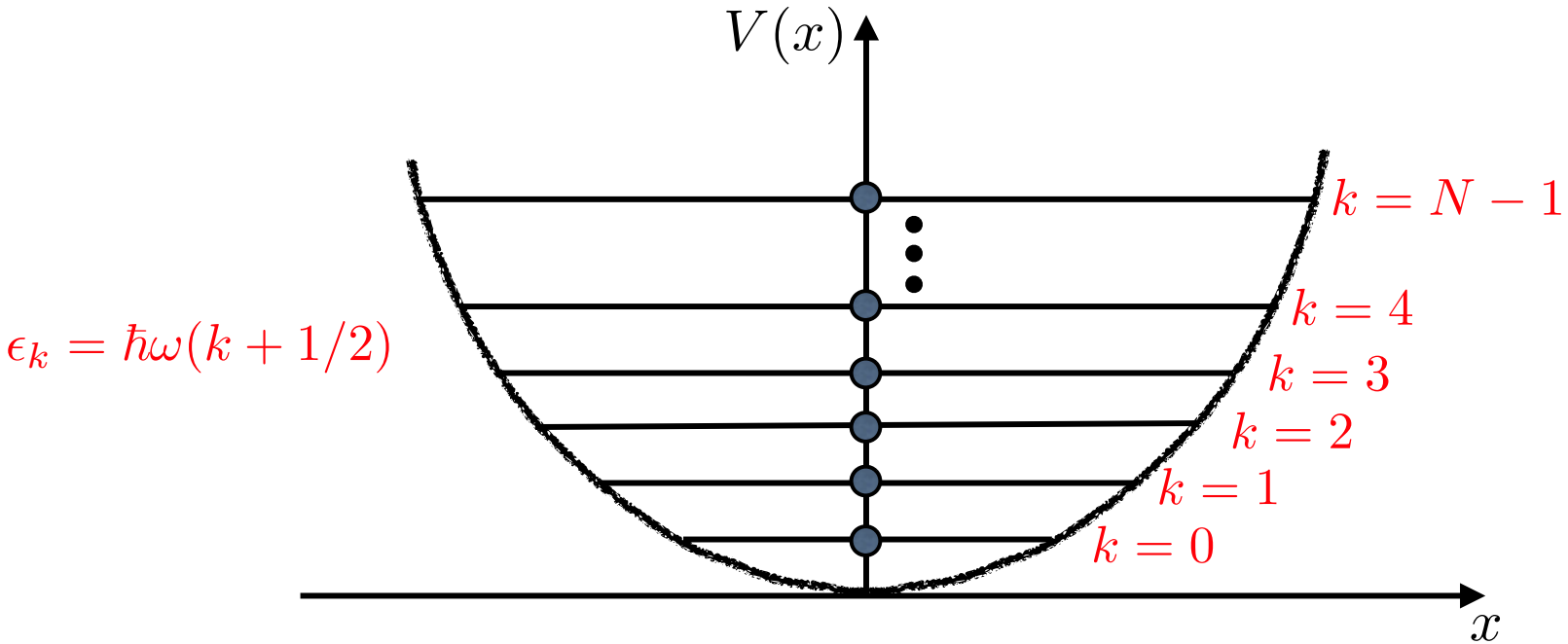

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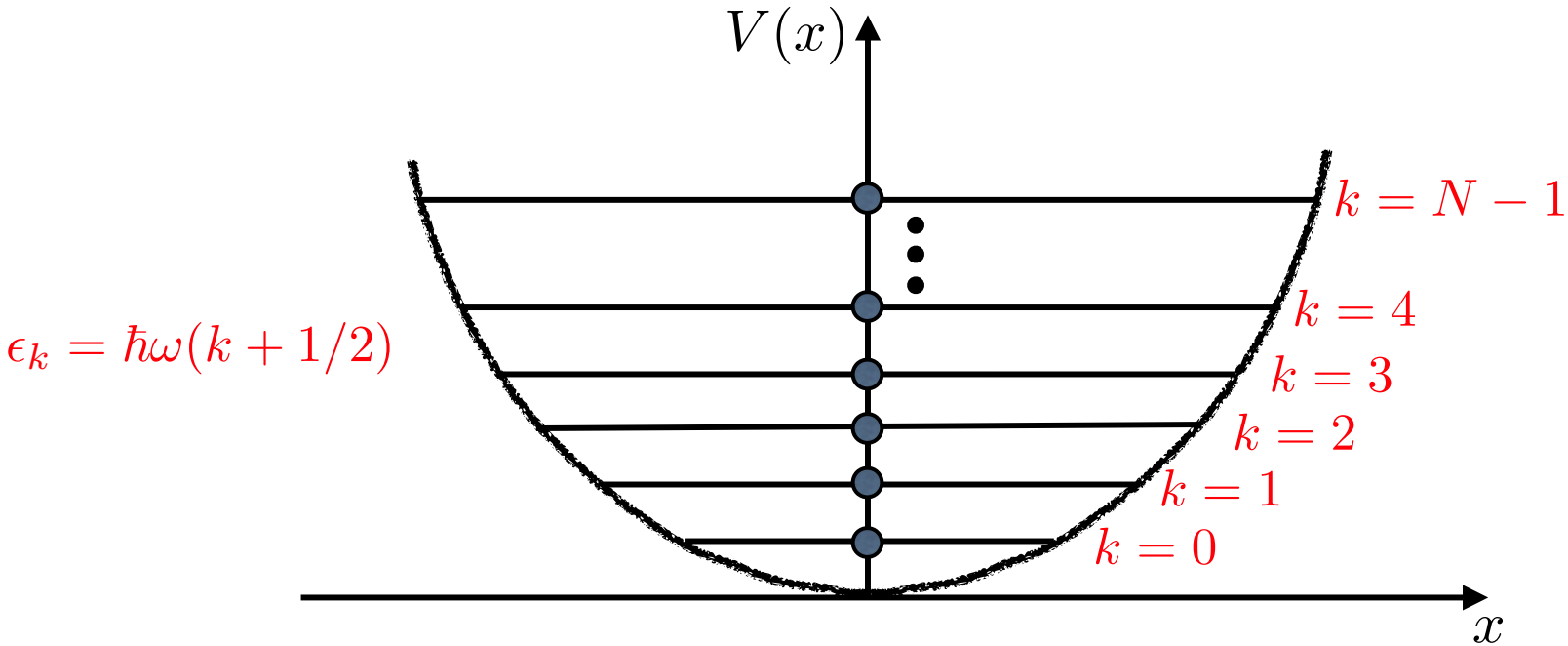
$$\Rightarrow \quad \Psi_0(x_1, x_2, \cdots, x_N) \propto e^{-\frac{\alpha^2}{2}(x_1^2 + \cdots + x_N^2)} \prod_{1 \le j < k \le N} (x_j - x_k)$$

N spinless free fermions in a 1d harmonic trap at T=0



- Probability density function (PDF) of the positions

$$|\Psi_0(x_1, \cdots, x_N)|^2 = \frac{1}{z_N(\alpha)} \prod_{i<j} (x_i - x_j)^2 e^{-\alpha^2 \sum_{i=1}^N x_i^2}$$

The positions of the free fermions behave statistically like the eigenvalues of GUE random matrices

Eisler '13/Marino, Majumdar, G. S., Vivo '14/Calabrese, Le Doussal, Majumdar '15

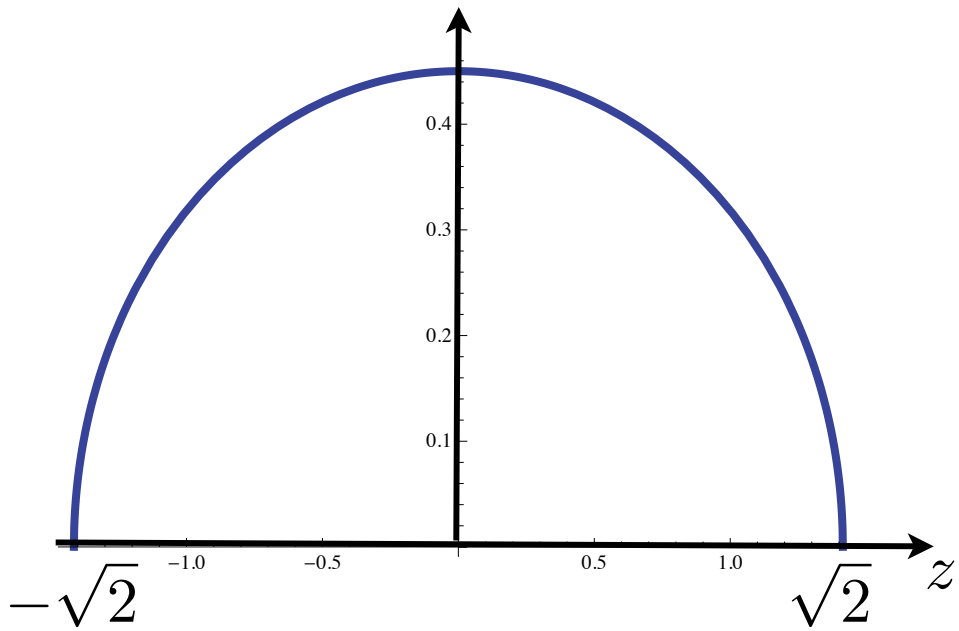
Properties of fermions in a 1d harmonic trap at T=0

- Average density of free fermions: Wigner semi-circle law

$$\rho_N(x, T=0) = \frac{1}{N}\sum_{i=1}^{N}\langle\delta(x-x_i)\rangle$$

for $N \gg 1$ $\quad \rho_N(x, T=0) \approx \frac{\alpha}{\sqrt{N}} f_W\left(\frac{\alpha\, x}{\sqrt{N}}\right)$, $\boxed{f_W(z) = \frac{1}{\pi}\sqrt{2-z^2}}$

See also Local Density (or Thomas-Fermi) Approx. in the literature on fermions



What about other Hermitian matrix models ?

- Laguerre Unitary Ensemble can be realized with a 1d singular potential

$$V(x) = \frac{\alpha(\alpha-1)}{x^2} + \beta x^2 \ , \ x > 0$$

Nadal, Majumdar, PRE '09

- Jacobi Unitary Ensemble can be realized with a 1d box potential

$$V(x) = \begin{cases} 0 \ , & -1 \leq x \leq +1 \\ +\infty \ , & |x| > 1 \end{cases}$$

Lacroix-A-Chez-Toine, Le Doussal, Majumdar, G. S., EPL '17

- For a review see Dean, Le Doussal, Majumdar, G. S., J. Phys. A '19

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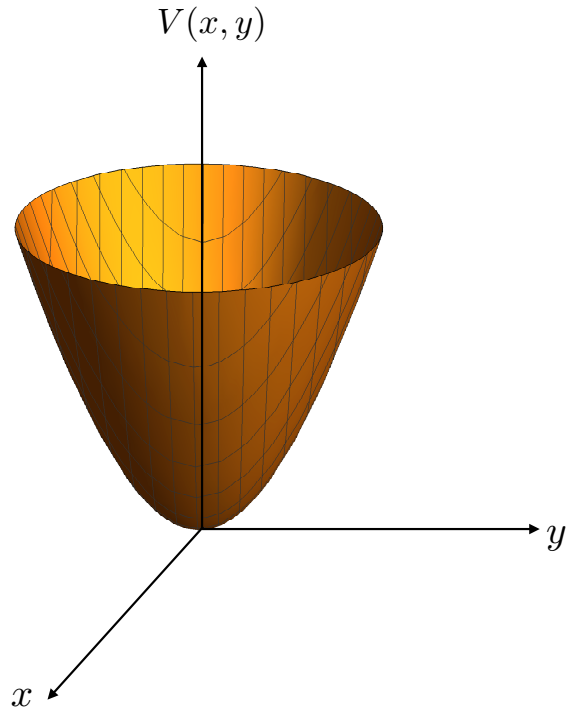
What about 2d (non Hermitian) matrix ensembles,
e.g., Ginibre ensemble ?

A natural candidate: fermions in a 2d harmonic trap

- Single particle Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}m\omega^2\underbrace{(x^2 + y^2)}_{r^2}$$



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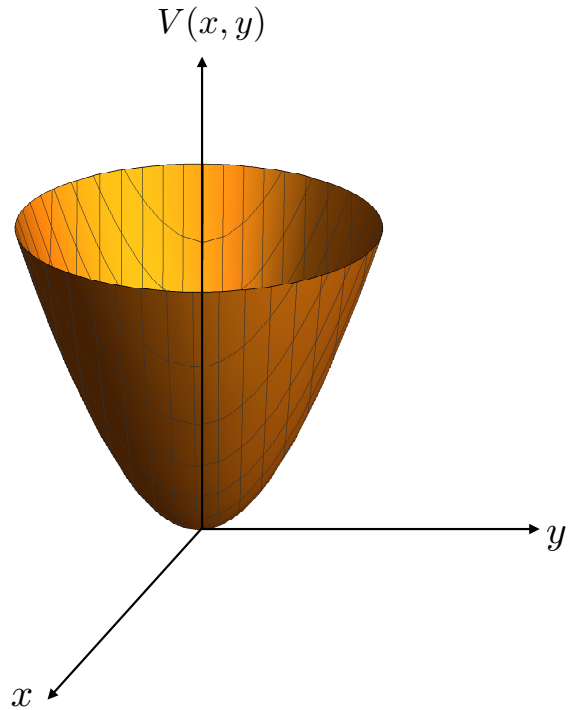
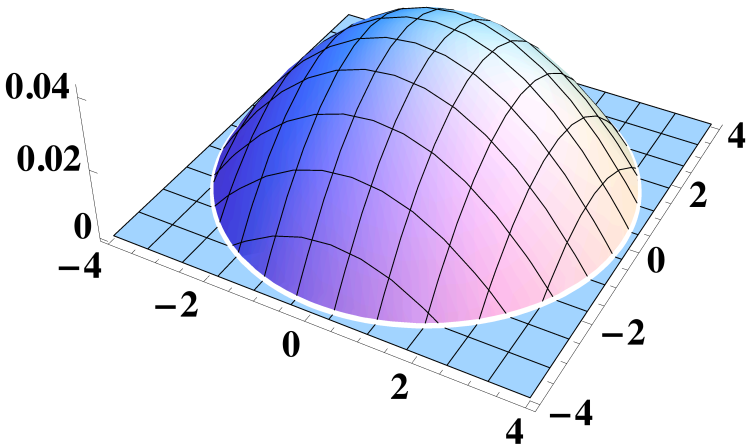


- Global density (at T=0)

$$\rho_N(r) \approx \frac{1}{N}\frac{m}{2\pi\hbar^2}\left(\mu - \frac{1}{2}m\omega^2 r^2\right)$$

with $\mu \approx \hbar\omega\sqrt{2N}$

spatial extent $R_e \propto N^{1/4}$



Dean, Le Doussal, Majumdar, G. S., EPL '15, PRA '16

see also G. Lambert 's talk

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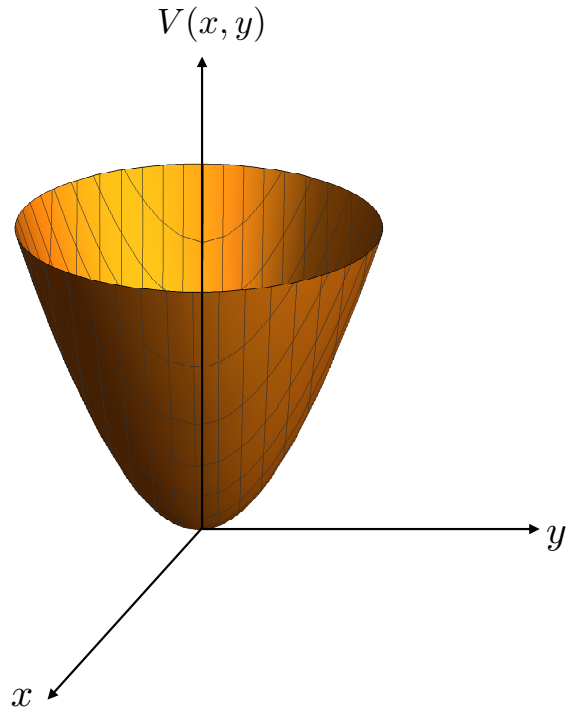

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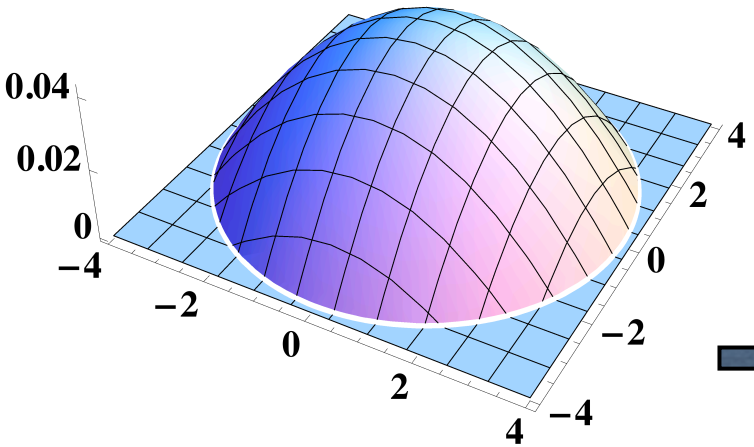

Dean, Le Doussal, Majumdar, G. S., EPL '15, PRA '16

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No apparent connection with known matrix models !

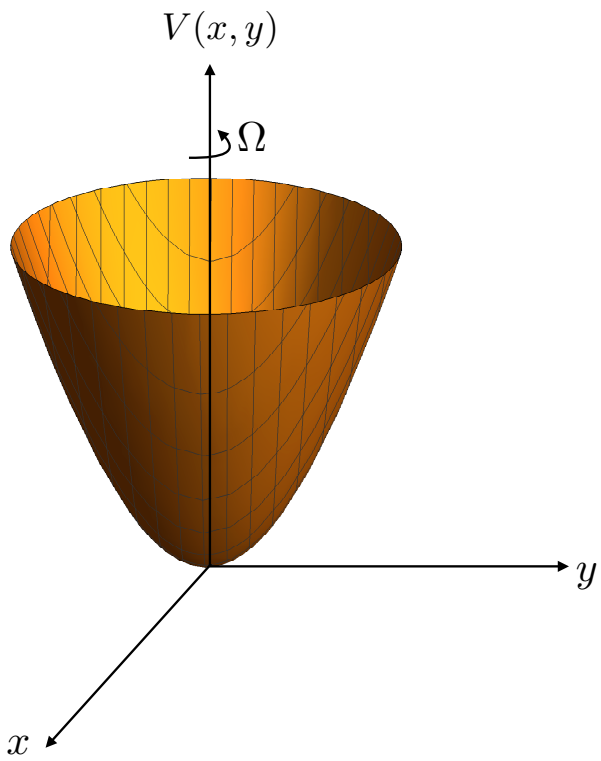
Fermions in a 2D rotating harmonic trap

- Single particle Hamiltonian in the rotating frame

$$\hat{H} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}\omega^2(x^2 + y^2) \quad -\Omega\, L_z$$

where $\mathbf{L} = (\mathbf{r} \wedge \mathbf{p})$: angular momentum

and $L_z = i(y\partial_x - x\partial_y)$



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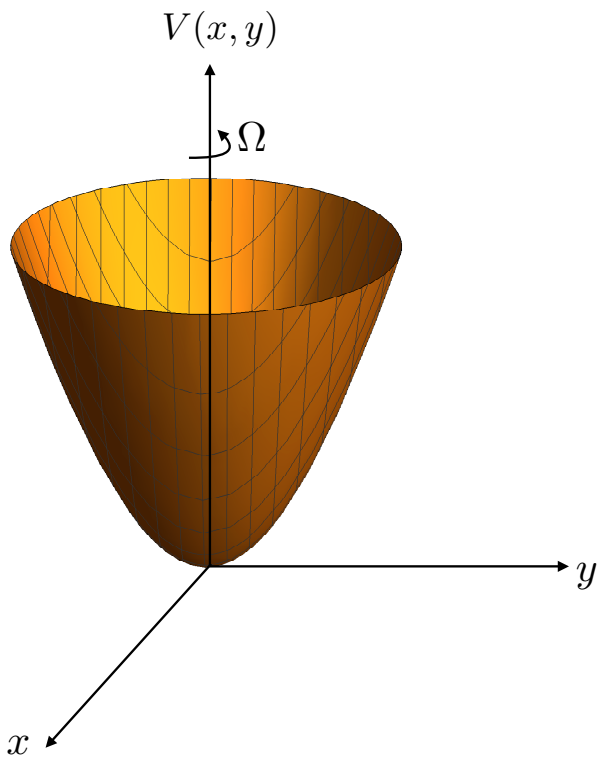

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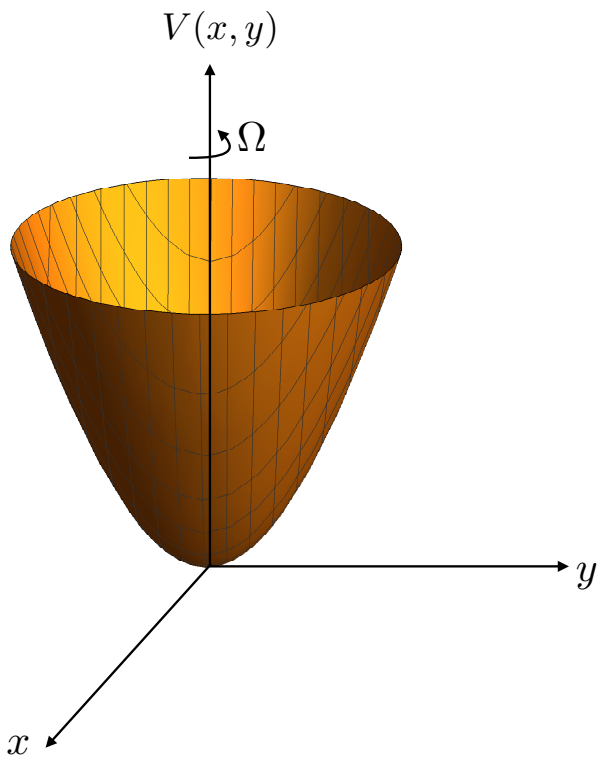

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- Analogy with a magnetic system

$$\hat{H} = \frac{1}{2}(\mathbf{p}-\mathbf{A})^2 + \frac{1}{2}\omega^2\mathbf{r}^2 - \frac{1}{2}\Omega^2\mathbf{r}^2$$

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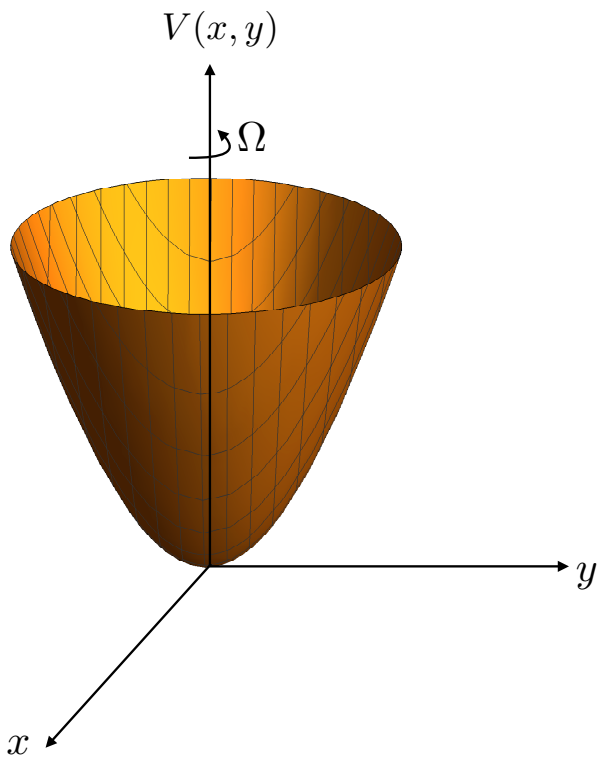

- Analogy with a magnetic system

Coriolis potential

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Fermions in a 2D rotating harmonic trap

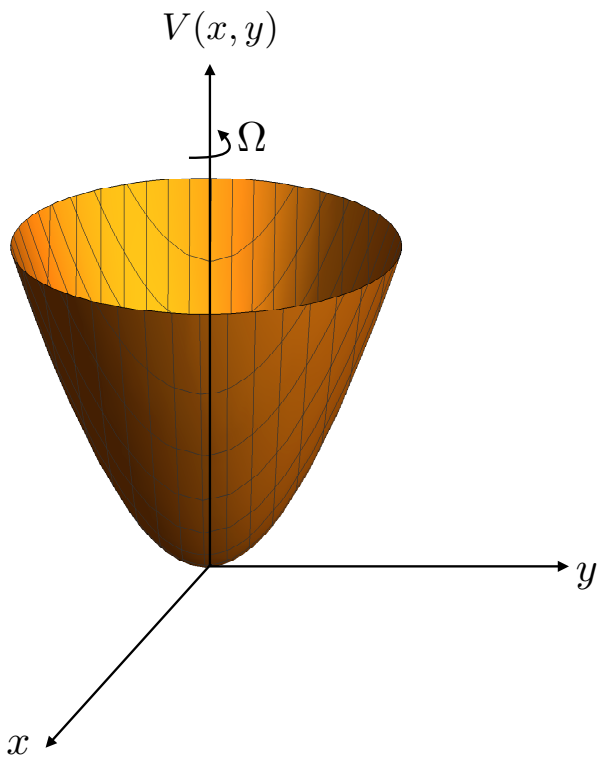
- Single particle Hamiltonian in the rotating frame

$$\hat{H} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}\omega^2(x^2 + y^2) \quad -\Omega\, L_z$$

where $\mathbf{L} = (\mathbf{r} \wedge \mathbf{p})$: angular momentum

and $L_z = i(y\partial_x - x\partial_y)$

$$\hat{H} = \frac{\mathbf{p}^2}{2} + \frac{1}{2}\omega^2\mathbf{r}^2 \quad -\Omega\, L_z$$



- Analogy with a magnetic system

Coriolis potential

centrifugal potential

$$\hat{H} = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2 + \frac{1}{2}\omega^2\mathbf{r}^2 - \frac{1}{2}\Omega^2\mathbf{r}^2$$

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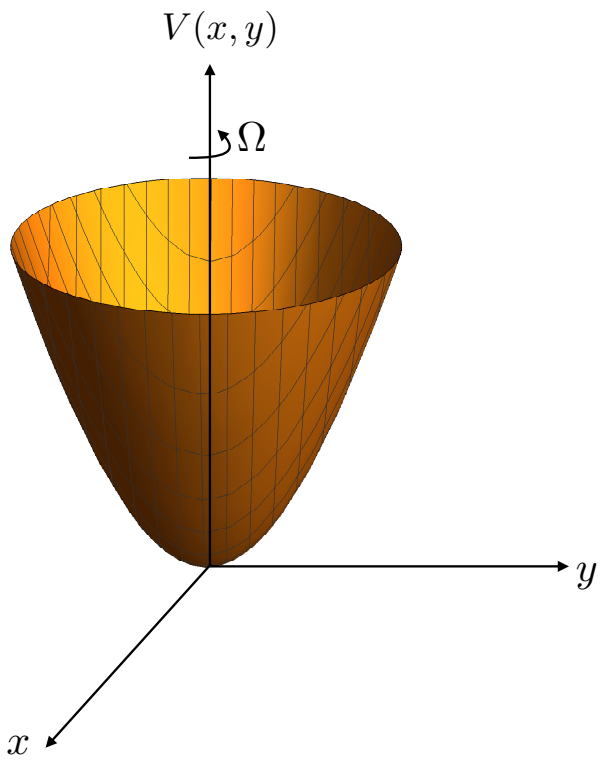

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effective mag. field

$$\mathbf{B}_{\text{eff}} = \nabla \wedge \mathbf{A} = \Omega\, \mathbf{u_z}$$

$$\mathbf{A} = \begin{pmatrix} \Omega\, y \\ -\Omega\, x \end{pmatrix}$$

symmetric gauge

Fermions in a 2D rotating harmonic trap

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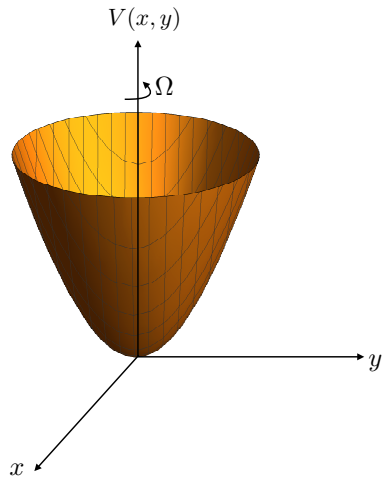

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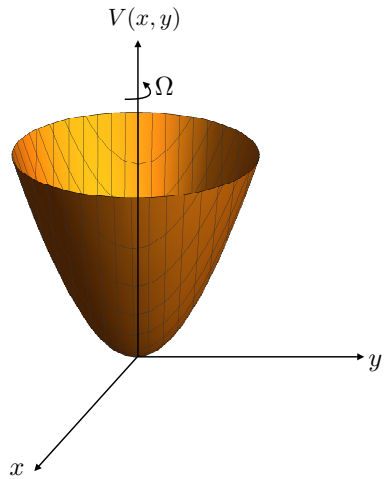

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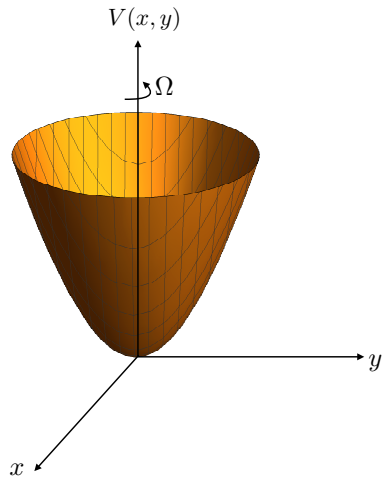

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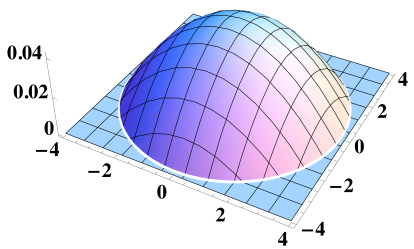

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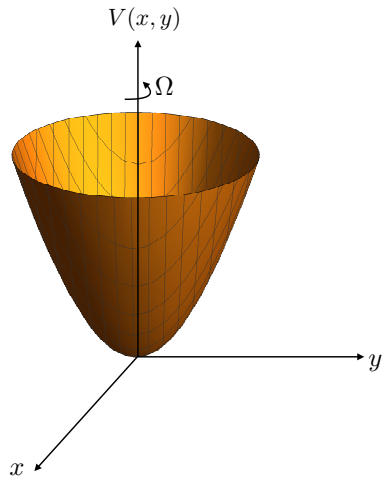

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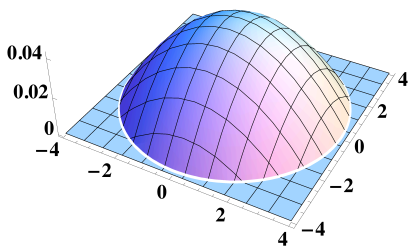

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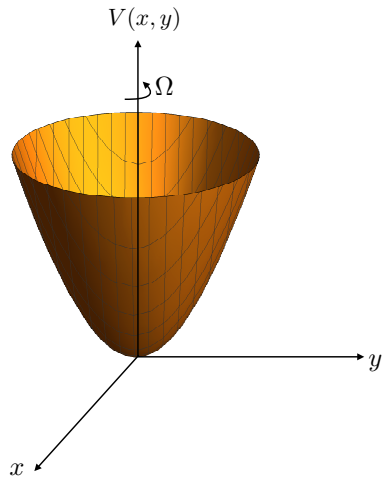

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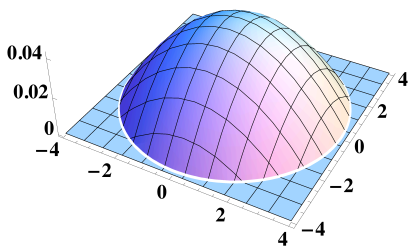
the Fermi gas becomes unstable

0 ω Ω

no rotation

free fermions in a magnetic field



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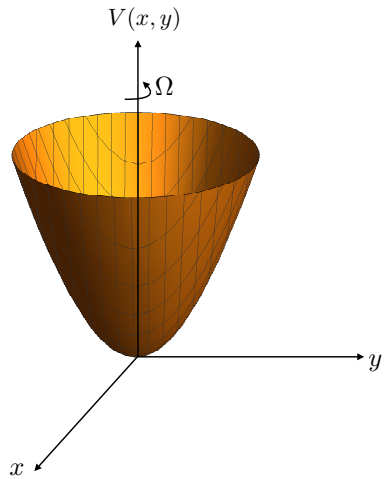

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$O(1/N)$

physics of the lowest Landau levels (LLL)

free fermions in a magnetic field

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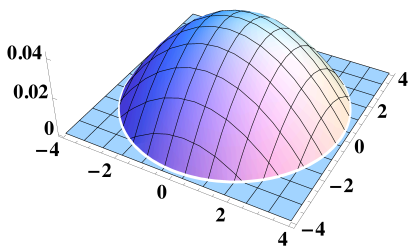

Fermions in a 2D rotating harmonic trap

$$\hat{H} = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2 + \frac{1}{2}\mathbf{r}^2 - \frac{1}{2}\Omega^2\mathbf{r}^2$$ we set $\hbar = m = \omega = 1$

- Single particle eigenfunctions $\psi_{k,\ell}(r,\theta)$ in polar coordinates are labelled by two quantum numbers (k,ℓ) with $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$

$$\hat{H}\psi_{k,\ell} = \epsilon_{k,\ell}\psi_{k,\ell}$$

$$\begin{cases} \psi_{k,\ell}(r,\theta) = a_{k,l} L_k^{|\ell|}(r^2) r^{|\ell|} e^{-r^2/2} e^{i\ell\theta} \\ \\ E_{k,\ell} = 2k + 1 + |\ell| - \Omega\ell \end{cases}$$

Fermions in a 2D rotating harmonic trap: Landau levels

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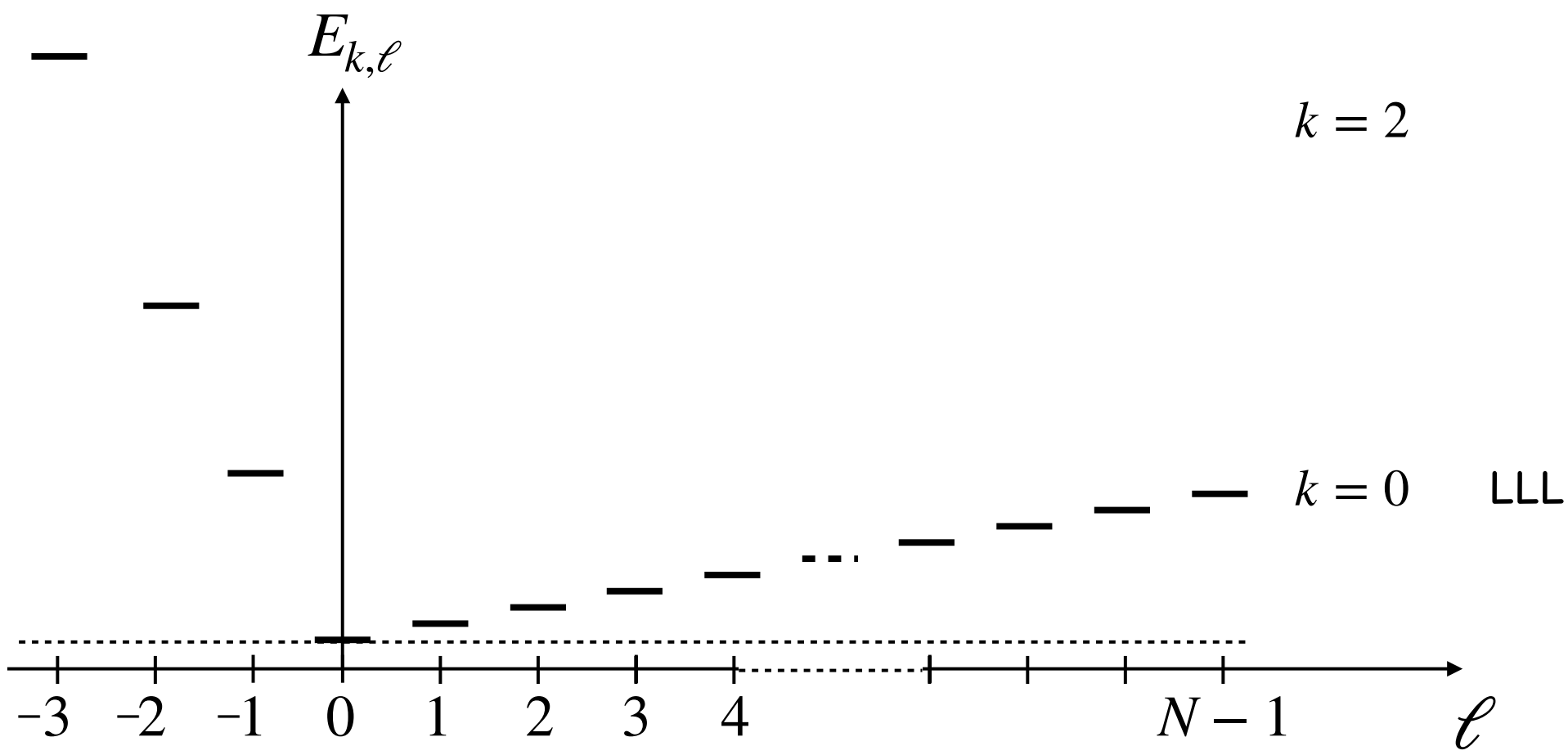

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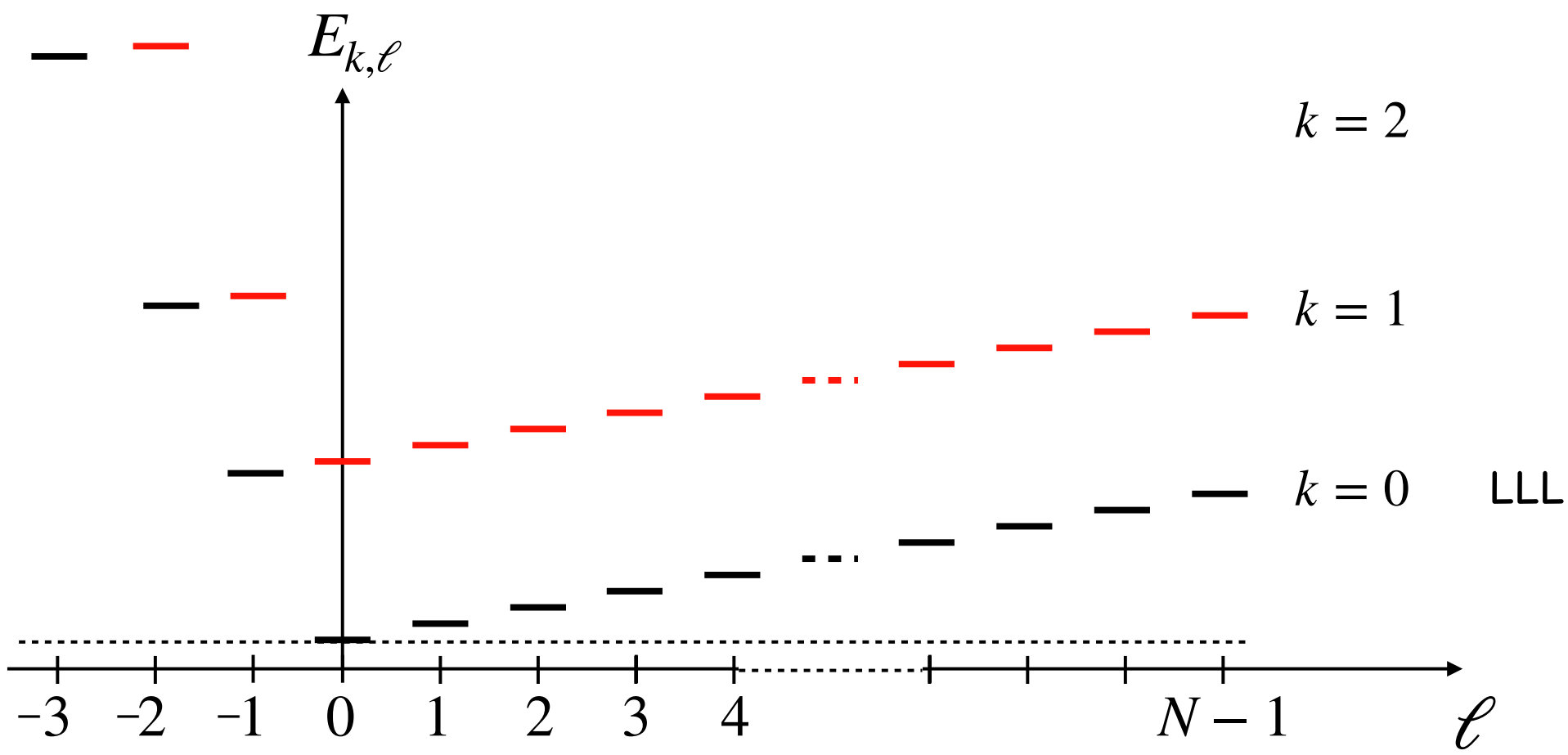

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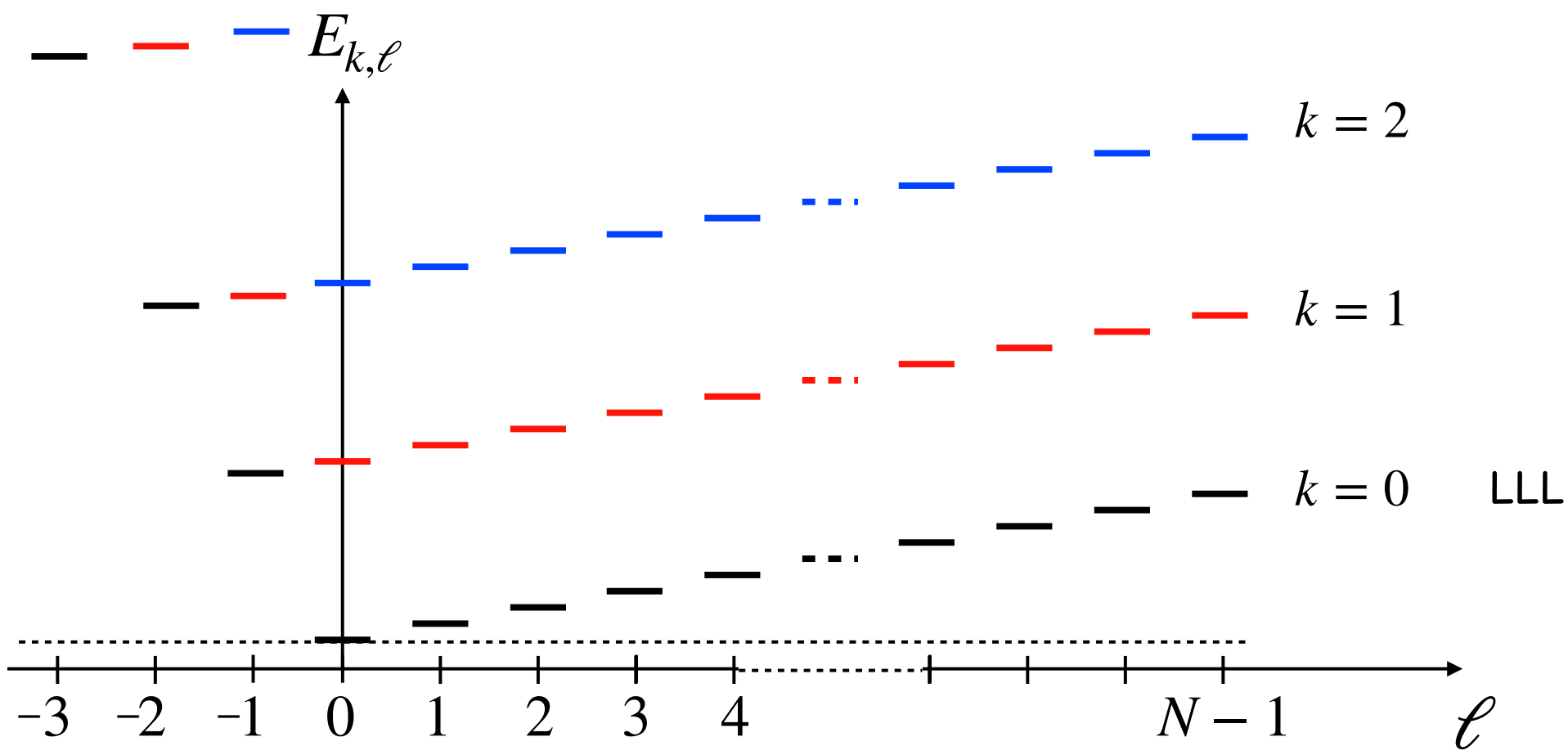

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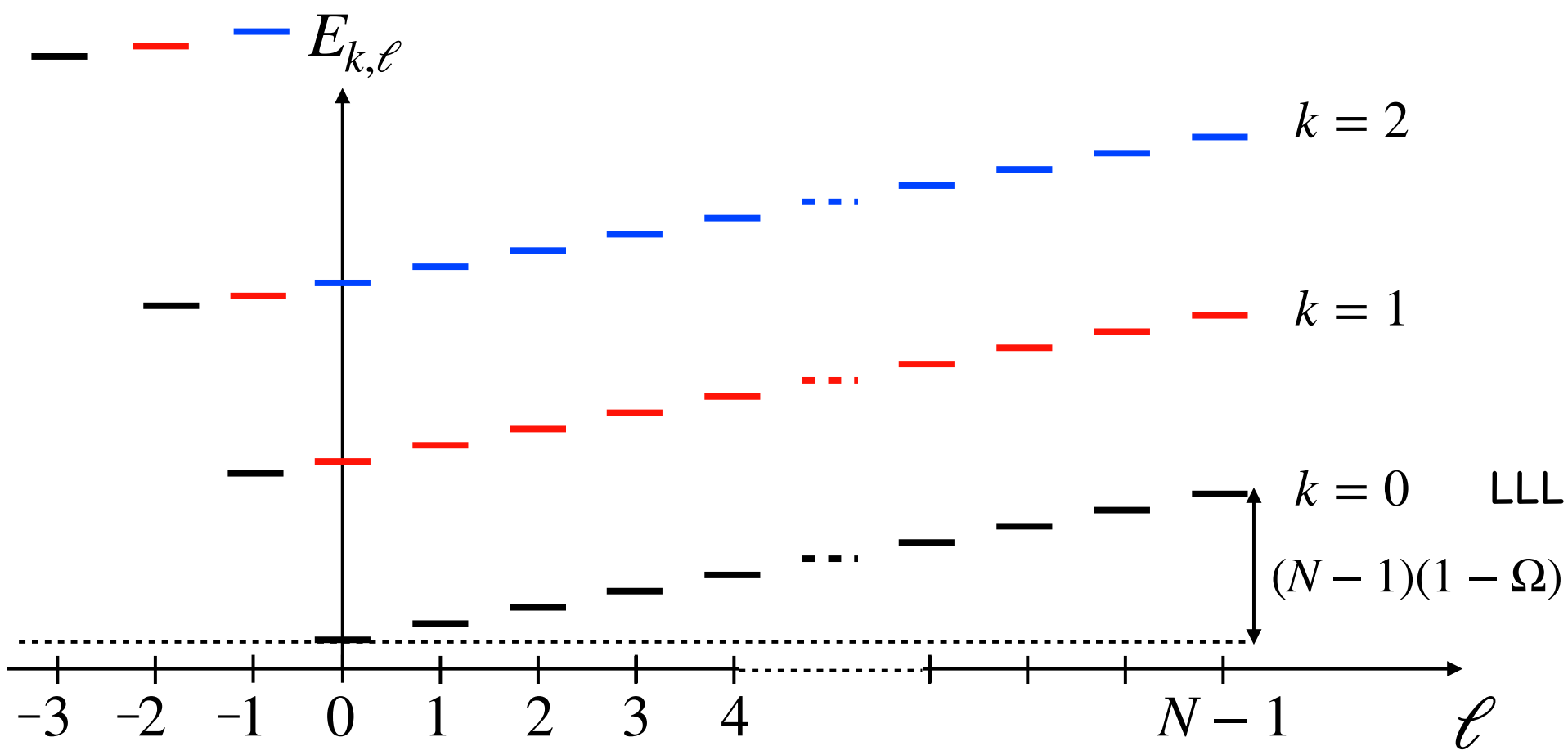

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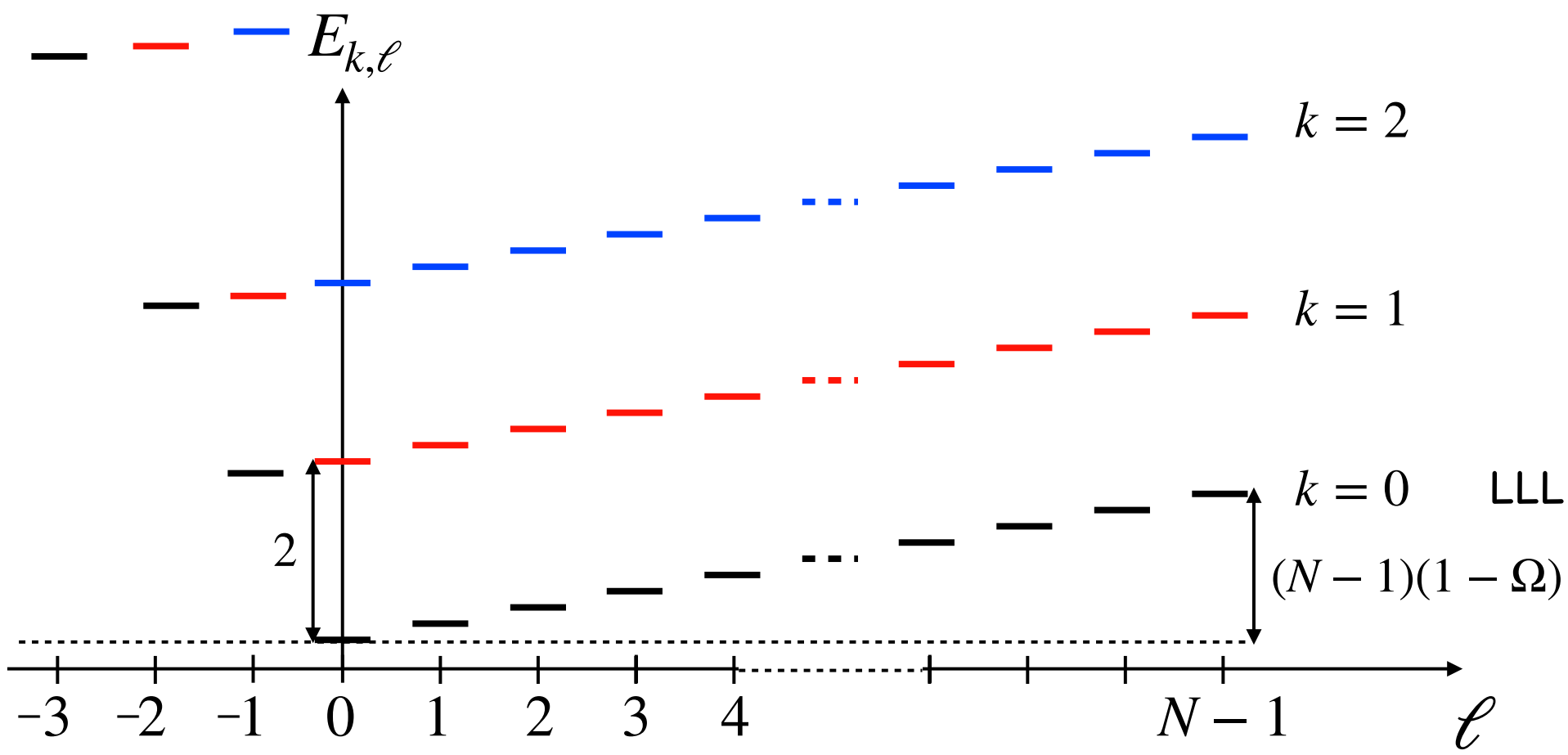

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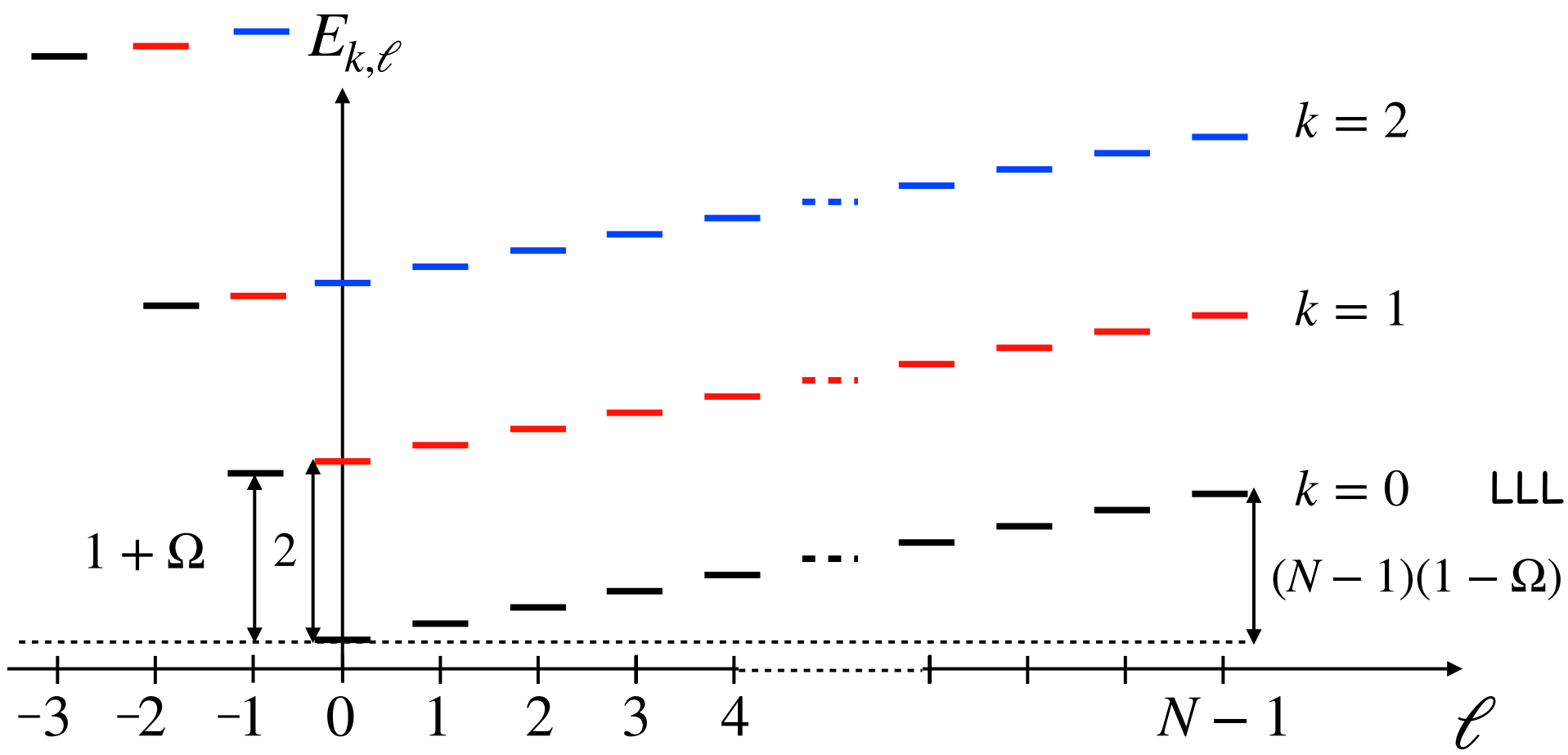
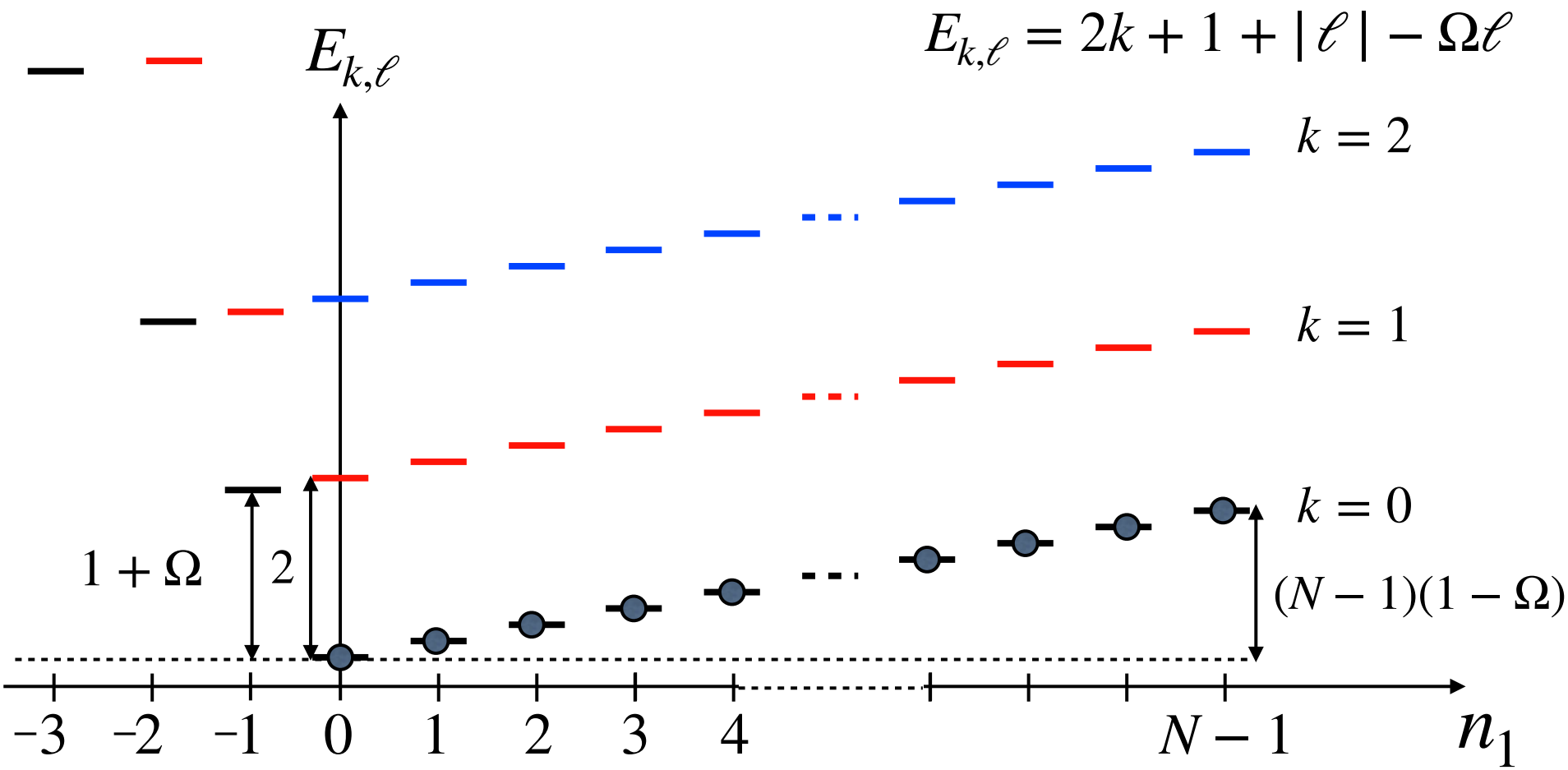


Fermions in a 2D rotating harmonic trap: ground state

- Ground state of N noninteracting spinless fermions



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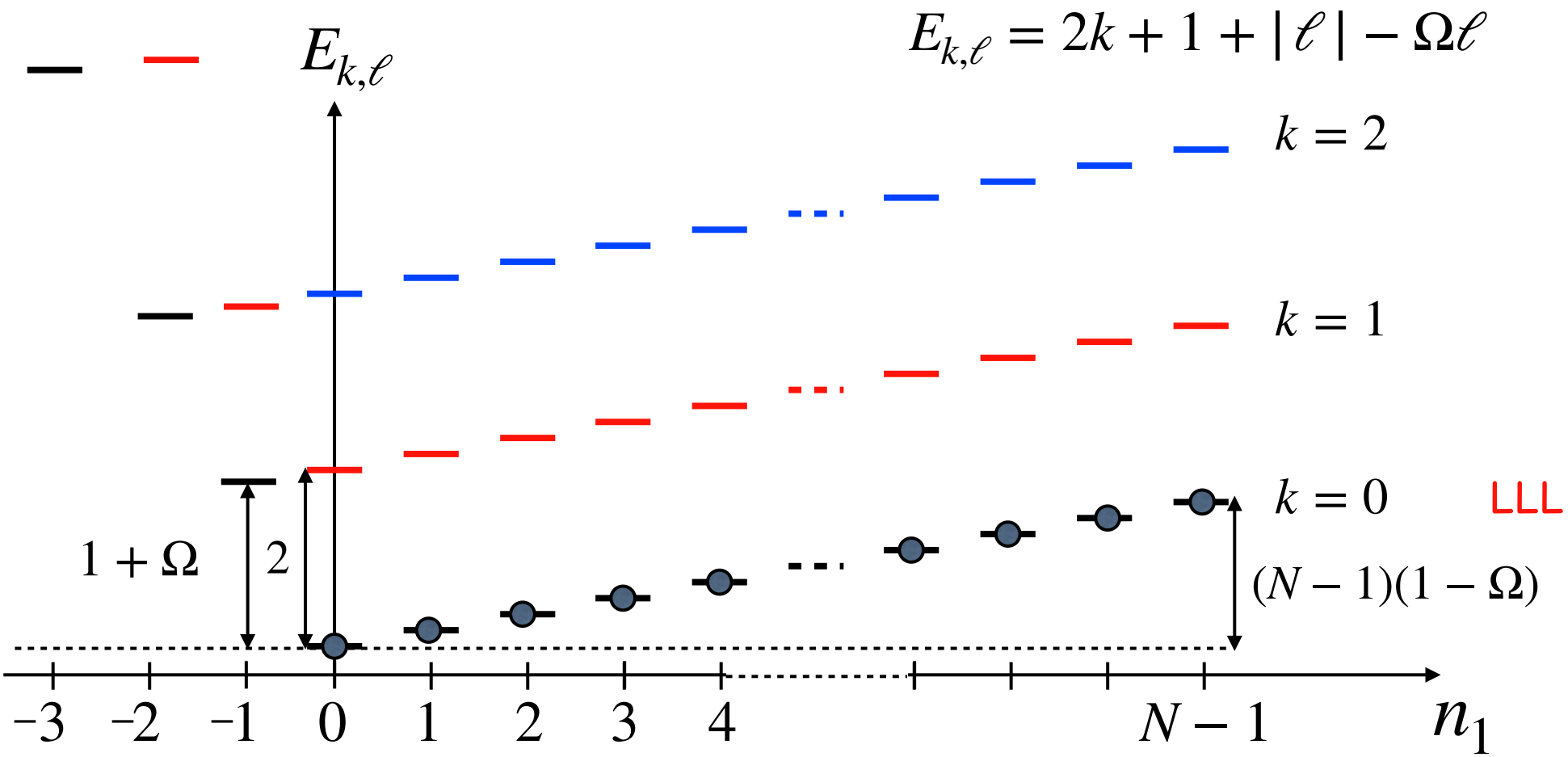

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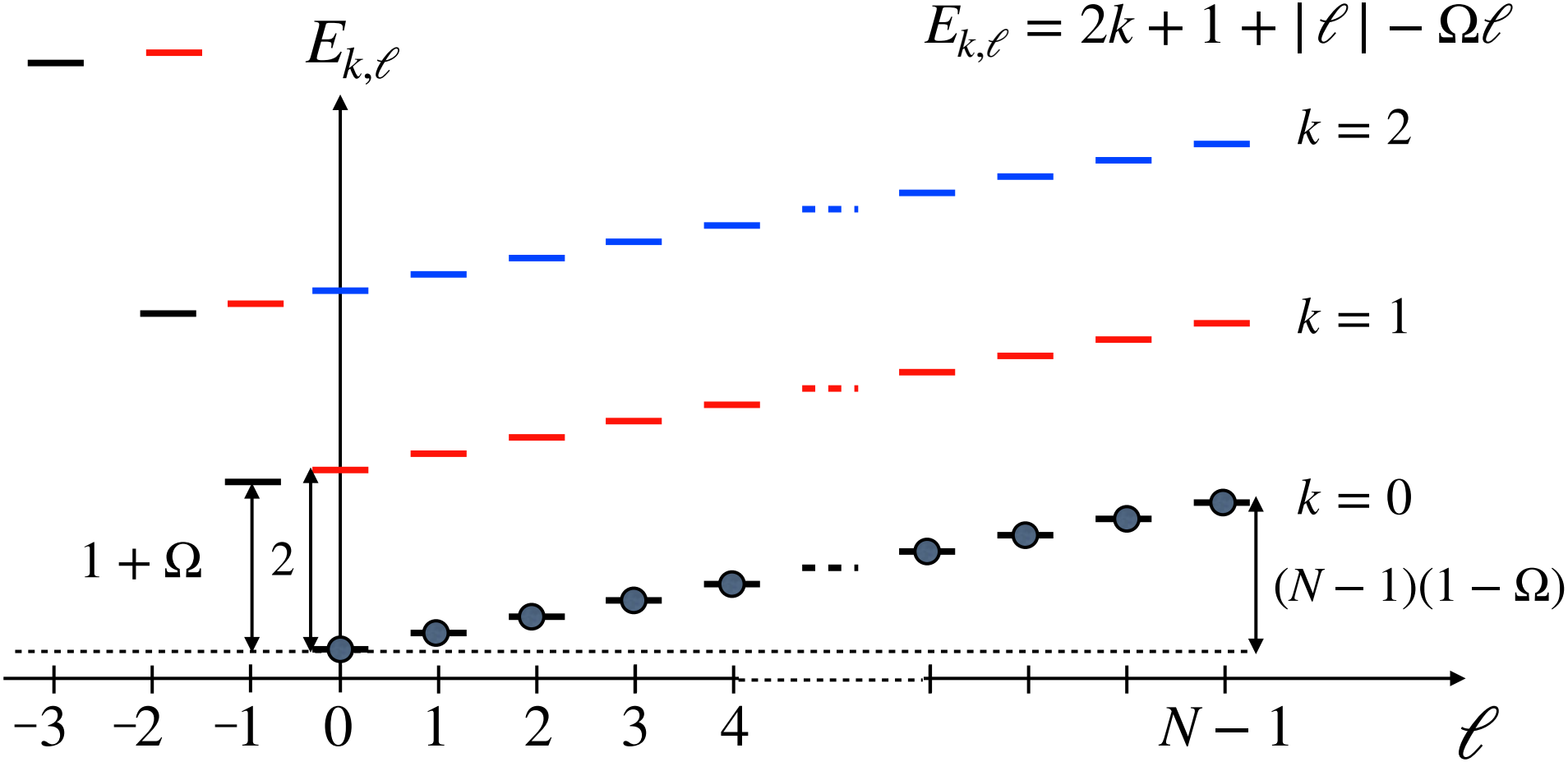

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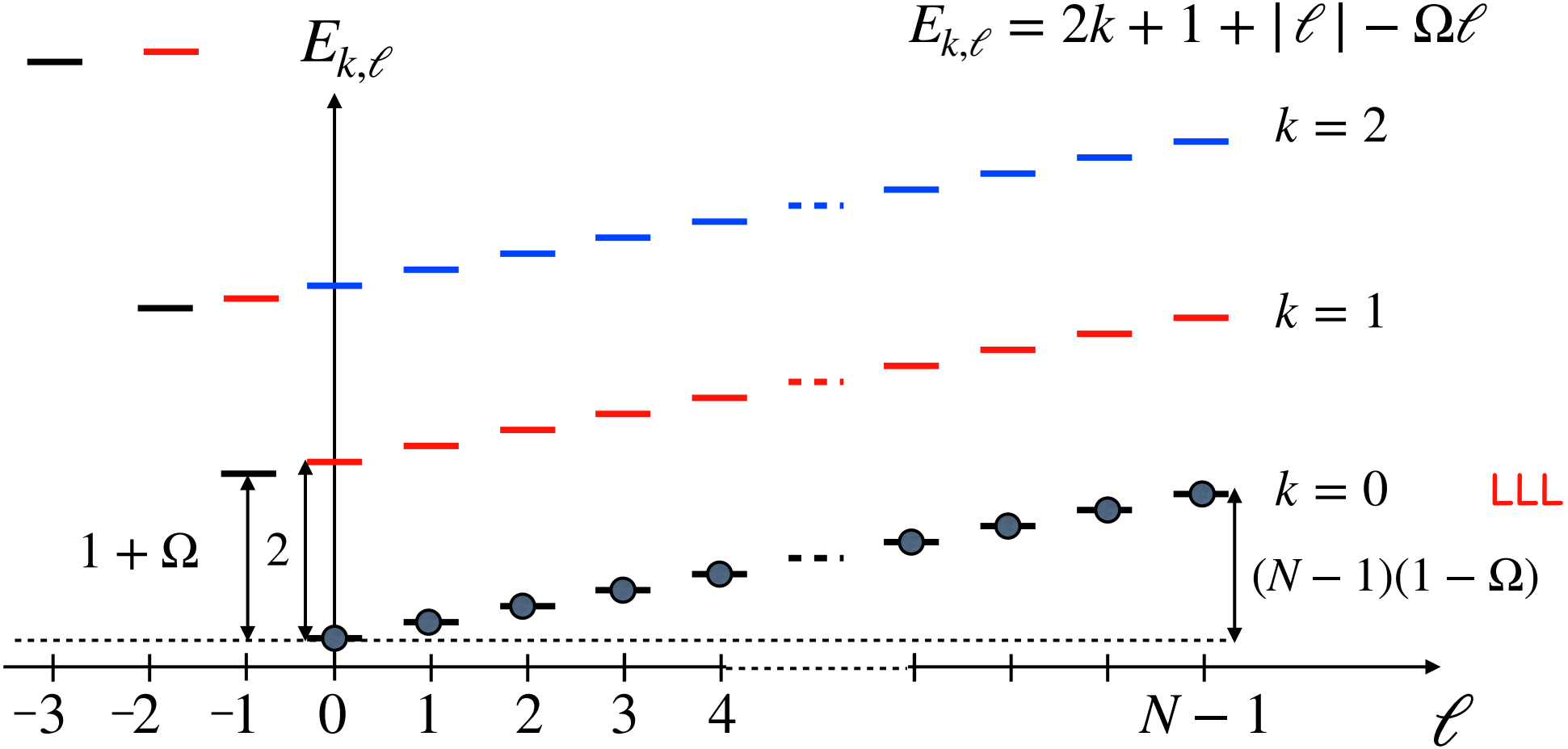

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- In the LLL, the single particle eigenstates/eigenvalues are

$$\psi_\ell(r,\theta,) \equiv \psi_{0,\ell}(r,\theta) = a_{0,\ell}\, r^\ell e^{i\ell\theta} e^{-r^2/2} \quad , \quad \ell \geq 0$$

$$\psi_\ell(z) = \frac{1}{\sqrt{\pi \ell\,!}} z^\ell\, e^{-\frac{1}{2}|z|^2} \quad \text{and} \quad E_\ell = 1 + (1-\Omega)\,\ell$$

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- Ground state many-body eigenfunction: Slater determinant

$$\Psi_0(z_1, z_2, \cdots, z_N) = \frac{1}{\sqrt{N!}} \det_{1\leq i,j\leq N} \psi_{i-1}(z_j)$$

$$\propto e^{-\frac{1}{2}\sum_{j=1}^n |z_j|^2} \det_{1\leq i,j\leq N} z_j^{i-1} \propto e^{-\frac{1}{2}\sum_{j=1}^n |z_j|^2} \prod_{1\leq i<j\leq N} (z_i - z_j)$$

Connection to the complex Ginibre ensemble GinUE

Lacroix-A-Chez-Toine, Majumdar, G. S., PRA (R) '19

- Joint proba. density function of the positions of the fermions

$$P_{\text{joint}}(z_1, z_2, \cdots, z_N) = |\Psi_0(z_1, z_2, \cdots, z_N)|^2 = A_N \prod_{1 \leq i,j \leq N} |z_i - z_j|^2 e^{-\sum_{i=1}^{N} |z_i|^2}$$

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→ the positions of the fermions are in one to one correspondence with the eigenvalues of a complex Ginibre random matrix

→ this provides a simple physical realisation of the Ginibre ensemble of RMT

→ this connection has been exploited to compute various observables, like the full counting statistics or the Rényi entanglement entropy

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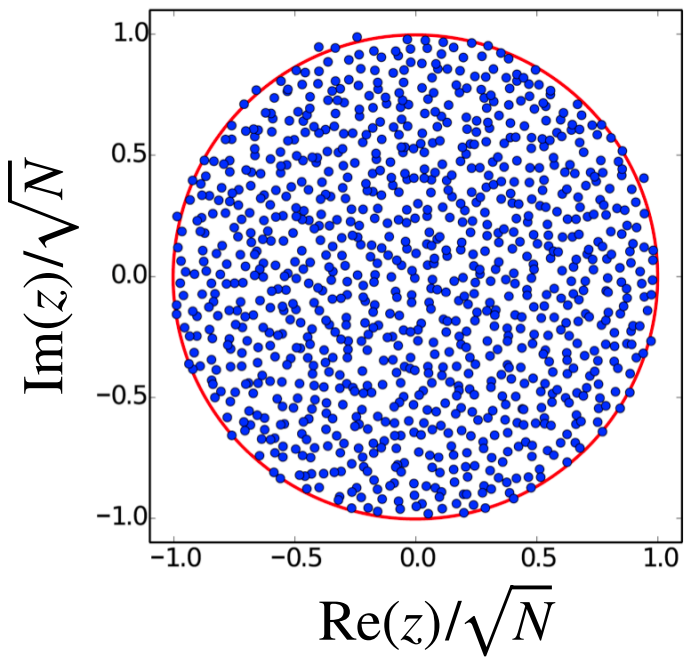
- Large N limit of the density: Girko 's law

$$\rho_N(z) = \sum_{i=1}^{N} \delta(z - z_i)$$

$$\xrightarrow[N \to \infty]{} \frac{1}{\pi} \Theta(\sqrt{N} - |z|)$$



spatial extent $R_e \propto N^{1/2}$

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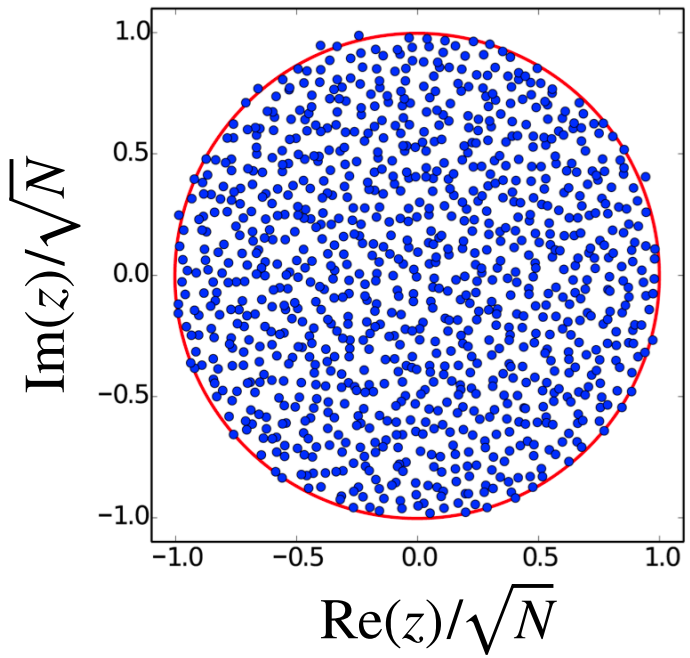
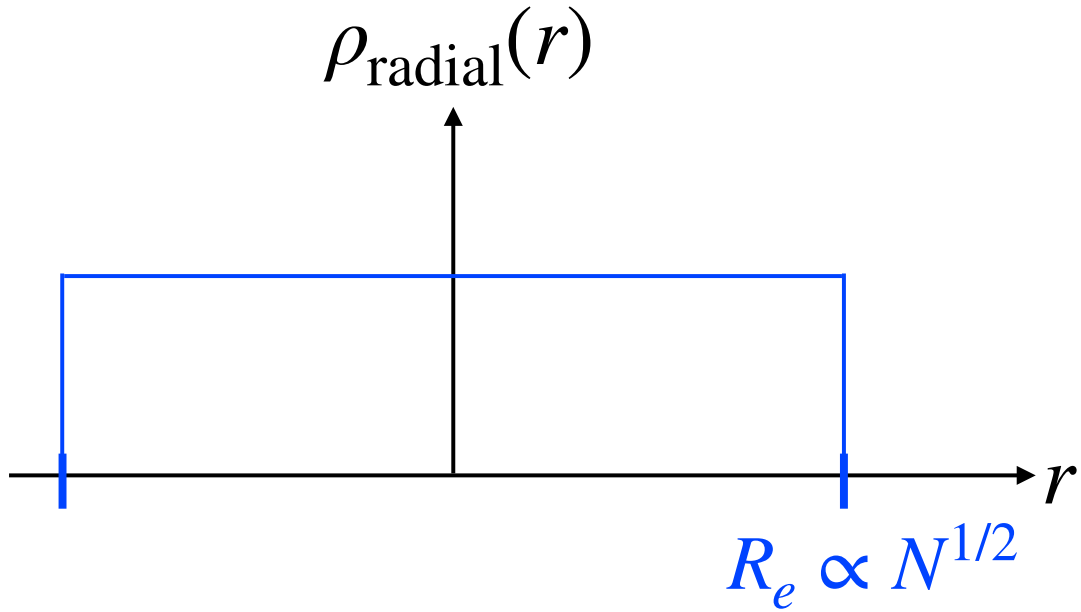
- Edge density profile





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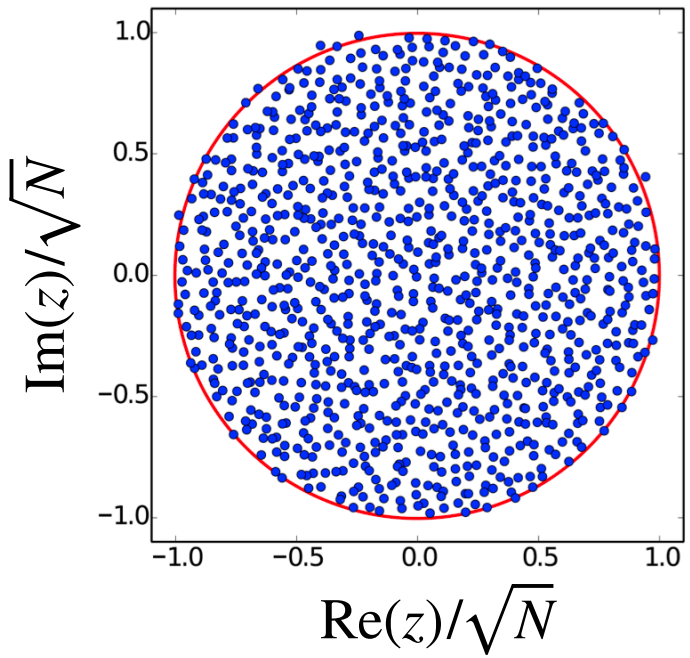

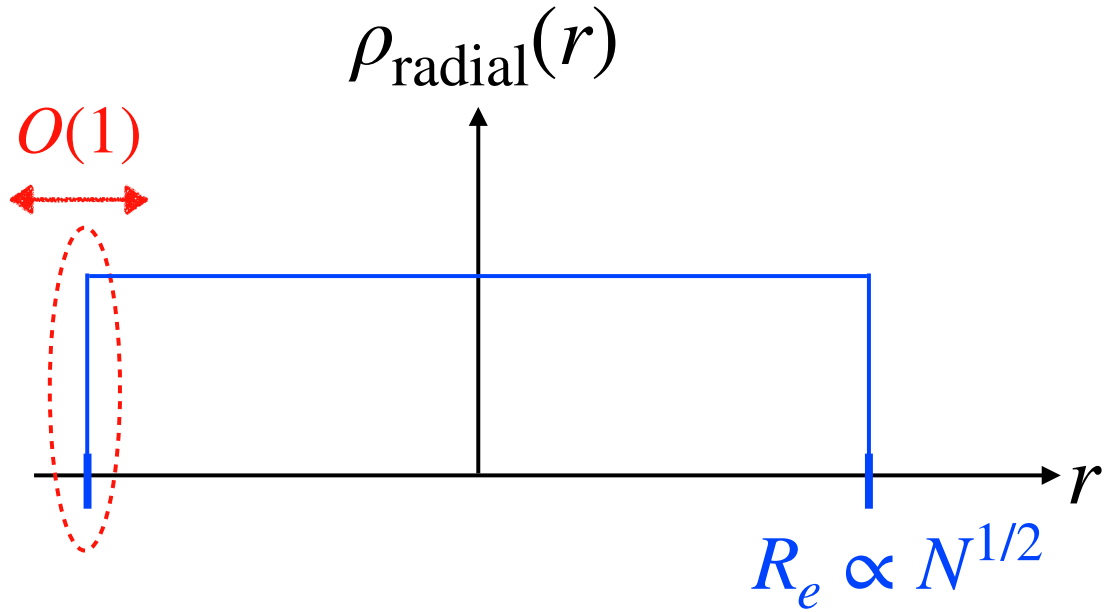

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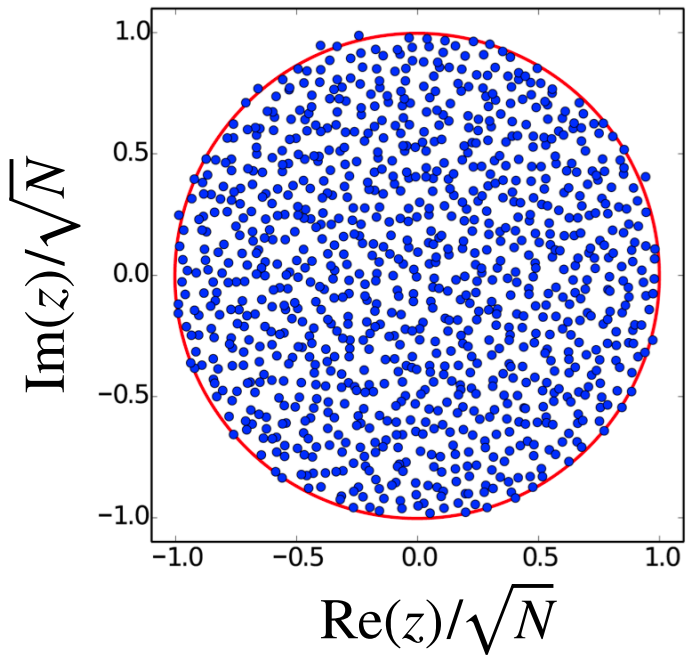

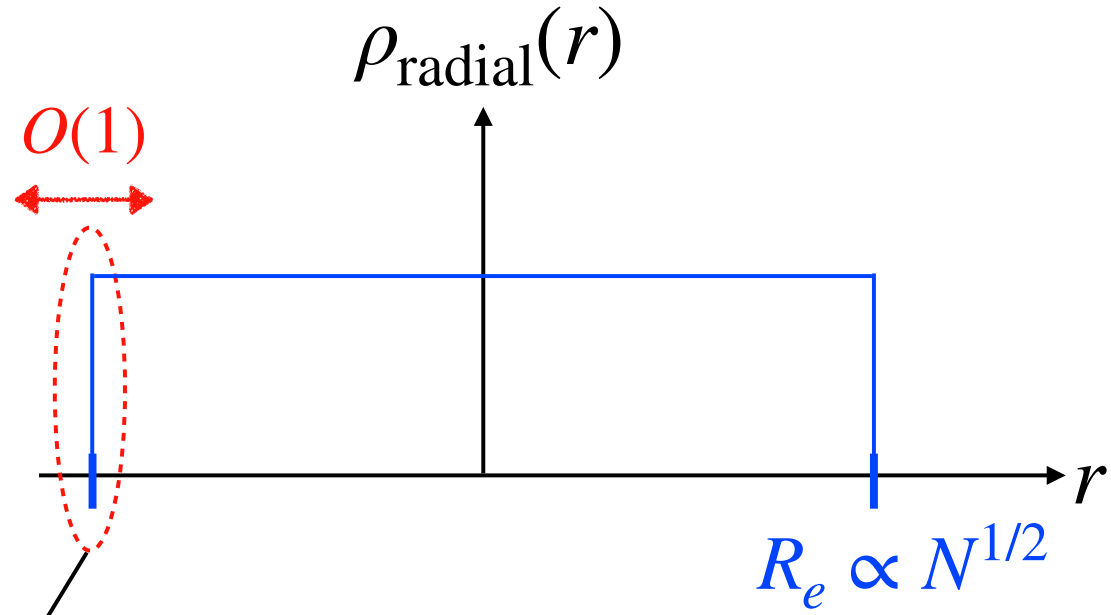

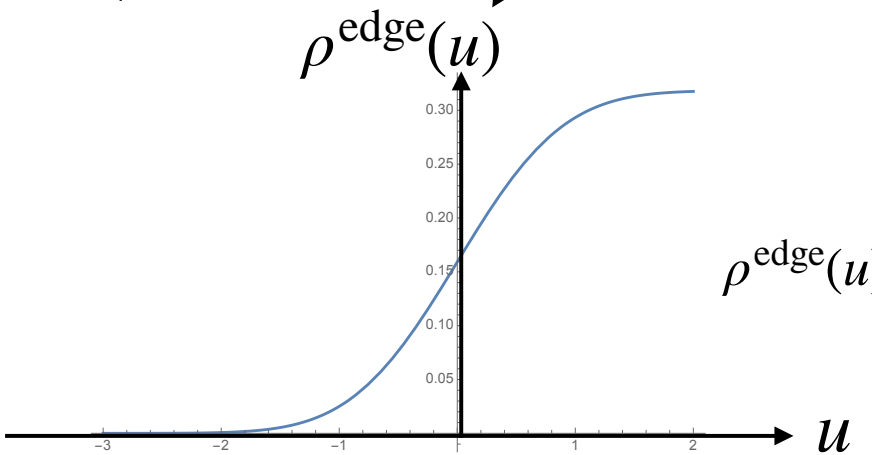

$$\rho^{\mathrm{edge}}(u) = \frac{1}{\pi^{3/2}} \int_{-u}^{\infty} e^{-x^2} dx = \frac{1}{2\pi} \mathrm{erfc}(-u)$$

Fermions in a 2D rotating harmonic trap

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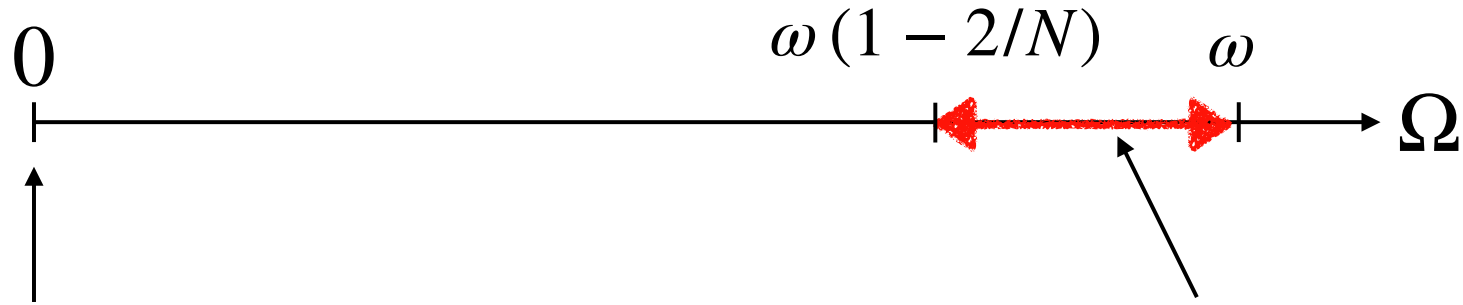

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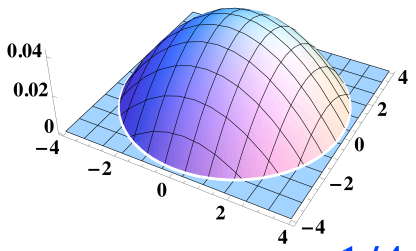
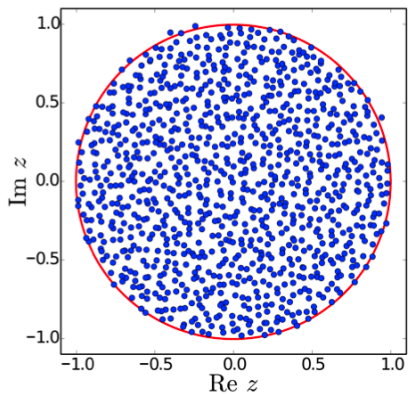


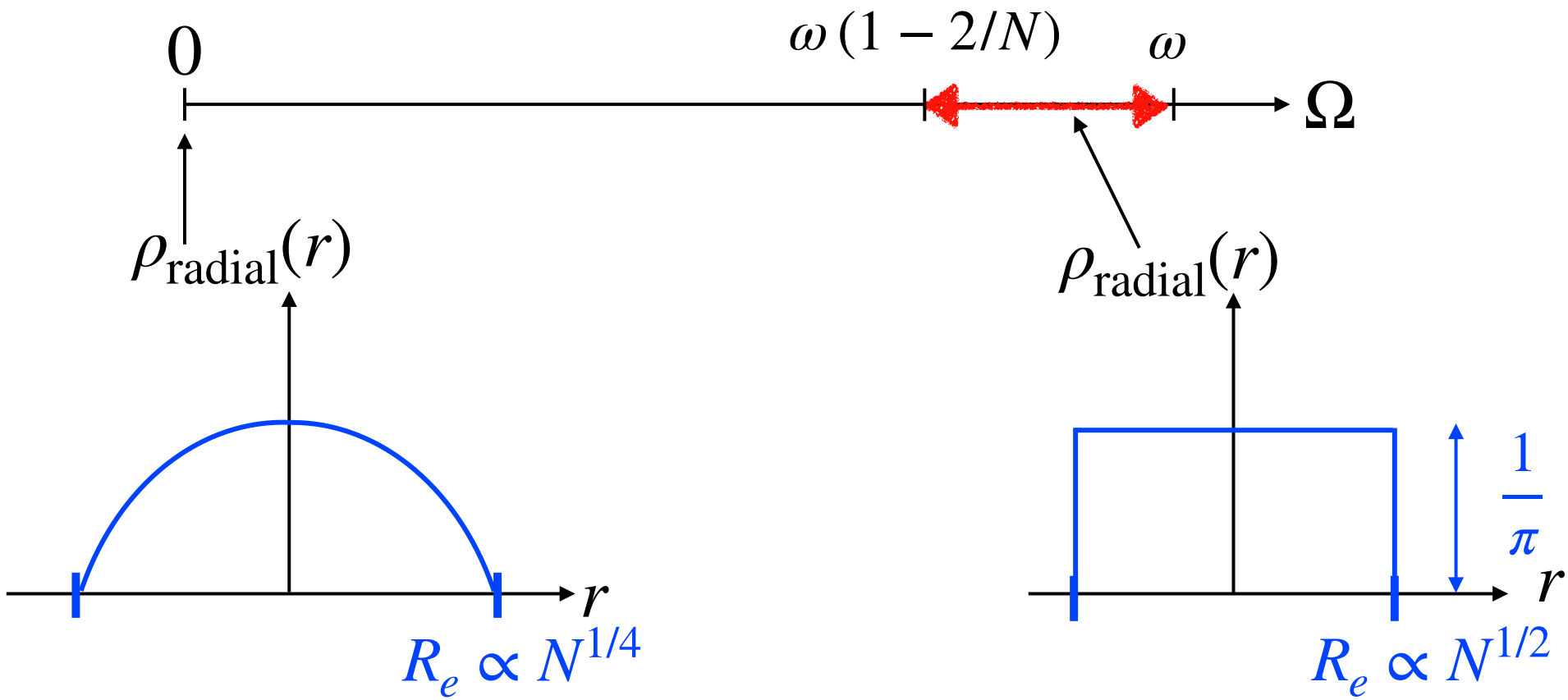
$R_e \propto N^{1/4}$



$R_e \propto N^{1/2}$

Fermions in a 2D rotating harmonic trap

$$\hat{H} = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2 + \frac{1}{2}\omega^2\mathbf{r}^2 - \frac{1}{2}\Omega^2\mathbf{r}^2$$



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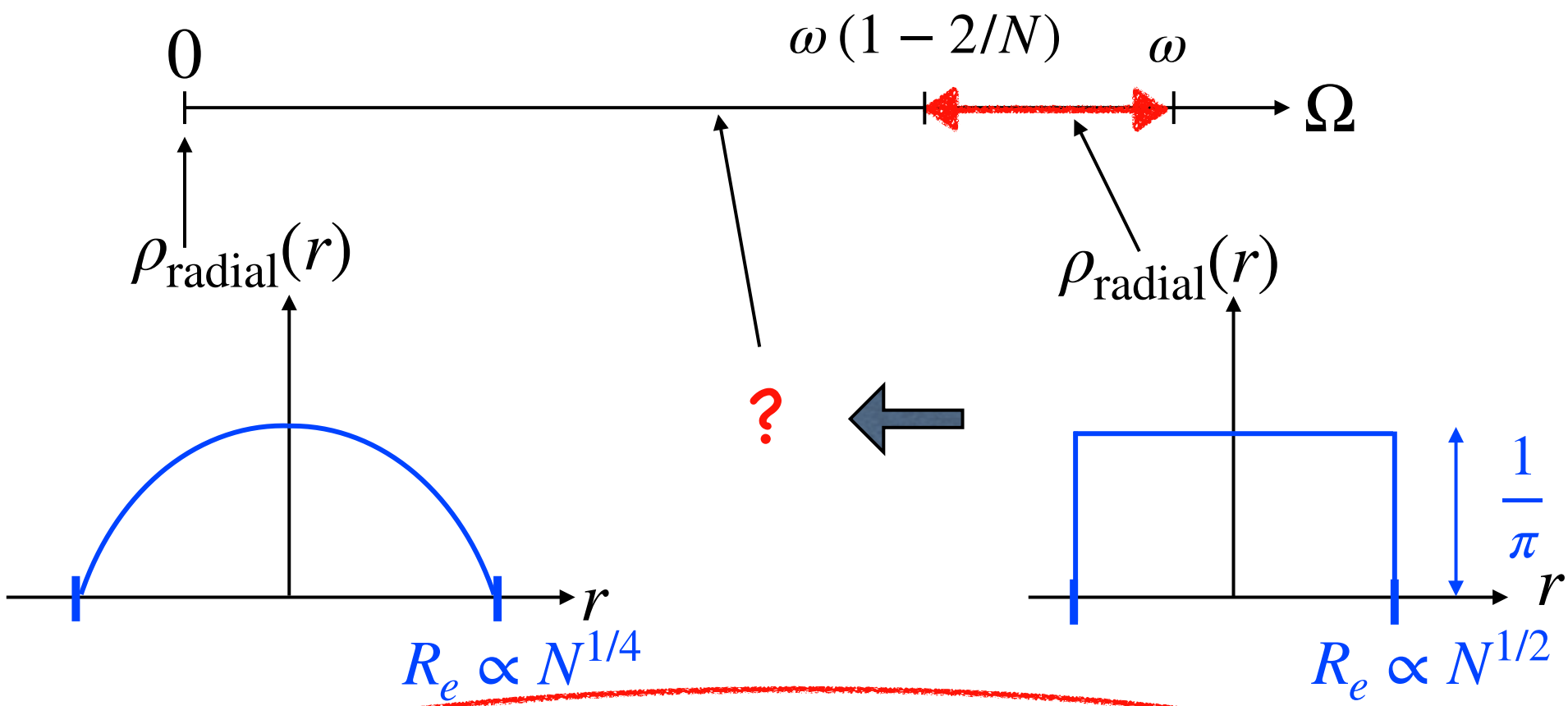
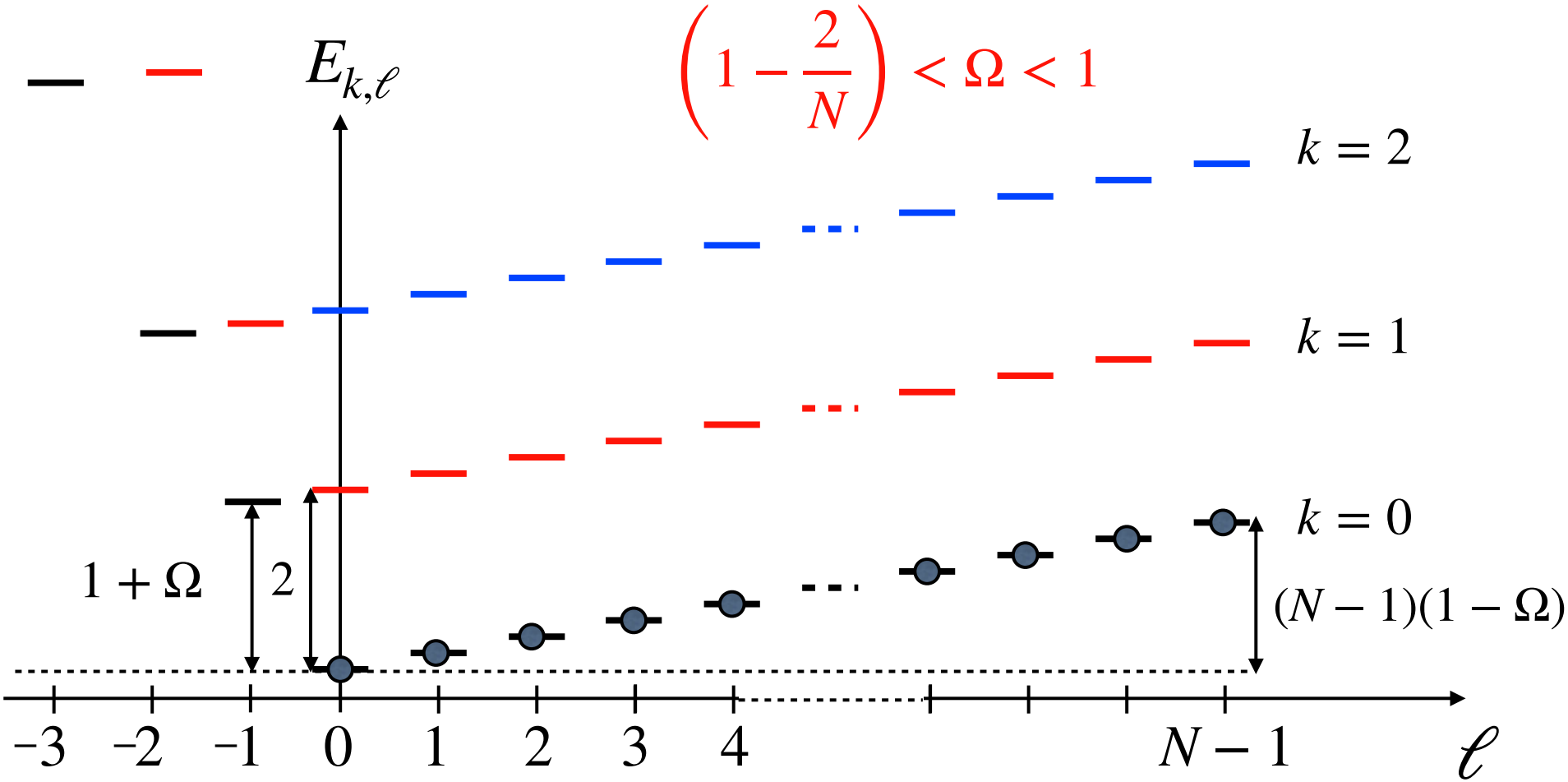


What happens when one decreases Ω ?

Decreasing the rotating frequency

- Ground state of N noninteracting spinless fermions

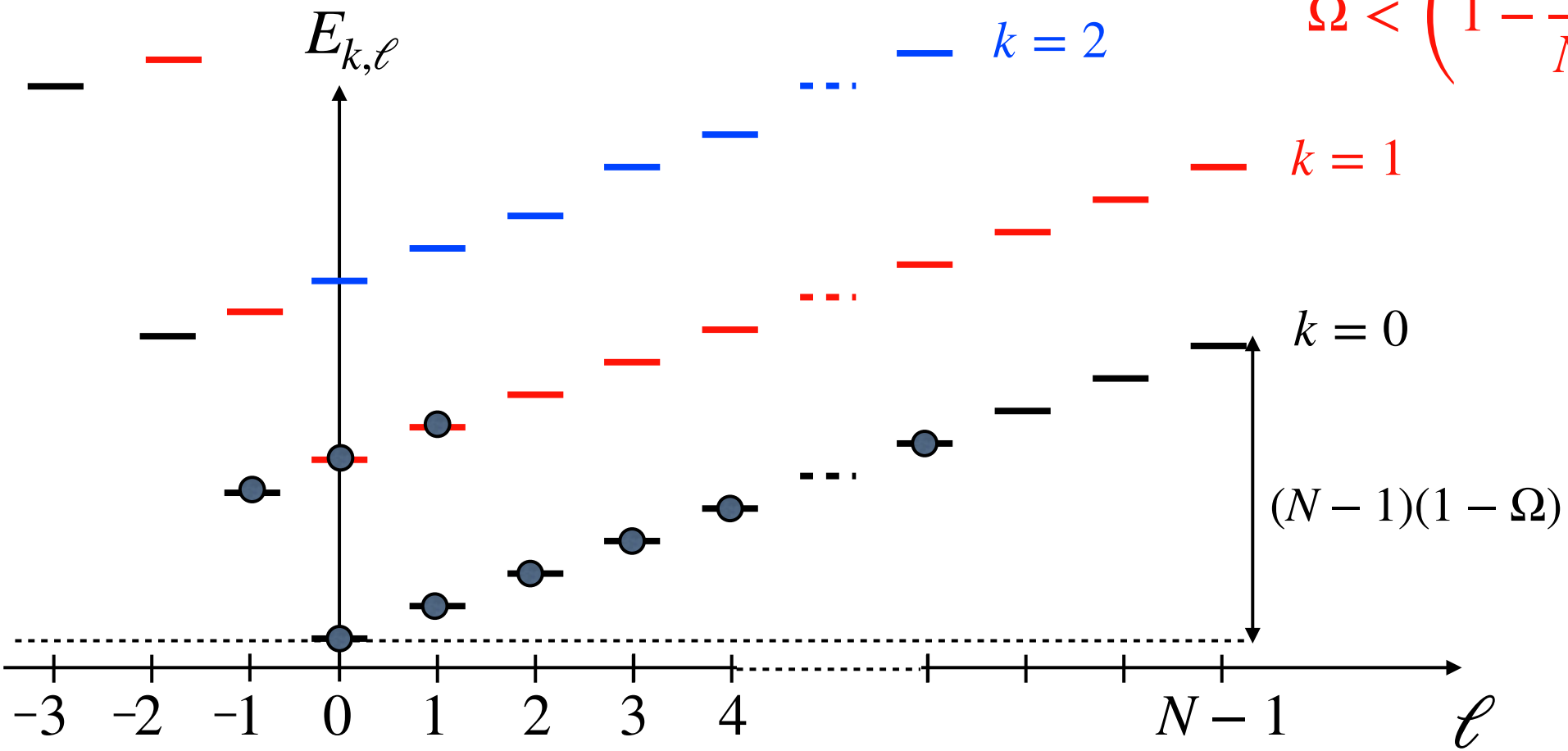


$$E_{k,\ell} = 2k + 1 + |\ell| - \Omega\ell$$

Decreasing the rotating frequency

- Ground state of N noninteracting spinless fermions

$$\Omega < \left(1 - \frac{2}{N}\right)$$



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Decreasing the rotating frequency: wedding cake structure

Kulkarni, Majumdar, G. S. '21

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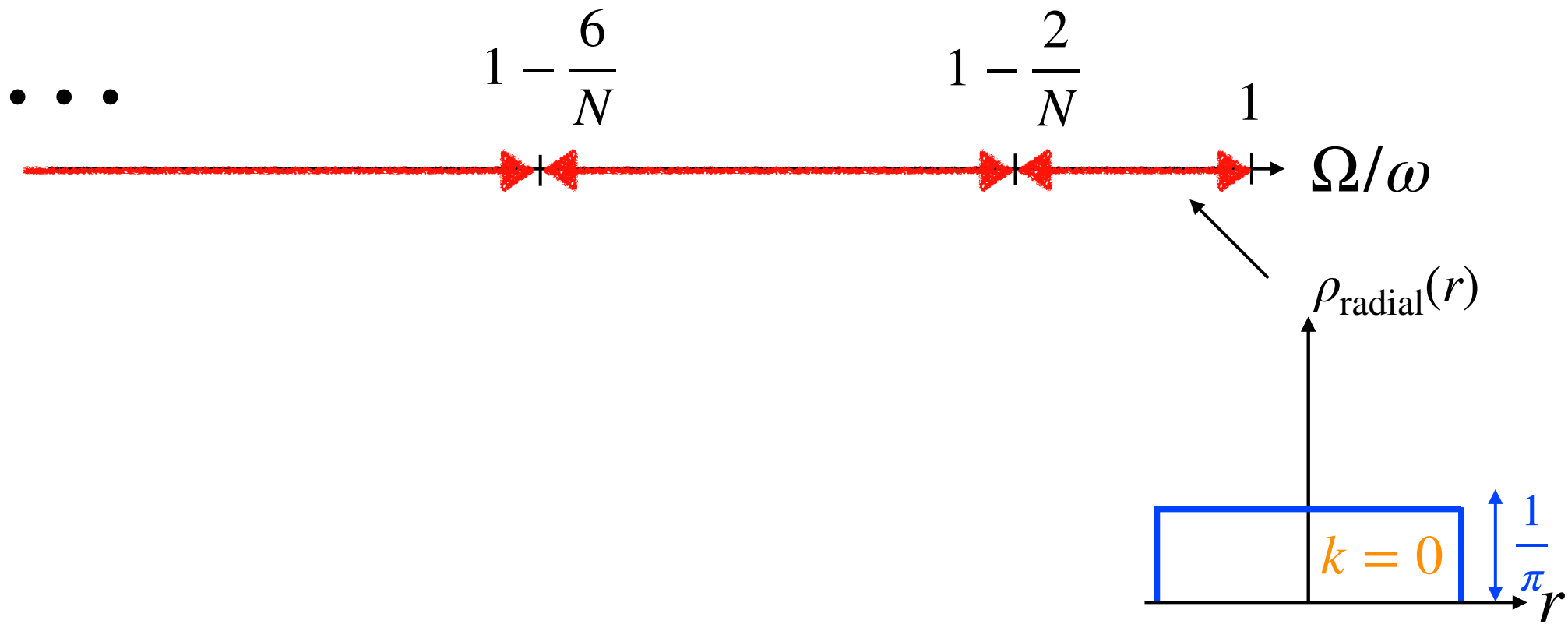

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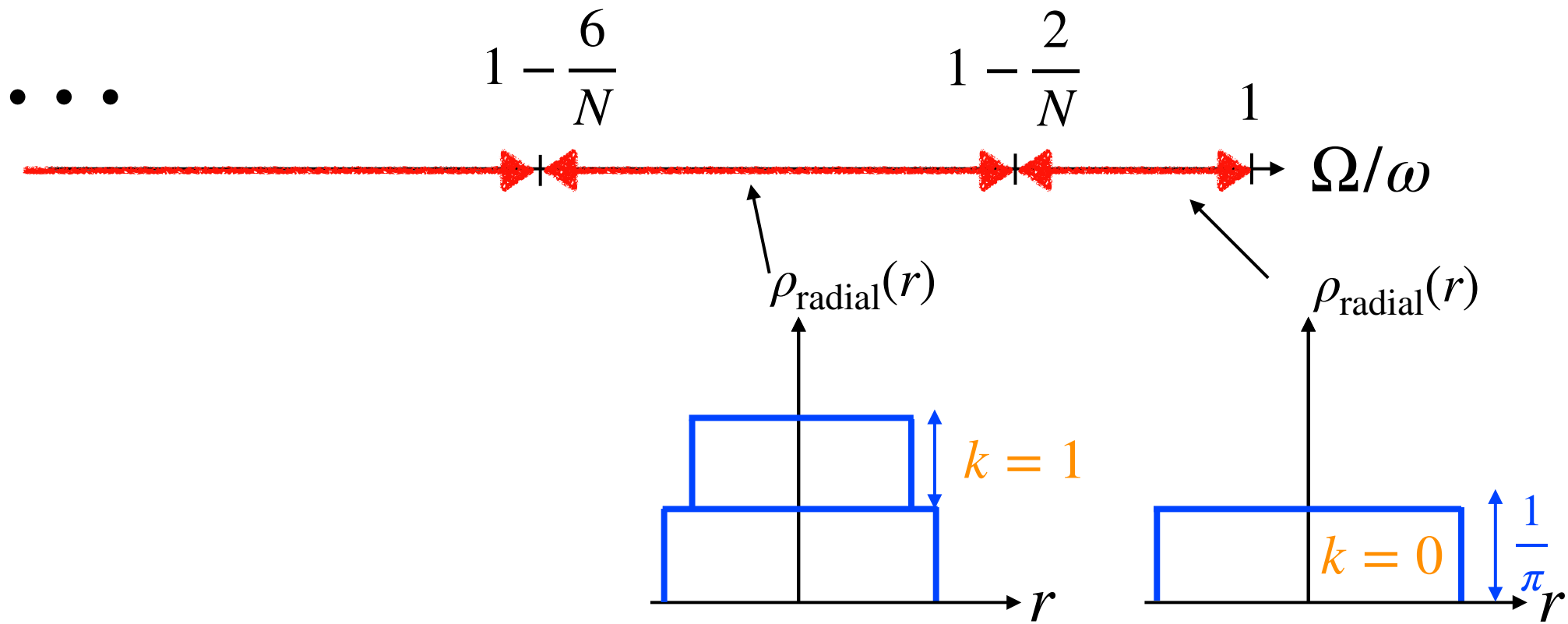

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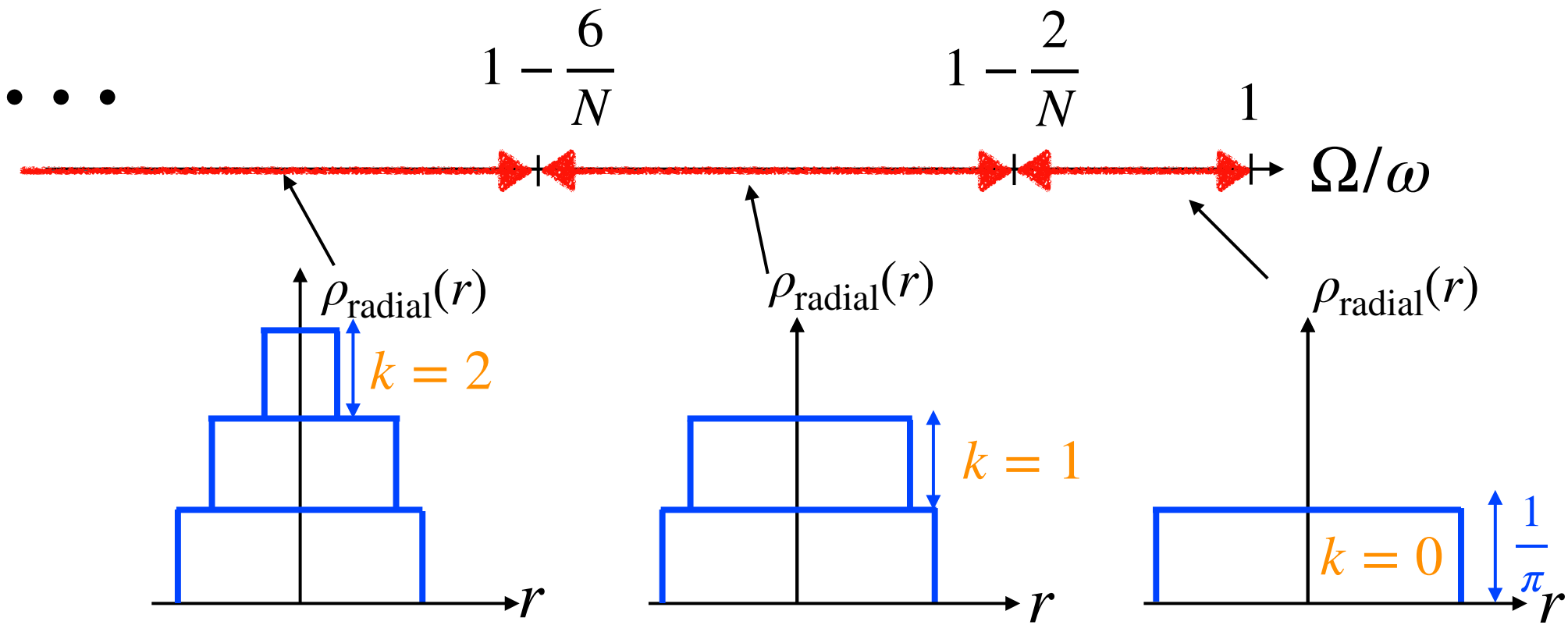

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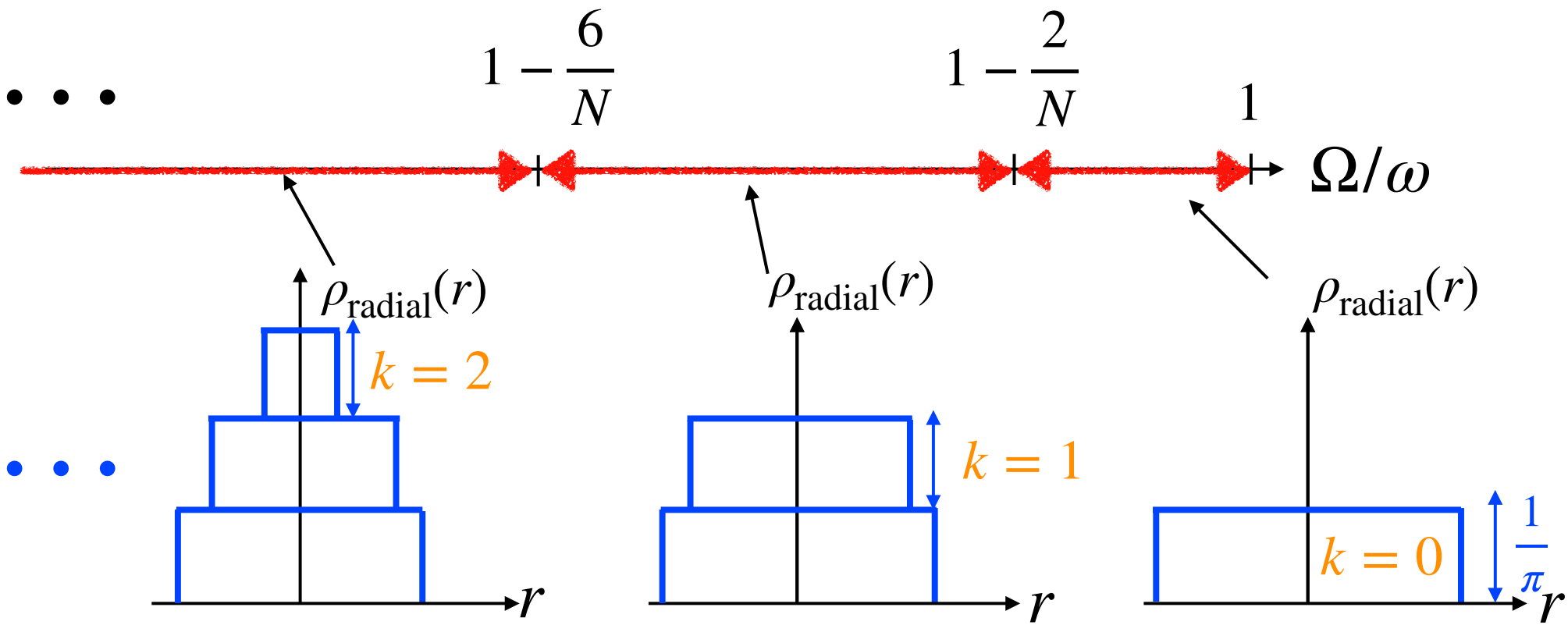

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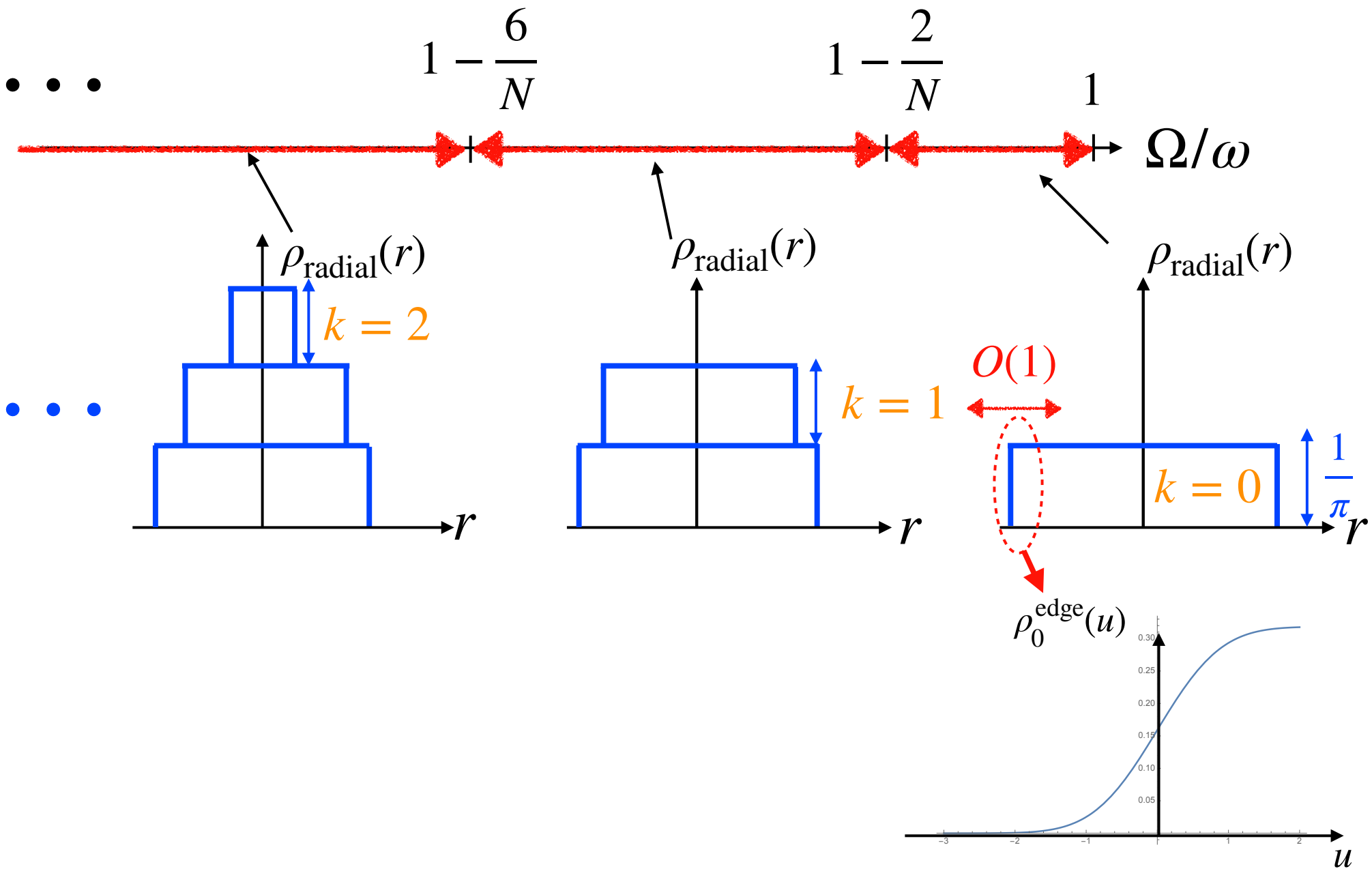

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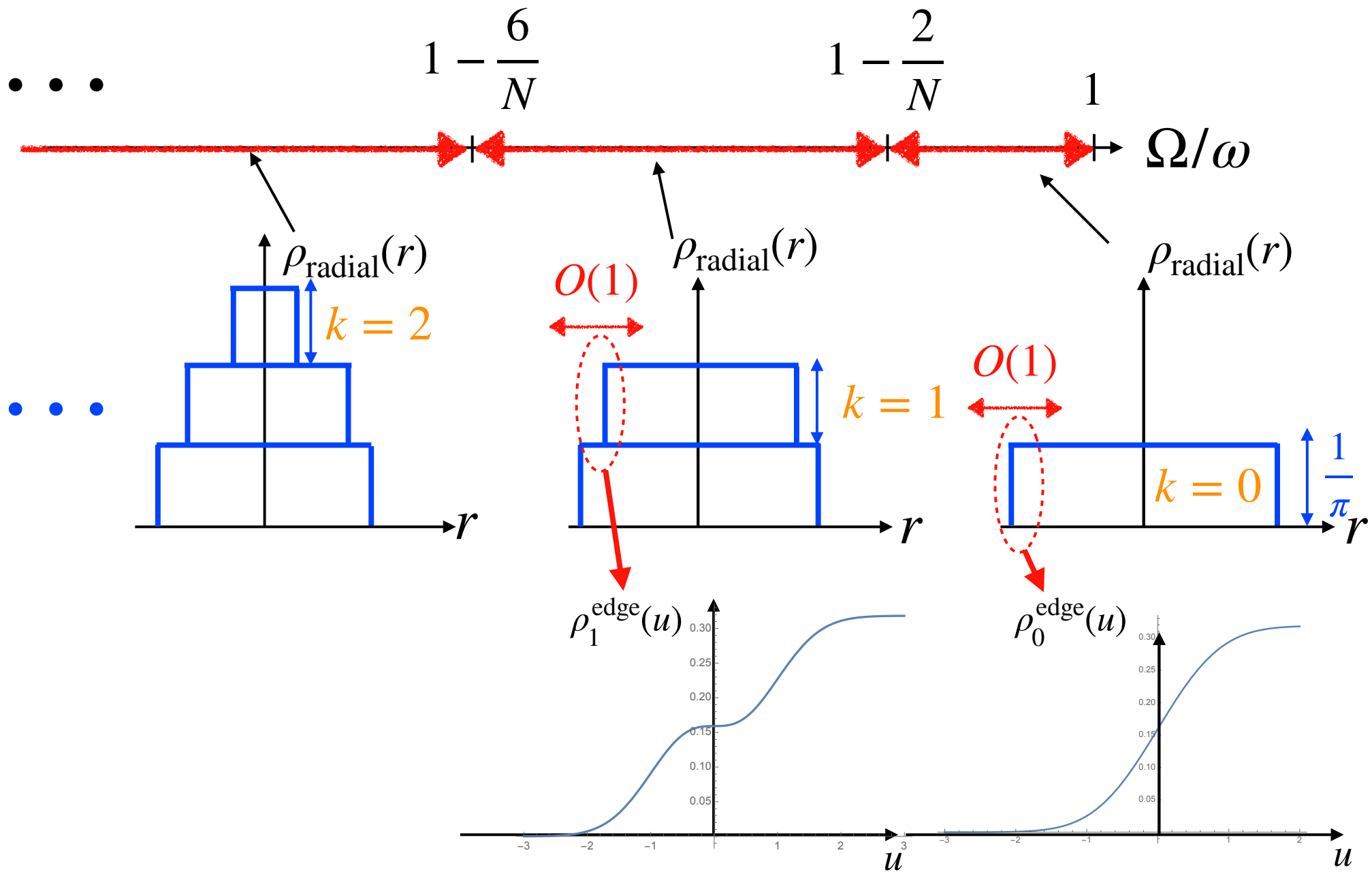

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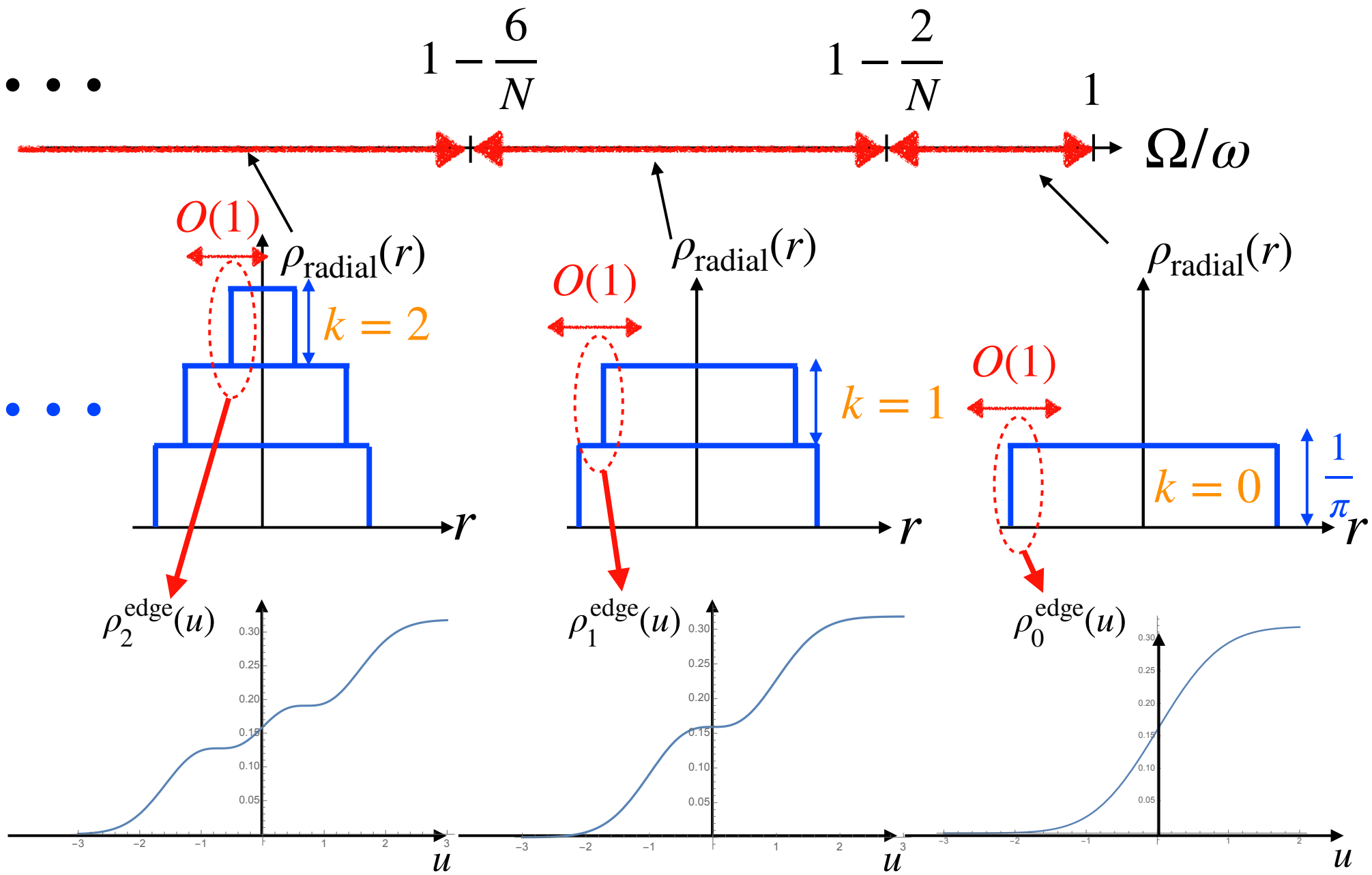
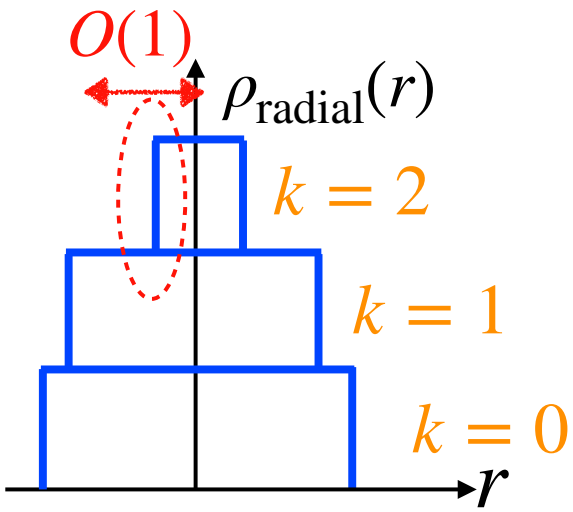
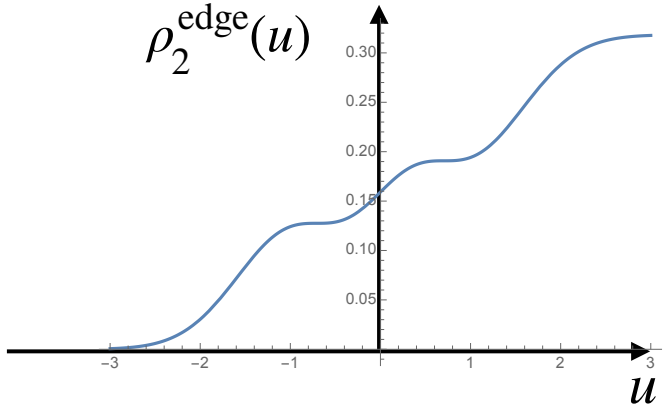


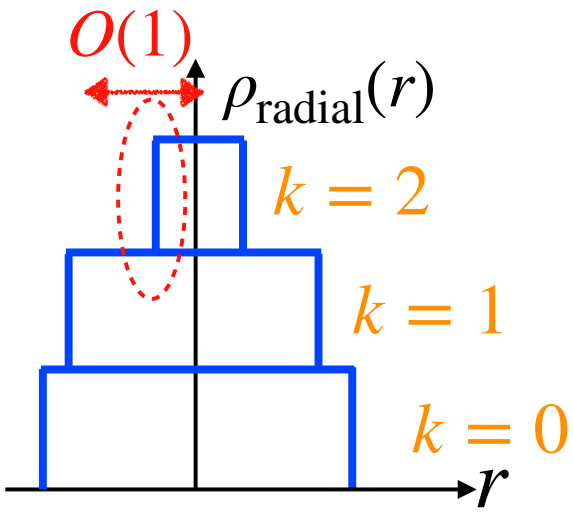
Decreasing the rotating frequency: higher Landau levels

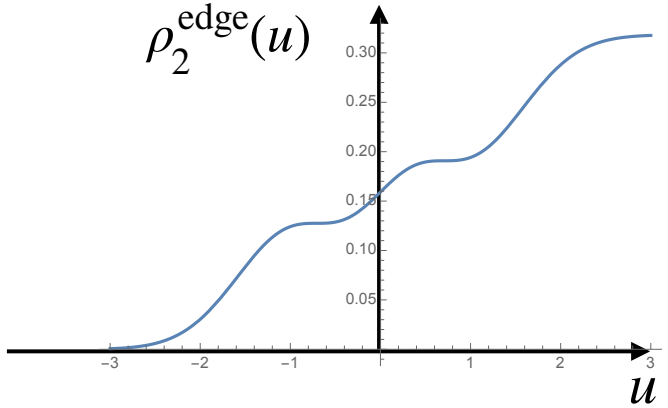




- Different layers correspond to different/higher Landau levels

Decreasing the rotating frequency: higher Landau levels





- Different layers correspond to different/higher Landau levels
- Edge density of the k-th step/Landau level

Kulkarni, Majumdar, G. S. '21
Smith, Le Doussal, Majumdar, G. S. '22

$$\rho_k^{\text{edge}}(u) = \frac{2^{-k}}{\pi^{3/2} k!} \int_{-u}^{\infty} e^{-x^2} \, [H_k(x)]^2 \, dx$$

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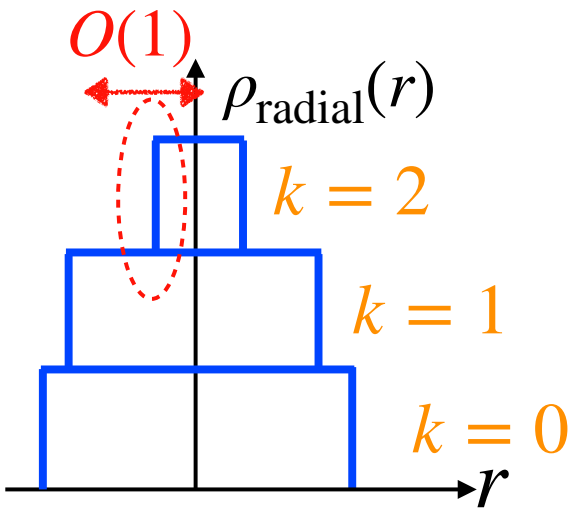

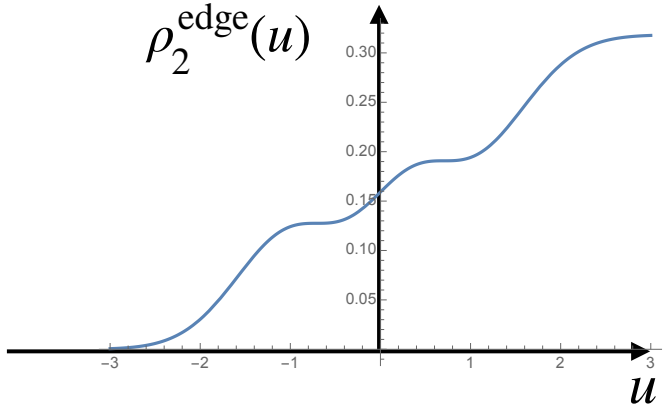

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see also Dunne '94

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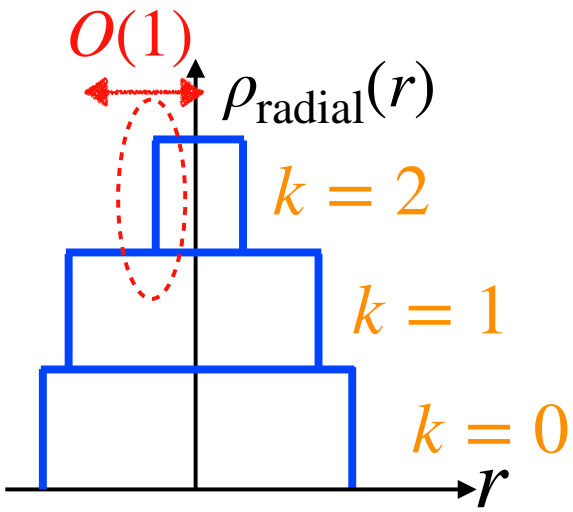

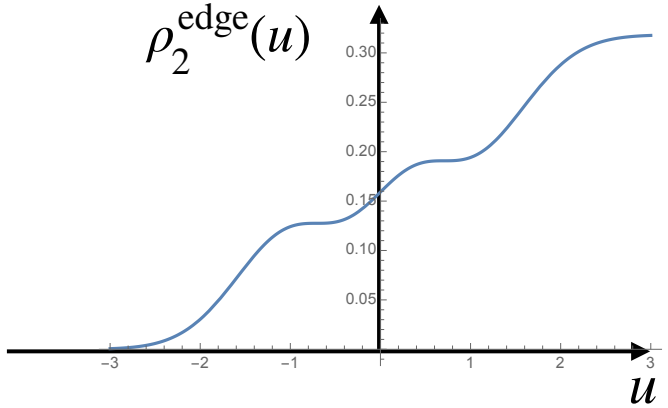

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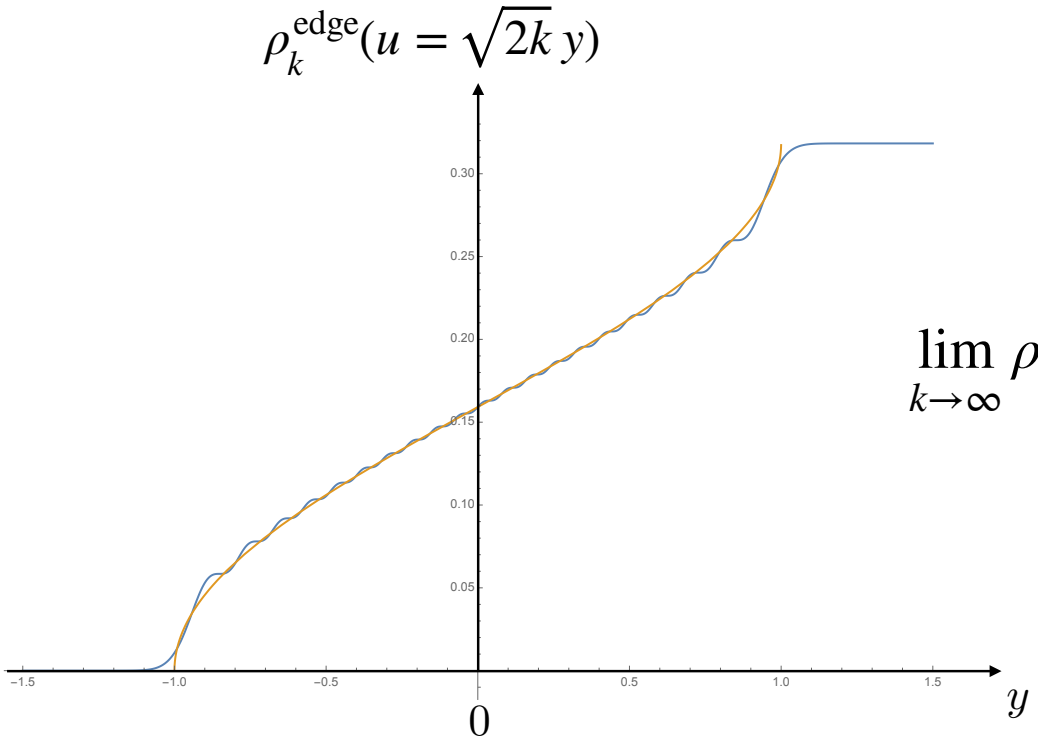
- Connection to Polyanalytic Ginibre Ensemble

see also Haimi, Hedenmalm '13
Fenzel, Lambert '20

Edge scaling limit for very high Landau levels

- Edge density profile of the k-th step/Landau level for $k \to \infty$

Kulkarni, Majumdar, G. S. '21



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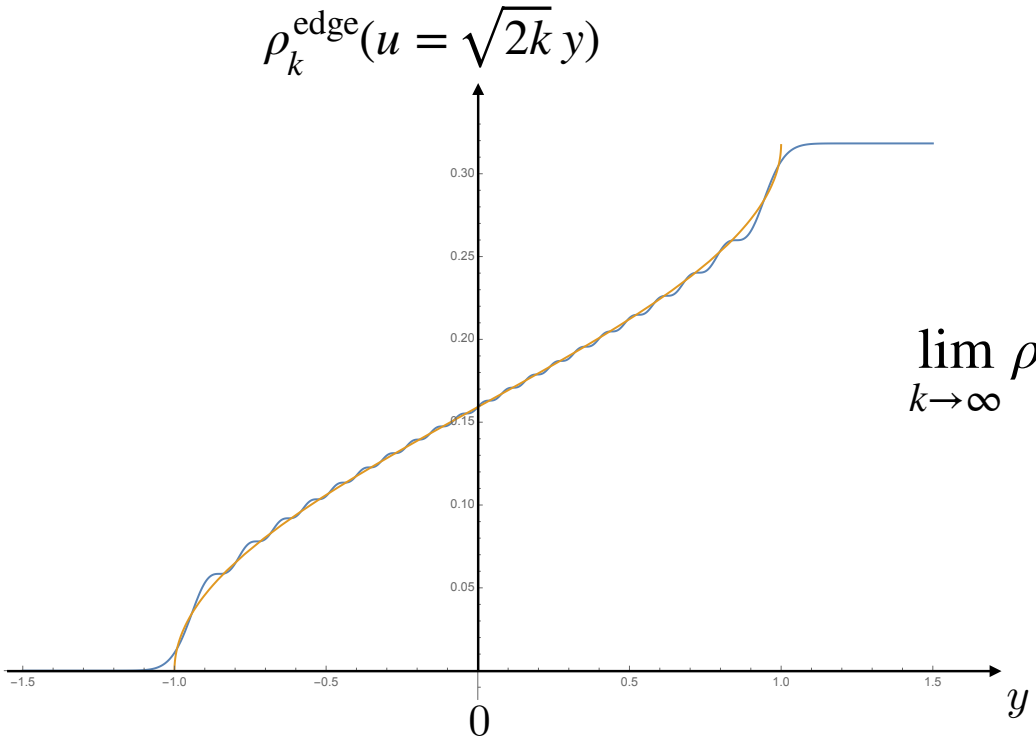

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- Airy universality class at the edge of the edge

$$\rho_k^{\text{edge}}\left(-\sqrt{2k} + \frac{w}{\sqrt{2}k^{1/6}}\right) \sim \frac{1}{k^{1/3}}\mathscr{F}(w) \quad , \quad \mathscr{F}(w) = \frac{1}{\pi}\int_0^\infty \text{Ai}^2(v - w)\, dv$$

Conclusion

- Connection between fermions in a rotating trap and the complex Ginibre ensemble of RMT

- « Wedding/birthday cake » structure as the rotating frequency decreases and higher Landau levels get populated

- Interesting edge behaviors and connection to the Airy kernel for very high Landau levels

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- Generalization to other types of potentials

$$\frac{1}{2}\omega^2 r^2 \longrightarrow \frac{1}{2}\omega^2 r^2 + v\left(\frac{r}{\sqrt{N}}\right) \quad , \quad \text{e.g.} \quad v(x) = \frac{\gamma}{x^2}$$

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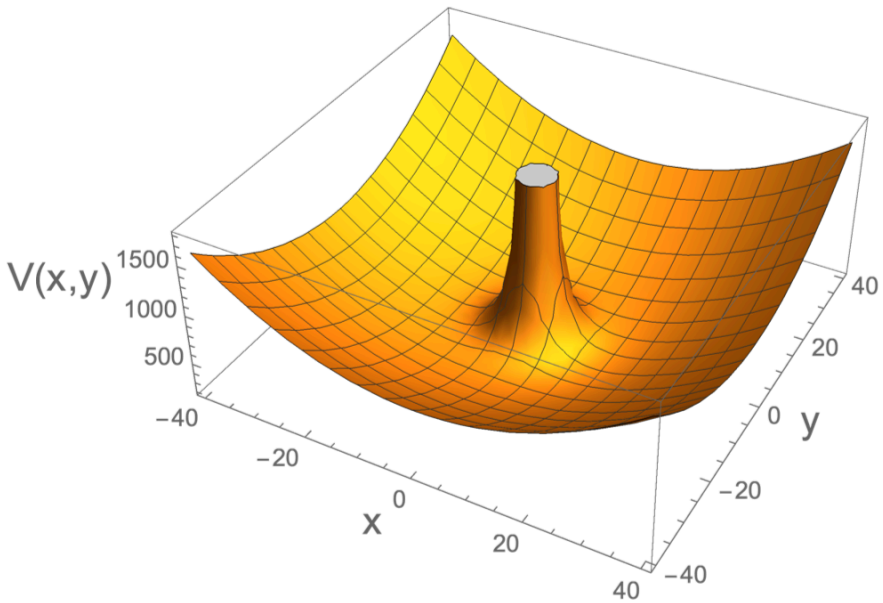
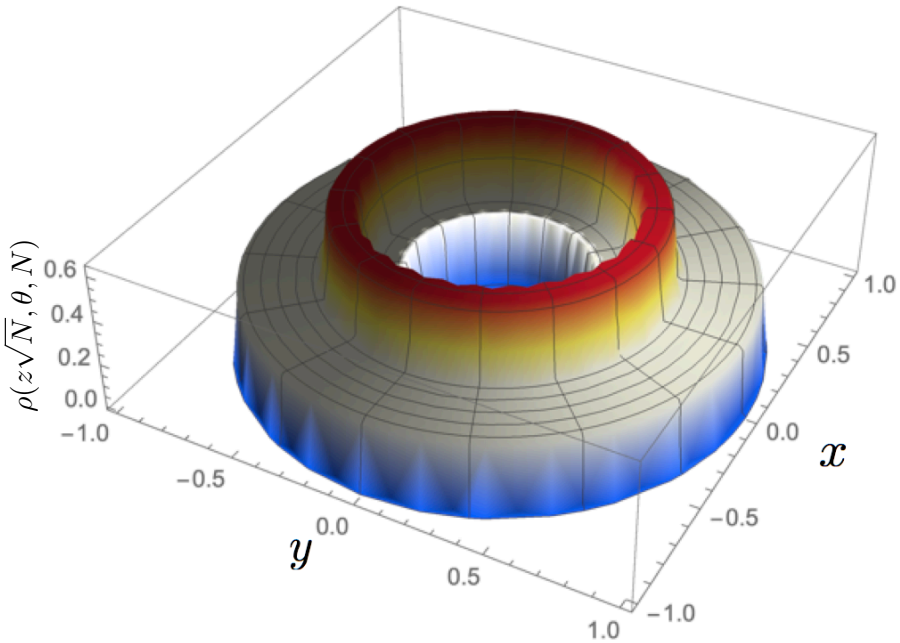
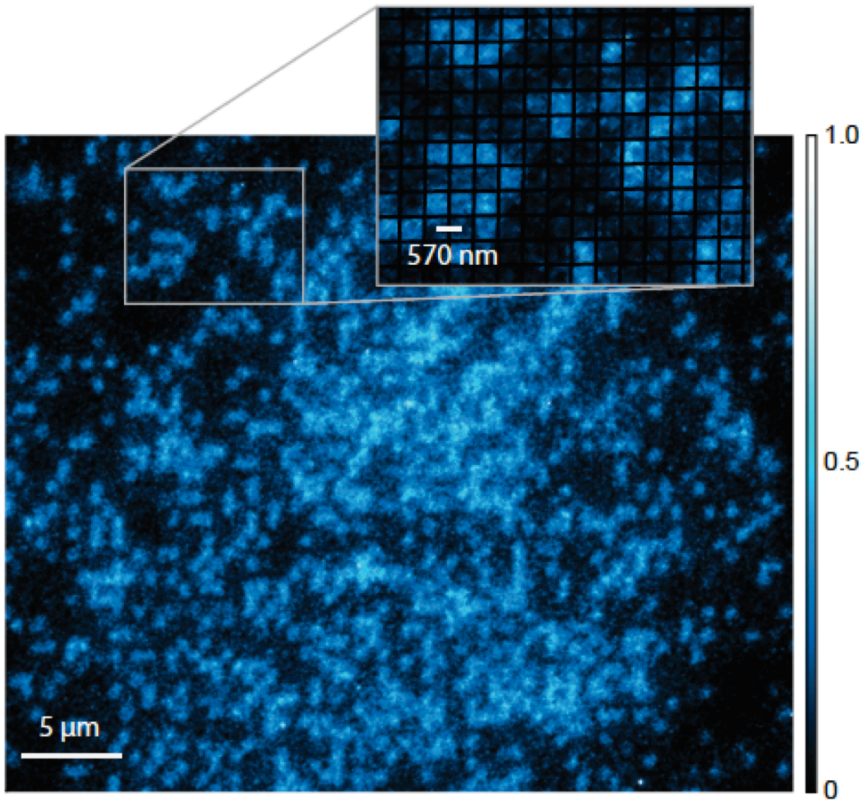




Happy Birthday Yan !

Quantum Fermi gas microscope

- Direct imaging of spatial fluctuations of the positions of fermions



M. Greiner et al., PRL 2015