Coefficients of random unitary matrices and Gaussian multiplicative chaos

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Unitary random matrices

The *Circular* β*-Ensemble* is the joint distribution on N points $e^{i\phi_1}, \ldots, e^{i\phi_N}$ on the unit circle, having density

$$P(e^{i\phi_1}, \ldots, e^{i\phi_N}) \propto \prod_{1 \le j < k \le N} |e^{i\phi_k} - e^{i\phi_j}|^\beta.$$

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For general $\beta > 0$, we define the characteristic polynomial as

$$\chi_N(z) = \prod_{j=1}^{N} (1 - e^{-i\phi_j} z) = \sum_{n=0}^{N} c_n^{(N)} z^n.$$

The $c_n^{(N)}$ are sometimes called *secular coefficients*.

They coincide with the elementary symmetric polynomials of degree n in the N eigenvalues.

Secular coefficients and magic squares

A remarkable paper of Diaconis and Gamburd '04 proves the following for $\beta = 2$.

Theorem (Diaconis, Gamburd '04)

Let $\vec{\mu}, \vec{\nu} \in \mathbb{N}_0^k$ be vectors of non-negative integers. If $\beta = 2$ and $N \geq \max\left(\sum_{j=1}^k \mu_j, \sum_{j=1}^k \nu_j\right)$, we have

$$\mathbb{E}\left(\prod_{j=1}^k c_{\mu_j}^{(N)} \overline{c_{\nu_j}^{(N)}}\right) = |\mathrm{Mag}_{\vec{\mu},\vec{\nu}}|,$$

where $\mathrm{Mag}_{\vec{\mu},\vec{\nu}}$ is the set of all $k \times k$ non-negative integer matrices with row sums $\vec{\mu}$ and column sums $\vec{\nu}$.

When $\vec{\nu} = \vec{\mu} = (n, \ldots, n)$, these are called *magic squares*.

Some further background

The polynomial $\chi_N(z)$ can be well approximated by $e^{\sqrt{\frac{2}{\beta}}G^{\mathbb{C}}(z)}$ where

$$G^{\mathbb{C}}(z) = \sum_{j=1}^{\infty} \frac{\mathcal{N}_j}{\sqrt{j}} z^j, \qquad |z| < 1,$$

and $\{\mathcal{N}_j\}_{j=1}^{\infty}$ are i.i.d. standard complex normal random variables.

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and $\{\mathcal{N}_j\}_{j=1}^{\infty}$ are i.i.d. standard complex normal random variables.

Theorem (Hughes, Keating and O'Connell '01)

For $\beta = 2$, and $|z| \leq 1$, we have the joint functional convergence

$$(\mathrm{Re}(\log \chi_N(z)), \mathrm{Im}(\log \chi_N(z))) \xrightarrow{d} (\mathrm{Re}(G^{\mathbb{C}(z)}), \mathrm{Im}(G^{\mathbb{C}}(z))),$$

as $N \to \infty$. For $|z| = 1$ the convergence occurs in a Sobolev space of negative regularity.

The same result for $|z| < 1$ and general $\beta > 0$ follows from work of Jiang and Matsumoto '15.

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Diaconis and Gamburd ask for an answer to Q2 in their 2004 paper, though they are strictly in the $\beta = 2$ setting.

All of these questions can be initially asked for a simpler quantity:

$$c_n := \lim_{N \to \infty} c_n^{(N)}$$

By the last slide, and exponentiating, we have the basic identity

$$c_n = [z^n] e^{\sqrt{\frac{2}{\beta}} G^{\mathbb{C}}(z)}.$$

Magic squares for general $\beta > 0$

Theorem (Najnudel, Paquette, S. '20)

Let $\vec{\mu}, \vec{\nu} \in \mathbb{N}_0^k$ *be vectors of non-negative integers. Then for any* $\beta > 0$ *we have*

$$\mathbb{E}\left(\prod_{j=1}^{k} c_{\mu_j}\overline{c_{\nu_j}}\right) = \sum_{A \in \mathrm{Mag}_{\vec{\mu},\vec{\nu}}} \prod_{i,j=1}^{k} \binom{A_{ij}+\theta-1}{\theta-1}$$

where $\theta = \frac{2}{\beta}$.

Note: When $\beta = 2$, the binomial coefficient is identically 1 and we recover the Diaconis Gamburd result.

Therefore, the general-β moments are *weighted* sums over the set of magic squares.

Proof sketch

Extracting coefficients, the moment we want is:

$$\mathbb{E}\left(\prod_{j=1}^{k} c_{\mu_j}\overline{c_{\nu_j}}\right)$$
$$= [z_1^{\mu_1}\dots z_k^{\mu_k}\overline{w}_1^{\nu_1}\dots\overline{w}_k^{\nu_k}]\mathbb{E}\big(e^{\sqrt{\theta}\sum_{\ell=1}^{k}(G^{\mathbb{C}}(z_\ell)+\overline{G^{\mathbb{C}}(w_\ell)})}\big)$$

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$$= [z_1^{\mu_1}\dots z_k^{\mu_k}\overline{w}_1^{\nu_1}\dots\overline{w}_k^{\nu_k}]\prod_{\ell_1=1,\ell_2=1}^{k}\frac{1}{(1-z_{\ell_1}\overline{w_{\ell_2}})^{\theta}}.$$

Now expand with the binomial formula to conclude.

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Question: Can we get insights into distribution of c_n or $c_n^{(N)}$ by studying the moments?

A 2×2 example

For example, taking $k = 2$ and $\mu_j = \nu_j = n$ for all j:

$$\mathbb{E}(|c_n|^4) = \sum_{q=0}^{n} \binom{q+\theta-1}{\theta-1}^2 \binom{n-q+\theta-1}{\theta-1}^2.$$

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The dominant contributions come from $q = O(1)$ and $q \to n-q$, leading to:

$$\begin{aligned} \frac{\mathbb{E}(|c_n|^4)}{\mathbb{E}(|c_n|^2)^2} &\sim 2\sum_{q=0}^{\infty} \binom{q+\theta-1}{\theta-1}^2 \\ &= 2\,\frac{1}{2\pi}\int_0^{2\pi} |1-e^{i\phi}|^{-2\theta}\, d\phi \\ &= 2\frac{\Gamma(1-2\theta)}{\Gamma(1-\theta)^2} \end{aligned}$$

for $\theta \in (0, \frac{1}{2})$. If $\theta \geq \frac{1}{2}$ the latter computation yields divergent results.

Asymptotic analysis of the moments

This example can be generalized to all higher moments. We show that

Theorem (Najnudel, Paquette, S. '20)

For any $\theta > 0$ such that $k\theta < 1$, we have for $k \in \mathbb{N}$:

$$\lim_{n\to\infty} \frac{\mathbb{E}(|c_n|^{2k})}{\mathbb{E}(|c_n|^2)^k} = k!\, \frac{\Gamma(1-k\theta)}{\Gamma(1-\theta)^k}.$$

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Remarks:

- The $k!$ are the Gaussian moments: $k! = \mathbb{E}(|\mathcal{Z}|^{2k})$.
- Fewer than $\lfloor \frac{1}{\theta} \rfloor$ moments are finite. The closer to $\theta = 1$ we get, the fewer moments exist.
- For $\theta < \frac{1}{2}$ ($\beta > 4$), we have a finite second moment.

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The ratio next to the $k!$ can be recognised and suitably interpreted.

The Fyodorov-Bouchaud theorem

Let $G(z) = 2\mathrm{Re}(G^{\mathbb{C}}(z))$. For $0 < r < 1$, consider the renormalized total mass

$$\mathcal{M}_{r,\theta} := \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\sqrt{\theta} G(re^{i\phi})}}{\mathbb{E}(e^{\sqrt{\theta} G(re^{i\phi})})} \, d\phi.$$

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In 2008 Fyodorov and Bouchaud conjectured the explicit law of $\mathcal{M}_\theta$ based on computing integer moments $k \in \mathbb{N}$ with $k\theta < 1$ and showed:

$$\mathbb{E}(\mathcal{M}_\theta^k) = \frac{\Gamma(1 - k\theta)}{\Gamma(1 - \theta)^k}.$$

The full proof including all negative moments was given by Remy in 2020.

L^2-phase limit for the coefficients

Theorem (Najnudel, Paquette, S. '20)

For any $0 < \theta < \frac{1}{2}$ and $n, N \to \infty$ with $n = o(N)$, we have

$$\frac{c_n^{(N)}}{\sqrt{\mathbb{E}(|c_n^{(N)}|^2)}} \xrightarrow{d} \mathcal{Z}\sqrt{\mathcal{M}_\theta}, \qquad n \to \infty,$$

where $\mathcal{Z}$ is a standard complex normal random variable, and $\mathcal{M}_\theta$ is the total mass of Gaussian multiplicative chaos on the unit circle, independently sampled.

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We can make this limit quite explicit. As mentioned by Fyodorov-Bouchaud and Remy, we have

$$\mathcal{M}_\theta \overset{d}{=} \frac{1}{\Gamma(1-\theta)}\,\mathcal{E}^{-\theta}$$

where $\mathcal{E}$ is a rate one exponential random variable.

The critical regime $\theta = 1$

When $\theta = 1$ we cannot get convergence in distribution, but we can establish tightness and order of magnitude estimates.

Although $\mathbb{E}(|c_n|^2) \equiv 1$, we actually have $c_n \to 0$ in probability.

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- We also establish 'anti-tightness': $(\log(n))^{-\frac{1}{4}} c_n^{-1}$ is tight.
- Independent proof by Soundararajan and Zaman '21.
- Zaman and collaborators '21: numerical estimation of

$$C := \lim_{n\to\infty} (\log(n))^{\frac{1}{4}} \mathbb{E}(|c_n|).$$

Ideas behind the convergence in law

Expanding out $c_n = [z^n]e^{\sqrt{\theta}G^{\mathbb{C}}(z)}$ we arrive at

$$c_n = \sum_{\vec{m}\in\mathbb{N}_0^n} \left(\mathbb{1}_{\sum_{k=1}^n km_k = n}\right) \left(\prod_{k=1}^n \mathcal{N}_k^{m_k} \left(\frac{\theta}{k}\right)^{m_k/2} \frac{1}{m_k!}\right).$$

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After some work, the total mass $\mathcal{M}_\theta$ arises from the bracket process of the martingale. The CLT then gives the limit for c_n as

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Conclusions

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- What happens when $n \approx N$? Haake, Kus, Schomerus, Sommers, Życzkowski '96 showed that

$$\mathbb{E}(|c_n^{(N)}|^2) = \frac{\Gamma(n+\theta)\Gamma(N-n+\theta)\Gamma(N+1)}{\Gamma(\theta)\Gamma(n+1)\Gamma(N-n+1)\Gamma(N+\theta)}$$

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Thanks for your attention. Happy birthday Yan!