Statistical Mechanics and Random Matrix Theory

Tom Spencer

July 25, 2022

Spectral properties of RMT via Statistical Mechanics

A) Review: Theorems and Conjectures about RMT

B) Spectral problems in RMT can be formulated in terms of spin systems - Statistical Mechanics.

F. Wegner (1980) Hyperbolic sigma model - replicas

K. Efetov (1982) Supersymmetric model $U(1,1|2)$ symmetry.

Classical Heisenberg spin system and determinants of Random Band Matrices - Fermion sector.

Saddle Manifold $\mathbb{S}^2$ and Fluctuations

Random Band matrices: spatial structure

$$\langle H_{ij} \rangle = 0, \quad \langle H_{ij} \bar{H}_{i'j'} \rangle = J_{ij}\, \delta_{ii'} \delta_{jj'} \quad i, j \in \Lambda_N \cap \mathbb{Z}^d$$

$$\Lambda_N = \text{box of side N}$$

where $J_{ij} \approx 0 \quad \mathit{for}\, |i - j| > \mathbf{W}, \quad \mathbf{W} = \mathit{width\, of\, band}, \sum_j J_{k,j} = 1$

Prediction: Fyodorov and Mirlin : d=1, GUE statistics provided $W \gg N^{1/2}$

Special case **proved** by M. Shcherbina and T. Shcherbina using **SUSY** Statistical mechanics.

Earlier work by Bougade-Yau-Yin, $W \gg N^{3/4}$

Examples where local statistics are fairly well understood

1) Unitarily invariant ensembles: $e^{-N\, tr\, P(H)}$ eg P(x) = x^4
Deift, Its, Gioev ; Pastur-Shcherbina

2) Wigner matrices - Mean-field but not necessarily Gaussian (Erdos-Schlein-Yau), Flow of random matrix by Dyson Brownian motion. Fast *local* equilibration of eigenvalues (time = 1/N).

Dyson: "The picture of the gas (eigenvalues interacting via log potential) coming into equilibrium in two well-separated stages, with microscopic and macroscopic time scales, is suggested with the help of physical intuition. A rigorous proof that this picture is accurate would require a much deeper mathematical analysis"
Also important contributions by Tao and Vu.

3) Adjacency matrices arising from random d-regular graphs on N vertices, d large, GOE statistics (Bauerschmidt, Huang, Knowles, Yau)

Examples - local statistics *not* well understood

4) Random Band matrices - on $\mathbb{Z}^d$ $d \geq 2$.

Wigner-Dyson local statistics expected for $d \geq 3$

Recent Theorem: Quantum Diffusion for RMB of width $W > N^{\epsilon}$, in high dimension. Yang-Yau-Yin

5) Random Schrödinger , $H = -\Delta + V(j),\ j \in \Lambda_N \cap Z^d$,

Anderson (1958): electrons scattered by random, independent impurities V(j), $< V(j)^2 >= 1$

W-D expected for $d \geq 3$. Need to show eigenstates are uniformly spread over Λ - Extended states.

Time evolution: Quantum diffusion due to scatterers - Conduction.

6) $H = -\Delta + V(j)$, V(j) = Quasi-periodic potential on Z^3,

$V(j) = \sum_i^3 \cos(\vec{\omega}_i \cdot j + \phi_i)$, $\vec{\omega_i} \in R^3$ incommensurate.

Devakul and Huse (2017): GUE statistics near center of spectrum.

Lemarié-Grémaud-Deland (2009) Kicked Rotator. Anderson Transition

7) Unitary matrices from Quantization of chaotic maps.
Bohigas-Gianonni-Schmidt conjecture: CUE

Quantization of $x \to 2x$ on [0,1] is given by $N \times N$ Unitary matrix:

$$1/\sqrt{2} \times \begin{pmatrix} 1 & -1 & 0 \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \cdots \end{pmatrix} = U_N$$

For large $N \neq 2^M$ CUE statistics observed.
Compose with random phase-more accessible?
(Recent discussions with L Shou)

Pakonski et al, Kottos-Smilanky
Altland, Zirnbauer, Gnutzmann-Keating-Piotet (SUSY- CUE)

Determinant of Random Band Matrix and the Classical Heisenberg spin model.

Moments of characteristic polynomial GUE, CUE, Fyodorov, Strahov, Keating...

Random band matrix with covariance J. In 1 Dimension

$$J_{jk} = [-W^2\Delta + 1]^{-1}(j,k) \approx e^{-|j-k|/W}/W, \quad 0 \le j,k \le N$$

$$Z_N(\delta) = \mathbb{E}[\det(H - E + \delta')\,\det(H - E - \delta')], \quad \delta' = \frac{\delta}{N\rho(E)}$$

Theorem (T Shcherbina). If $\quad W \gg N^{1/2}$ then

$$Z_N(\delta)/Z_N(0) \to \frac{\sin(\delta)}{\delta} = \int_{S^2} e^{i\delta S^{(3)}} d\mu.$$

When $W << N^{1/2}$, $\; Z_N(\delta)/Z_N(0) \to 1$

Idea of Proof: Saddle Manifold + control of fluctuations

Let $X(j), 1 \leq j \leq N$ be 2×2 hermitian matrices.

$$Z_N(\delta) = \int e^{-\sum_j^N tr[W^2(\nabla X)^2(j)+X^2(j)]} \prod_j \det(X_j - E + \delta' \sigma_3) dX(j)$$

$$\delta' = \frac{\delta}{N\rho(E)}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Saddle independent of j.

$$X_s = iE\, I_2/2 + \sqrt{1-(E/2)^2}\,\sigma_3$$

For $\delta = 0$, $\quad U^* X_s U$ - Saddle Manifold$= \mathbb{S}^2$.

Sigma Model Approximation:

Approximate

$$X_j = iE\, I_2/2 + \sqrt{1-(E/2)^2}\; U_j^* \sigma_3 U_j.$$

Then $\det(X_j - iE) = 1$

We get the Classical Heisenberg model:

$$Z = \int e^{-\beta \sum_j^N [\nabla S]^2(j)} d\mu(S(j)), \quad S(j) \in \mathbb{S}^2,$$

$$\beta = W^2 \sqrt{1-(E/2)^2}$$

.

$$Z(\delta) = \int e^{-\beta \sum_j^N [\nabla S]^2(j)} \prod_j^N e^{iN^{-1}\delta S_j^{(3)}} d\mu(S(j))$$

In 1D, Spins are aligned provided $\beta = W^2\sqrt{1-(E/2)^2} > N$.

For $d \geq 3$, Heisenberg is ordered for $\beta \geq 1$ and
Conjecture

$$\langle e^{i|\Lambda_N|^{-1}\delta \sum S_j^{(3)}|}\rangle_{\Lambda_N}(\beta) = \frac{\sin M\delta}{M\delta}(1+\frac{c\delta^2}{\beta N^{d-2}})$$

Where $M = M(\beta)$ is the magnetization.

In general corrections to saddle in SUSY sigma model (Kravtsov-Mirlin) are smaller and give precise corrections to Wigner-Dyson pair correlation.

Question: If H = Random Schrödinger with complex Hermitian random potential does the same formula hold?

Happy Birthday Yan!