Multiple partition structures

E. Strahov
Department of Mathematics,
The Hebrew University of Jerusalem

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§1. A representation theorem for multiple partition structures

What are multiple partition structures?

- Multiple partition structures are sequences $\left(\mathcal{M}_n^{(k)}\right)_{n=1}^{\infty}$ of probability measures on families of Young diagrams (multiple partitions) such that for each n, $\mathcal{M}_n^{(k)}$ and $\mathcal{M}_{n+1}^{(k)}$ are connected by a certain consistency relation.
- Multiple partition structures can be understood as generalizations of the Kingman partition structures (Kingman, J.F.C. Random partitions in population genetics. Proc. R. Soc. London A 361, 1-20, (1978))

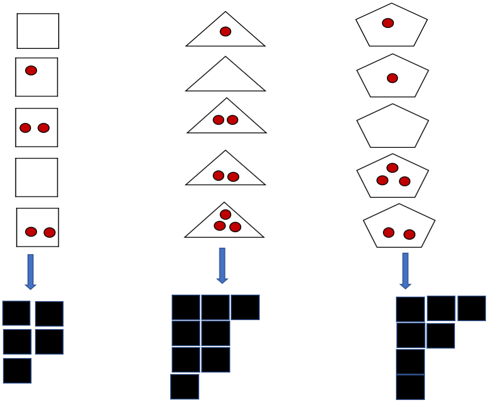
Multiple partitions

- Let $\lambda^{(1)}$, ..., $\lambda^{(k)}$ be Young diagrams such that the condition

$$|\lambda^{(1)}| + \ldots + |\lambda^{(k)}| = n$$

  is satisfied (here $|\lambda^{(l)}|$ denotes the number of boxes on $\lambda^{(l)}$). The family $\Lambda_n^{(k)} = \left(\lambda^{(1)}, \ldots, \lambda^{(k)}\right)$ is called a multiple partition of n into k components.
- Multiple partitions in representation theory of finite groups: Let G be a finite group with k conjugacy classes and k irreducible characters. Both the conjugacy classes and the irreducible characters of the wreath product $G \sim S(n)$ are parameterized by multiple partition $\Lambda_n^{(k)}$.
- Each $\Lambda_n^{(k)}$ corresponds to a configuration of n identical balls partitioned into boxes of k different types and vice versa.

Example: balls configurations as multiple partitions



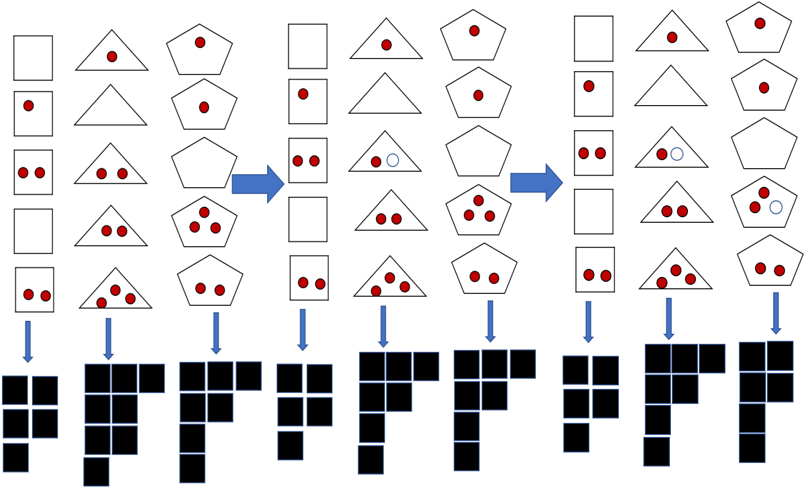
$$\Lambda_{20}^{(3)} = \left(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\right),$$

$$\lambda^{(1)} = (2, 2, 1),\ \lambda^{(2)} = (3, 2, 2, 1),\ \lambda^{(3)} = (3, 2, 1, 1).$$

Definition of a multiple partition structure

- A *random* multiple partition of n is a random variable $\Lambda_n^{(k)}$ with values in the set $\mathbb{Y}_n^{(k)} = \left\{ \left(\lambda^{(1)}, \ldots, \lambda^{(k)}\right) : \, |\lambda^{(1)}| + \ldots + |\lambda^{(k)}| = n \right\}.$
- A multiple partition structure is a sequence $\mathcal{M}_1^{(k)}, \mathcal{M}_2^{(k)}, \ldots$ of distributions for $\Lambda_1^{(k)}, \Lambda_2^{(k)}, \ldots$ which is consistent in the following sense: if n balls are partitioned into boxes of k different types such that their configuration is $\Lambda_n^{(k)}$, and a ball is deleted uniformly at random, independently of $\Lambda_n^{(k)}$, then the multiple partition $\Lambda_{n-1}^{(k)}$ describing the configuration of the remaining balls is distributed according to $\mathcal{M}_{n-1}^{(k)}$.

Example: each time a ball is deleted uniformly at random



Multiple partition structures in population genetics

- If $k = 1$ then a multiple partition structure is a partition structure in the sense of Kingman. Kingman considers a sample of n representatives from a population, and studies the probability that there A_1 alleles (versions of a specific gene appeared due to mutations) represented once in the sample, A_2 alleles represented twice, ...
- In our language boxes correspond to alleles, so A_1 is interpreted as the number of boxes with exactly one ball, A_2 is interpreted as the number of boxes with exactly two balls, and so on.
- For an arbitrary k, boxes of k different types can be interpreted as alleles of k *different* genes under considerations. A model for the allelic partition is a probability distribution $\mathcal{M}_n^{(k)}$ over the set of all multiple partitions of the integer n into k components. Since n may be chosen at the experimenter's convenience, the consistency condition should be satisfied, and $\left(\mathcal{M}_n^{(k)}\right)_{n=1}^{\infty}$ is a multiple partition structure.

The representation theorem for multiple partition structures (E.S., 2022)

(A) There is a bijective correspondence between multiple partition structures $\left\{\mathcal{M}_n^{(k)}\right\}_{n=1}^{\infty}$ and probability measures P on the space

$$\overline{\nabla}^{(k)} = \Bigg\{ (x,\delta) \Bigg| x = \left(x^{(1)},\ldots,x^{(k)}\right), \delta = \left(\delta^{(1)},\ldots,\delta^{(k)}\right);$$

$$x^{(l)} = \left(x_1^{(l)} \geq x_2^{(l)} \geq \ldots \geq 0\right), \ \ \delta^{(l)} \geq 0, \ \ 1 \leq l \leq k,$$

$$\text{where } \sum_{i=1}^{\infty} x_i^{(l)} \leq \delta^{(l)}, \ 1 \leq l \leq k, \text{ and } \sum_{l=1}^{k} \delta^{(l)} = 1 \Bigg\}.$$

(B) The correspondence is determined by

$$\mathcal{M}_n^{(k)}\left(\Lambda_n^{(k)}\right) = \int_{\overline{\nabla}^{(k)}} \mathbb{K}\left(\Lambda_n^{(k)},\omega\right) P(d\omega),$$

where $\mathbb{K}: \ \mathbb{Y}_n^{(k)} \times \overline{\nabla}^{(k)} \to \mathbb{R}$ is a kernel function.

(C) The kernel function $\mathbb{K}\left(\Lambda_n^{(k)}, \omega\right)$ is given by

$$\mathbb{K}\left(\Lambda_n^{(k)}, \omega\right) = \frac{n!}{\prod_{j=1}^{k} |\lambda_1^{(j)}|! \dots |\lambda_n^{(j)}|!} \times M_{\lambda^{(1)}}\left(x^{(1)}, \delta^{(1)}\right) \dots M_{\lambda^{(k)}}\left(x^{(k)}, \delta^{(k)}\right),$$

where $M_{\lambda^{(l)}}\left(x^{(l)}, \delta^{(l)}\right)$ are extended monomial symmetric functions defined in terms of the usual monomial symmetric functions $m_{\lambda^{(l)}}(x_1, x_2, \dots)$ as

$$M_{\lambda^{(l)}}\left(x^{(l)}, \delta^{(l)}\right) = \sum_{p=0}^{\varrho_1^{(l)}} \frac{\left[\delta^{(l)} - \sum_{j=1}^{\infty} x_j^{(l)}\right]^p}{p!} m_{\left(1^{\varrho_1^{(l)}-p} \, 2^{\varrho_2^{(l)}} \dots\right)}\left(x_1^{(l)}, x_2^{(l)}, \dots\right)$$

where $\lambda^{(l)} = \left(1^{\varrho_1^{(l)}} \, 2^{\varrho_2^{(l)}} \dots\right)$.

Remark (The Kingman result (1978) as a particular case)

If $k = 1$ then $\delta^{(1)} = 1$, and

$$\overline{\nabla}^{(1)} = \left\{ x = \left(x_1^{(1)} \geq x_2^{(1)} \geq \ldots \geq 0 \right), \sum_{i=1}^{\infty} x_i^{(1)} \leq 1 \right\}.$$

We obtain a bijective correspondence between the sequence $\left\{ \mathcal{M}_n^{(1)} \right\}_{n=1}^{\infty}$ of measures on partitions satisfying the consistency condition and probability measures P on the space $\overline{\nabla}^{(1)}$. The correspondence is determined by

$$\mathcal{M}_n^{(1)}(\lambda) = \int_{\overline{\nabla}^{(1)}} \mathbb{K}(\lambda, \omega) P(d\omega), \quad \forall \lambda \in \mathbb{Y}_n,$$

where

$$\mathbb{K}(\lambda, \omega) = \sum_{p=0}^{\varrho_1} \frac{\left[1 - \sum_{j=1}^{\infty} x_j \right]^p}{p!} m_{\left(1^{\varrho_1 - p} 2^{\varrho_2} \ldots\right)}(x_1, x_2, \ldots).$$

§2. Multiple partition structures and harmonic functions on branching graphs

Algebra $Sym(G)$

- Let G be a finite group, $G_* = \{c_1, \ldots, c_k\}$ be the set of conjugacy classes in G, and $G^* = \{\gamma^1, \ldots, \gamma^k\}$ be the set of irreducible characters of G.
- The algebra $Sym(G)$ is defined by

$$Sym(G) = \mathbb{C}\left[p_{r_1}(c_1), \ldots, p_{r_k}(c_k) : \ r_1 \geq 1, \ldots, r_k \geq 1\right],$$

  where $p_{r_l}(c_l)$ is r_lth power sum in variables $x_{1c_l}, x_{2c_l}, \ldots$.
- For each multiple partition $\Lambda_n^{(k)} = \left(\lambda^{(1)}, \ldots, \lambda^{(k)}\right)$ define

$$\mathbb{P}_{\Lambda_n^{(k)}}\left(\gamma^1, \ldots, \gamma^k; \theta\right) = \prod_{j=1}^{k} P_{\lambda^{(j)}}\left(\gamma^j; \theta\right), \quad \theta > 0,$$

  where $P_{\lambda^{(j)}}\left(\gamma^j; \theta\right)$ is the Jack symmetric function expressed as a polynomial in variables $\left\{p_r\left(\gamma^j\right) : \ r \geq 1\right\}$,

$$p_r\left(\gamma^j\right) = \sum_{i=1}^{k} \frac{|G|}{|c_i|} \gamma^j(c_i)\, p_r(c_i).$$

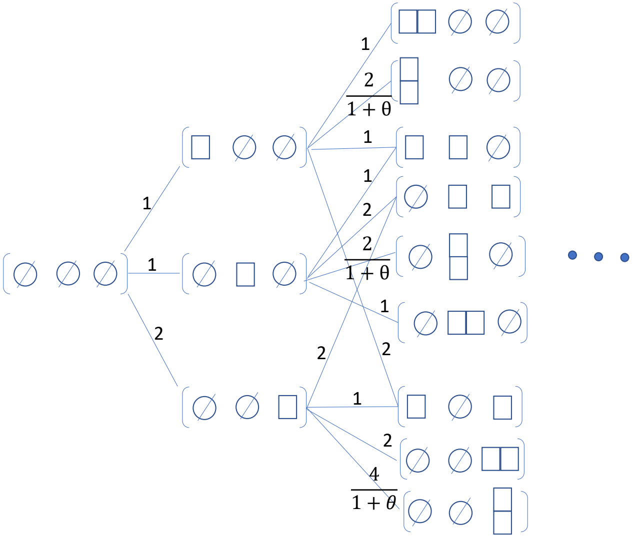
A Pieri-type formula. The branching graph $\Gamma_\theta(G)$

- The following Pieri-type rule holds for the functions $\mathbb{P}_{\Lambda_n^{(k)}}\left(\gamma^1,\ldots,\gamma^k;\theta\right)$

$$p_1(c_1)\mathbb{P}_{\Lambda_n^{(k)}}\left(\gamma^1,\ldots,\gamma^k;\theta\right)$$
$$=\sum_{\widetilde{\Lambda}_{n+1}^{(k)}\in\mathbb{Y}_{n+1}^{(k)}}\Upsilon_\theta\left(\Lambda_n^{(k)},\widetilde{\Lambda}_{n+1}^{(k)}\right)\mathbb{P}_{\widetilde{\Lambda}_{n+1}^{(k)}}\left(\gamma^1,\ldots,\gamma^k;\theta\right),$$

  where $\Upsilon_\theta:\mathbb{Y}_n^{(k)}\times\mathbb{Y}_{n+1}^{(k)}\to\mathbb{R}_{\geq 0}$ is the multiplicity function which can be written explicitly.
- Denote by $\Gamma_\theta(G)$ the branching graph associated with the Pieri-type rule for $\mathbb{P}_{\Lambda_n^{(k)}}\left(\gamma^1,\ldots,\gamma^k;\theta\right)$. The vertex set of this graph is the set of all multiple partitions $\mathbb{Y}^{(k)}$.

Example: the branching graph $\Gamma_\theta(S(3))$



Harmonic functions on $\Gamma_\theta(G)$ and multiple partition structures

- A function $\varphi: \ \Gamma_\theta\left(G\right) \to \mathbb{R}_{\geq 0}$ is called harmonic on $\Gamma_\theta(G)$ if
$$\varphi\left(\Lambda_{n-1}^{(k)}\right) = \sum_{\widetilde{\Lambda}_n^{(k)} \in \mathbb{Y}_n^{(k)}} \Upsilon_\theta\left(\Lambda_{n-1}^{(k)}, \widetilde{\Lambda}_n^{(k)}\right) \varphi\left(\widetilde{\Lambda}_n^{(k)}\right),$$
and $\varphi\left(\emptyset\right) = 1$.
- Let φ be the harmonic function on $\Gamma_\theta\left(G\right)$. Set
$$\mathcal{M}_n^{(k)}\left(\Lambda_n^{(k)}\right) = DIM\left(\Lambda_n^{(k)}\right)\varphi\left(\Lambda_n^{(k)}\right),$$
where $DIM\left(\Lambda_n^{(k)}\right)$ is the dimension function of $\Gamma_\theta(G)$ (if the multiplicities are integers, the dimension function at $\Lambda_n^{(k)}$ is equal to the number of ways to get $\Lambda_n^{(k)}$ form the empty set). **As $\theta = 0$, the sequence $\left(\mathcal{M}_n^{(k)}\right)_{n=1}^{\infty}$ is a multiple partition structure.**
- Conversely, each multiple partition structure gives rise to a harmonic function on $\Gamma_{\theta=0}(G)$.

Potential theory on the branching graph $\Gamma_\theta(G)$

Harmonic functions on $\Gamma_\theta(G)$ can be described in terms of potential theory. Namely, consider an analogue $\triangle$ of the Laplace operator on $\Gamma_\theta(G)$ defined by its action on a function on $\Gamma_\theta(G)$ as

$$(\triangle f)\left(\Lambda_n^{(k)}\right) = -f\left(\Lambda_n^{(k)}\right) + \sum_{\widetilde{\Lambda}_{n+1}^{(k)} \in \mathbb{Y}_{n+1}^{(k)}} \Upsilon_\theta\left(\Lambda_n^{(k)}, \widetilde{\Lambda}_{n+1}^{(k)}\right) f\left(\widetilde{\Lambda}_{n+1}^{(k)}\right).$$

Then each harmonic function φ on $\Gamma_\theta(G)$ satisfies the Laplace equation

$$(\triangle\varphi)\left(\Lambda_n^{(k)}\right) = 0.$$

Moreover, the dimension function of $\Gamma_\theta(G)$ can be understood as the Green function of the operator $\triangle$.

The generalized Thoma set $\Omega(G)$

- Set

$$\Omega(G)=\Big\{(\alpha,\beta,\delta)\Big|\alpha=\left(\alpha^{(1)},\ldots,\alpha^{(k)}\right),$$
$$\beta=\left(\beta^{(1)},\ldots,\beta^{(k)}\right),\quad \delta=\left(\delta^{(1)},\ldots,\delta^{(k)}\right);$$
$$\alpha^{(l)}=\left(\alpha_1^{(l)}\geq\alpha_2^{(l)}\geq\ldots\geq 0\right),\beta^{(l)}=\left(\beta_1^{(l)}\geq\beta_2^{(l)}\geq\ldots\geq 0\right),$$
$$\delta^{(1)}\geq 0,\ldots,\delta^{(k)}\geq 0,$$
$$\text{where}\quad \sum_{i=1}^{\infty}\alpha_i^{(l)}+\beta_i^{(l)}\leq\delta^{(l)},\ 1\leq l\leq k,\ \text{and}\ \sum_{l=1}^{k}\delta^{(l)}=1\Big\}.$$

- For each l, $1\leq l\leq k$, denote by $p^o_{r,l}(\alpha,\beta,\delta;\theta)$ the θ-extended power sum symmetric function evaluated on $\Omega(G)$,

$$p^o_{r,l}(\alpha,\beta,\delta;\theta)=\begin{cases}\sum\limits_{i=1}^{\infty}\left(\alpha_i^{(l)}\right)^r+(-\theta)^{r-1}\sum\limits_{i=1}^{\infty}\left(\beta_i^{(l)}\right)^r, & r=2,3,\ldots\\ \delta^{(l)}, & r=1.\end{cases}$$

Main result (E.S. 2022)

- There is a bijective correspondence between the set of the normalized harmonic functions on $\Gamma_\theta(G)$, and the set of probability measures on the generalized Thoma set $\Omega(G)$. This correspondence is determined by

$$\varphi\left(\Lambda_n^{(k)}\right)=\int_{\Omega(G)}\mathbb{K}_\theta\left(\Lambda_n^{(k)},\omega\right)P(d\omega),\ \ \forall\Lambda_n^{(k)}=\left(\lambda^{(1)},\ldots,\lambda^{(k)}\right),$$

  where P is a probability measure on $\Omega(G)$, and $n=1,2,...$, and k is the number of conjugacy classes in G.

- The kernel $\mathbb{K}_\theta\left(\Lambda_n^{(k)},\omega\right)$ is given by

$$\mathbb{K}_\theta\left(\Lambda_n^{(k)},\omega\right)=\prod_{l=1}^{k}\frac{1}{d_l^{|\lambda^{(l)}|}}P^o_{\lambda^{(l)}}\left(\alpha,\beta,\delta;\theta\right).$$

  Here $P^o_{\lambda^{(l)}}\left(\alpha,\beta,\delta;\theta\right)$ denotes the Jack symmetric function parameterized by $\lambda^{(l)}$ which is obtained as a polynomial in the variables $\left\{p^o_{r,l}\left(\alpha,\beta,\delta;\theta\right):\ r\geq 1\right\}$, and d_l are the dimensions of irreps of G.

Remarks

a) Particular cases

- $k = 1$, $\theta = 0$ - Kingman (1978) (partition structures)
- $k = 1$, $\theta = 1$ - Thoma (1964), Vershik and Kerov (1981) (characters of the infinite symmetric group)
- $k = 1$, $\theta \geq 0$ - Okounkov (1997), Kerov, Olshanski, and Okounkov (1998) (different motivations, the case $\theta = 1/2$ is relevant for the harmonic analysis for the Gelfand pair $(S(2\infty), H(2\infty))$, $H(2\infty)$ is hyperoctahedral subgroup of $S(2\infty)$).
- k is an arbitrary, $\theta = 1$-Hora, Hirai (2014) (characters of the infinite wreath products).

b) **The case k is an arbitrary, $\theta = 0$ corresponds to the multiple partition structures.**

§3. A multiple partition structure related to the wreath products $G \sim S(n)$

The wreath product $G \sim S(n)$

Our aim is to construct an example of a multiple partition structure. We will use probability measures on the wreath product $G \sim S(n)$. The wreath product $G \sim S(n)$ is the group whose underlying set is

$$\{((g_1, \ldots, g_n), s) : \; g_i \in G, s \in S(n)\},$$

where G is a finite group, and $S(n)$ is the symmetric group. The multiplication in $G \sim S(n)$ is defined by

$$((g_1, \ldots, g_n), s)\,((h_1, \ldots, h_n), t) = \left(\left(g_1 h_{s^{-1}(1)}, \ldots, g_n h_{s^{-1}(n)}\right), st\right).$$

When $n = 1$, $G \sim S(1)$ is G. The order of $G \sim S(n)$ is $2^n n!$. The important fact is that both **the conjugacy classes and the irreducible representations of $G \sim S(n)$ are parameterized by multiple partitions $\Lambda_n^{(k)}$**, where k is the number of conjugacy classes in G.

The Ewens probability measure on $G \sim S(n)$

Assume that $x = ((g_1, \ldots, g_n), s) \in G \sim S(n)$. A cycle $(i_1, \ldots, i_r)$ of s is called of type c_i if the product $g_{i_r} \ldots g_{i_2} g_{i_1}$ belongs the the conjugacy class c_i of G, $1 \leq i \leq k$.

Proposition

Fix $t_1 > 0, \ldots, t_k > 0$, and set

$$P^{Ewens}_{t_1,\ldots,t_k;n}(x) = \frac{t_1^{[x]_{c_1}} t_2^{[x]_{c_2}} \ldots t_k^{[x]_{c_k}}}{|G|^n \left(\frac{t_1}{\zeta_{c_1}} + \ldots + \frac{t_k}{\zeta_{c_k}}\right)_n}, \; x \in G \sim S(n),$$

where $c_1, \ldots, c_k$ are the conjugacy classes of G, $[x]_{c_i}$ is the number of cycles of type c_i in x, $\zeta_{c_i} = \frac{|G|}{|c_i|}$, and $(a)_n$ is the Pochhammer symbol. Each $P^{Ewens}_{t_1,\ldots,t_k;n}$ is a probability measure on $G \sim S(n)$.

The measure $P^{Ewens}_{t_1,\ldots,t_k;n}$ is called the **Ewens probability measure** on $G \sim S(n)$.

The multiple partition structure $\left(\mathcal{M}^{Ewens}_{t_1,\ldots,t_k;n}\right)_{n=1}^{\infty}$

To an element $x \in G \sim S(n)$ we assign a multiple partition $\Lambda_n^{(k)}$ describing the conjugacy class of x. Since $P^{Ewens}_{t_1,\ldots,t_k;n}(x)$ is invariant under action of $G \sim S(n)$ on itself by conjugations, the projection $x \to \Lambda_n^{(k)}$ takes $P^{Ewens}_{t_1,\ldots,t_k;n}$ to a probability measure $\mathcal{M}^{Ewens}_{t_1,\ldots,t_k;n}$ on the set of multiple partitions $\mathbb{Y}_n^{(k)}$.

Proposition

The sequence $\left(\mathcal{M}^{Ewens}_{t_1,\ldots,t_k;n}\right)_{n=1}^{\infty}$ *is a multiple partition structure.*

The representation theorem for multiple partition structures gives

$$\mathcal{M}^{Ewens}_{t_1,\ldots,t_k;n}\left(\Lambda_n^{(k)}\right) = \int_{\overline{\nabla}^{(k)}} \mathbb{K}\left(\Lambda_n^{(k)}, \omega\right) P^{Ewens}_{t_1,\ldots,t_k}(d\omega),$$

where $P^{Ewens}_{t_1,\ldots,t_k}$ is a probability measure on $\overline{\nabla}^{(k)}$.

Problem. Describe the probability measure $P^{Ewens}_{t_1,\ldots,t_k}$.

We will see that $P^{Ewens}_{t_1,\ldots,t_k}$ can be understood as **the multiple Poisson-Dirichlet distribution**.

§4. The multiple Poisson-Dirichlet distribution. The representation theorem for $\left(\mathcal{M}^{Ewens}_{t_1,\dots,t_k;n}\right)_{n=1}^{\infty}$.

The Poisson-Dirichlet distribution $PD(t)$

The Poisson-Dirichlet distribution $PD(t)$ can be understood as the Poisson-Dirichlet limit of the Dirichlet distribution $D(T_1, \ldots, T_M)$ with density

$$\frac{\Gamma\left(T_1 + \ldots + T_M\right)}{\Gamma\left(T_1\right) \ldots \Gamma\left(T_M\right)} x_1^{T_1 - 1} x_2^{T_2 - 1} \ldots x_M^{T_M - 1}$$

relative to the $(M-1)$- dimensional Lebesgue measure on the simplex

$$\triangle_M = \left\{(x_1, \ldots, x_M) : \; x_i \geq 0, \; x_1 + \ldots + x_M = 1\right\},$$

where T_1, ..., T_M are strictly positive parameters. Assume that $(x_1, \ldots, x_M)$ has the Dirichlet distribution with equal parameters,

$$T_1 = \ldots = T_M = \frac{t}{M-1}.$$

If $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(M)}$ denote the x_j arranged in descending order, then $x_{(1)}$, $x_{(2)}$, ... converge in joint distribution as $M \to \infty$, the limit is $PD(t)$.

The multiple Poisson-Dirichlet distribution $PD(t_1, \ldots, t_k)$

Definition

Let $t_1 > 0$, …, $t_k > 0$. For each l, $1 \leq l \leq k$, let $x^{(l)} = \left(x_1^{(l)}, x_2^{(l)}, \ldots\right)$ be independent sequences of random variables such that

$$x^{(l)} \sim PD(t_l), \ l = 1, \ldots, k.$$

Furthermore, let $\delta^{(1)}$, …, $\delta^{(k)}$ be random variables independent of $x^{(1)}$, …, $x^{(k)}$, and such that joint distribution of $\delta^{(1)}$, …, $\delta^{(k)}$ is the Dirichlet distribution $D(t_1, \ldots, t_k)$. The joint distribution of the sequences $\delta^{(1)}x^{(1)}$, …, $\delta^{(k)}x^{(k)}$ is called the *multiple Poisson-Dirichlet distribution* $PD(t_1, \ldots, t_k)$.

Remarks

- The distribution $PD(t_1, \ldots, t_k)$ is concentrated on

$$\overline{\nabla}_0^{(k)} = \left\{ (x, \delta) \middle| x = \left(x^{(1)}, \ldots, x^{(k)}\right), \delta = \left(\delta^{(1)}, \ldots, \delta^{(k)}\right); \right.$$
$$x^{(l)} = \left(x_1^{(l)}, x_2^{(l)}, \ldots\right), x_1^{(l)} \geq x_2^{(l)} \geq \ldots \geq 0,\ 1 \leq l \leq k,$$
$$\delta^{(1)} \geq 0, \ldots, \delta^{(k)} \geq 0,$$
$$\left. \text{where } \sum_{i=1}^{\infty} x_i^{(l)} = \delta^{(l)},\ 1 \leq l \leq k, \text{ and } \sum_{l=1}^{k} \delta^{(l)} = 1 \right\}.$$

- If $k = 1$, the multiple Poisson-Dirichlet distribution turns into the usual Poisson-Dirichlet distribution $PD(t_1)$.
- A point process can be associated with $PD(t_1, \ldots, t_k)$, and its correlation functions can be explicitly computed.

The representation theorem for $\left(\mathcal{M}^{Ewens}_{t_1,\ldots,t_k;n}\right)_{n=1}^{\infty}$. (E. S. (2022))

The multiple partition structure $\left(\mathcal{M}^{Ewens}_{t_1,\ldots,t_k;\,n}\right)_{n=1}^{\infty}$ has the following representation

$$\mathcal{M}^{Ewens}_{t_1,\ldots,t_k;\,n}\left(\Lambda_n^{(k)}\right) = \frac{n!}{\prod\limits_{j=1}^{k} |\lambda_1^{(j)}|!\ldots|\lambda_n^{(j)}|!}$$

$$\times \int\limits_{\overline{\nabla}_0^{(k)}} m_{\lambda^{(1)}}(x^{(1)})\ldots m_{\lambda^{(k)}}(x^{(k)})PD(T_1,\ldots,T_k)\,(d\omega)\,.$$

Here $\Lambda_n^{(k)} = \left(\lambda^{(1)},\ldots,\lambda^{(k)}\right)$ is a multiple partition, $m_{\lambda^{(l)}}\left(x^{(l)}\right)$ are the monomial symmetric functions, $T_1 = \frac{t_1}{\zeta_{c_1}}$, …, $T_k = \frac{t_1}{\zeta_{c_k}}$, and $PD(T_1,\ldots,T_k)$ is the multiple Poisson-Dirichlet distribution.