Thick points of log-correlated fields and multiplicative chaos

Random Matrices and Random Landscapes, Ascona, July 25, 2022

Christian Webb – University of Helsinki.

Based on joint work with J. Junnila and G. Lambert

A conjecture by Fyodorov and Keating

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Freezing transitions and extreme values: random matrix theory, $\zeta(\frac{1}{2}+it)$ and disordered landscapes

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Conjecture (Fyodorov and Keating, 2014)

Let U_N be a Haar distributed, $N \times N$, random unitary matrix, and $p_N(z) = \det(z - U_N)$ its characteristic polynomial. Then, for any $x \in (0,1)$ and large N,

$$\frac{1}{2\pi}|\{\theta \in [0,2\pi] : |p_N(e^{i\theta})| \geq N^x\}| \approx N^{-x^2} \frac{1}{\sqrt{\pi \log N}} \frac{G(1+x)^2}{2xG(1+2x)} \frac{1}{\Gamma(1-x^2)} \Xi_x,$$

and the law of Ξ_x has density

$$\mathcal{P}_x(\xi) = \frac{1}{x^2} \xi^{-1-\frac{1}{x^2}} \exp(-\xi^{-1/x^2}).$$

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- First some background and heuristics about why the conjecture should be true.
- Then a hint of the type of ideas going into the proof.
- The approach to the proof is through the theory of log-correlated fields and multiplicative chaos.

Log-correlated fields

Random (generalized) functions: for $\Omega \subset \mathbb{R}^d$, $X : \Omega \to \mathbb{R}$:

$$\mathbb{E}X(x) = 0 \qquad \text{and} \qquad \mathbb{E}X(x)X(y) = \log|x-y|^{-1} + h(x,y)$$

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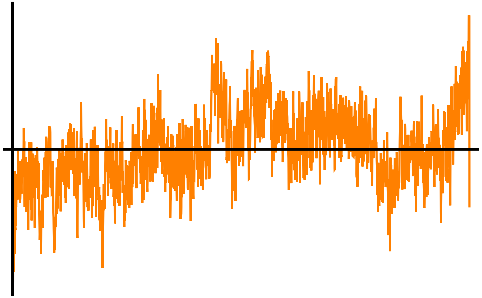

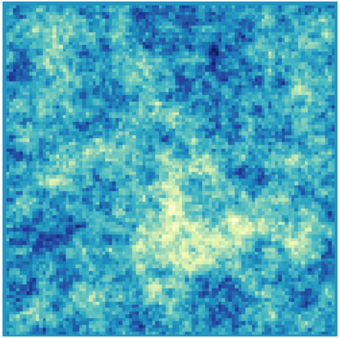

Left: $X(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}(Z_n e^{in\theta} + Z_n^* e^{-in\theta})$. Right: GFF.

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- GFF plays a role in stochastic homogenization of 2d PDEs.
- For this talk, the most important example is random matrix theory.

Log-correlated fields in random matrix theory

Theorem (Szegő; Johansson; Diaconis and Shahshahani; Hughes, Keating, and O'Connell, ...)

Let U_N be a Haar distributed, $N \times N$ random unitary matrix. Then as $N \to \infty$,

$$\sqrt{2} \log |\det(U_N - e^{i\theta})| \xrightarrow{d} X(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}} (Z_n e^{in\theta} + Z_n^* e^{-in\theta}),$$

where $(Z_n)_{n=1}^{\infty}$ are i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$.

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- Also for any trigonometric polynomial f with zero mean,

$$\int_0^{2\pi} \sqrt{2}\log|\det(U_N - e^{i\theta})| f(e^{i\theta}) \frac{d\theta}{2\pi} = \sqrt{2}\,\mathrm{Tr}\, f(U_N),$$

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- Log-cor. in RMT: GUE counting function, log-char. polynomial of Ginibre, ...

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- Also some aspects of "thick/high points of the field" are understood. E.g. for $K \subset \Omega$ compact and $\gamma \in (0, \sqrt{2d})$, one expects

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- This type of results proven under assumptions under assumptions on asymptotics of exponential moments, e.g. $\mathbb{E}e^{\gamma X_\epsilon(x) + \gamma X_\epsilon(y)}$ with $\gamma \in (0, \sqrt{2d})$.

Multiplicative chaos

One approach to studying log-correlated fields is through multiplicative chaos measures.

Theorem (Kahane 85; Berestycki 17; Claeys, Fahs, Lambert, W 21; ...)

For a nice enough regularization X_ϵ (actually, assumptions on exponential moments), for any $\gamma \in (0, \sqrt{2d})$, as $\epsilon \to 0$, the random measure

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Results above about thick points and extrema are corollaries of this type of theorems.

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- In some special models such finer asymptotics for thick points have been related to multiplicative chaos measures:
  - Branching Brownian motion (Genz, Kistler, and Schmidt 18)
  - The discrete GFF (Biskup and Louidor 19)
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- Such things known to physicists (e.g. Yan) way before any of these results and for general models.
- The proofs are all specific to the models – no general approach exists. Our goal: provide one.

The main result

Theorem (JLW 22+)

For a nice regularization X_ϵ *(satisfying rather elaborate exponential moment assumptions) for any* $\gamma \in (0, \sqrt{2d})$, *and* $K \subset \Omega$ *compact,*

$$\lim_{\epsilon \to 0} \mathbb{E} \left| \int_K \frac{\mathbf{1}\{X_\epsilon(x) > \gamma \log \epsilon^{-1}\}}{\mathbb{P}(X_\epsilon(x) > \gamma \log \epsilon^{-1})} dx - \int_K \frac{e^{\gamma X_\epsilon(x)}}{\mathbb{E} e^{\gamma X_\epsilon(x)}} dx \right| = 0.$$

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Some comments about the proof

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The real proof is a kind of second moment method (barriers) and uses exponential moment estimates to justify Fourier techniques to control the tail probabilities.