Log-correlated structures for random Jacobi matrices

Ofer Zeitouni

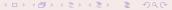
Joint with Raphael Butez and with Fanny Augeri

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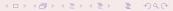


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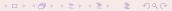
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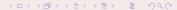
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If it is log-correlated, what about the extrema?



Set
$$M_N(\theta) = \log |P_N(e^{i\theta})|, M_N^* = \max_{\theta \in [0,2\pi]} M_N(\theta).$$

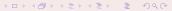


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$$M_N^* = \log N - \frac{3}{4} \log \log N + W$$

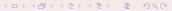
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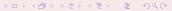


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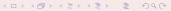


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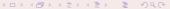
Both use in essential way CUE (aka $\beta = 2$), where joint distribution of eigenvalues is

$$\prod_{i < j} |\lambda_i - \lambda_j|^2$$

for which Gaussianity of traces follows from Diaconis-Shashahani and moments of determinant (=exponential moments of $M_N(z)$) are Toeplitz determinants.

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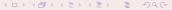


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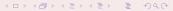


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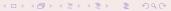


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 $\alpha_k = B_k e^{2\pi i \theta_k}$, $EB_k^2 \sim 2/\beta k$, beta variable. $\alpha_k \sim g_k + i g_k'$, Gaussian.



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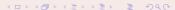
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 $\alpha_k = B_k e^{2\pi i \theta_k}$, $EB_k^2 \sim 2/\beta k$, beta variable. $\alpha_k \sim g_k + i g_k'$, Gaussian. In addition, $\sup_{|z|=1} |\log |M_N(z)| - \log |\Phi_k^*(z)||$ is tight.

$$\log \Phi_k^*(e^{i\theta}) - \log \Phi_{k-1}^*(e^{i\theta}) = \log(1 - \alpha_j e^{i\Psi_{k-1}(\theta)}) \sim -\alpha_j e^{i\Psi_{k-1}(\theta)}$$

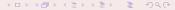
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Thus, marginal of $\log |\Phi_N^*(e^{i\theta})|$ is essentially Gaussian, of variance $(2/\beta) \log N$.



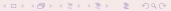
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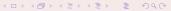


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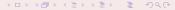
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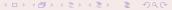


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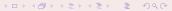
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Central limit theorem $f: \mathbb{R} \to \mathbb{R}$ compactly supported, smooth. Consider

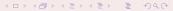
$$W_{f,N} = \sum_{i=1}^{N} f(\lambda_i) - N \int f d\sigma.$$



Theorem (Johansson '98; β ensembles)

 $W_{f,N}$ satisfies CLT, mean $(2/\beta - 1) \int f d\nu$, variance

$$\frac{(2/\beta)}{4\pi^2} \iint_{-2}^2 f(t)f'(s) \frac{\sqrt{4-s^2}}{(t-s)\sqrt{4-t^2}} ds dt.$$



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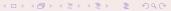
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The variance has an alternative expression

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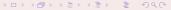
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CLT's of this type go back at least to CLT of Jonsson for moments ('82), Pastur and co-workers, Bai-Silverstein, Shcherbina, For Coulomb gas models, see Sylvia Serfaty's talk; also mesoscopic.

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Actually, optimal error rates, local law, convergence of exponential moments, imaginary part, LLN for maximum,...

Proof uses loop equations, and Coulomb gas structure. Works for non-Gaussian invariant ensembles.

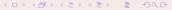


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$$f_N(z) = |P_N(z)| = |\det(zI - X_N)|.$$

Goal is to study recursions. $G\beta E$ give a 3-diagonal matrix model:

$$J_{n,\mathbf{a},\mathbf{b}} = \begin{pmatrix} b_n & a_{n-1} & 0 & \cdots & 0 \\ a_{n-1} & b_{n-1} & a_{n-2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_2 & b_2 & a_1 \\ 0 & \cdots & \cdots & a_1 & b_1 \end{pmatrix}.$$
(1



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Assumption

 $v = 2/\beta > 0$, $\{b_k\}_{k \ge 1}$, $\{g_k\}_{k \ge 1}$ - independent sequences of independent centered random variables of variance v + O(1/k), a.c. laws wrt Lebesgue measure on \mathbb{R} . Let a_{k-1} be such that, for a deterministic sequence c_k satisfying $c_k = O(1/k)$ and $c_{k+1} - c_k = O(1/k^2)$,

$$\frac{a_{k-1}^2}{\sqrt{k(k-1)}} = 1 - c_k + \frac{g_k}{\sqrt{k}}.$$

Further, there exist $\lambda_0 > 0$ and M > 0 independent of k such that

$$E(e^{\lambda_0|b_k|}) \le M, \ E(e^{\lambda_0|g_k|}) \le M. \tag{2}$$

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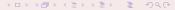
Theorem (Augeri-Butez-Z. '20)

Fix $z \in (-2,2) \setminus \{0\}$. Then, the sequence of random variables

$$w_n(z) = \frac{\log |\det(zI_n - J_n/\sqrt{n})| - n(z^2/4 - 1/2)}{\sqrt{v \log n/2}}$$
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converges in distribution to a standard Gaussian law.

The centering $n(z^2/4 - 1/2)$ is the *n* multiple of the logarithmic potential of the semi-circle distribution.



10/21

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Theorem (Augeri-Z. '22, in progress)

 $\sqrt{v \log n/2} w_n(z)$ converges to a log-correlated Gaussian field.

In GUE case, access to max through R-H methods, Lambert-Paquette '18; also, link to GMC: Berestycki-Webb-Wong '18.

Claeys, Fahs, Lambert, Webb '21: sharp CLT's for counting functions, GMC convergence.

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CLT for log determinant $G\beta E$

The case z = 0 is special.

Theorem (Tao-Vu '11)

 $\log M_N(0)/\sqrt{\beta \log N}$ converges (for Wigner matrices, 4 matching moments) to standard Gaussian.

Bourgade-Mody '19: extends w/out matching 4 moments.



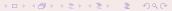
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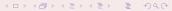
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The Dumitriu-Edelman representation

Theorem (Dumitriu-Edelman '05)

 X_N from $G\beta E$ is unitarily equivalent to the following 3-diagonal Jacobi matrix

$$\frac{1}{\sqrt{N}}X_N = \frac{1}{\sqrt{N}} \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \mathbf{0} & a_{N-1} & b_N \end{pmatrix}$$

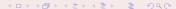
where $b_i \sim N(0, \sqrt{2/\beta})$, $a_i \sim \chi_{i\beta}/\sqrt{\beta}$.

Here $a_i \sim \chi_{i\beta}/\sqrt{\beta}$; here $\chi^2_{i\beta}$ has chi-square distribution with $i\beta$ degrees of freedom, ie $\chi_{i\beta}/\sqrt{\beta} \sim \sqrt{i\beta} + G/\sqrt{2\beta} + O(1/i)$.



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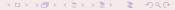
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13/21

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$$\varphi_k(z\sqrt{N}) = (z\sqrt{N} - b_k)\varphi_k(z\sqrt{N}) - a_{k-1}^2\varphi_{k-1}(z\sqrt{N}), \varphi_{-1} = 0, \varphi_0 = 1.$$



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We set

$$\Psi_k(z) = \phi_k(z\sqrt{N}) \frac{1}{\sqrt{k!} \prod_{i=1}^k \alpha(\omega_i)}$$

and then

$$\Psi_k(z) = \frac{z\sqrt{N} - b_k}{\sqrt{k}\alpha(\omega_k)} \Psi_{k-1}(z) - \frac{a_{k-1}^2}{\sqrt{k(k-1)}\alpha(\omega_k)\alpha(\omega_{k-1})} \Psi_{k-2}(z).$$

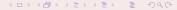


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Recall that k_0 satisfies $\omega_{k_0}=2$ (if z=0 then $k_0=1$). In matrix form, for $k\geq k_0$,

$$\begin{pmatrix} \Psi_{k+1}(z) \\ \Psi_{k}(z) \end{pmatrix} \sim \begin{pmatrix} \omega_{k} & -1 + 1/2k \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_{k}(z) \\ \Psi_{k-1}(z) \end{pmatrix} + \begin{pmatrix} b_{k}/\sqrt{k} & g_{k}/\sqrt{k} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi_{k}(z) \\ \Psi_{k-1}(z) \end{pmatrix}$$

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Tao-Vu show that $\Psi_k(z)^2 + \Psi_{k-1}(z)^2$ (essentially) forms a martingale with quadratic variation process of increment $\sim 1/k$. This gives the CLT.



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$$\begin{pmatrix} \Psi_{k+1}(z) \\ \Psi_{k}(z) \end{pmatrix} = A_{k} \begin{pmatrix} \Psi_{k}(z) \\ \Psi_{k-1}(z) \end{pmatrix} + E_{k} \begin{pmatrix} \Psi_{k}(z) \\ \Psi_{k-1}(z) \end{pmatrix}$$

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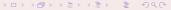
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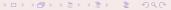
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For $k < k_0 = z^2 n/4$, eigenvalues real and smaller that 1.

For $k > k_0$, eigenvalues imaginary, of modulus roughly 1.



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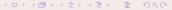
Recursions - general z

There are several regimes to consider. Fix $\epsilon > 0$, recall that $k_0 = z^2 N/4$.

- $k < (1 \epsilon)k_0$: one checks that $\Psi_k(z) \sim 1$.
- $k \in [(1 \epsilon)k_0, k_0]$: write

$$X_k = \Psi_k/\Psi_{k-1} = 1 + \delta_k, \quad X_k = A_k + B_k/X_{k-1}$$

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for appropriate A_k, B_k . In this regime, $\delta_k \sim 0$ and one obtains a recursion

$$\delta_k \sim u_k + v_k \delta_{k-1}$$

where, with $\alpha_k = \alpha(\omega_k)$,

$$u_k \sim \frac{b_k}{\sqrt{k\alpha_k^2}} + \frac{1}{2k\alpha_k^2} - \frac{g_k}{\sqrt{k\alpha_k^4}}, \quad v_k = \frac{1 - \frac{1}{2k} + \frac{g_k}{\sqrt{k}}}{\alpha_k^2}.$$

A much finer analysis by Lambert-Paquette: hyperbolic regime (up to $k_0 - k_0^{1/3} (\log k_0)^{2/3}$) arXiv:2001.09042 and edge regime (up to k_0) arxiv:2009.05003.

• $k > k_0$: Oscillatory regime, most interesting.

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$$X_k = \begin{pmatrix} \Psi_{k+1} \\ \Psi_k \end{pmatrix}, k > k_0.$$

We have

$$X_{k+1} = (A_k + W_k)X_k,$$

where

$$A_k = \left(\begin{array}{cc} \omega_k & -1 + \frac{1}{2k} \\ 1 & 0 \end{array}\right), \ W_k = \left(\begin{array}{cc} \frac{-b_k}{\sqrt{k}} & \frac{g_k}{\sqrt{k}} \\ 0 & 0 \end{array}\right),$$

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Eigenvalues of A_k for $k > k_0$ are complex of (essentially) unit norm. Change basis to eigenvector basis, get

$$\hat{X}_k = Q_k \prod_{i=k_0}^{k-1} Q_{i+1}^{-1} Q_i (R_i + \hat{W}_i) Q_{k_0}^{-1} \hat{X}_{k_0},$$

where R_i are rotation matrices of angle $\theta_k \sim \sqrt{k/k_0-1}$.

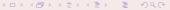
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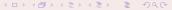
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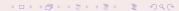
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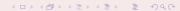
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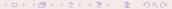
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For lower bound on norm, use anti-concentration.



Problem 2: Noncommutative product - control

$$\hat{X}_k = Q_k \prod_{i=k_0}^{k-1} Q_{i+1}^{-1} Q_i (R_i + \hat{W}_i) \hat{Y}_{k_0 + Ck_0^{1/3}}$$



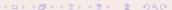
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First order approximation: divide to blocks of length $\ell_{i+1} - \ell_i = (k_0/i)^{1/3}$, linearize in each block, and get contribution to variance of order 1/i.



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Ofer Zeitouni Log-cor Jacobi June 2022

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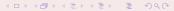
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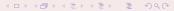
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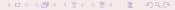
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Easy to compute effect of linearization, get that $\rho_i \sim 1 + g_i + c'/i$ where g_i has variance c/i.

Caveat: Complication when blocks get too small - cannot ensure the approximation, e.g. if block is of length 1; But variance is small there, so can combine blocks!

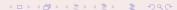
Computing correlation between different z's:

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For
$$|z-z'| \ll N^{-2/3}$$
 get correlation until block i corresponding to $\log(|z-z'|N^{2/3})$ which gives log-correlated structure.

