Log-correlated structures for random Jacobi matrices

Ofer Zeitouni
Joint with Raphael Butez and with Fanny Augeri

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If it is log-correlated, what about the extrema?

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Both use in essential way CUE (aka $\beta = 2$), where joint distribution of eigenvalues is

$$\prod_{i<j} |\lambda_i - \lambda_j|^2$$

for which Gaussianity of traces follows from Diaconis-Shashahani and moments of determinant (=exponential moments of $M_N(z)$) are Toeplitz determinants.

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$\alpha_k = B_k e^{2\pi i \theta_k}$, $EB_k^2 \sim 2/\beta k$, beta variable. $\alpha_k \sim g_k + i g_k'$, Gaussian.

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In addition, $\sup_{|z|=1} |\log|M_N(z)| - \log|\Phi_k^*(z)||$ is tight.

Recursions - Circular ensembles

$$\log \Phi_k^*(e^{i\theta}) - \log \Phi_{k-1}^*(e^{i\theta}) = \log(1 - \alpha_j e^{i\Psi_{k-1}(\theta)}) \sim -\alpha_j e^{i\Psi_{k-1}(\theta)}$$

$$\Psi_k(\theta) = \Psi_k(\theta) + \theta - 2\Im \log(1 - \alpha_j e^{i\Psi_{k-1}(\theta)}).$$

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Some new phenomena for log-determinant of random permutations: Cook-Z. ’20

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Central limit theorem $f:\mathbb{R}\to\mathbb{R}$ compactly supported, smooth. Consider

$$W_{f,N} = \sum_{i=1}^N f(\lambda_i) - N\int f d\sigma.$$

CLT

Theorem (Johansson '98; β ensembles)

$W_{f,N}$ satisfies CLT, mean $(2/\beta - 1)\int f d\nu$, variance

$$\frac{(2/\beta)}{4\pi^2}\iint_{-2}^{2} f(t)f'(s)\frac{\sqrt{4-s^2}}{(t-s)\sqrt{4-t^2}}dsdt.$$

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CLT's of this type go back at least to CLT of Jonsson for moments ('82), Pastur and co-workers, Bai-Silverstein, Shcherbina, ... For Coulomb gas models, see Sylvia Serfaty's talk; also mesoscopic.

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$\log|P_N(z)| - N\int \log|z-x|\sigma(dx)$, $z \in (-2,2)$ converges to a log-correlated field.

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Actually, optimal error rates, local law, convergence of exponential moments, imaginary part, LLN for maximum,..
Proof uses loop equations, and Coulomb gas structure. Works for non-Gaussian invariant ensembles.

The CLT

$f_N(z) = |P_N(z)| = |\det(zI - X_N)|$.

Goal is to study recursions. GβE give a 3-diagonal matrix model:

$$J_{n,\mathbf{a},\mathbf{b}} = \begin{pmatrix} b_n & a_{n-1} & 0 & \cdots & 0 \\ a_{n-1} & b_{n-1} & a_{n-2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_2 & b_2 & a_1 \\ 0 & \cdots & \cdots & a_1 & b_1 \end{pmatrix}. \quad (1)$$

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Assumption

$v = 2/\beta > 0$, $\{b_k\}_{k\geq 1}$, $\{g_k\}_{k\geq 1}$ - independent sequences of independent centered random variables of variance $v + O(1/k)$, a.c. laws wrt Lebesgue measure on $\mathbb{R}$. Let a_{k-1} be such that, for a deterministic sequence c_k satisfying $c_k = O(1/k)$ and $c_{k+1} - c_k = O(1/k^2)$,

$$\frac{a_{k-1}^2}{\sqrt{k(k-1)}} = 1 - c_k + \frac{g_k}{\sqrt{k}}.$$

Further, there exist $\lambda_0 > 0$ and $M > 0$ independent of k such that

$$E(e^{\lambda_0 |b_k|}) \leq M, \; E(e^{\lambda_0 |g_k|}) \leq M. \tag{2}$$

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Theorem (Augeri-Butez-Z. '20)

Fix $z \in (-2,2) \setminus \{0\}$. Then, the sequence of random variables

$$w_n(z) = \frac{\log|\det(zI_n - J_n/\sqrt{n})| - n(z^2/4 - 1/2)}{\sqrt{v \log n/2}} \tag{3}$$

converges in distribution to a standard Gaussian law.

The centering $n(z^2/4 - 1/2)$ is the n multiple of the logarithmic potential of the semi-circle distribution.

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Theorem (Augeri-Z. '22, in progress)

$\sqrt{v \log n/2}\, w_n(z)$ *converges to a log-correlated Gaussian field.*

In GUE case, access to max through R-H methods, Lambert-Paquette '18; also, link to GMC: Berestycki-Webb-Wong '18.

Claeys, Fahs, Lambert, Webb '21: sharp CLT's for counting functions, GMC convergence.

CLT for log determinant GβE

The case $z = 0$ is special.

Theorem (Tao-Vu '11)

$\log M_N(0)/\sqrt{\beta \log N}$ *converges (for Wigner matrices, 4 matching moments) to standard Gaussian.*

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By replacement principle, the key step in the TV proof is the result for GβE, $\beta = 1, 2$. Their proof extends to general $\beta > 0$, and is based on recursions.

The Dumitriu-Edelman representation

Theorem (Dumitriu-Edelman '05)

X_N from $G\beta E$ is unitarily equivalent to the following 3-diagonal Jacobi matrix

$$\frac{1}{\sqrt{N}} X_N = \frac{1}{\sqrt{N}} \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \mathbf{0} & a_{N-1} & b_N \end{pmatrix}$$

where $b_i \sim N(0, \sqrt{2/\beta})$, $a_i \sim \chi_{i\beta}/\sqrt{\beta}$.

Here $a_i \sim \chi_{i\beta}/\sqrt{\beta}$; here $\chi^2_{i\beta}$ has chi-square distribution with $i\beta$ degrees of freedom, ie $\chi_{i\beta}/\sqrt{\beta} \sim \sqrt{i\beta} + G/\sqrt{2\beta} + O(1/i)$.

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We set

$$\Psi_k(z) = \phi_k(z\sqrt{N})\frac{1}{\sqrt{k!}\prod_{i=1}^k \alpha(\omega_i)}$$

and then

$$\Psi_k(z) = \frac{z\sqrt{N} - b_k}{\sqrt{k}\alpha(\omega_k)}\Psi_{k-1}(z) - \frac{a_{k-1}^2}{\sqrt{k(k-1)}\alpha(\omega_k)\alpha(\omega_{k-1})}\Psi_{k-2}(z).$$

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Recall that k_0 satisfies $\omega_{k_0} = 2$ (if $z = 0$ then $k_0 = 1$). In matrix form, for $k \geq k_0$,

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Tao-Vu show that $\Psi_k(z)^2 + \Psi_{k-1}(z)^2$ (essentially) forms a martingale with quadratic variation process of increment $\sim 1/k$. This gives the CLT.

$$\begin{pmatrix} \Psi_{k+1}(z) \\ \Psi_k(z) \end{pmatrix} = A_k \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix} + E_k \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix}$$

where

$$A_k = \begin{pmatrix} \omega_k & -1 + 1/2k \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix}, \quad \omega_k = z\sqrt{n/k}$$

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The eigenvalues of A_k are roughly $\frac{1}{2}\omega_k \pm \frac{1}{2}\sqrt{\omega_k^2 - 4}$.
For $k < k_0 = z^2 n/4$, eigenvalues real and smaller that 1.
For $k > k_0$, eigenvalues imaginary, of modulus roughly 1.

Recursions - general z

There are several regimes to consider. Fix $\epsilon > 0$, recall that $k_0 = z^2 N/4$.

- $k < (1-\epsilon)k_0$: one checks that $\Psi_k(z) \sim 1$.
- $k \in [(1-\epsilon)k_0, k_0]$: write

$$X_k = \Psi_k/\Psi_{k-1} = 1 + \delta_k, \quad X_k = A_k + B_k/X_{k-1}$$

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for appropriate A_k, B_k. In this regime, $\delta_k \sim 0$ and one obtains a recursion

$$\delta_k \sim u_k + v_k \delta_{k-1}$$

where, with $\alpha_k = \alpha(\omega_k)$,

$$u_k \sim \frac{b_k}{\sqrt{k\alpha_k^2}} + \frac{1}{2k\alpha_k^2} - \frac{g_k}{\sqrt{k\alpha_k^4}}, \quad v_k = \frac{1 - \frac{1}{2k} + \frac{g_k}{\sqrt{k}}}{\alpha_k^2}.$$

A much finer analysis by Lambert-Paquette: hyperbolic regime (up to $k_0 - k_0^{1/3}(\log k_0)^{2/3}$) arXiv:2001.09042 and edge regime (up to k_0) arxiv:2009.05003.

- $k > k_0$: Oscillatory regime, most interesting.

Recursions - general z - the oscilatory regime

$$X_k = \begin{pmatrix} \Psi_{k+1} \\ \Psi_k \end{pmatrix}, k > k_0.$$

We have

$$X_{k+1} = (A_k + W_k) X_k,$$

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Eigenvalues of A_k for $k > k_0$ are complex of (essentially) unit norm. Change basis to eigenvector basis, get

$$\hat{X}_k = Q_k \prod_{i=k_0}^{k-1} Q_{i+1}^{-1} Q_i \left(R_i + \hat{W}_i\right) Q_{k_0}^{-1} \hat{X}_{k_0},$$

where R_i are rotation matrices of angle $\theta_k \sim \sqrt{k/k_0 - 1}$.

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For lower bound on norm, use anti-concentration.

Recursions - general z - the oscilatory regime

Problem 2: Noncommutative product - control

$$\hat{X}_k = Q_k \prod_{i=k_0}^{k-1} Q_{i+1}^{-1} Q_i (R_i + \hat{W}_i) \hat{Y}_{k_0 + Ck_0^{1/3}}$$

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We have $\ell_{j+1} - \ell_j \sim (k_0/j)^{1/3}$, and variance computation as in sketch.
Of course, cannot achieve exactly $(1,0)^T$, but can control error by choosing when to stop.

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Within a block, linearization is a good approximation:

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$$= \sum_{k=\ell_i+1}^{\ell_{i+1}} \mathbf{R_k}(I + \Delta_k + \hat{W}_k)\mathbf{R_k}^{-1} + \text{error terms} = I + \sum_{k=\ell_i+1}^{\ell_{i+1}} \mathbf{R_k}(\Delta_k + \hat{W}_k)\mathbf{R_k}^{-1} + \text{error term}$$

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where $\mathbf{R_k}$ is a rotation by an angle between 0 and 2π.
Easy to compute effect of linearization, get that $\rho_i \sim 1 + g_i + c'/i$ where g_i has variance c/i.
Caveat: Complication when blocks get too small - cannot ensure the approximation, e.g. if block is of length 1; But variance is small there, so can combine blocks!

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Computing correlation between different z's:

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So noise looks like $\sum_{k=\ell_i}^{\ell_{i+1}} \frac{g_k}{\sqrt{k}} \cos(\omega_i(k-\ell_i))$. Recall that a given $k > k_0(z)$ translates to block i so that $\ell_i = k - k_0$. If $|z-z'| \gg N^{-2/3}$ and $z' < z$, at first block you have $i = 1$ but $i' = |z-z'|N \gg 1$. Thus $\omega_{i'} \gg \omega_i$ which gives no correlation!

Correlation

Computing correlation between different z's:

1. Scalar regime: follows directly from linearized equation.
2. Oscilatory regime: depends on $|z-z'|$. Easy case: $|z-z'| > N^{-2/3}$.

Recall the blocks: Oscilatory, in block i have frequency $\omega_i := 1/(\ell_{i+1} - \ell_i) \sim (\frac{i}{N})^{1/3}$.
So noise looks like $\sum_{k=\ell_i}^{\ell_{i+1}} \frac{g_k}{\sqrt{k}} \cos(\omega_i(k-\ell_i))$. Recall that a given $k > k_0(z)$ translates to block i so that $\ell_i = k - k_0$. If $|z-z'| \gg N^{-2/3}$ and $z' < z$, at first block you have $i = 1$ but $i' = |z-z'|N \gg 1$.
Thus $\omega_{i'} \gg \omega_i$ which gives no correlation!
For $|z-z'| \ll N^{-2/3}$ get correlation until block i corresponding to $\log(|z-z'|N^{2/3})$ which gives log-correlated structure.