Tropical series, sandpiles and symplectic area

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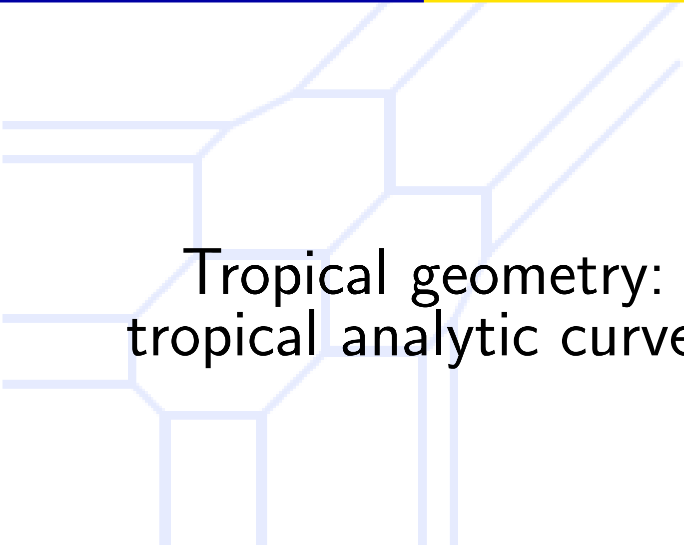
Tropical geometry: tropical analytic curves



A quartic curve.

Tropical curves

Definition

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^2$. A tropical polynomial is a function $h(x,y) = \min_{(i,j)\in\mathcal{A}} (a_{ij} + ix + jy), a_{ij} \in \mathbb{R}$. Note that $h : \mathbb{R}^2 \to \mathbb{R}$.

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Consider a family of polynomials $F_t(x,y) = \sum_{(i,j)\in\mathcal{A}} t^{a_{ij}} x^i y^j$.

Define the corresponding curves $C_t = \{(x,y) | F_t(x,y) = 0\} \subset \mathbb{C}^2$.

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Proposition

Then $C = \lim_{t\to 0} log_t C_t$ *is the tropical curve defined by* $h(x,y) = \min_{(i,j)\in\mathcal{A}} (a_{ij} + ix + jx)$.

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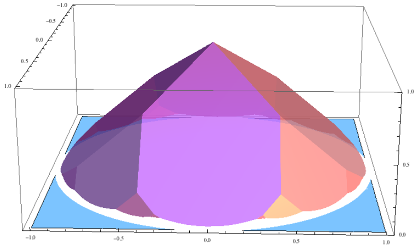
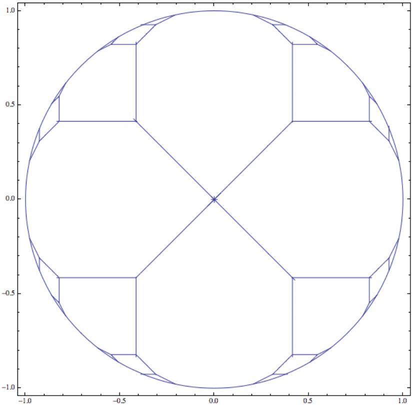
Proposition

If Ω is a convex set then a tropical series f on it can be written as $f(x,y)|_{\Omega^\circ} = \min_{(i,j)\in\mathbb{Z}^2}(a_{ij} + ix + jy)|_{\Omega^\circ}$ *for some $a_{ij} \in \mathbb{R}\cup\{+\infty\}$.*

Example





The graph of a tropical series on the circle $\Omega = \{(x, y) | x^2 + y^2 \leq 1\}$ and the corresponding tropical analytic curve.

$f(x, y) = \min_{(i,j) \in \mathbb{Z}^2}(a_{ij} + ix + jy)$ where a_{ij} is $-\inf_\Omega(ix + jy)$.

Sandpile models: emergence of tropical objects

Sandpile model

Definition

A sandpile is a collection of indistinguishable sand grains distributed among a finite subset Γ of $\mathbb{Z}^2$, that is a function $\varphi : \Gamma \to \mathbb{N}_0$. A vertex v is unstable if $\varphi(v) \geq 4$. An unstable vertex can topple by sending one grain of sand to each of 4 neighbours.

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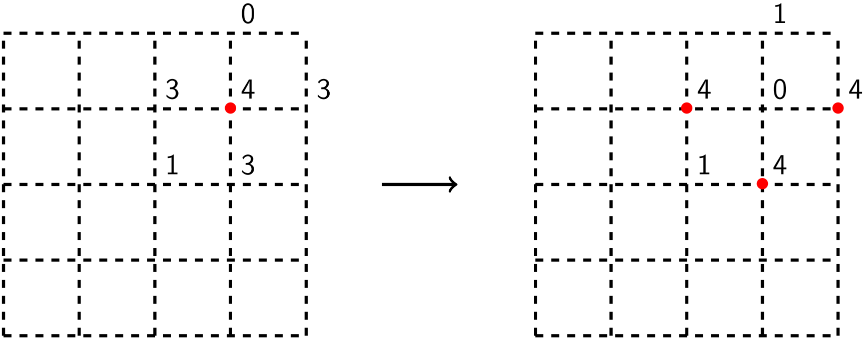

Topplings, relaxation and toppling function

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The final configuration after the relaxation does not depend on the order of topplings.

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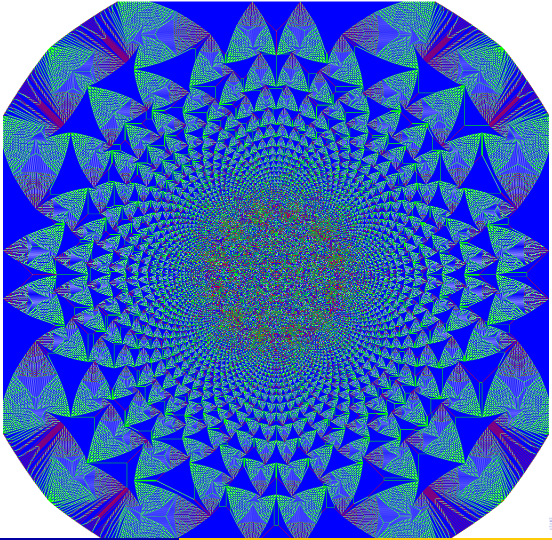
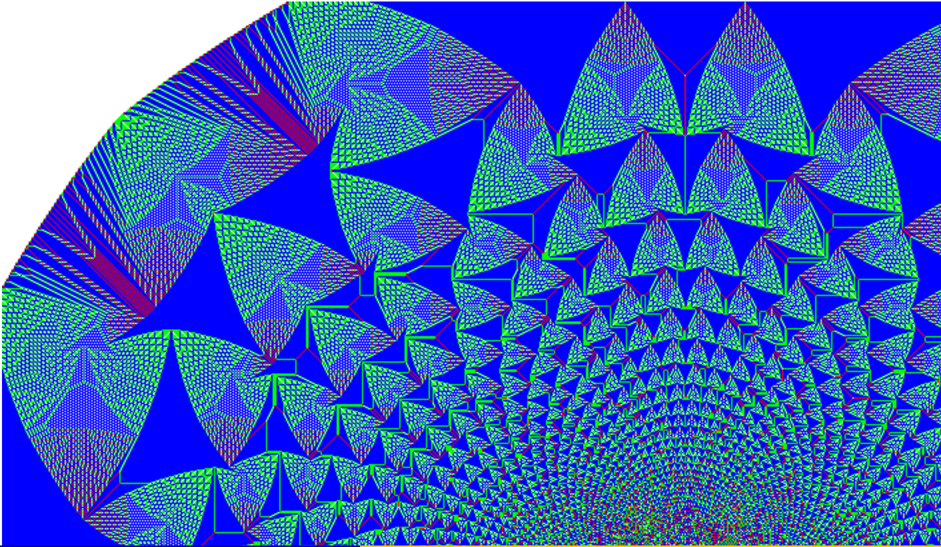
Neither the number of topplings at a given vertex depends on the order.

28 million grains dropped at one point

We start from the empty grid $\mathbb{Z}^2$, add a lot of sand to $(0, 0)$ and relax...



Hidden tropical curves



The problem

Definition

We chose a boundary $\partial\Gamma \subset \Gamma$, where we never do topplings. For example, $\Gamma = [-N..N] \times [-N..N]$, $\partial\Gamma$ is the set (i,j) where $i^2 + j^2 \geq N^2$.

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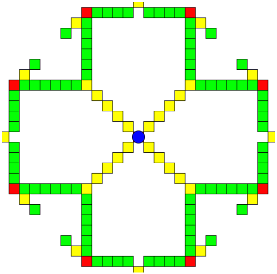
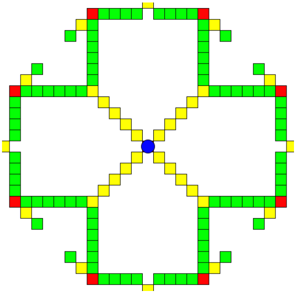
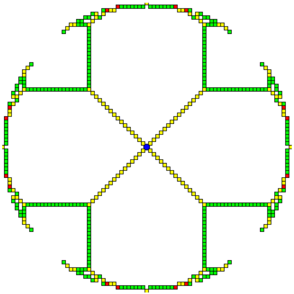


White: 3 grains, Green: 2 grains, Yellow: 1 grain, Red: 0 grains

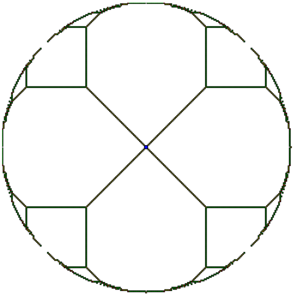
Sandpile on a disk: a point in the center



Extra grain at the center



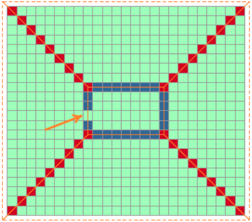
The grid is 3 times finer

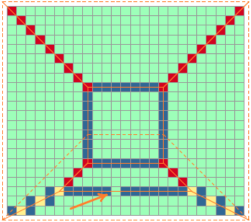


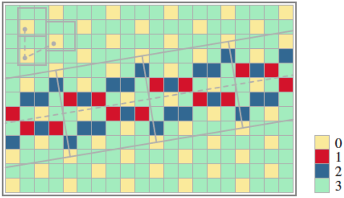
The grid is 9 times finer

White: 3 grains, Green: 2 grains, Yellow: 1 grain, Red: 0 grains

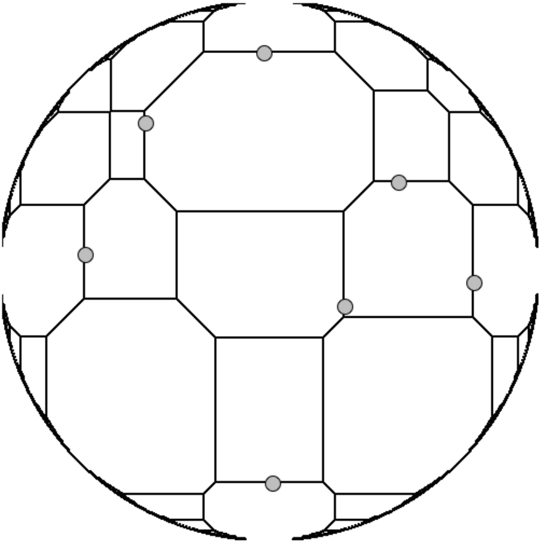
The origin of this problem: works by S. Caracciolo, G. Paoletti and A. Sportiello

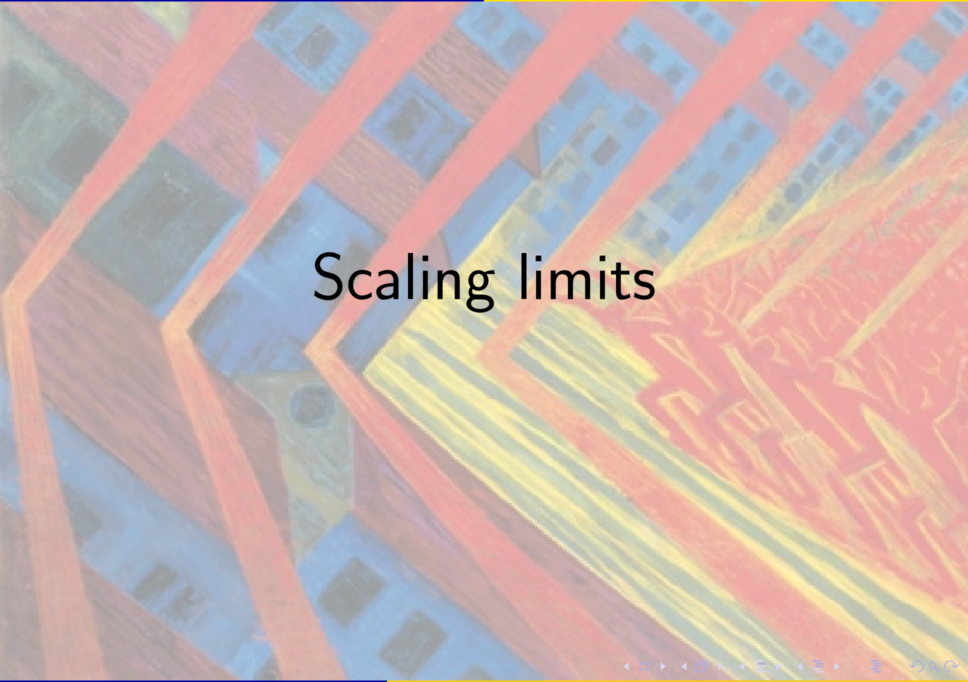






Sandpile on a disk: several points





Scaling limits

The existence of a scaling limit

Consider a convex closed set $\Omega \subset \mathbb{R}^2$ and a collection of distinct points $\bar{p} = \{p_1, ..., p_n\}$ in the interior of Ω. For each integer $N > 0$ we define a graph Γ_N as the subgraph of $\mathbb{Z}^2$ bounded by $N \cdot \Omega$. Define the state φ_N as result of adding extra grains of sand at the vertices $[Np_i]$ to the maximal stable state. Let $\tilde{\varphi}_N$ be the result of relaxation for the state φ_N.

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Definition

Let C_N be the set of vertices where $\tilde{\varphi}_N \neq 3$.

Theorem

If Ω is not $\mathbb{R}^2$ and $\partial\Omega$ does not contain a line with irrational slope, then the sequence of sets $\frac{1}{N}C_N$ converges to a tropical analytic curve $C_{\bar{p}}^{\Omega} \subset \Omega$, passing through the points $p_1, p_2, \ldots, p_n$.

Toppling function

Let h_N be the toppling function of the relaxation process $\varphi_N \to \tilde{\varphi}_N$, i.e. the value of h_N at $v \in \mathbb{Z}^2$ is the number of topplings at v during the relaxation. Define a function $F_N \colon \Omega \to \mathbb{R}$ by $F_N(x) = \frac{1}{N} h_N([Nx])$.

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Theorem

The function F_N converges to a tropical series $G_{\bar{p}}^{\Omega}$ vanishing at $\partial\Omega$. Moreover, $C_{\bar{p}}^{\Omega} \subset \Omega$ is defined by $G_{\bar{p}}^{\Omega}$.

Harmonic functions

Explanation: look at the number of topplings

Consider the number $h(x,y)$ of topplings at a point (x,y) during the relaxation.

Proposition

If the number of sand grains at (x,y) after relaxation is the same as before, then
$h(x-1,y)+h(x+1,y)+h(x,y-1)+h(x,y+1)-4h(x,y)=0.$

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Proof.

Indeed, count the number of incoming and outgoing grains. □

That is, h is harmonic at (x, y). In fact, $h(x, y)$ is a "piece-wise" linear function.

Conclusions

Recap:

- The function h of topplings is (almost everywhere) a piece-wise linear (Theorem).
- On its linear parts h is harmonic, therefore we have 3 grains there.
- Finally, we have 0,1,2 grains in a neighbourhoods of non-linearity locus.

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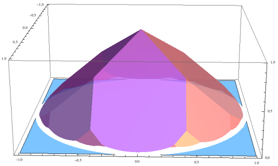

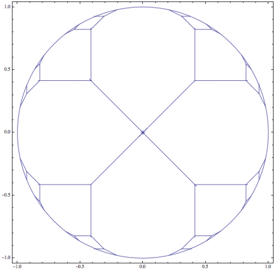

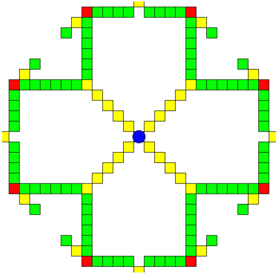

Hence, coloured vertices constitute a tropical curve!

Tropical symplectic area

Limiting curve minimizes a functional

Denote by $V(\Omega, \bar{p})$ the set of tropical series F on Ω such that $F|_\Omega \geq 0$ and F is not smooth at each of the points p_i.

Theorem

The functional

$$F \mapsto \int_\Omega F dxdy$$

defined on the space $V(\Omega, \bar{p})$ has the unique minimum which coincides with the function $G^\Omega_{\bar{p}}$.

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Remark

It is sort of Least Action Principle: sandpiles are "lazy".

Sandpile on a polygon with rational slopes: symplectic area

Definition

The tropical symplectic area of an edge e (of a direction $(p,q)\in\mathbb{Z}^2$ and multiplicity $m(e)$) of a tropical curve is given by
$EuclideanLength(e)\cdot m(e)\cdot\sqrt{p^2+q^2}$.

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Let $v = (p, q)$ be a primitive integer vector. Let C_t^{pq} be the lift of the interval $[0, v]$ to the torus $(\mathbb{C}^*)^2$ under Log_t, i.e. $C_t^{pq} = \{(z^p, z^q) | 1 \leq |z| \leq t\}$. Then

$$\int_{C_t^{pq}} d\log(z_1) \wedge d\log(\bar{z}_1) = -4i\pi^2 p^2 \log t.$$

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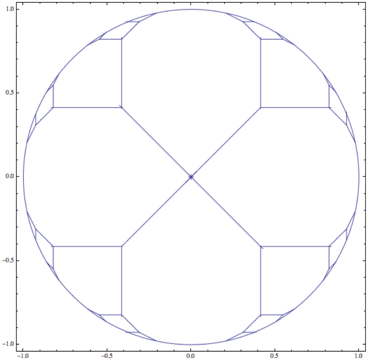
Theorem (N. K.,M. S. (arXiv:1502.06284))

If Ω is a polygon with sides of rational slopes then $C_{\overline{P}}^{\Omega}$ minimizes the tropical symplectic area.

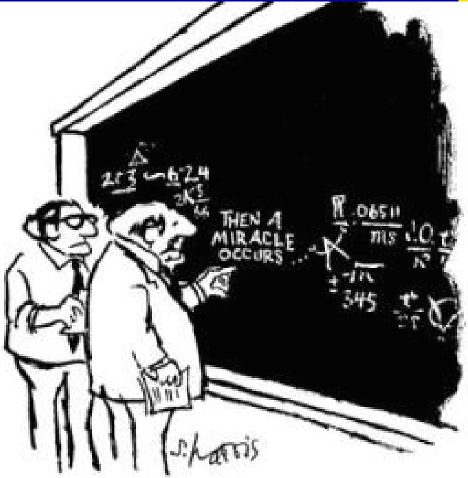
Symplectic area for tropical analytic curves

Remark

Symplectic area of a tropical analytic curve in general is infinite.



Question: How can we properly state the problem of minimization of tropical symplectic area on an arbitrary convex domain?



"I think you should be more explicit here in step two."

Ideas of proofs

Proofs: superharmonic functions

Definition

Let $(\Delta f)(x,y) = \frac{f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)}{4} - f(x,y)$

Proofs: superharmonic functions

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The toppling function h is the pointwise minimal integer valued function with the conditions:

$$h|_{\partial\Omega} = 0,$$

$$\Delta h < 0 \ \text{ at } \ p_1, p_2, \ldots, p_n,$$

$$\Delta h \leq 0, \text{elsewhere}$$

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- now, $\Delta h = 0$ and less than a linear function,
- therefore, h is linear almost everywhere.

Proofs: waves and sand dynamics

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- Let us make a wave from a point in a part where $a + ix + jy$ dominates. This results as $a \to a + 1$.
- So, we have better than a scaling limit, because we have a continuous dynamic on tropical series, and sand pictures are its discrete approximations.

Thank you for your attention!