Chapter 1

A guide to tropical modifications

Substance is by nature prior to its modifications.
... nothing is granted in addition to the understanding,
except substance and its modifications.
Ethics. Benedictus de Spinoza.

This is a draft of an expository chapter in my thesis. I would be really glad to receive any comments and suggestions (Nikita.Kalinin at unige.ch). Probably, this text is intended to be published somewhere, but definitely not in this state and not soon. If you know some appearance of tropical modification, not mentioned here, or some practical advices about the exposition, do not hesitate to contact me. The most recent version is available at https://www.unige.ch/math/tggroup/lib/exe/fetch.php?media=guide.pdf

This chapter is dedicated to *tropical modifications*, which have already become a folklore in tropical geometry. Tropical modifications are used in tropical intersection theory, the study of singularities, and admit interpretations in various contexts, such as hyperbolic geometry, Berkovich spaces, and non-standard analysis.

We cite [10]: "*Tropical modifications ... can be seen as a refinement of the tropicalization process, and allows one to recover some information about X sensitive to higher order terms.*"

One must say that the name "modification" is used in two different senses: the modification as a well-defined *operation*; and a modification along X as a *method* that reveals a behaviour of other varieties in an infinitesimal neighborhood of X. Namely, performing the modification of Y along $X \subset Y$, we know how Y changes, but the objects of codimension 1 in Y may behave differently, depending on their behavior near X. We will clarify this distinction with examples.

Our main goal is to mention different points of view, give references, and demonstrate the ability of tropical modifications which are not covered in the literature. We assume that the reader have already met "tropical modifications" somewhere and wants to understand them better. There are novelties here: a tropical version of Weil's reciprocity law and the study of non-transversal intersections. For a preliminary introduction to tropical geometry, see [8], [9] and [30], where tropical modifications are also discussed. We would like to mention two other texts, promoting modifications from different perspectives: [3] (repairing the j-invariant of elliptic curves), and [36] (intersection theory on tropical surfaces).

1.1 Motivation

Tropical modifications were introduced in the seminal paper [28] as the main ingredient in the tropical equivalence relation. Namely, two tropical varieties are *equivalent* if they are related by a chain of tropical modifications and reverse operations[1].

The underlying idea is the following. Recall, that a tropical variety V can be decomposed into a disjoint union of a compact part V_c and a non-compact part V_∞, and $V = V_c \cup V_\infty$. Moreover, V retracts on V_c. Then, the set V_∞ consists of “tree-like” unions of hyperplanes’ parts. We call these parts *legs*, by analogy with the one-dimensional case. For tropical curves, V_∞ is a union of half-lines. For example, for a tropical elliptic curve (see Fig.1.1, left side) the set V_c is the ellipse, V_∞ is the set of trees growing on the ellipse.

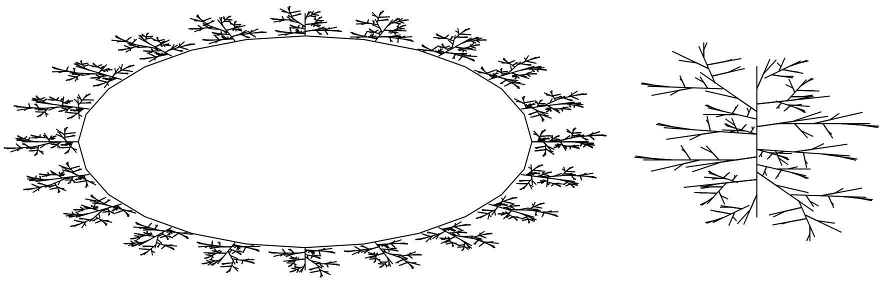


Figure 1.1: On the left side we see a tropical elliptic curve V which is a part of the analytification of an elliptic curve. The ellipse is V_c and the union of tree-like pieces is V_∞. On the right side we see a tropical rational curve V, which is equal to V_∞. We can chose each point of V as V_c, because V contracts onto any of its point $x \in V$.

Remark 1.1.1. The set V_c exists, but it is not canonically defined. On a tropical rational[2] variety V, each point can serve as V_c, see Fig. 1.1 right side.

Consider the tropical limit V of algebraic varieties $W_{t_i} \subset (\mathbb{C}^*)^n$, i.e. $V = \lim_{t_i\to\infty} \mathrm{Log}_{t_i}(W_{t_i})$, where we apply the map $\mathrm{Log}_{t_i} : \mathbb{C}^* \to \mathbb{R}, x \to \log_{t_i}|x|$ coordinate-wise. In this case the set V_∞ encodes the topological way of how W_i approach some compactification of $(\mathbb{C}^*)^n$. For the moment, the particular choice of the compactification does not matter[3].

Example 1.1.2. If curves $C_i, i = 1, 2, \dots$ in $(\mathbb{C}^*)^2$ all have branches with asymptotic (s^k, s^l) with a local parameter $s \to 0$, then the tropical limit C of this family lies in $\mathbb{R}^2$, and C has the infinite leg (half-line) in the lattice direction $(-k, -l)$.

Besides, for i big enough the Bergman fan $B(W_i) := \lim_{t\to\infty} \mathrm{Log}_t(W_i)$ of W_i is equal to $\lim_{t\to\infty} \frac{1}{t}V$. The latter limit is obtained by contracting the compact part V_c of V, so the Bergman fan can be restored by V_∞. Note, that V came here with a particular immersion to $\mathbb{R}^n$.

[1] For the full definition of an abstract tropical variety, see [30] and [29].

[2] Basically, rational tropical varieties are the contractible ones. They are not well studied even in small dimensions. For example, there exist 3 dimensional cubic hypersurfaces which are not rational. It is not known whether we can see this tropically.

[3] For a fixed compactification, see the notion of sedentarity in [38] and [8], p.44.

Let us suppose that we have an algebraic map $f : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$, and f is in general position with respect to the family $\{W_i\}$, i.e. for each i big enough, the image $f(W_i)$ is birationally equivalent to W_i. Let V' be the tropical limit of the family $\{f(W_i)\}$. One can prove that V'_∞ differs from V_∞ by adding new half-planes and contracting other half-planes, see Section 1.1.1. These half-planes grow along the tropicalization of zeros and poles of f on W_i. This consideration suggests the ideas of *modification* and *tropical birational equivalence*. The name "modification" was borrowed from complex analysis, and tropical modification is sometimes called "tropical blow-up".

In Section 1.4.2 we see how the notion of modifications allows us to define the *category of tropical curves*. This category keeps track of birational isomorphism in the category of complex algebraic curves. See also §1.3, where making modifications for curves simplifies a proof to some extent.

Tropical geometry can be thought as studying of skeletas of analytifications of algebraic varieties. We can obtain a tropical variety V as the non-Archimedean amoeba of an algebraic variety W over a non-Archimedean field. This approach (see section §1.2.2) finally suggests the same idea of equivalence up to modification, because the analytification W^{an} should be thought as the injective limit of all "affine" tropical modifications (i.e. along only principal divisors) of V([34]). Berkovich proved that W^{an} retracts on a finite polyhedral complex, so V_0 is a deformation retract of W^{an}. Even better, the metric on W^{an} agrees with the metric on V for the case of curves[4] ([5]).

Connection between tropical geometry and analytic geometry leads to the questions of *lifting* or *realisability*, i.e. what could be the intersection of two varieties if we know the intersection of their tropicalizations? In fact, if their tropicalizations intersect transversally, it is relatively simple [32]. If the intersection is non-transverse, then we can lift *the stable intersection* of these tropical varieties [33],[**?**].

This raised the following question: to what extent the only condition for a divisor on a curve to be realizable as an intersection is to be rationally equivalent to the stable intersection (cf. [31], Conjecture 3.4)?

Tropical modification (as a *method*) helps dealing with such questions. It is known ([10], Lemma 3.15) that being rationally equivalent to the stable intersection is not enough. We consider other existing obstructions (in fact, equivalent to Vieta theorem) for what can happen in non-transverse tropical intersections, and prove, for that occasion, the tropical Weil reciprocity law by using the tropical momentum.

Consequently, modifications are used in tropical intersection theory ([36, 37]), to define the intersection product. Nevertheless, one must use modifications along non-Cartier divisors (Examples 1.1.37, 3.4.18 in [37], for moduli space of five points on rational curve) and even along non-realizable subvarieties – for a proof that they are non-realizable as tropical limits.

As we stated before, one should think that a tropical modification along X reveals asymptotical behaviour of objects near X. We can find an analogy in non-standard analysis: the tropical line is the hyperreal line, the modification at a point is an approaching this point with *an infinitesimal telescope*, see Fig. 1.5 and Section 1.2.2.

Given a surface with hyperbolic structure, we can make a puncture at x. This changes the hyperbolic structure and x goes, in a sense, to "infinity". A tropical curve can be obtained as a degeneration of hyperbolic structures, and making a puncture at x results as the modification at

[4]That should be true for varieties of any dimension, modulo integer affine transformations, but no proof has appeared yet. For the skeletas in higher dimensions see [13, 14]

the limit of x, see Section §1.2.1.

A modification can be described as a graph of a function, if we use the convention about multi-valued addition, brought in tropical geometry by Oleg Viro ([41]), see the next section.

The other applications of tropical modification as a *method* are following. Passing to tropical limit squashes a variety, and some local features become invisible. In order to reveal them back we can do a modification[5]. For example, modifications allow us to restore transversality between lines if we have lost it during tropicalization (§1.4.3), then it allows us to see (-1)-curves on del Pezzo surfaces ([35]). Methods of lifting non-transverse intersections leads us to use modifications in questions about singularities: inflection points – [10], singular points – [26]. As an example, we use modification in the study of singular points of order m (but obtain weaker results than in [17]).

1.1.1 Tropical modification via the graphs of functions

Consider two algebraic curves $C_1, C_2 \subset (\mathbb{C}^*)^2$ defined by equations $F_1(x,y) = 0, F_2(x,y) = 0$, respectively. We build the map $m_{F_2} : (x,y) \to (x,y,F_2(x,y)) \in (\mathbb{C}^*)^3$. The set $m_{F_2}((\mathbb{C}^*)^2)$ is the graph of F_2. Now the intersection $C_1 \cap C_2$ can be easily recovered as $m_{F_2}(C_1) \cap \{(x,y,0)\}$. For the complex curves this seems to be not very interesting, but during the tropicalization process the plane $(x,y,0)$ goes to infinity, and the intersection of tropical curves will be represented by certain rays going to infinity.

Namely, given two tropical curves $C_1, C_2 \subset \mathbb{T}^2$, we start with $C_{1,t}, C_{2,t}$ – two families of plane algebraic curves, which tropicalize to C_1, C_2, i.e. $C_1 = \lim \mathrm{Log}_t(C_{1,t}), C_2 = \lim \mathrm{Log}_t(C_{2,t})$, where we apply $\mathrm{Log}_t : \mathbb{C} \to \mathbb{T}$ coordinate-wise.

Define $F_2 = \lim \mathrm{Log}_t F_{2,t}$, where $F_{2,t}$ is the equation of $C_{2,t}$; so F_2 is the equation of C_2. Then,

Definition 1.1.3. *The tropical modification $m_{F_2}\mathbb{T}^2$ of $\mathbb{T}^2$ along C_2* is the limit of surfaces $S_t = \mathrm{Log}_t\{(x,y,F_{2,t}(x,y)) \in \mathbb{C}^3 | x,y, \in C^*\}$, i.e. $m_{F_2}\mathbb{T}^2 = \lim_t S_t$.

Definition 1.1.4. *The tropical modification $m_{F_2}C_1$ of C_1 along C_2* is the tropical limit of the curves $m_{F_{2,t}}(C_{1,t}) \subset m_{F_{2,t}}((\mathbb{C}^*)^2)$, i.e. $m_{F_2}C_1 = \mathrm{Log}_t m_{F_{2,t}}(C_{1,t})$.

Note, that the families $C_{1,t}, C_{2,t}$ are included in the data. For given tropical curves C_1, C_2 we can construct different families $C_{1,t}, C_{2,t}$ and the result $m_{F_2}C_1$ also can be different.

That is why a modification of a curve along another curve is rather *a method*. The strategy is the following: given two tropical curves, we lift them in Puiseux series (or present as limits of complex curves), then we construct the graph as above and tropicalize the result. Depending on the conditions we imposed on lifted curves (be smooth or singular, be tangent to each other, etc), we have a set of possible results for modification of one curve along the second, see examples below.

1.1.2 Multivalued addition

Remark 1.1.5. We can obtain the same object $m_{F_2}\mathbb{T}^2 = \lim_t S_t$ as the graph of the tropical function, using the following convention about multivalued addition. This method does not allow to use modification as a methods, it conveys only the properties of modification as a well-defined operation.

[5] so the metaphor "look in an infinitesimal microscope" grasps the essence.

Definition 1.1.6. [41]. Define tropical addition $+_{trop}$ and multiplication $\cdot_{trop}$ as follows:

- $a \cdot_{trop} b = a + b$,
- $a +_{trop} b = \max(a, b)$ if $a \neq b$, and
- $a +_{trop} a = \{x | x \leq a\}$.

We can say, equivalently, that the operation max is redefined to be multivalued in the case of equal arguments, i.e. $\max(a, a) = \{x | x \leq a\}$.

Now, if $F_2 = \lim \log_t F_{2,t}$ is the tropical function[6], defining C_2 (i.e. C_2 is the set on non-smooth points of the piece-wise linear function F_2), then the graph $\{(a, b, F_2(a, b))\}$ of $F_2 : \mathbb{T}^2 \to \mathbb{T}$ is exactly the modification $m_{F_2}\mathbb{T}^2$ of the tropical plane $\mathbb{T}^2$ along C_2.

Remark 1.1.7. There is a natural projection $p_{F_2} : m_{F_2}\mathbb{T}^2 \to \mathbb{T}P^2, (x, y, z) \to (x, y)$, which is one-to-one everywhere except C_2. Over C_2 it locally looks like contracting half-planes (*legs*) on lines.

Definition 1.1.8. A modification $m_{F_2}(C_1)$ of a plane tropical curve C_1 along a tropical curve C_2 is **a** tropical curve C_1' inside the tropical surface $m_{F_2}\mathbb{T}^2$ such that $p_{F_2}(C_1') = C_1$.

Remark 1.1.9. As it is stressed above, there is a choice for C_1'.

It means, that given **only** tropical curves C_1, C_2 it is often not possible to uniquely "determine" the image of C_1 after the modification along C_2. Still, as it is proved below (and equivalent to the tropical Weil reciprocity law), we know the sum of coordinates of all the legs of $m_{F_2}(C_1)$ going to minus infinity, i.e. for each connected component C of $C_1 \cap C_2$ we know the valuation of the product of intersections of C_1, C_2 which are tropicalized to C, just looking on behavior of C_1 and C_2 near C.

1.1.3 Examples.

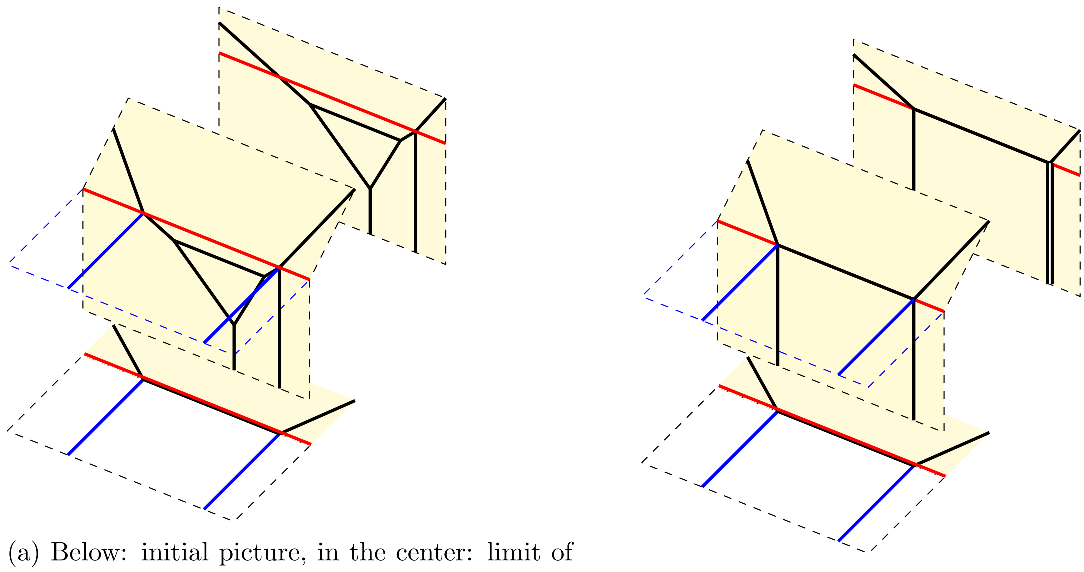
In this section we calculate examples of the modification, treated as a method. Do not be scared with these horrific equations which are reverse-engineered, starting from the pictures: all the calculations are quite straightforward.

First of all, we consider how modifications resolve indeterminacy that occured in a non-transversal intersection of tropical objects. That promotes the point of view that tropical modification is the same as adding a new coordinate.

In the second example, a modification helps to recover the position of the inflection point. Also, the power of tropical momentum is demonstrated. The tropical Weil theorem which shortens the combinatorial descriptions of possible results of a modification is proven in Section 1.3.

In the third example we study the influence of a singular point on the Newton polygon of a curve. The same method suits for higher dimension and different types of singularities, but nothing is yet done there, due to complicated combinatorics.

[6]If $F_{2,t} = \cdots + a_{ij}(t)x^i y^j + a_{kl}(t)x^k y^l + \ldots$, then $F_2 = \cdots +_{trop} \lim \log_t(a_{ij}) \cdot_{trop} ix \cdot_{trop} jy +_{trop} \lim \log_t(a_{kl}) \cdot_{trop} kx \cdot_{trop} ly +_{trop} \ldots$ because $x \cdot_{trop} x \cdot_{trop} x \cdots \cdot_{trop} x$ is ix if x appears i times.



(a) Below: initial picture, in the center: limit of classical modifications, behind: projection to the plane xz.

(b) Modification for the stable intersection

Figure 1.2: Example of a modification along a line

The forth example describes how to find all possible valuations of the intersections of a line with a curve, knowing only their stable tropical intersection – the answer is Vieta theorem. The same arguments may be applied for non-transversal intersections of tropical varieties of any dimension.

Example 1.1.10. Modification, root of big multiplicity, Figure 1.2a.

In this example we see two tropical curves with non-transverse intersection which hides tangency and genus. Consider the plane curve C, given by the following equation: $F(x,y)=0$,

$$F(x,y)=(x-t^{1/3})^3(x-t^{-2})+t^{-4}xy^2+(t^{-4}+2t^{-5})xy+(t^{-5}+t^{-6})x.$$

Its tropicalization[7] is the curve, given by the set of non-smooth points of

$$\text{Trop}(F)=\max(1,6+x,5+x+y,4+x+2y,5/3+2x,2+3x,4x).$$

We want to know what is the intersection of C with the line l given by the equation $y+t^{-1}=0$. Tropicalizations of C and l are drawn on Figure 1.2a, below. The intersection is not transverse, hence we do not know the tropicalization of $C\cap l$.

Then, let us consider the map $m_F:(x,y)\to(x,y,y+t^{-1})$. On Figure 1.2a, in the middle, we see the tropicalization of the set $\{(x,y,y+t^{-1})\}$ and the tropicalization of the image of C

[7]One can think that we have a family of curves C_t with parameter t and its tropicalization is the limit of amoebas $\lim_{t\to 0}\log_t(\{(x,y)|F(x,y)=0\})$, or that we have a curve C over Puiseux series and the non-Archimedean amoeba of $\sum a_{ij}x^iy^j=0$ is given by the set of non-smooth points of the function $\max_{ij}(val(a_{ij})+ix+jy)$. Both ways lead to the same result.

under the map m_F. $F(x,y)=0$ implies that for $z=y+t^{-1}$ we have $G(x,z)=0, G(x,z)=(x-t^{1/3})^3(x-t^{-2})+t^{-4}xz+t^{-4}xz^2$. That is the curve $C'=pr_{xz}m_F(C)$, given by the set of non-smooth points of $\max(1,4+x+y,4+x+2y,2+3x,4x)$, we see it on the projection onto the plane xz on the left part of Figure 1.2a. One can notice, that in order to have transversal intersection of non-Archimedean amoebas we did nothing else as a change of coordinates, .

Now, on C' we see a tropical root of multiplicity 3, i.e. z coordinate is zero at this point, and amoeba goes to minus infinity. That can mean tangency of order 3 between C and l (and that is indeed the case), and that this point is a singular point of C, but the curve C (according to criteria of [26] or, more generally [17]) has no singular points, though the tropicalization of C has an edge of multiplicity 2.

Thus, this new tropicalization restores the valuations of intersection. We see that the modification of the plane (i.e. amoeba of the set $\{(x,y,y+t^{-1})\}$) is defined, but in codimension one this procedure shows order of roots and more unapparent structures like hidden genus. It was close to intersection, but after change of coordinates it is visible on the picture, look at this cycle in the amoeba of C'.

Remark 1.1.11. Nevertheless, for a general choice of representative in Puiseux series for these two tropical curves, after modification we will have Fig. 1.2b, which represents stable intersection of the curves.

Example 1.1.12. Modification, inflection point, momentum map.

We consider a curve and its tangent line at an inflection point. Suppose, that the intersection of their tropicalizations is not transverse. How can we recover the presence of the inflection point?

We consider the curve with the equation $F(x,y)=0$ where

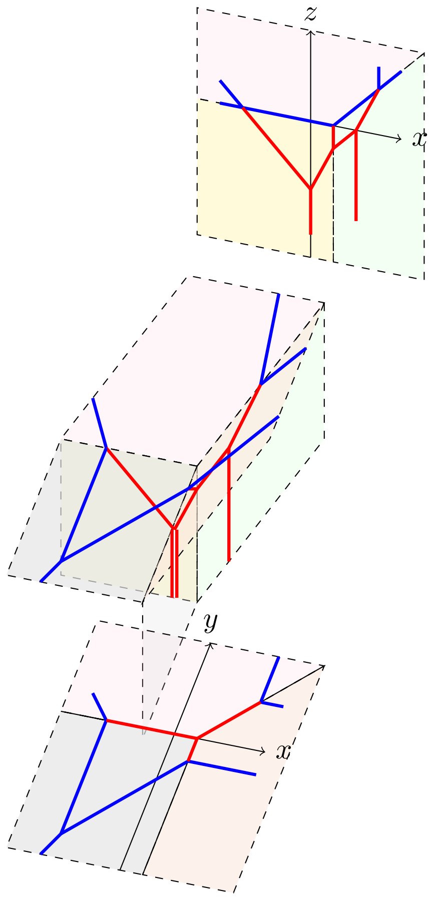
$$F(x,y)=y+t^{-3}xy+(t^{-1}+4+6t+4t^2+t^3)x^2+(-t^{-3}-3-t-t^2)xy^2+(t^{-2}-t^{-1}-2+t^2+t^3)x^2y+x^2y^2,$$

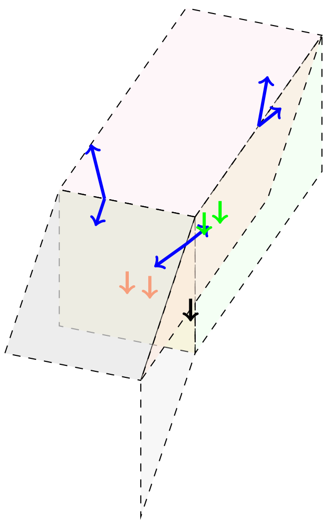
and the line with equation $y=1+tx$. The equation of the curve is chosen just in such a the way that its restriction on the line is $t^2(x-1)^3(x-t^{-1})$, i.e. the point $(1,1+t)$ is the inflection point of the curve and the line is tangent at it.

Tropicalization of the curve is given by the following equation: $\mathrm{Trop}(F)=\max(y,x+y+3,2x+1,2x+y+2,x+2y+3,2x+2y)$.

On the Fig. 1.4.1 we see the non-Archimedean amoeba of the image of the curve under the map $(x,y)\to(x,y,y-1-tx)$.

In order to find the x-coordinates of the possible legs we can apply the tropical momentum: see Fig.**??**. We take the vertex O of the tropical plane, and sum up the vector products $OX_i\times X_iY_i$ where X_iY_i are blue edges (which we know),beige, black, and green edges (which are projected to points on the initial curve). This sum is zero (see §1.3.1). Computation gives us $(-4,0,0)\times(-1,1,1)+(-4,0,0)\times(0,-1,0)+(0,-1,0)\times(-1,-1,0)+(0,-1,0)\times(1,0,1)+(2,2,2)\times(1,0,1)+(2,2,2)\times(0,1,1)+(\sum x-coord,0,0)\times(0,0,-1)+(0,\sum y-coord,0)\times(0,0,-1)+(\sum z-coord,\sum z-coord,0)\times(0,0,-1)=0$, i.e. $(1,-2,0)+(\sum y-coord+\sum z-coord,\sum x-coord+\sum z-coord,0)$ On the left picture we see that in fact we have one beige leg (which represent the inflection point), one green leg, and no black legs. But, since modification of a tropical curve C along a tropical





(a) Application of the tropical moment: we know the direction of the infinite blue rays emanating from the tropical curve, therefore an application of tropical momentum gives the sum of x- and y- coordinates of legs, going vertically to the bottom (these legs present exactly the intersection of two considered curves.)

Figure 1.3: Example of modification in the case of inflection point. The point $(0, 0)$ on the bottom picture is the tropicalization of the inflection point.

curve C' is not canonically defined[8], that is useful to know at least something about the legs of a modification.

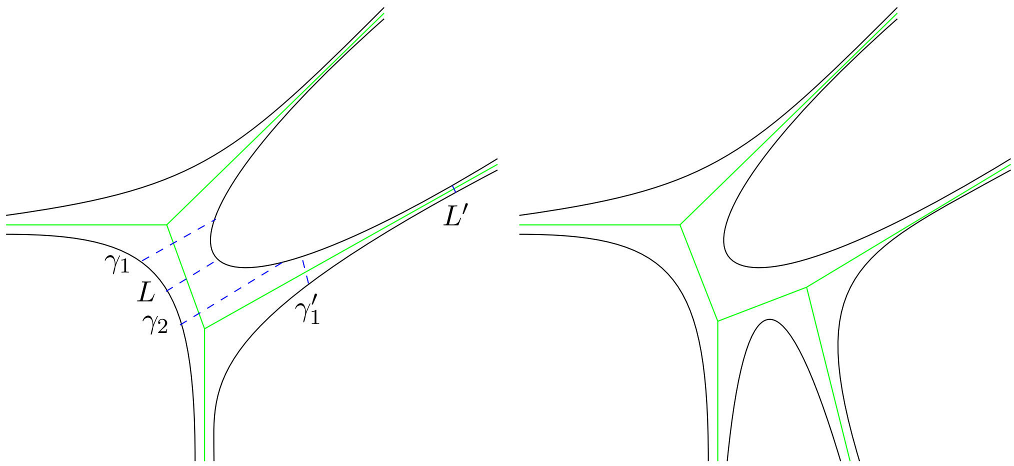
For example, the modification for thee tropical curves could differ from the initial curve by adding vertical legs at four vertices of the blue curve: this would correspond to stable intersection (which is always realizable in the sense that there exist curve in Puiseux series, such that, etc.)

1.2 Interpretations

1.2.1 Hyperbolic approach and moduli spaces

Consider a tropical curve C given as the limit of complex curves C_i. From the point of view of hyperbolic geometry, a modification at a point x of tropical curve C means just making a puncture x_i in C_i, with condition that $x_i \to x$. To explain this we need to know how to directly construct tropical curves via limits of abstract surfaces with hyperbolic structure on them, without any immersions[9].

So, for details how tropical geometry can be built on on the ground of hyperbolic geometry, see [25]. Here we briefly sketch the construction.



(a) Blue dashed lines γ_1, γ_2 depict the collar of geodesic L, γ_1' is a part of L''s collar.

(b) Modification subdivides old edge and adds a new edge of infinite lenght.

Figure 1.4: We draw the limits of hyperbolic surfaces, i.e. tropical curves. Modification adds a puncture to each curve in the family and a leg to the tropical curve.

The approach, proposed by L. Lang, uses the collar lemma ([11]). This lemma simply says that any closed geodesic of length l has a collar of width $\log coth(l/4)$ and what is more important, for different closed geodesics their collars do not intersect, see Fig. 1.4. That is also important that smaller geodesics have bigger collars (and, intuitively, a puncture has the collar if infinite width).

[8] If the intersection $C \cap C'$ is transverse, then the modification is uniquely defined.

[9] Usually people consider curves C_i in toric variety X and then they consider degeneration of complex structures on X.

Thus, given a family of curves C_i (of the same genus), we consider a fixed pair-of-pants decomposition by geodesics L_i. The tropical curve is constructed as follows: its vertices are in one-to-one correspondence with the pair-of-pants, each shared boundary component between two pairs-of-pants correspond to an edge of the tropical curve, and the collar lemma furnishes us with the length of the edges of the tropical curve as the logarithms (with base t, and $t \to \infty$ as the hyperbolic structure degenerates) of widths of the collars of L_i's. Compare this approach with [7].

What will happen if we make a puncture ? A puncture is the limit of small geodesic circles. Cutting out a disk with radius t^l add a leaf of finite length, as it is seen from the above description. Therefore, cutting out a point results in adding an infinite edge, i.e. a modification.

That explains why a permanent using of graphs for moduli space problems actually work ([24], then compare with tropical interpretation [20]). Tropical curves describe the part of boundary of a moduli space, and modification corresponds to marking a point (read [12] to see the hyperbolic view on moduli space problems), which are punctures from the hyperbolic point of view (see applications to moduli space of points [29]). Tropical differential forms are also defined in this manner while taking a limit of hyperbolic structure [30].

1.2.2 Berkovich spaces, non-standard analysis

Non-standard analysis appeared as an attempt to formalize the notion of "infinitesimally small" variables (see §4 [39] for nice and short exposition).

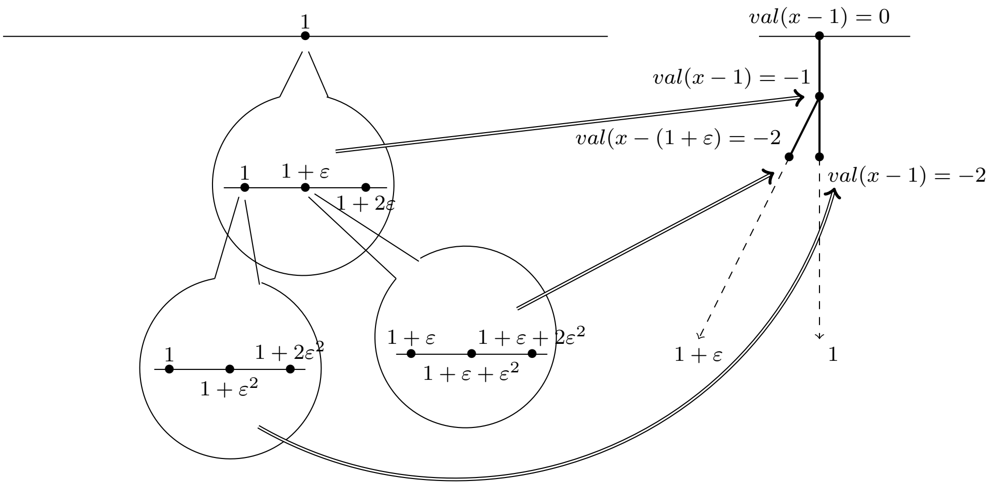


Figure 1.5: Similarity in the pictures of infinitesimal microscope (left) and tropical modification at points 1 and $1+\varepsilon$ (right).

There is an approach to tropical geometry via nonstandard analysis (cf. §1.4 [16]) and the following Fig. 1.5 shows that tropical modifications is similar to "infinitesimal microscope" for the hyperreal line in the terminology of [19], and this interpretation in computational sense is the same as for Berkovich spaces: doing modification at the point $x=1$ on a curve is adding a leg to the tropical curve, which ranges points according their asymptotical distance to $x=1$, i.e. $\mathrm{val}(x-1)$, these pictures are also similar to the hyperbolic ones.

It is worth noting that there are still no applications of this point of view, neither in tropical geometry, nor in non-standard analysis. Still, Berkovich spaces can be treated as a more modern version of non-standard analysis, and tropical modification has applications there.

We should say that an important feature of tropical geometry is that it erects a bridge from a very geometric things (hyperbolic geometry) to very discrete things as p-adic valuations and non-Archimedean analysis.

See Figure 1.1, the analytification of an elliptic curve on the left, the analytification of $\mathbb{P}^1$ on the right. Ends of leaves represent the norms with “zero” radius; . Berkovich spaces appeared as a wish to have an analytic geometry on discrete spaces. The analytification X^{an} of a variety X is the set of all seminorms on functions on X. Each point $x \in X$ defines such a seminorm by measuring the order of vanishing of a function at x, on Fig.1.1 these points are represented by the ends of leafs. For the sake of shortness, we refer the reader to a nice introduction in Berkovich spaces, with a bit of pictures [4],[40] and to [5] to see how it has been applied to tropical geometry (also, see on the page 7 in [5], using of log reminds hyperbolic approach). Also there exists Berkovich skeletas of analytifications, they correspond to the compact part V_c of a tropical variety, for example, for elliptic curves that will be a circle in both tropical and analytical cases, and its length is prescribed by j-invariant of a curve ([3]). The analytification of an elliptic curve is the injective limit of all modifications of its tropicalization, see Fig. 1.1.

1.3 Tropical Weil reciprocity law and momentum map

The aim of this section is to state a new example of the correspondence between tropical geometry and classical algebraic geometry.

Weil reciprocity law can be formulated as

Theorem 1.3.1. Let C be a complex curve and f, g are two meromorphic functions on C. Then $\prod\limits_{x\in C} f(x)^{ord_g x} = \prod\limits_{x\in C} g(x)^{ord_f x}$, where $ord_f x$ is the minimal degree in the Taylor expansion (in local coordinates) of the function f at a point x: $f(z) = a_0(z-x)^{ord_f x} + a_1(z-x)^{ord_f x+1} + \dots, a_0 \neq 0$.

Remark 1.3.2. The products in this theorem seems to be infinite, but all terms except finite number, are ones, because $ord_g x, ord_f x$ are zeros everywhere except finite number of points.

Remark 1.3.3. If f and g share some points in their zeros and poles sets, then we state this theorem as $\prod\limits_{x\in C} [f,g]_x = 1$ and define the term $[f,g]_x =$ “$\frac{f(x)^{ord_g x}}{g(x)^{ord_f x}}$” $= \frac{a_n^m}{b_m^n} \cdot (-1)^{ij}$ at a point x, where $f(z) = a_n(z-x)^n + \dots, g(z) = b_m(z-x)^m + \dots$ are Taylor expansions at the point x.

Khovanskii studied various generalizations of Weil reciprocity law and reformulated them in terms of logarithmic differentials [22],[21],[23]. The final formulation is for toric surfaces and seems like a tropical balancing condition, what is, indeed, the case. The symbol $[f,g]_x$ is related with Hilbert character and link coefficient, and is generalized by Parshin residues. Mazin treated them [27] in geometric context of resolutions of singularities[10].

[10]Unfortunately, tropical analog of this problem has no big interest: Parshin residues are destined for non-transversal intersection, in order to define local residue. That suggests that Mazin's resolution of singularities is a classical version of tropical modifications. Probably, tropical approach can repeat classical results, and better visualize the different types of non-transversality for higher-dimensional varieties.

In order to study what happens after a modification we consider a tropical version of Weil theorem. We need to define tropical meromorphic function and $ord_f x$, see also see [30].

Definition 1.3.4. A tropical meromorphic function f on a tropical curve C is a piece-wise linear function with integer slope on C. The points, where the balancing condition is not satisfied, are roots and zeroes and $ord_f x$ is by definition the defect in the balancing condition.

Theorem 1.3.5. Let C be a tropical curve and f, g are two meromorphic tropical functions on C. Then $\sum\limits_{x\in C} f(x)\cdot ord_g x = \sum\limits_{x\in C} g(x)\cdot ord_f x$.

Weil reciprocity is obvious for polynomials on tropical line $\mathbb{T}P^1$, essentially as in the complex case. Indeed, in the complex case the product of values of the polynomial $A\prod(x-a_i)$ at the roots of the polynomial $B\prod(x-b_i)$ is just $A^kB^l\prod(b_i-a_j)$. Almost the same we will have got for the symmetric calculation, and the difference will be killed by the common root of polynomials, at infinity. Word-by-word repetition proofs this case in tropical context.

For the general statement there are many proofs (and one can proceed by studying piece-wise linear functions on a graph), we give here the shortest[11] one, via so-called *tropical momentum.*

Suppose C is a planar tropical curve, its infinite edges are $E_1, \dots, E_k$ with directions given by primitive[12] integer vectors $v_1, \dots, v_n$. Suppose that each edge E_i has weight w_i and the direction of each v_i is chosen to be "to infinity", because there are two choices and for us the orientation of v_i will be important. Let A be a point on the plane. Let EB_i be the perpendicular from A to the line l_i containing E_i and $B_i\in l_i$.

Definition 1.3.6. (Due to G. Mikhalkin) Tropical momentum for the point A with respect to C is given by $\rho(A)=\sum det(v_i, AB_i)\cdot w_i$.

Lemma 1.3.7. For each point A in the plane $\rho(A)=0$

Proof. First of all, $\rho(A)$ does not depend on point A, because if we translate A by some vector u, each summand in $\rho(A)$ will change by $det(v_i, u)\cdot w_i$ and the sum of changes is zero because of the balancing condition. Therefore, if C is a curve consisting of only one vertex and some number of edges then $\rho(A)=0$, because we can place A in the vertex of this curve. This implies the lemma for any curve, because the moment for a curve is the sum of moments for all vertices (a summand corresponding to an edge between two vertices will appear two times with different signs.) □

Remark 1.3.8. It is easy to see that the same proof works for points and curves in $\mathbb{T}P^3$, if we define momentum map not as $det(v_i, AB_i)$, but as the vector product, $v_i\times AB_i$.

This remark was demonstrated in Example 1.1.12.

We carry on with a proof of the tropical Weil theorem. Two tropical meromorphic functions f, g on a tropical curve C define the map $C\to\mathbb{T}P^2, x\to(f(x), g(x))$. Now

$$\sum_{x\in X} f(x)\cdot ord_g x - \sum_{x\in X} g(x)\cdot ord_f x = \rho((0,0)) = 0.$$

[11] And which is using tropical modification

[12] i.e. non-multiple of another integer vector

Here we used tropical modification, because a priori, the image of tropical curve under the map $(f,g): C \to \mathbb{T}^2$ with f,g tropical meromorphic functions, is **not** a plane tropical curve: balancing condition is not satisfied near zeroes and poles of f and g, we need to add legs, going to infinity at these points. Namely, if $ord_f x = k, ord_g x = l$ and at least one of k,l is not zero, we should add the leg from this point in the direction (k,l).

Example 1.3.9. Consider a tropical meromorphic function f on an edge E of a tropical curve. Let a be a point where $ord_f a \neq 0$. Add to the tropical curve an infinite ray l, emanating from a and define the extension $\bar{f}$ of f on l as $f(a) - ord_f a \cdot x$ where x is a coordinate on l such that $x=0$ at a and then grows. At all others points of the tropical curve define $\bar{f} = f$. One can verify that $\bar{f}$ has no zero at a on the new tropical curve, instead it has a zero of order $ord_a f$ at the point at infinity on l.

Remark 1.3.10. If f,g come as limits of complex functions f_i, g_i, then the tropicalization of $\{(f_i(x), g_i(x)) | x \in C_i\}$ might not have this leg in the direction k,l but a tree of legs, growing from this point, with sum of slops be still equal (k,l). Nevertheless, because of the tropical momentum and the balancing condition, it does not matter.

1.3.1 Application of the tropical momentum to modifications.

On the Fig. 1.2b,1.4.1, *a priori* we know only the sum of directions of edges with endpoints on the modified curve. We know that there is no horizontal infinite edges (in these examples). Generally it is possible, if the intersection of two initial curves is infinite. Therefore by Weil theorem (or tropical moment map, they are the same) we know the sum of x-coordinates of vertical infinite edges. The sum of weights for vertical edges equals the sum of vertical components of the green edges on the Figure. As we will see in the next section, the tropical momentum provides basically the same information as Vieta theorem.

1.4 Examples of applications for a modification along a line or a quadric

1.4.1 Inflection points.

An inflection point of a curve is either its singular point, or a point where the tangent line has order of tangency at least 3. It was known before that the number of real inflection points is no more than $d(d-2)$ and the maximum is attainable. The question, attacked in [10] is *which topological types of planar real algebraic curves admits the maximal number of real inflection points?* Using classical way to construct algebraic curves – Viro's patchworking method – they construct examples, using local tropical pictures, for what they study local view of possible tropicalizations of inflection points. The property to be verified is tangency, but intersection of tropical curve with a tangent line at some point in most cases is not transversal and it is not visible what is the actual order of tangency. To see that, the authors do tropical modifications.

1.4.2 The category of tropical curves

See also [1, 2]. Namely, Mikhalkin defines the morphisms in the category of tropical curves as all the maps, satisfying the balancing and Riemann-Hurwitz conditions (see, for example [6]) and subject to the *modifiability* condition. It says that a morphism $f : A \to B$ of tropical curves A, B is modifiable if for any modification B' of B there exists a modification A' of A and a lift f' of f which makes the obtained diagram commutative.

The modifiability condition ensures that a morphism came as a degeneration of maps between complex curves. The proof is straightforward: firstly, after a number of modifications we make the map f' to be contracting no cycles. Then we construct a family of complex curves B_i such that $\lim B_i = B'$ in hyperbolic sense (see section 1.2.1). Finally, since f' is a covering, the complex curves A_i with $\lim A_i = A'$ are constructed as coverings over $f_i : A_i \to B_i$ where the combinatorics of f_i is prescribed by f'.

1.4.3 Collection of lines.

Which configuration of lines and points in $\mathbb{P}^2$ with given incidence relation are possible? That is a classical question and even for seemingly easy data the answer is not clear.

Definition 1.4.1. A $(4, d)$-net in $\mathbb{P}^2$ is four collections by d lines each of them, such that through any point of intersection of two lines from different collections exactly four lines pass, all from different collections.

It is not clear whether a $(4, d)$-net exists for $d \geq 5$. In [15] the authors proved, using tropical geometry, that there is no $(4, 4)$-net.

The one of the key ingredients is the following: if some net exists in the classical world, then it exists in tropical world. The problem is the following: if we have more than three tropical lines through a point on a plane, that the intersection will be non-transversal. But thanks to modifications we always can have transversal intersection, but probably in the space of bigger dimension. For that we just do modification along lines which has non-transversal intersection, after this modification, all intersections with it become transversal and the modified lines goes to infinity. Therefore, the following theorem, announced by the authors of [15] has a very clear idea: if for some combinatorial data of intersection of linear spaces can be realized in $\mathbb{P}^k$, then there is a tropical configuration of tropical linear spaces which realize the same data, in $T\mathbb{P}^n$ with $n \geq k$.

1.4.4 A point of big multiplicity on a planar curve.

In its most general form, this question could be formulated as follows: given a type of subvariety S in a bigger variety, how many singularities S may have? For example, is it possible for a surface of degree 4 in $\mathbb{C}P^4$ to have four double points and three two fold lines?

There are several reasons for tropical geometry could provide tools for such questions. We will demonstrate these tools in the case of curves, where the work has already done. Combinatorics of a tropical curves is encoded in the subdivision of its Newton polygon. In fact, a singular point of multiplicity m influences a part of the subdivision of area of order m^2, what is in accordance with the order of the number of linear conditions ($\frac{m(m+1)}{2}$) that a point of multiplicity m imposes on the coefficients of the curve's equation. For general treatment, see [17],[18].

In this section we will only demonstrate how to apply modification technic in this problem, though we will obtain weaker estimation – but still of order m^2.

The idea is the following: if a curve C has a point p of multiplicity m, then for each curve D, passing through p, the local intersection of C and D at p is at least m. The multiplicity of a local intersection of C and D can be estimated from above by studying the connected component, containing $\mathrm{Val}(p)$, of the stable intersection $\mathrm{Val}(C) \cap \mathrm{Val}(D)$ for the non-Archimedean amoebas of C and D.

So, the method: we take the polynomial F defining D, and use the fact that the image of C under the map $m_F : (x, y) \to (x, y, F(x, y))$ intersects the plane $z = 0$ with multiplicity at least m. In its turn, that implies **existence** of a modification of $\mathrm{Trop}(C)$ along $\mathrm{Trop}(D)$, which has a leg of multiplicity m going in direction $(0, 0, -1)$, exactly under the point $\mathrm{Val}(p)$. The latter modification is obtained just by taking the non-Archimedean amoeba of $m_F(C) \subset m_F(\mathbb{P}^2)$.

Now we reduce the problem for its combinatorial counterpart: is it possible for two given tropical curves, that after the modification along the second, the first curve will have a leg of multiplicity m, which projects exactly on the given point $\mathrm{Val}(p)$? After some work with intrinsically tropical objects, we will get an estimate of this point's influence on the Newton polygon of the curve.

We are not going to consider this problem in the full generality, so we will have a close look at the simplest interesting example. Suppose that $\mathrm{Val}(p)$ is inside some edge E of the tropical curve $\mathrm{Trop}(C)$ and this edge is horizontal. Let $\mathfrak{E}$ be the stable intersection of $\mathrm{Trop}(C)$ and the horizontal line; clearly $E \subset \mathfrak{E}$.

Suppose that p is of multiplicity m for C. Let us take a line D through p, whose non-Archimedean amoeba contains $\mathrm{Val}(p)$ inside its vertical edge. Clearly the intersection $\mathrm{Val}(C) \cap \mathrm{Val}(D)$ is one point, and of multiplicity m. That immediately implies that the weight of E is at least m. Hence the lattice length of $d(E)$ is at least m.

If we consider the modification along the horizontal line, then the contribution to the edge of direction $(0, 0, -1)$ consists of the horizontal components of the edges which intersect $\mathfrak{E}$ at exactly one point.

What to do if there is a rational component through $\mathrm{Val}(p)$? We do the modification along the horizontal line L. If a part of the curve goes to the minus infinity, that means that we can divide the equation of F by some degree of L. That affects the Newton polygon of C in a manageable way. The components which do not go to minus infinity do not contribute to the singularity.

Now, let's compute sum of areas of faces corresponding to the singular point p. By that, we mean the sum of areas of $d(V)$ where V runs over all vertices on E. It can be possible that to one singular point correspond more than two faces, if edge with the singular point has an extension.

First of all, we consider the simplest case.

Proposition 1.4.2. Suppose an horizontal edge E contains a point $\mathrm{Val}(p)$ of multiplicity m. Suppose that on the dual subdivision of the Newton polygon the vertical edge $d(E)$ is dual to E. Let the endpoints of E be A_1, A_2 and two faces $d(A_1), d(A_2)$ adjacent to $d(E)$ have no other vertical edges. Therefore the sum of widths of the faces $d(A_1), d(A_2)$ is at least m, so their total area is at least $m^2/2$.

Proof. Let L be a tropical line containing E and with vertex not coinciding with the endpoints of E. Making the modification along the line l we see that the sum S of vertical components of edges going upward from A_1, A_2 equals the sum S of the y-components of them.

Then, the sum of vertical components of edges going downward equals S by the balancing condition for tropical curves. Sum of y-components of edges in the vertex v is exactly the width in the $(1,0)$ direction of the dual to v face $d(v)$ in the Newton polygon. Let us look at the dual picture in the Newton polygon. Two faces $d(A_1), d(A_2)$ adjacent to the vertical edge have the sum of width in the $(1,0)$ direction at least m, $d(E_1)$ has length m, so the sum of the areas of $d(A_1), d(A_2)$ is at least $m^2/2$. $\square$

Remark 1.4.3. Suppose a tropical curve has edges $A_1A_2, A_2A_3, \dots, A_{k-1}A_k$ and $A_1, A_2, \dots, A_k$ are situated on a horizontal interval $A_1A_k = \mathfrak{E}$. Suppose that p, point of multiplicity m, is on the edge A_sA_{s+1}. The method above doesn't work.

Explanation. Making a modification along **a line** containing A_1A_k in the horizontal ray we estimate only common width of faces corresponding to $A_1, A_2, \dots A_k$, which gives no good estimate for the sum of areas of $d(A_i)$.

The new idea is to make a modification along a quadric.

Lemma 1.4.4. In the hypothesis above the sum of areas of all faces $d(A_1), d(A_2), \dots, d(A_k)$ is at least $m/2 + m^2/4$.

Proof. Let a_i be the width of i-th face on the right of the egde dual to A_sA_{s+1}, b_i be the width of i-th face on the left, c_i be the length of i-th vertical edge on the right, d_i be the length of the i-th vertical edge on the left, $\sum\limits_{i=1}^{k} a_i = A_k, \sum\limits_{i=1}^{k} b_i = B_k$.

With the same calculations as above, making the modification along a piece of a quadric (see Figure??) with vertices on $A_{s-j}A_{s+1-j}$ and $A_{s+i}A_{s+1+i}$ we get $A_i + c_i + B_j + d_j \geq m$. Denote $\min\limits_i (c_i + A_i) = A, \min\limits_i (d_j + B_j) = B$, so $A + B \geq m$.

Then, $c_i \geq A - A_i, d_j \geq B - B_j$. Sum S of areas can be estimated as

$$2S \geq (m + c_1)A_1 + \sum (A_{i+1} - A_i)(c_i + c_{i+1}) + (m + d_1)B_1 + \sum (B_{i+1} - B_i)(d_i + d_{i+1})$$

$$2S \geq (m+A-A_1)A_1 + \sum (A_{i+1}-A_i)(A-A_i+A-A_{i+1}) + (m+B-B_1)B_1 + \sum (B_{i+1}-B_i)(B-B_i+B-B_{i+1}) \geq$$

$$\geq A_1(m-A) + A^2 + B_1(m-B) + B^2 \geq m + m^2/2.$$

So, $S \geq m/2 + m^2/4$.

1.4.5 Difference between stable intersection and any other realizable intersection

After examples considered, one may ask if the only obstruction for a modification is the tropical momentum theorem? As we will see in this section, not at all.

Let us start with a variety $X' \subset \mathbb{K}^n$ and a hypersurface $F' \subset \mathbb{K}^n$ and their non-Archimedean amoebas $X, F \subset \mathbb{T}^n$. We suppose that the intersection of X with a tropical hypersurface F is not transverse. We ask: how does the non-Archimedean amoeba of of intersections of $X' \cap F'$ looks like?

First of all, as a divisor on X (or F) it should be rationally equivalently to the stable intersection of X and F, as it shown for the case of curves in [31]. In the general case in follows from the results of this section.

It it easy to find some additional necessary conditions. Let us restrict on X' the equation g of F', and take the valuations of all these objects X', F', g. We get some function $\text{Trop}(g)$ whose behavior on a neighborhood of $X \cap F$ is fixed but its behavior on F is under the question.

Definition 1.4.5. Let X be an abstract tropical variety and $\iota : X \to \mathbb{T}^n$ be its realization as an tropical subvariety of $\mathbb{T}^n$. Let f is a tropical function on $\mathbb{T}^n$. We define the pull-back of $\iota^*(f)$ to X as $f \circ \iota$. We call $\iota^*(f)$ *frozen* at a point $p \in X$ if f is smooth at $\iota(p)$.

Note that in general, the slopes of f along $\iota(X)$ does not coincide with slopes of $\iota^*(f)$ on X. From now on we consider tropical functions which have frozen points, the motivation is explained in the following defintition.

Definition 1.4.6. A principal divisor P on an abstract tropical variety X is called *positively* equivalent to a principal divisor Q, which is defined by a rational function f, if P can defined by a rational function h, which satisfies $h \le f$ and $h = f$ at the points where f is frozen.

Remark 1.4.7. As it is easy to see, the fact of positively equivalence depends only on P, Q, and does not depend on particular choice of f, h as long as the sets of frozen points for f and h coincide.

Now we prove the following theorem whose proof consists only in a reformulation of the statement on the language of tropical modifications.

Theorem 1.4.8. For an abstract tropical variety X, $\iota : X \to \mathbb{T}^n$ and a tropical hypersurface $F \subset \mathbb{T}^n$, given by a tropical function f, the pullback of the divisor of the stable intersection of $\iota(X)$ and F is given by $\iota^*(f)$. Furthermore, if F' and X are such that $\text{Trop}(F') = F$, F' is given by an equation $g = 0$, and $\text{Trop}(X') = X$, then the pullback of $\text{Trop}(F' \cap X')$ is positively equivalent to the divisor of $\iota^*(f)$.

Proof. Let us make the modification of $\mathbb{T}^n$ along F. Look at the image $m_f(X)$ of X under this map. The natural projection $m_f(X) \to X$ can be interpreted as a function on X. Note that this function is exactly $\iota^*(f)$. Now we look at the tropicalization of the restriction $g|_X$ of the equation of F' on X'. Clearly, $\text{Trop}(g|_X)$ coincides with f at the points where f is smooth. At other points, the graph of $\text{Trop}(g|_X)$ belongs to $m_F(\mathbb{T}^2)$. Therefore the pullback of $\iota^*(\text{Trop}(g|_x))$ is at most $\iota^*(f)$ everywhere, and $\iota^*(\text{Trop}(g|_x)) = \iota^*(f)$ at the points where $\iota^*(f)$ is frozen. So, the divisor of $\iota^*(\text{Trop}(g|_x))$ on X is positively equivalent to the pullback of the stable intersection. The graph of $\text{Trop}(g|_X)$ can be less than the graph of F_X because when we substitute the points on X' to g, some cancellation can occur, but not in the general case. $\square$

Question 1. Is it true that for the case of rational curves the positive equivalence is the only restriction?

This reasoning can be applied to the intersection of any two tropical varieties, if one of them is a complete intersection. We restrict the equations of the second variety on the first, that give us a stable intersection, then we have a situation similar to Definition 1.4.6.

Question 2. The hypothesis is that for a rational variety (or at least rational curves), that is, contractible, this is the only condition.

Theorem 1.4.9. For a tropical line in $\mathbb{T}^2$ the positive equivalence is the only condition.

Remark 1.4.10. Already for elliptic tropical curves there are additional conditions.

Example 1.4.11. Difference between multiplicity leg and root For example, $1+(t^{-1}+t)x+(2t^{-1}+t^2+t^4)x^2+(t^3+2t^4)x^3+t^{-1}xy+2t^{-1}x^2y$ in intersection with the line $t^5x+y+1=0$. The same example can be constructed for the similar Newton polygon $(0,0)-(1,1)-(n,1)-(n+1,0)$.

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