QUANTUM INDICES OF REAL PLANE CURVES AND REFINED ENUMERATIVE GEOMETRY

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Abstract. We associate a half-integer number, called *the quantum index*, to algebraic curves in the real plane satisfying to certain conditions. The area encompassed by the logarithmic image of such curves is equal to π^2 times the quantum index of the curve and thus has a discrete spectrum of values. We use the quantum index to refine real enumerative geometry in a way consistent with the Block-Göttsche invariants from tropical enumerative geometry.

1. Introduction.

1.1. **Quantum index.** Geometry of real algebraic curves in the plane is one of the most classical subject in Algebraic Geometry.

It is easy to see that the logarithmic image $\mathrm{Log}(\mathbb{R}C^\circ) \subset \mathbb{R}^2$ of any real algebraic curve $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2 \subset \mathbb{R}^2$ under the map (2) bounds a region of finite area in $\mathbb{R}^2$ (see Figures 1, 2, 3 for some examples of $\mathrm{Log}(\mathbb{R}C^\circ)$ in degrees 1 and 2). Furthermore, this area is universally bounded from above for all curves of a given degree by the Passare-Rullgård inequality [19] for the area of amoebas.

E.g. if $\mathbb{R}C^\circ$ is a circle contained in the positive quadrant $(\mathbb{R}_{>0})^2$ then it bounds a disk $D \subset (\mathbb{R}_{>0})^2$, $\partial D = \mathbb{R}C^\circ$. The area of the disk D is

$$\int\limits_D dxdy = \pi r^2,$$

where r is its radius. Clearly, $\mathrm{Area}(D)$ may be arbitrarily large. In the same time it is easy to see that the area of $\mathrm{Log}\, D$ is

$$\int\limits_D \frac{dx}{x}\frac{dy}{y} < \pi^2$$

and thus stays bounded no matter how large is the radius r. In the same time it is clear that $\mathrm{Area}(\mathrm{Log}(D))$ can assume any value between 0 and π^2.

In this paper we impose the following conditions on an algebraic curve $\mathbb{R}C \subset \mathbb{RP}^2$ (in the main body of the paper it is also formulated for

other toric surfaces in place of $\mathbb{RP}^2$) so that such continuous behavior of the logarithmic area is no longer possible.

Namely, we assume that $\mathbb{R}C$ is an irreducible curve of type I (see subsection 2.1). Then according to [20] $\mathbb{R}C$ comes with a canonical orientation (defined up to simultaneous reversal in all components of $\mathbb{R}C$). This enables us to consider the signed area (with multiplicities) $\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C)$ bounded by $\operatorname{Log}(\mathbb{R}C^\circ) \subset \mathbb{R}^2$. Unless one of the two possible complex orientations of $\mathbb{R}C$ is chosen, $\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C)$ is only well-defined up to sign.

The paper starts from the following result asserting that if all coordinate intersections of $\mathbb{R}C$ are real or purely imaginary then $\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C)$ must be divisible by $\frac{\pi^2}{2}$ and thus cannot vary continuously. The number $k = \frac{\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C)}{\pi^2}$ is thus a half-integer number naturally associated with the curve. We call it *the quantum index* of $\mathbb{R}C$.

Theorem 1 (special case for $\mathbb{R}\Delta = \mathbb{RP}^2$)**.** *Let $\mathbb{R}C \subset \mathbb{RP}^2$ be a real curve of degree d and type I enhanced with a complex orientation. If $\mathbb{R}C$ has pure imaginary or real coordinate intersection then*

$$\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C) = k\pi^2$$

with $k \in \frac{1}{2}\mathbb{Z}$ and $-\frac{d^2}{2} \le k \le \frac{d^2}{2}$.

To our knowledge this classical-looking result is new even in the case $d = 2$. Meanwhile the special case of $d = 1$ is well-known. The identity $|\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C)| = \frac{\pi^2}{2}$ in the case of lines was used by Mikael Passare [18] in his elegant new proof of Euler's formula $\zeta(2) = \frac{\pi^2}{6}$. Another known special case of Theorem 1 is the case of the so-called *simple Harnack curves* introduced in [13]. As it was shown in [17] these curves have the maximal possible value of $|\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C)|$ for their degree (equal to $\frac{d^2}{2}\pi^2$). Simple Harnack curves have a number of geometric properties [13]. By now these curves have appeared in a number of situation outside of real algebraic geometry, in particular in random perfect matchings of bipartite doubly periodic planar graphs of Richard Kenyon, Andrei Okounkov and Scott Sheffield [9]. The quantum index of Theorem 1 can be interpreted as a measure of proximity of a real curve to a simple Harnack curve.

Half-integrality of the quantum index k may be explained through appearance of $2k$ as the degree of some map as exhibited in Proposition 3. In accordance with this interpretation Theorem 2 computes the quantum index through the degree of the real logarithmic Gauß map of $\mathbb{R}C$.

Theorem 3 studies the quantum index in the special case when $\mathbb{R}C$ is not only of type I, but also of *toric type I* (Definition 7). This condition implies that all coordinate intersections of $\mathbb{R}C$ are real. In this case the quantum index may be refined to the *index diagram* (Definition 9), a closed broken lattice curve $\Sigma \subset \mathbb{R}^2$ well-defined up to a translation by $2\mathbb{Z}^2$.

The broken curve Σ is an immersed multicomponent curve with each component corresponding to a component of the compactification $\mathbb{R}\bar{C}$ of $\mathbb{R}C^\circ$ defined by its Newton polygon Δ. The complex orientation of $\mathbb{R}C$ induces an orientation of the closed broken curve Σ so that we may compute the signed area $\operatorname{Area}\Sigma$ inside Σ which is a half-integer number as the vertices of Σ are integer.

Theorem 3 (simplified version)**.** *If $\mathbb{R}C \subset \mathbb{RP}^2$ is a real algebraic curve of toric type I enhanced with a choice of its complex orientation then its quantum index k coincides with* $\operatorname{Area}\Sigma$.

Each edge of Σ corresponds to an intersection of $\mathbb{R}\bar{C}$ with a toric divisor of the toric variety $\mathbb{R}\Delta$ corresponding to the Newton polygon Δ and thus to a side $E \subset \partial\Delta$. If this intersection is transversal then the corresponding oriented edge of Σ is given by the primitive integer outer normal vector $\vec{n}(E)$. More generally it is given by $\vec{n}(E)$ times the multiplicity of the intersection. This makes finding the index diagram Σ and thus the quantum index k especially easy at least in the case of rational curves with real coordinate intersections (cf. e.g. Figures 2 and 4).

The index diagram Σ can be viewed as a non-commutative version of the Newton polygon Δ: it is made from the same elements (the vectors $\vec{n}(E)$ taken $\#(E \cap \mathbb{Z}^2) - 1$ times) as $\partial\Delta$, but the real structure on $\mathbb{R}\bar{C}$ gives those pieces a cyclic order (in the case of connected $\mathbb{R}\bar{C}$) or divides these elements into several cyclically ordered subsets.

Recall that Mikael Forsberg, Mikael Passare and August Tsikh in [4] have defined the amoeba index map which is a locally constant map on $\mathbb{R}^2 \setminus \mathcal{A}$, the complement of the *amoeba* $\mathcal{A} = \operatorname{Log}(\mathbb{C}C^\circ)$ of the complexification $\mathbb{C}C^\circ$ of $\mathbb{R}C^\circ$. Each connected component of $\mathbb{R}^2 \setminus \mathcal{A}$ is associated a lattice point of the Newton polygon Δ.

For toric type I curves the formula (6) defines the real index map so that each connected component of the normalization $\mathbb{R}\tilde{C}^\circ$ or a solitary real singularity of $\mathbb{R}C^\circ$ acquires a real index which is a lattice point of the convex hull of the index diagram Σ. Theorem 4 computes the amoeba index map ind in terms of the linking number with the curve $\mathbb{R}C^\circ$ enhanced with the real indices.

1.2. **Refined real enumerative geometry in the plane.** The second part of the paper is devoted to applications of the quantum index of real curves introduced in this paper to refined enumerative geometry. According to the seminal vision of Lothar Göttsche [5] as well as his recent works in collaboration with Florian Block [1] (in the tropical geometry set-up) and with Vivek Shende [6] (in the set-up of algebraic topology of complex moduli spaces) the number of curves in surfaces may often be viewed as a q-number (a Laurent polynomial in q with positive integer coefficients invariant under $q \mapsto \frac{1}{q}$) rather than a bare integer number. The value at $q = 1$ is capable to recover the number of complex curves while the value at $q = -1$ should be capable to recover the number of real curves in the same enumerative problem. E.g. there are $q + 10 + q^{-1}$ many of rational cubic curves passing through 8 generic points in $\mathbb{RP}^2$. In the same time there are 12 curves over $\mathbb{C}$ and 8 curves over $\mathbb{R}$ (if we count real curves with the Welschinger sign [22]).

The quantum index allows us to obtain a refined enumeration of planar curves entirely within real algebraic geometry with the help of Theorem 5. Once again for simplicity we discuss only the special case of the projective plane here in the introduction while in the main body of the paper the theorem is formulated for other toric surfaces as well.

Recall that the space of rational curves of degree d in $\mathbb{RP}^2$ is $(3d-1)$-dimensional. Thus we expect a finite number of such curves if we impose on them $3d-1$ conditions. Let us choose a generic configuration of $3d-1$ points on the three coordinate axes of $\mathbb{RP}^2$ (the x-axis, the y-axis and the ∞-axis) so that each axis contains no more than d points: e.g. there are d generic points on the x- and y-axis and $3d-1$ generic points on the ∞-axis. The elementary generalization of the classical Menelaus theorem (see Figure 7) ensures that there is a unique $3d$th point on the ∞-axis such that any irreducible curve of degree d passing through our $3d-1$ points also passes through the $3d$th point. The resulting configuration of $3d$ points on the union $\partial\mathbb{RP}^2$ of three coordinate axes varies in a $(3d-1)$-dimensional family of *Menelaus configurations*.

Any irreducible rational curve passing through $\mathcal{P}$ is of toric index I and thus has a well-defined quantum index k (as well as a well-defined index diagram Σ). We define the square map $\mathrm{Fr}_2 : \mathbb{CP}^2 \to \mathbb{CP}^2$ by $\mathrm{Fr}_2(z_0 : z_1 : z_2) = (z_0^2 : z_1^2 : z_2^2)$. An irreducible rational curve $\mathbb{R}C \subset \mathbb{RP}^2$ such that $\mathrm{Fr}_2(\mathbb{R}C)$ passes through $\mathcal{P}$ is of type I and has purely imaginary or real coordinate intersections. Thus the quantum index k is well-defined.

In (20) we define $R_{d,k}(\mathcal{P})$ (here we write d instead of Δ since we restrict ourselves to the special case of $\mathbb{RP}^2$ in the introduction) as one quarter of the number of irreducible oriented rational curves $\mathbb{R}C \subset \mathbb{RP}^2$ of degree d and quantum index k such that $\mathrm{Fr}_2(\mathbb{R}C)$ passes through $\mathcal{P}$. Each curve $\mathbb{R}C$ here is taken with the sign (18) which is a modification of the Welschinger sign [22]. Note that such curves come in quadruples thanks to the action of the deck transformation of the 4-1 covering $\mathrm{Fr}_2\,|_{(\mathbb{R}^\times)^2} : (\mathbb{R}^\times)^2 \to (\mathbb{R}^\times)^2$. This is the reason for including $\frac{1}{4}$ in the definition of $R_{d,k}(\mathcal{P})$. The points of $\mathcal{P}$ contained in the closure of the positive quadrant $(\mathbb{R}_{>0})^2$ (positive points) correspond to real coordinate axes intersections of $\mathbb{R}C$, other (negative) points correspond to purely imaginary coordinate axes intersections.

The image of each component of $\mathbb{C}C \smallsetminus \mathbb{R}C$ under Fr_2 may be viewed as an open holomorphic disk F in $\mathbb{CP}^2$ with the boundary contained in the closure $L = \overline{(\mathbb{R}_{>0})^2}$ of the positive quadrant. The subspace $L \subset \mathbb{CP}^2$ is a *Lagrangian submanifold with boundary*. The positive points of $\mathcal{P}$ correspond to tangencies of ∂F and ∂L while the negative points of $\mathcal{P}$ correspond to intersections of the open disk F with $\partial\mathbb{RP}^2 \smallsetminus L$. Note that such viewpoint provides a connection with the viewpoint on refinement of enumerative geometry of Maxim Kontsevich and Yan Soibelman [10].

The number of negative points on three coordinate axes is given by $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_j \leq d$.

Theorem 5. *The number $R_{d,k,\lambda} = R_{d,k}(\mathcal{P})$ is invariant of the choice of $\mathcal{P}$ and depends only on d, k and λ.*

In particular, $R_{d,k}(\mathcal{P})$ depends only on d and k when all points of $\mathcal{P}$ are positive.

For a positive point $p \in \mathcal{P}$ the inverse image $\mathrm{Fr}_2^{-1}(p)$ consists of two points a positive point $p_+ \in \partial\mathbb{RP}^2$ and a negative point $p_- \in \partial\mathbb{RP}^2$. The condition $\mathrm{Fr}_2(\mathbb{R}C) \ni p$ is equivalent to the condition that $\mathbb{R}C$ passes through p_+ or p_-. Note that this invariance result depends on including into $R_{d,k}(\mathcal{P})$ both these possibilities. If we leave out only the curves passing through p_+ (or p_-) as in (21) then the resulting sum $\tilde{R}_{d,k}(\mathcal{P})$ is no longer invariant under deformations of $\mathcal{P}$. Nevertheless, a partial invariance result for $\tilde{R}_{d,k}(\mathcal{P})$ is provided by Theorem 6.

The generating function $R_d(\lambda) = \sum\limits_k R_{d,k,\lambda} q^k$ defined in (24) is a Laurent polynomial in $q^{\frac{1}{2}}$. As such it can be compared with the Block-Göttsche refined tropical invariants BG_d where each tropical curve is counted with the multiplicity equal to the product of q-numbers corresponding to the multiplicities of its vertices ([1],[7]). The last theorem

of the paper is an identity between $R_d = R_d(0)$ (for $\lambda_1 = \lambda_2 = \lambda_3 = 0$) and BG_d.

Theorem 7 (special case of $\mathbb{RP}^2$)**.**

$$R_d = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{3d-2} BG_d.$$

This theorem has a surprising corollary. As the number of irreducible rational complex curves $\mathbb{C}C \subset \mathbb{CP}^2$ of degree d passing through $\mathcal{P}$ coincides with $BG_d(1)$, this number is completely determined by R_d, the number accounting only for curves defined over $\mathbb{R}$. Note that for this purpose it is crucial to use the quantum refinement by q^k as for $q = 1$ we would have to divide by 0 (the value of $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{3d-2}$ at $q^{\frac{1}{2}} = 1$) to recover $BG_d(1)$.

2. Conventions and notations

2.1. **Real curves of type I and their complex orientation.** A real curve $\mathbb{R}C \subset \mathbb{RP}^2$ is given by a single homogeneous polynomial equation $F(z_0, z_1, z_2) = \sum_{j,k,l} a_{k,l} z_0^j z_1^k z_2^l = 0$, $j + k + l = d$, $a_{k,l} \in \mathbb{R}$. The locus $\mathbb{C}C \subset \mathbb{CP}^2$ of complex solutions of $F = 0$ is called the *complexification* of $\mathbb{R}C$. The normalization

$$\nu : \mathbb{C}\tilde{C} \to \mathbb{C}C \tag{1}$$

defines a parameterization of $\mathbb{C}C$ by a Riemann surface $\mathbb{C}\tilde{C}$. The antiholomorphic involution of complex conjugation conj acts on $\mathbb{C}C$ in an orientation-reversing way so that its fixed point locus is $\mathbb{R}C$. The restriction of conj to the smooth locus of $\mathbb{C}C$ lifts to an antiholomorphic involution $\tilde{\text{conj}} : \mathbb{C}\tilde{C} \to \mathbb{C}\tilde{C}$ on the normalization. We denote with $\mathbb{R}\tilde{C}$ the fixed point locus of $\tilde{\text{conj}}$. Clearly, $\nu(\mathbb{R}\tilde{C}) \subset \mathbb{R}C$. In this paper we assume that $\mathbb{C}C$ is irreducible, i.e. that $\mathbb{C}\tilde{C}$ is connected.

Following Felix Klein we say that $\mathbb{R}C$ is *of type I* if $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$ is disconnected. In such case it consists of two connected components S and $S' = \tilde{\text{conj}}(S)$ which are naturally oriented by the complex orientation of the Riemann surface $\mathbb{C}\tilde{C}$. We have $\mathbb{R}\tilde{C} = \partial S = \partial S'$, so a choice of one of these components, say S, induces the boundary orientation on $\mathbb{R}\tilde{C}$. The resulting orientation is called *a complex orientation of* $\mathbb{R}\tilde{C}$ and is subject to Rokhlin's complex orientation formula [20]. If we choose S' instead of S then the orientations of all components of $\mathbb{R}C$ will reverse simultaneously. Thus any orientaion of a component of $\mathbb{R}\tilde{C}$ determines a component of $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$.

2.2. **Toric viewpoint and reality of coordinate intersections.** The projective plane $\mathbb{CP}^2$ can be thought of as the topic compactification of the torus $(\mathbb{C}^\times)^2$. The curve $\mathbb{C}C \subset \mathbb{CP}^2$ is the closure of its toric part $\mathbb{C}C^\circ = \mathbb{C}C \cap (\mathbb{C}^\times)^2$. The complement $\partial\mathbb{CP}^2 = \mathbb{CP}^2 \smallsetminus (\mathbb{C}^\times)^2$ is the union of three axes: the x-axis, the y-axis and the ∞-axis. These axes intersect pairwise at the points $(1:0:0), (0:1:0), (0:0:1) \in \mathbb{RP}^2$. If the coefficients $a_{0,0}, a_{d,0}, a_{0,d}$ are non-zero then $\mathbb{C}C$ is disjoint from the intersection points of the axes. In the general case it is reasonable to consider other toric surfaces compactifying $(\mathbb{C}^\times)^2$, so that the closure of $\mathbb{C}C^\circ$ is disjoint from pairwise intersections of toric divisors.

Let us consider the (non-homogeneous) polynomial $f(x,y) = F(1,x,y)$ and its Newton polygon

$$\Delta = \mathrm{ConvexHull}\{(j,k) \in \mathbb{R}^2 \mid a_{j,k} \neq 0\}.$$

The fan dual to Δ defines a toric compactification $\mathbb{C}\Delta \supset (\mathbb{C}^\times)^2$. The toric divisors of $\mathbb{C}\Delta$ correspond to the sides of Δ. Their pairwise intersections correspond to the vertices of Δ and are disjoint from the compactification $\mathbb{C}\bar{C}$ of the curve $\mathbb{C}C^\circ$. We denote with $\partial\mathbb{C}\Delta \subset \mathbb{C}\Delta$ the union of toric divisors. Accordingly, we denote with $\mathbb{R}\Delta$ (resp. $\partial\mathbb{R}\Delta$, $\mathbb{R}C^\circ$, $\mathbb{R}\bar{C}$) the real part of $\mathbb{C}\Delta$ (resp. $\partial\mathbb{C}\Delta$, $\mathbb{C}C^\circ$, $\mathbb{C}\bar{C}$). E.g. we have $\mathbb{RP}^2 = \mathbb{R}\Delta$ for the triangle $\Delta = \mathrm{ConvexHull}\{(0,0),(1,0),(0,1)\}$ or a positive integer multiple of this triangle.

Let $\mathrm{Fr}_2 : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2$ be the map defined by $\mathrm{Fr}_2(x,y) = (x^2, y^2)$. This map extends to a map $\mathrm{Fr}_2^\Delta : \mathbb{C}\Delta \to \mathbb{C}\Delta$.

We call a point $p \in \mathbb{C}\Delta$ *purely imaginary or real* if $\mathrm{Fr}_2^\Delta(p) \in \mathbb{R}\Delta$. We say that a curve $\mathbb{R}C \subset \mathbb{RP}^2$ has *pure imaginary or real coordinate intersection* if every point of $\mathbb{C}C \cap \partial\mathbb{C}\Delta$ is purely imaginary or real.

2.3. **Logarithmic area and other numbers associated to a real curve of type I.** Let $\mathbb{R}C$ be a real curve of type I enhanced with a choice of a complex orientation. Consider the image $\mathrm{Log}(\mathbb{R}C^\circ) \subset \mathbb{R}^2$, where $\mathrm{Log} : (\mathbb{C}^\times)^2 \to \mathbb{R}^2$ the map defined by

$$\mathrm{Log}(x,y) = (\log|x|, \log|y|). \tag{2}$$

For a point $p \in \mathbb{R}^2 \smallsetminus \mathrm{Log}(\mathbb{R}C^\circ)$ we define $\mathrm{ind}(p) \in \mathbb{Z}$ as the intersection number of an oriented ray $R \subset \mathbb{R}^2$ emanating from x in a generic direction and the oriented curve $\mathrm{Log}(\mathbb{R}C^\circ)$ (this number can be considered as the linking number of p and $\mathrm{Log}(\mathbb{R}C^\circ)$).

Definition 1. The integral

$$\mathrm{Area}_{\mathrm{Log}}(\mathbb{R}C) = \int_{\mathbb{R}^2} \mathrm{ind}_{\mathbb{R}C}(x)dx$$

is called the *logarithmic area* of $\mathbb{R}C$.

This is the signed area encompassed by $\mathrm{Log}(\mathbb{R}C^\circ)$, where the area of each region of $\mathbb{R}^2 \smallsetminus \mathrm{Log}(\mathbb{R}C^\circ)$ is taken with the multiplicity equal to the linking number of $\mathrm{Log}(\mathbb{R}C^\circ)$.

Let $S \subset \mathbb{C}\tilde{C} \smallsetminus \mathbb{R}\tilde{C}$ be the component corresponding to the complex orientation of $\mathbb{R}\tilde{C}$. The intersection points $\nu(S) \cap (\mathbb{R}^\times)^2$ are the so-called *solitary real singularities* of $\mathbb{R}C^\circ$. The *multiplicity* of a solitary real singularity $p \in \nu(S) \cap (\mathbb{R}^\times)^2$ is the local intersection number of S and $(\mathbb{R}^\times)^2$ at p. Here the orientation of S is induced by the inclusion $S \subset \mathbb{C}\tilde{C}$, while the orientation of $(\mathbb{R}^\times)^2$ is induced by the covering $\mathrm{Log}\,|_{(\mathbb{R}^\times)^2} : (\mathbb{R}^\times)^2 \to \mathbb{R}^2$. In other words, the quadrants $\mathbb{R}^2_{>0}$ and $\mathbb{R}^2_{<0}$ are counterclockwise-oriented while the quadrants $\mathbb{R}_{>0} \times \mathbb{R}_{<0}$ and $\mathbb{R}_{<0} \times \mathbb{R}_{>0}$ are clockwise-oriented. *The toric solitary singularities number* $E(\mathbb{R}C) \in \mathbb{Z}$ is the sum of multiplicities over all solitary real singularities of $\mathbb{R}C^\circ$, i.e. the total intersection number of S and $(\mathbb{R}^\times)^2$ (enhanced with our choice of orientation).

The logarithmic Gauß map sends a smooth point of $\mathbb{R}C^\circ$ to the tangent direction of the corresponding point on $\mathrm{Log}(\mathbb{R}C) \subset \mathbb{R}^2$. This map uniquely extends to a map

$$\gamma : \mathbb{R}\tilde{C} \to \mathbb{RP}^1,$$

cf. [8], [13]. *The logarithmic rotation number* $\mathrm{Rot}_{\mathrm{Log}}(\mathbb{R}C) \in \mathbb{Z}$ is the degree of γ.

3. Quantum indices of real curves.

Theorem 1. *Let $\mathbb{R}C \subset \mathbb{RP}^2$ be a real curve of type I enhanced with a complex orientation. If $\mathbb{R}C$ has pure imaginary or real coordinate intersection then*

$$\mathrm{Area}_{\mathrm{Log}}(\mathbb{R}C) = k\pi^2$$

where $k \in \frac{1}{2}\mathbb{Z}$,

$$-\mathrm{Area}(\Delta) \le k \le \mathrm{Area}(\Delta)$$

and $k \equiv \mathrm{Area}(\Delta) \pmod 1$.

Note that as $\Delta \subset \mathbb{R}^2$ is a lattice polynomial, its area is a half-integer number.

Definition 2. We say that $k(\mathbb{R}C) = \frac{1}{\pi^2}\mathrm{Area}_{\mathrm{Log}}(\mathbb{R}C)$ is *the quantum index* of $\mathbb{R}C$.

If $\mathbb{R}C$ is a real curve of type I with pure real or imaginary coordinate intersection, but the complex orientation of $\mathbb{R}C$ is not fixed then its quantum index is well-defined up to sign.

The quantum index $k(\mathbb{R}C)$ can also be expressed without computing the logarithmic area.

Proposition 3. *The integer number* $2k(\mathbb{R}C)$ *coincides with the degree of the map*

$$2\operatorname{Arg} : \mathbb{C}C^\circ \smallsetminus \mathbb{R}C^\circ \to (\mathbb{R}/\pi\mathbb{Z})^2,$$

i.e. with the number of inverse images at a generic point of the torus $(\mathbb{R}/\pi\mathbb{Z})^2$ *counted with the sign according to the orientation. (In particular, this number does not depend on the choice of a point in* $(\mathbb{R}/\pi\mathbb{Z})^2$ *as long as this choice is generic.) Here the orientation of of* $\mathbb{C}C^\circ \smallsetminus \mathbb{R}C^\circ$ *is defined by the condition that it coincides with the complex orientation of* $\mathbb{C}C$ *on the component* $S \subset \mathbb{C}C^\circ \smallsetminus \mathbb{R}C^\circ$ *determined by the orientation of* $\mathbb{R}C$ *and is opposite to the complex orientation of* $\mathbb{C}C$ *on the component* $\operatorname{conj}(S) \subset \mathbb{C}C^\circ \smallsetminus \mathbb{R}C^\circ$. *The map* $2\operatorname{Arg}$ *is defined by* $2\operatorname{Arg}(x, y) = (2\arg(x), 2\arg(y))$.

We say that $\mathbb{R}\bar{C}$ is *transversal to* $\partial\mathbb{R}\Delta$ if for any $p \in \mathbb{R}C \cap \partial\mathbb{R}\Delta$ we have $\nu^{-1}(p) \subset \mathbb{R}\tilde{C}$ and the composition $\mathbb{R}\tilde{C} \to \mathbb{R}\bar{C} \subset \mathbb{R}\Delta$ is an immersion near $\nu^{-1}(p) \subset \mathbb{R}\tilde{C}$ which is transversal to $\partial\mathbb{R}\Delta$.

Theorem 2. *Let* $\mathbb{R}C$ *be a curve of type I with pure real or imaginary coordinate intersections such that* $\mathbb{R}\bar{C}$ *is transversal to* $\partial\mathbb{R}\Delta$. *Then*

$$k(\mathbb{R}C) = -\frac{1}{2}\operatorname{Rot}_{\operatorname{Log}}(\mathbb{R}C) + E(\mathbb{R}C).$$

If $\mathbb{R}\bar{C}$ is not transversal to $\partial\mathbb{R}\Delta$ then an adjustment of the right-hand side according to the order of tangency and the orientation of $\mathbb{R}C$ should be added to the formula of Theorem 2.

Example 4 (Simple Harnack curves). If $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ is a simple Harnack curve (see [13]) then $k(\mathbb{R}C) = \pm\operatorname{Area}(\Delta)$. Vice versa, if $k(\mathbb{R}C) = \pm\operatorname{Area}(\Delta)$ then $\mathbb{R}C^\circ$ is a simple Harnack curve, see [17]. This characterizes real curves of highest and lowest quantum index.

Example 5 (Quantum indices of real lines). Any real line is a curve of type I and has real coordinate intersections. The quantum index of a real line in $\mathbb{RP}^2$ disjoint from the points $(1:0:0), (0:1:0), (0:0:1)$ is $\pm\frac{1}{2}$ (depending on its orientation), see Figure 1. The quantum index of a line passing through exactly one of these points is 0.

Example 6 (Quantum indices of real conics). A smooth nonempty real conic is a curve of type I. Figure 2 depicts real conics in $\mathbb{RP}^2$ that intersect the coordinates axes in 6 real points.

Note that a circle in $\mathbb{R}^2$ intersects the infinite axis of $\mathbb{RP}^2$ at the points $(0:1:\pm i)$. Thus a circle intersecting the coordinate axes of $\mathbb{R}^2$

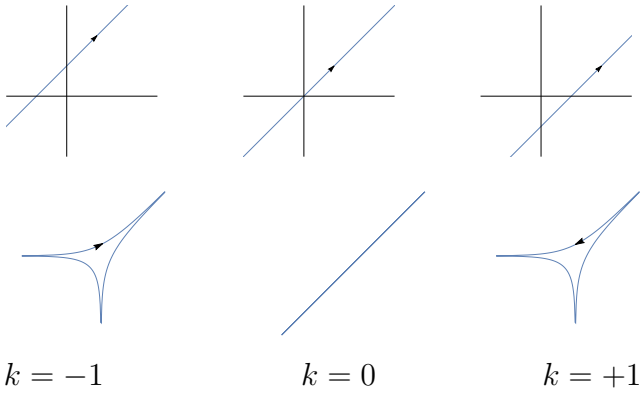

FIGURE 1. Oriented lines, their logarithmic images and quantum indices.

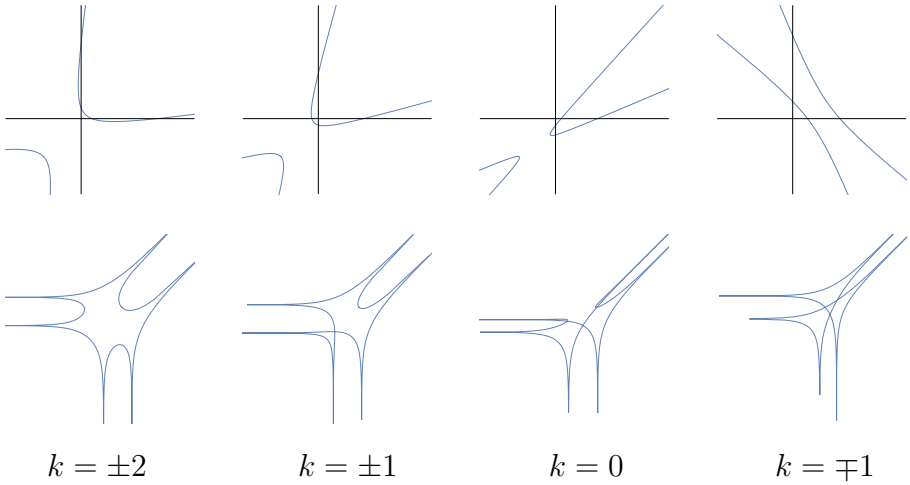

FIGURE 2. Projective hyperbolas, their logarithmic images and quantum indices.

in 4 real points has pure real or imaginary coordinate intersections, see Figure 3. A circle passing through the origin in $\mathbb{R}^2$ has quantum index $\pm\frac{1}{2}$. Otherwise, the quantum index of a circle is 0 or ± 1.

4. Toric type I curves: quantum indices and diagrams

4.1. **Definition of toric type I curves and their index diagrams.** Denote with $\mathbb{C}\tilde{C}^\circ \subset \mathbb{C}\tilde{C}$ the normalization of an algebraic curve $\mathbb{C}C^\circ \subset (\mathbb{C}^\times)^2$ and with $\mathbb{R}\tilde{C}^\circ$ its real part. The composition of the normalization and the inclusion map induces a map $\mathbb{C}\tilde{C}^\circ \smallsetminus \mathbb{R}\tilde{C}^\circ \to (\mathbb{C}^\times)^2$.

Definition 7. We say that an irreducible real algebraic curve $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ has *toric type I* if $\mathbb{R}C$ is of type I (in the sense of Klein) and the

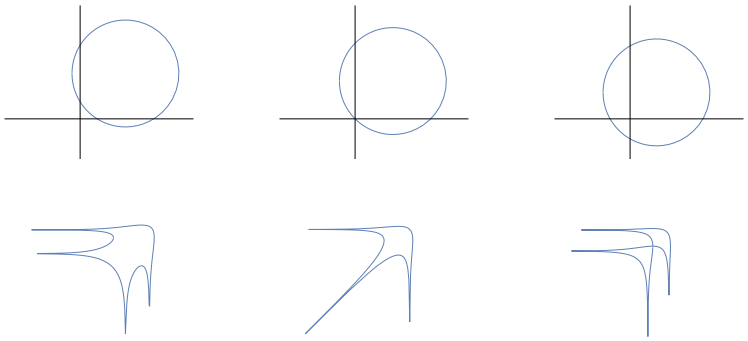

FIGURE 3. Circles, their logarithmic images and quantum indices.

induced homomorphism

$$H_1(\mathbb{C}\tilde{C}^\circ \smallsetminus \mathbb{R}\tilde{C}^\circ) \to H_1((\mathbb{C}^\times)^2) = \mathbb{Z}^2 \tag{3}$$

is trivial.

Each side $E \subset \Delta$ is dual to a unique primitive integer vector $\vec{n}(E) \subset \mathbb{Z}^2$ (which sits in the space dual to the vector space containing the Newton polygon Δ) oriented away from Δ. We refer to $\vec{n}(E)$ as *the normal vector to* $E \subset \partial\Delta$.

Proposition 8. *If* $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ *is of toric type I then*

$$\mathbb{C}\bar{C} \cap \partial\mathbb{C}\Delta \subset \mathbb{R}\bar{C} \subset \mathbb{R}\Delta.$$

In other words $\mathbb{R}C$ *has real coordinate intersection. Thus it has a well-defined quantum index.*

Proof. The homology class in $H_1((\mathbb{C}^\times)^2) = \mathbb{Z}^2$ of a small loop in $\mathbb{C}C^\circ$ around a point of $\mathbb{C}\bar{C} \cap \partial\mathbb{C}\Delta$ is a positive multiple of $\vec{n}(E)$ for a side $E \subset \Delta$. Therefore it is non-zero. Thus such loop must intersect $\mathbb{R}C^\circ$ if $\mathbb{R}C^\circ$ is of toric type I. □

Definition 9. An oriented broken line $\Sigma \subset \mathbb{R}^2$ is called *the index diagram* of the curve $\mathbb{R}C^\circ$ of toric type I enhanced with a choice of the complex orientation $S \subset \mathbb{C}\tilde{C} \smallsetminus \mathbb{R}\tilde{C}$ if the following conditions hold.

- The vertices Σ are in 1-1 correspondence with the components of $K^\circ \subset \mathbb{R}\tilde{C}^\circ$. We write $v = v(K^\circ)$.
- Each vertex $v(K^\circ) = (a, b) \in \mathbb{Z}^2$ is a lattice point in $\mathbb{R}^2$ such that K is contained in the $((-1)^a, (-1)^b)$-quadrant of $(\mathbb{R}^\times)^2$.
- The vertices v_1 and v_2 are connected with an oriented straight edge $e = v_1\vec{v}_2$ if the corresponding (arc) components K_1° and

K_2° are adjacent at a point $p_e \in \mathbb{R}\tilde{C}$ in the order defined by the complex orientation of $\mathbb{R}\tilde{C}$. Furthermore, we have

$$v_1\vec{v}_2 = m_e\vec{n}(E), \tag{4}$$

where $p_e \in \mathbb{C}E$ and m_e is the local intersection number of $\mathbb{C}\tilde{C}$ and $\partial\mathbb{C}\Delta$ at p_e.

- There exists a continuous map

$$\tilde{l} : \tilde{S} = (S \smallsetminus \partial\mathbb{C}\Delta) \cup \mathbb{R}\tilde{C}^\circ \to \mathbb{C}^2 \tag{5}$$

holomorphic on $\tilde{S} \smallsetminus \mathbb{R}\tilde{C}^\circ$ such that $e^{\pi\tilde{l}}$ coincides with the tautological map $\tilde{S} \to (\mathbb{C}^\times)^2$ while for every connected component $K^\circ \subset \mathbb{R}\tilde{C}^\circ$ we have

$$\operatorname{Im}\tilde{l}(K^\circ) = v(K^\circ).$$

Here both the exponent map and the imaginary part are understood coordinatewise.

We write $\Sigma = \Sigma(\mathbb{R}C) \subset \mathbb{R}^2$ and denote with $\bar{\Sigma} \subset \mathbb{R}^2$ the convex hull of Σ. The map

$$\alpha : K^\circ \mapsto \operatorname{Im}\tilde{l}(K^0) = v(K^\circ) \in \bar{\Sigma} \cap \mathbb{Z}^2 \tag{6}$$

is called the *real index map*.

Since the map $\tilde{l}$ is holomorphic its imaginary part $\operatorname{Im}\tilde{l}$ is harmonic and thus $\alpha(K^\circ) \in \bar{\Sigma}$.

Proposition 10. *Any curve $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ of toric type I admits an index diagram $\Sigma(\mathbb{R}C) \subset \mathbb{R}^2$ which is unique up to a translation by $2\mathbb{Z}^2$ in $\mathbb{R}^2$.*

Proof. Since $\mathbb{R}C$ is of toric type I the surface $\tilde{S} \subset (\mathbb{C}^\times)^2$ lifts under the exponent map $\mathbb{C}^2 \to (\mathbb{C}^\times)^2$. Rescaling this lift by π ensures that $(a,b)+2i\mathbb{Z}^2 \subset \mathbb{C}^2$ corresponds to the lift of the $((-1)^a,(-1)^b)$-quadrant in $(\mathbb{C}^\times)^2$. The diagram Σ is given by the union of the image of $\mathbb{R}\tilde{C}^\circ$ and the accumulation sets of the ends of $\tilde{S}$.

Each end e of $\tilde{S}$ corresponds to an intersection point with a toric divisor in $\mathbb{C}\Delta \supset (\mathbb{C}^\times)^2$. Changing coordinates in $(\mathbb{C}^\times)^2$ multiplicatively we may assume that this toric divisor is an x-axis so that the end wraps m_e times around the geodesic $\operatorname{Arg}(y) = \text{const}$ in $S^1 \times S^1$. The inverse coordinate change maps this geodesic to the one defined by $\vec{n}(E)$. □

Note that for each connected component $K \subset \mathbb{R}\tilde{C}$ (which is necessarily closed) with $K \cap \partial\mathbb{R}\Delta \neq \emptyset$ the formula (4) already determines the part $\Sigma(K)$ of the index diagram Σ corresponding to K up to a translation in $\mathbb{R}^2$. Indeed it suffices to choose arbitrarily $v(K^\circ)$ of an arc $K^\circ \subset K \smallsetminus \partial\mathbb{R}\Delta$ and proceed inductively.

Proposition 11. *If $\mathbb{R}C^\circ$ is a curve of toric type I then the broken line $\Sigma(K)$ resulting from inductive application of (4) is closed for any connected component K of $\mathbb{R}\tilde{C}$ with $K \cap \partial\mathbb{R}\Delta = \emptyset$.*

Conversely, suppose that $\mathbb{R}\tilde{C}$ is an M-curve (i.e. the number of its components is one plus the genus of $\mathbb{C}\tilde{C}$) with $\mathbb{C}\tilde{C} \smallsetminus \mathbb{R}\tilde{C} \cap \partial\mathbb{C}\Delta = \emptyset$ such that $\Sigma(K)$ is closed for all connected components $K \subset \mathbb{R}\tilde{C}$. Then $\mathbb{R}C^\circ$ has toric type I.

Proof. The first part of the statement is a corollary of Proposition 10. Conversely, for an M-curve $\mathbb{R}\tilde{C}$ each component of $\mathbb{C}\tilde{C} \smallsetminus \mathbb{R}\tilde{C}$ is a sphere with punctures corresponding to the components of $\mathbb{R}\tilde{C}$. The loops around the punctures are trivial if all $\Sigma(K)$ are closed. □

4.2. **Quantum index and toric complex orientation formula.** Let

$$\operatorname{Area}\Sigma(\mathbb{R}C) \in \frac{1}{2}\mathbb{Z} \tag{7}$$

be the signed area (with multiplicities) enclosed by $\Sigma(\mathbb{R}C)$.

Recall the definition of the linking number $\operatorname{lk}(p,\Sigma)$ between a point $p \in \mathbb{R}^2 \smallsetminus \Sigma$. The integer number $\operatorname{lk}(p,\Sigma)$ is defined as the intersection number $\operatorname{lk}_\epsilon(p,\Sigma)$ of the closed oriented curve $\Sigma(\mathbb{R}C) \subset \mathbb{R}^2$ and a ray R_ϵ in $\mathbb{R}^2$ emanating from p in a generic direction $\epsilon \in \tilde{\mathbb{RP}}^1$. Here $\tilde{\mathbb{RP}}^1 \approx S^1$ is the oriented projective line (the non-trivial double covering of $\mathbb{RP}^1$). Clearly, $\operatorname{lk}_\epsilon(p,\Sigma)$ does not depend on the choice of ϵ if $p \notin \Sigma$.

Consider now the case $p \in \Sigma$. Let $\epsilon \in \tilde{\mathbb{RP}}^1 \smallsetminus \tilde{\mathbb{QP}}^1$ be an irrational direction. The number $\operatorname{lk}_\epsilon(p,\Sigma)$ (where we ignore the intersection of R_ϵ and Σ at p itself) coincides with $\operatorname{lk}(p_\epsilon,\Sigma)$ for the point $p_\epsilon \in \mathbb{R}^2$ obtained by a very small translation of p in the direction of ϵ.

For each quadrant $Q = ((-1)^a,(-1)^b)\mathbb{R}^2_{>0}$ we define

$$\operatorname{lk}_\epsilon(Q,\Sigma) = \sum_{k_a,k_b\in\mathbb{Z}} \operatorname{lk}_\epsilon((a+2k_a, b+2k_b),\Sigma) \in \mathbb{Z}. \tag{8}$$

Any connected component $K \subset \mathbb{R}\tilde{C}$ disjoint from $\partial\mathbb{R}\Delta$ is contained in a single quadrant Q. Such K is a closed component of $\mathbb{R}\tilde{C}$. The image $\operatorname{Log}(K)$ is a closed oriented curve in $\mathbb{R}^2$. Let $\lambda(K) \in \mathbb{Z}$ be the rotation number of $\operatorname{Log}(K)$ that is the degree of the logarithmic Gauß map of $K \subset \mathbb{R}^2$. E.g. if $K \subset \mathbb{R}^2$ is a positively oriented embedded circle contained in the $(+,+)$- or $(-,-)$-quadrant (resp. in the $(+,-)$- or $(-,+)$-quadrant) then $\lambda(K) = 1$ (resp. $\lambda(K) = -1$). Any point of $S \cap Q$ is a real isolated singular point $p \in \mathbb{R}C^\circ$. We denote with $\lambda(p) \in \mathbb{Z}$ the intersection number of S (enhanced with the complex

orientation) and Q (enhanced with the orientation pulled back from the standard orientation of $\mathbb{R}^2$ by $\operatorname{Log}|_Q$).

If $K^\circ \subset \mathbb{R}\tilde{C}^\circ$ is a connected component (not necessarily compact) then the local degree of the oriented logarithmic Gauß map $\tilde{\gamma}|_K : K \to \tilde{\mathbb{RP}}^1$ at a point $\epsilon \in \tilde{\mathbb{RP}}^1$ may depend on the choice of ϵ. For $(a,b) \in \mathbb{Z}^2$ and $\epsilon \in \tilde{\mathbb{RP}}^1 \smallsetminus \tilde{\mathbb{QP}}^1$ we set

$$\lambda_\epsilon(a,b) = -\sum_K \lambda(K) + \sum_p \lambda(p), \tag{9}$$

where the sums are taken over all components $K^\circ \subset \mathbb{R}\tilde{C}^\circ$ with $\alpha(K^\circ) = (a,b)$ and all isolated singular points p of $\mathbb{R}C^\circ$ with $\operatorname{Im}\tilde{l}(p) = (a,b)$. The maximum principle for $\operatorname{Im}\tilde{l}$ implies that λ_ϵ can only take non-zero values in the convex hull $\bar{\Sigma}$ of Σ. We get the following statement.

Proposition 12. *The number $\lambda_\epsilon(a,b)$ does not depend on ϵ if $(a,b) \notin \Sigma$. If $(a,b) \notin \bar{\Sigma}$ then $\lambda_\epsilon(a,b) = 0$.*

For each quadrant $Q = ((-1)^a, (-1)^b)\mathbb{R}^2_{>0} \subset (\mathbb{R}^\times)^2$ we may take the sum

$$\lambda_\epsilon(Q) = \sum_{k_a,k_b \in \mathbb{Z}} \lambda_\epsilon(a+2k_a, b+2k_b). \tag{10}$$

The result is independent of the definition of the real index map as we take the degree of all the components K° and solitary singular points p contained in the quadrant Q without looking at its real indices.

Theorem 3. *If $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ is a real algebraic curve of toric type I enhanced with a choice of its complex orientation then*

$$k(\mathbb{R}C) = \operatorname{Area}\Sigma(\mathbb{R}C). \tag{11}$$

Furthermore, for each $(a,b) \in \mathbb{Z}^2$ and $\epsilon \in \tilde{\mathbb{RP}}^1 \smallsetminus \tilde{\mathbb{QP}}^1$ we have

$$\lambda_\epsilon(a,b) = \operatorname{lk}_\epsilon((a,b),\Sigma). \tag{12}$$

for each lattice point $(a,b) \in \mathbb{Z}^2$.

Corollary 13. *For a curve of toric type I with the index diagram Σ we have*

$$\lambda_\epsilon(Q) = \operatorname{lk}_\epsilon(Q,\Sigma) \tag{13}$$

for each quadrant $Q \subset (\mathbb{R}^\times)^2$.

The equality (13) may be viewed as *the toric complex orientation formula* for toric type I curves.

Corollary 14. *The total number of closed components of a curve* $\mathbb{R}\tilde{C}^\circ$ *of toric type I and its solitary real singularities is not less than the number of lattice points* $(a,b) \in \mathbb{Z}^2 \smallsetminus \Sigma$ *with* $\mathrm{lk}((a,b),\Sigma) \neq 0$.

Proof. If $\mathrm{lk}((a,b),\Sigma) \neq 0$ then by (12) $\lambda(a,b) \neq 0$ and thus there exists a closed component or a solitary real singularity of $\mathbb{R}C^\circ$ of real index (a,b). □

Example 15. All real rational curves which intersect $\partial\Delta$ in $\#(\partial\Delta \cap \mathbb{Z}^2)$ points (counted with multiplicity) have toric type I as $\mathbb{C}\tilde{C}^\circ \smallsetminus \mathbb{R}\tilde{C}^\circ$ is the disjoint union of two open disks. Therefore we may compute the quantum index of such curves with the help of Theorem 3.

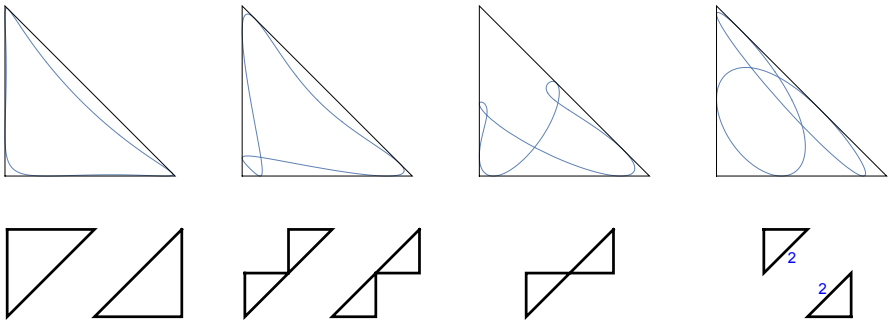

FIGURE 4. Squares of the real conics from Figure 2 and their diagrams Σ for both possible orientations.

Figure 4 depicts the images of the real conics from Figure 2 under Fr_2^Δ (reparameterized with the help of the moment map). Each of these conics may be oriented in two different ways producing two different diagrams. For one of these conics the diagrams for the two opposite orientations coincide. For the other three conics they are different.

Note that in the case of $\mathbb{C}\Delta = \mathbb{CP}^2$ the orientation can be uniquely recovered from the diagram as the edges correspond to the normals to Δ. E.g. the vertical edges are always directed downwards.

Remark 16. The diagram $\Sigma = \Sigma(\mathbb{R}C)$ may be viewed as a non-commutative version of the polygon Δ. Here the set of normal vectors is given an order.

Note that $-\operatorname{Area}\Delta \leq \operatorname{Area}\Sigma \leq \operatorname{Area}\Delta$ for any (possibly multicomponent) closed broken curve Σ whose oriented edges are normal vectors to Δ so that each side $E \subset \Delta$ contribute to $\#(E \cap \mathbb{Z}^2) - 1$ normal vectors (counted with multiplicity). Furthermore, we have $\operatorname{Area}\Sigma = \pm\operatorname{Area}\Delta$ if and only if $\Sigma \subset \mathbb{R}^2$ is a single-component broken curve coinciding with the polygon Δ itself rotated by 90 degrees (as we can represent the

primitive normal vector to a vector $(a,b) \in \mathbb{Z}^2$ by $(-b,a)$ identifying the vector space $\mathbb{R}^2$ with its dual).

Recall the notion of cyclically maximal position of $\mathbb{R}\bar{C} \subset \mathbb{R}\Delta$ in $\mathbb{R}\Delta$ from Definition 2 of [13]. It can be rephrased that $\mathbb{R}\bar{C}$ has a connected component K intersecting $\partial\mathbb{R}\Delta$ in $m = \#(\partial\Delta \cap \mathbb{Z}^2)$ points counted with multiplicity, and the cyclic order of the intersection points on K agrees with the cyclic order of the correwsponding normal vectors to $\partial\Delta$. This condition is equivalent to the equality (14).

The number of points in $(\Delta \smallsetminus \partial\Delta) \cap \mathbb{Z}^2$ is equal to the arithmetic genus g of $\mathbb{R}\bar{C}$. On the other hand, Corollary 14 implies that the total number of closed connected components of $\mathbb{R}C^\circ$ and its isolated real singular points is at least g. Thus all closed components of $\mathbb{R}C^\circ$ are smooth ovals and all singular points of $\mathbb{R}C^\circ$ are solitary double points, and the curve $\mathbb{C}C^\circ$ is a nodal M-curve. We get the following statement.

Corollary 17. *If $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ is a curve of toric type I with*

$$\operatorname{Area}\Sigma(\mathbb{R}C) = \pm\operatorname{Area}\Delta \tag{14}$$

then it is an M-curve whose only singularities are solitary real nodes.

Furthermore, the topological type of $(\mathbb{R}\Delta; \mathbb{R}\bar{C}, \partial\mathbb{R}\Delta)$ is obtained from the unique topological type of a smooth real curve $\mathbb{R}\bar{A}$ transversal to $\partial\mathbb{R}\Delta$ by contracting some of the ovals of $\mathbb{R}A^\circ$ to solitary double points and replacing some n-tuples of consecutive transversal intersection points of $\mathbb{R}\bar{A}$ and $\partial\mathbb{R}\Delta$ (sitting on the same toric divisor) with points of nth order of tangency.

Proof. The curve $\mathrm{Fr}_2^\Delta(\mathbb{C}C^\circ)$ also has toric type I. Its diagram is obtained from $\Sigma(\mathbb{R}C)$ by scaling both coordinates by 2, so the quantum index of $\mathrm{Fr}_2^\Delta(\mathbb{C}C^\circ)$ is equal to $\pm\operatorname{Area}(2\Delta)$. Corollary 14 implies that the only singularities of $\mathrm{Fr}_2^\Delta(\mathbb{C}C^\circ)$ are solitary real nodes, so that $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{C})$ does not have self-intersections. Therefore for each toric divisor $\mathbb{R}E$ the order of intersection points on $\mathbb{R}E$ and that on the component $K \subset \mathbb{R}\tilde{C}$ agree.

Let us look at the compact components of $\mathbb{R}C^\circ$. Their number and distribution among the quadrants of $(\mathbb{R}^\times)^2$ is determined by the lattice points of $\Sigma(\mathbb{R}C)$ and thus by Δ. Furthermore, Corollary 14 implies that in each quadrant of $(\mathbb{R}^\times)^2$ all ovals and solitary real nodes of $\mathbb{R}C^\circ$ have the same orientation. The complex orientation formula [20] ensures that these components cannot be nested among themselves and that they are arranged with respect to K so that their complex orientation is coherent with the complex orientation of K. □

Remark 18. The proof of Corollary 14 is applicable also for pseudoholomorphic, and even the so-called *flexible* (see [21]) real curves of toric

type I. Thus Corollary 17 may be considered as a further generalization of the topological uniqueness theorem for simple Harnack curves [13] from its version for pseudoholomorphic curves [2] recently found by Erwan Brugallé.

Example 19. The curve sketched on Figure 5 is isotopic to a smooth real quartic curve of type I. Namely, it can be obtained as a perturbation of a union of 4 lines. However, it is not an M-curve while its diagram coincides with the diagram of the simple Harnack curve of the same degree (i.e. the triangle with vertices $(0,0),(0,-4),(-4,-4)$ for one of the orientations). By Corollary 17 this curve is not of toric type I. In other words there is a cycle in $\mathbb{C}C^\circ \smallsetminus \mathbb{R}C^\circ$ that is homologically non-trivial in $H_1((\mathbb{C}^\times)^2)$. Also we can deduce this from the toric complex orientation formula (13).

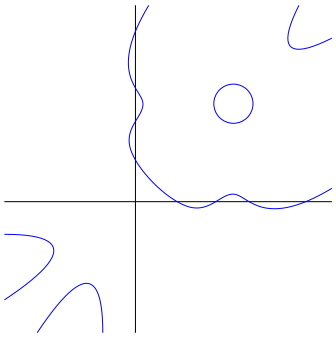

FIGURE 5. A quartic curve of type I, but not of toric type I.

4.3. **The real index map vs. the amoeba index map.** To forward the viewpoint of the index diagram Σ as a non-commutative version of the Newton polygon Δ it is interesting to compare the real index map (6) for toric type I curves and the *amoeba index map*

$$\mathrm{ind} : \mathbb{R}^2 \smallsetminus \mathrm{Log}(\mathbb{C}C^\circ) \to \Delta \cap \mathbb{Z}^2 \tag{15}$$

of Forsberg-Passare-Tsikh [4]. The map (15) is locally constant and thus it indexes the components K of the amoeba complement $\mathbb{R}^2 \smallsetminus \mathrm{Log}(\mathbb{C}C^\circ)$ by lattice points of Δ.

One obvious distinction between ind and α is that they take values in dual spaces: the Newton polygon Δ belongs to the dual vector space to $\mathbb{R}^2 = \mathrm{Log}(\mathbb{C}^\times)^2$. But thanks to the symplectic form $\omega((a,b),(c,d)) = ad - bc$, $a,b,c,d \in \mathbb{R}$ we have a preferred isomorphism between these spaces. Denote with $(a,b)^* = (b,-a)$ the corresponding identification.

As usual, we fix a complex orientation on $\mathbb{R}\tilde{C}$ and consider the corresponding component $S^\circ \subset \mathbb{C}\tilde{C}^\circ \smallsetminus \mathbb{R}\tilde{C}^\circ$. Let

$$p, p' \in \mathbb{R}^2 \smallsetminus \mathrm{Log}(\mathbb{C}C^\circ)$$

and $\gamma \subset \mathbb{R}^2$ be a smooth path between p and p'. We assume γ to be in general position with respect to $\mathbb{R}C^\circ$. The intersection number $\#(\gamma, \mathrm{Log}\,\mathbb{R}C) \in \mathbb{Z}$ is well-defined as $\mathrm{Log}(\mathbb{R}C) \subset \mathbb{R}^2$ is proper.

Proposition 20. *We have* $\#(\gamma, \mathrm{Log}\,\mathbb{R}C) = 0$.

Proof. The local degree of the map $\mathrm{Log}\,|_{S^\circ} : S^\circ \to \mathbb{R}^2$ changes along γ according to the intersection with $\mathbb{R}C^\circ$. Since the local degree at the endpoints of γ vanishes we have $\#(\gamma, \mathrm{Log}\,\mathbb{R}C) = 0$. □

Using the real index map (6) we may refine the intersection number $\#(\gamma, \mathrm{Log}\,\mathbb{R}C) = 0$ to

$$\#_\alpha(\gamma, \mathrm{Log}\,\mathbb{R}C^\circ) = \sum_{q \in \mathrm{Log}^{-1}(\gamma) \cap \mathbb{R}C^\circ} \#_q(\gamma, \mathrm{Log}\,\mathbb{R}C^\circ)\alpha(q) \in \mathbb{Z}^2 \tag{16}$$

Here $\#_q(\gamma, \mathrm{Log}\,\mathbb{R}C^\circ) = \pm 1$ is the local intersection number between γ and $\mathrm{Log}\,\mathbb{R}C^\circ$ and $\alpha(q) \in \mathbb{Z}^2$ is the real index of the component of $\mathbb{R}\tilde{C}^\circ$ containing the point q.

Theorem 4. *Let $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ be an algebraic curve of toric type I. For any two points $p, p' \in \mathbb{R}^2 \smallsetminus S^\circ \cup \mathbb{R}C^\circ$ and a generic smooth path $\gamma \subset \mathbb{R}^2$ connecting p and p' we have*

$$(\mathrm{ind}(p') - \mathrm{ind}(p))^* = \#_\alpha(\gamma, \mathrm{Log}\,\mathbb{R}C^\circ). \tag{17}$$

Proof. Consider the 1-dimensional submanifold

$$M = \mathrm{Log}^{-1}(\gamma) \cap \mathbb{C}C^\circ.$$

Its orientation is induced by that of γ (together with the complex orientation of $\mathbb{C}C^\circ$). Since γ is chosen generically M is a smooth 1-submanifold.

Any component of M disjoint from $\mathbb{R}C^\circ$ is null-homologous in $H_1((\mathbb{C}^\times)^2)$ as $\mathbb{R}C^\circ$ is a toric type I curve. A component $L \subset M$ intersecting $\mathbb{R}C^\circ$ consists of two arcs interchanged by conj. Suppose that the arc δ from S° is directed from q to q', $q, q' \in \mathbb{R}C^\circ$. We have

$$\mathrm{Im}\,\tilde{l}(q') - \mathrm{Im}\,\tilde{l}(q) = \alpha(q') - \alpha(q)$$

by (6) and therefore $[L] = \alpha(q') - \alpha(q) \in \mathbb{Z}^2 = H_1((\mathbb{C}^\times)^2)$ so that $[M] \in H_1((\mathbb{C}^\times)^2)$ is given by the right-hand side of (17).

By [13] we may interpret $\mathrm{ind}(p)$ as the linear functional on

$$H_1(\mathrm{Log}^{-1}(p)) = \mathbb{Z}^2$$

that associates to each oriented loop $N \subset \mathrm{Log}^{-1}(p)$ the linking number of N and the closure of the surface $\mathbb{C}C^\circ$ in $\mathbb{C}^2$. Suppose that $N' \subset \mathrm{Log}^{-1}(p')$ is a loop homologous to N in $(\mathbb{C}^\times)^2$ so that $N' - N = \partial P$ for a surface $P \subset (\mathbb{C}^\times)^2$. Then the difference of the linking numbers of N' and N coincides with the intersection number of P and $\mathbb{C}C^\circ$. Choosing the membrane P to be contained in $\mathrm{Log}^{-1}(\gamma)$ we identify the difference with the intersection number of $[N]$ and $[M]$ in

$$\mathbb{Z}^2 = H_1(\mathrm{Log}^{-1}(p)) = H_1((\mathbb{C}^\times)^2) = H_1(S^1 \times S^1).$$

□

5. Refined real enumerative geometry

5.1. **Invariance of real refined enumeration.** Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon. Let $E_j \subset \partial\Delta$, $j = 1, \dots, n$, be its sides of integer length $m_j = \#(E_j \cap \mathbb{Z}^2) - 1$. Let $\mathbb{C}E_j$ be the corresponding toric divisors. Let $\mathcal{P} = \{p_j\}_{j=1}^m$, $m = \sum_{j=1}^n m_j$, be a configuration of points on $\partial\mathbb{R}\Delta$ such that we have exactly m_j points on the toric divisor $\mathbb{R}E_j$.

We require that the first $m-1$ points in $\mathcal{P}$ are chosen on $\partial\mathbb{R}\Delta$ generically, and that the whole configuration $\mathcal{P}$ is subject to the condition of the Menelaus theorem (cf. discussion after Proposition 28). This means that the linear system defined by the divisor $\mathcal{P}$ on the (singular) elliptic curve $\partial\mathbb{C}\Delta$ is $\mathcal{O}(\Delta)$, i.e. any curve with the Newton polygon Δ passing through the point $\{p_j\}_{j=1}^{m-1}$ also passes through p_m. For any subset of $m-1$ points on $\mathcal{P}$ there exists a unique remaining point with this condition. We say that $\mathcal{P}$ is a generic configuration of m points on $\partial\mathbb{R}\Delta$ subject to *the Menelaus condition*.

The configuration $(\mathrm{Fr}_2^\Delta)^{-1}(\mathcal{P})$ consists of purely real or imaginary points. Thus any oriented real rational curve $\mathbb{R}\bar{C} \subset \mathbb{R}\Delta$ with the Newton polygon Δ such that $\mathrm{Fr}_2^\Delta(\mathbb{R}C)$ passes through $\mathcal{P}$ has the quantum index $k(\mathbb{R}C) \in \frac{1}{2}\mathbb{Z}$. (Note that rationality over the real numbers imply, in particular, that $\mathbb{R}\tilde{C}$ is non-empty and that $\mathbb{R}C$ is of type I.)

Define

$$\sigma(\mathbb{R}C) = (-1)^{\frac{m - \mathrm{Rot}_{\mathrm{Log}}(\mathbb{R}C)}{2}}. \tag{18}$$

Since the parities of $\mathrm{Rot}_{\mathrm{Log}}(\mathbb{R}C)$ and m coincides we have $\sigma(\mathbb{R}C) = \pm 1$.

Remark 21. Note that if $\mathbb{R}C^\circ$ is nodal then its toric elliptic number $E(\mathbb{R}C)$ has the same parity as the number of solitary nodes of $\mathbb{R}C^\circ$. Thus the Welschinger sign $w(\mathbb{R}C)$ (see [22]) coincides with $(-1)^{E(\mathbb{R}C)}$. Since the curve $\mathbb{R}\bar{C}$ intersects the union $\partial\mathbb{R}\Delta$ of toric divisors in m

distinct points and $\frac{m}{2} \equiv \mathrm{Area}(\Delta) \pmod 1$ by Pick's formula, we have

$$\sigma(\mathbb{R}C) = (-1)^{Area(\Delta)-k(\mathbb{R}C)} w(\mathbb{R}C) \tag{19}$$

by Theorem 2.

We define

$$R_{\Delta,k}(\mathcal{P}) = \frac{1}{4} \sum_{\mathbb{R}\bar{C}} \sigma(\mathbb{R}C), \tag{20}$$

where the sum is taken over all oriented real rational curves $\mathbb{R}\bar{C}$ with the Newton polygon Δ and $k(\mathbb{R}\bar{C}) = k$ such that $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{C}) \supset \mathcal{P}$. We have the coefficient $\frac{1}{4}$ in the right-hand side of (20) as the group of the deck transformations of $\mathrm{Fr}_2^\Delta : \mathbb{C}\Delta \to \mathbb{C}\Delta$ is $\mathbb{Z}_2^2$, so each curve $\mathbb{R}\bar{C}$ comes in four copies with the same image $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{C})$. (Alternatively, we can take a sum over different oriented images $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{C})$ without the coefficient $\frac{1}{4}$.)

We call a point in $\partial\mathbb{R}\Delta$ *positive* if it is adjacent to the quadrant $(\mathbb{R}_{>0})^2$ and *negative* otherwise. Note that $(\mathrm{Fr}_2^\Delta)^{-1}(p)$ consists of real points if p is positive and of purely imaginary points if it is negative. Let λ_j be the number of negative points in $\mathcal{P} \cap \mathbb{R}E_j$. Denote $\lambda = (\lambda_j)_{j=1}^n$.

Theorem 5. *The number $R_{\Delta,k}(\lambda) = R_{\Delta,k}(\mathcal{P})$ is invariant of the choice of $\mathcal{P}$ and depends only on Δ, k and λ.*

If all points of $\mathcal{P}$ are negative (i.e. $m_j = \lambda_j$) the number $R_{\Delta,k}(\lambda)$ is the number of real curves $\mathbb{R}\bar{C}$ contained in the positive quadrant $\mathbb{R}^2_{>0} \subset (\mathbb{R}^\times)^2$ and passing through *all* points of the purely imaginary configuration $(\mathrm{Fr}_2^\Delta)^{-1}(\mathcal{P})$. For a positive point $p \in \mathcal{P}$ a curve $\mathbb{R}\bar{C}$ should passes through one of the two real points in $(\mathrm{Fr}_2^\Delta)^{-1}(p)$.

Define

$$\tilde{R}_{\Delta,k}(\mathcal{P}) = \sum_{\mathbb{R}\bar{C}} \sigma(\mathbb{R}\bar{C}) \tag{21}$$

where the sum is taken over all curves $\mathbb{R}\bar{C}$ with the Newton polygon Δ and $k(\mathbb{R}C) = k$ passing through $\mathcal{P}$. Let

$$\Delta_d = \mathrm{ConvexHull}\{(0,0),(d,0),(0,d)\}. \tag{22}$$

We have $\mathbb{C}\Delta_d = \mathbb{CP}^2$.

Example 22. The curves $\mathbb{R}C$ with the Newton polygon Δ_2 are projective conics. In this case $n = 3$, $m_1 = m_2 = m_3 = 2$, $m = 6$. For any generic $\mathcal{P} \subset \partial\mathbb{CP}^2$ subject to the Menelaus condition we have a unique conic through $\mathcal{P}$. This gives us two oriented curves through $\mathcal{P}$ of opposite quantum index.

We may assume (applying the reflections in x and y axes if needed) that $\mathcal{P}$ contains a positive point in the x-axis and a positive point in the y-axis. As the positivity of the last point of $\mathcal{P}$ will be determined by the Menelaus condition we have 3 possibility for the non-decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. The possible values of $k(\mathbb{R}C)$ are listed in the following table, cf. Figure 2. In particular in this case $\tilde{R}_{\Delta_2,k}(\mathcal{P})$ is

λ_1	λ_2	λ_3	$k(\mathbb{R}C)$
0	0	0	± 2
0	0	2	± 1 or 0
0	1	1	± 1 or 1

TABLE 1. Quantum indices for conics

different for different configurations with the same λ for two last rows of Table 1. Thus the numbers $\tilde{R}_{\Delta,k}(\mathcal{P})$ may vary when we deform $\mathcal{P}$.

Define

$$\tilde{R}_{\Delta,even}(\mathcal{P}) = \sum_{k\in\frac{m}{2}+2\mathbb{Z}} \tilde{R}_{\Delta,k}(\mathcal{P}); \quad \tilde{R}_{\Delta,odd}(\mathcal{P}) = \sum_{k\in\frac{m}{2}+1+2\mathbb{Z}} \tilde{R}_{\Delta,k}(\mathcal{P}). \tag{23}$$

Theorem 6. *The numbers $\tilde{R}_{\Delta_d,even}(\mathcal{P})$ and $\tilde{R}_{\Delta_d,odd}(\mathcal{P})$ do not depend of $\mathcal{P}$ as long as d is even and all the points of $\mathcal{P}$ are positive (i.e. $\lambda = (0,0,0)$).*

5.2. **Refined real and refined tropical enumerative geometry.** We return to the study of the invariant $R_{\Delta,k}$ from Theorem 5.

Definition 23. The sum

$$R_\Delta(\lambda) = \sum_{k=-\operatorname{Area}(\Delta)}^{\operatorname{Area}(\Delta)} R_{\Delta,k}(\lambda) q^k \tag{24}$$

is called *the real refined enumerative invariant* of $(\mathbb{R}^\times)^2$ in degree Δ.

If $\lambda = 0$ (i.e. all $\lambda_j = 0$) then all points of $(\mathrm{Fr}_2^\Delta)^{-1}(\mathcal{P})$ are real. In such case we use notations $R_{\Delta,k} = R_{\Delta,k}(0)$ and $R_\Delta = R_\Delta(0)$.

Recall that the Block-Göttsche invariant is a symmetric (with respect to the substitution $q \mapsto q^{-1}$) Laurent polynomial with positive integer coefficients. This polynomial is responsible for the enumeration of the tropical curves with the Newton polygon Δ of genus g, passing through a generic collection of points in $\mathbb{R}^2$, see [7]. The expression BG_Δ defined by (43) may be viewed as the counterpart for the case when the tropical curves pass through a collection of m points on the boundary of the toric tropical surface $\mathbb{T}\Delta$ which are generic among those satisfying to the tropical Menelaus condition (38).

Theorem 7.

$$R_\Delta = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{m-2} BG_\Delta.$$

Corollary 24. *The value BG_Δ at $q = 1$ corresponding to the number of complex rational curves in $\mathbb{C}\Delta$ with the Newton polygon Δ passing through $\mathcal{P}$ is determined by R_Δ.*

Let us reiterate that R_Δ accounts only for curves in $(\mathbb{C}^\times)^2$ defined over $\mathbb{R}$.

5.3. **Holomorphic disk interpretation.** Recall that an orientation of a real rational curve $\mathbb{R}C$ defines a connected component $S \subset \mathbb{C}\tilde{C} \smallsetminus \mathbb{R}\tilde{C}$. Let D be the topological closure of the image of S in $\mathbb{C}\Delta$. The disk $D \subset \mathbb{C}\Delta$ is a holomorphic disk whose boundary $\partial D = \mathbb{R}C$ is contained in the Lagrangian subvariety $\mathbb{R}\Delta \subset \mathbb{C}\Delta$.

Let L be the topological closure of the quadrant $\mathbb{R}^2_{>0}$ in $\mathbb{C}\Delta$. Note that L is a Lagrangian subvariety of $\mathbb{C}\Delta$ with boundary. The image $\mathrm{Fr}_2^\Delta(D)$ is a holomorphic disk whose boundary is contained in L.

Thus the expression (24) may also be interpreted as a refined enumeration of holomorphic disks with boundary in L, passing through $\mathcal{P}$, and tangent to $\partial\mathbb{R}\Delta$ at the points of $\mathcal{P}$.

These disks are images under Fr_2^Δ of disks D with boundary in $\mathbb{R}\Delta$ and

$$\mathrm{Fr}_2^\Delta(D) \cap \partial\mathbb{C}\Delta = \mathcal{P} \subset \partial\mathbb{R}\Delta. \tag{25}$$

Let $\mathbb{C}\hat{\Delta}$ be the result of blowup of the toric variety $\mathbb{C}\Delta$ at $\mathcal{P}$. Let $\hat{L} = \overline{(\mathbb{R}^\times)^2} \smallsetminus \partial\hat{\mathbb{R}}\Delta$ where $\overline{(\mathbb{R}^\times)^2}$ is the topological closure of $(\mathbb{R}^\times)^2$ in $\mathbb{C}\hat{\Delta}$ and $\partial\hat{\mathbb{R}}\Delta$ is the proper transform of $\partial\mathbb{R}\Delta$ in $\mathbb{C}\hat{\Delta}$. Then a holomorphic disk D lifts to a holomorphic disk $\hat{D}$ with the boundary in the non-compact Lagrangian subvariety $\hat{L} \subset \mathbb{C}\hat{\Delta}$ without boundary. Furthermore, the Maslov index of $\hat{D}$ is 0. This construction appears in the Kontsevich-Soibelman wall-crossing approach [10].

6. Proofs

6.1. **Proof of Proposition 3 and Theorems 1, 2 and 3.** Consider the map $\mathrm{Arg} : (\mathbb{C}^\times)^2 \to (\mathbb{R}/2\pi\mathbb{Z})^2$ defined by

$$\mathrm{Arg}(z, w) = (\arg(z), \arg(w)) \tag{26}$$

and the map $2\,\mathrm{Arg} : (\mathbb{C}^\times)^2 \to (\mathbb{R}/\pi\mathbb{Z})^2$ obtained by multiplication of Arg by 2, cf. [16]. The involution of complex conjugation in $(\mathbb{C}^\times)^2$ descends to $(\mathbb{R}/\pi\mathbb{Z})^2$ as the involution $\sigma : (a, b) \mapsto (-a, -b)$. Denote with

$$P = (\mathbb{R}/\pi\mathbb{Z})^2/\sigma \tag{27}$$

the quotient space. The orbifold P is the so-called *pillowcase*. The projections of the four points $(0,0), (\frac{\pi}{2},0), (0,\frac{\pi}{2}), (\frac{\pi}{2},\frac{\pi}{2})$ form the $\mathbb{Z}_2$-orbifold locus of P (the corners of the pillowcase). All other points are smooth. (The orbifold P can be viewed as the real part of the Kummer surface.) We denote with $0 \in P$ the origin of P, i.e. the projection of $(0,0)$. Note that $(2\operatorname{Arg})^{-1}(0,0) = (\mathbb{R}^\times)^2$. Since the involution σ preserves the orientation, the standard volume form on $(\mathbb{R}/\pi\mathbb{Z})^2$ induces the standard volume form $d\operatorname{Vol}_P$ on the orbifold P.

Let $\mathbb{R}C$ be a real curve of type I with purely real or imaginary coordinate intersection. Consider the surface $S^\circ = S \smallsetminus \nu^{-1}(\partial\mathbb{C}\Delta)$, where S is the component of $\mathbb{C}\tilde{C} \smallsetminus \mathbb{R}\tilde{C}$ corresponding to the orientation of $\mathbb{R}C$ and ν is the normalization map (1). Denote with

$$\beta : S^\circ \to P \tag{28}$$

the composition of the map $2\operatorname{Arg}|_{S^\circ} : S^\circ \to (\mathbb{R}/\pi\mathbb{Z})^2$ and the projection $(\mathbb{R}/\pi\mathbb{Z})^2 \to P$.

Let $p \in P$ be a regular point of β. A point $q \in \beta^{-1}(p)$ is called *positive* (resp. *negative*) if locally near q the map β is an orientation-preserving (resp. orientation-reversing) open embedding. The difference between the number of positive and negative points in $\beta^{-1}(p)$ is called *the degree at* p. A priori, since β is a non-proper map, the degree at different points could be different.

Lemma 25. *We have*

$$\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C) = \int\limits_{S^\circ} \beta^* d\operatorname{Vol}_P\,. \tag{29}$$

Furthermore, the degree of β at a generic point of P is $2k(\mathbb{R}C)$.

Proof. Consider the form

$$\begin{aligned}\frac{dx}{x} \wedge \frac{dy}{y} = (d\log|x| + id\arg(x)) \wedge (d\log|y| + id\arg(y)) = \\ d\log|x| \wedge d\log|y| - d\arg(x) \wedge d\arg(y) + \\ i(d\log|x| \wedge d\arg(y) + d\arg(x) \wedge d\log|y|)\end{aligned} \tag{30}$$

on $(\mathbb{C}^\times)^2$. As it is a holomorphic 2-form, it must restrict to the zero form on any holomorphic curve in $(\mathbb{C}^\times)^2$. In particular, the real part of this form must vanish everywhere on S°, so that $d\log|x| \wedge d\log|y| = d\arg(x) \wedge d\arg(y)$ on S°, and thus (29) holds, cf. [12].

The smooth map $\beta : S^\circ \to P$ extends to a continuous map

$$\bar{\beta} : \bar{S} \to P \tag{31}$$

for a surface with boundary $\bar S \supset S^\circ$ such that $S^\circ = \bar S \smallsetminus \partial\bar S$. Each $p \in \mathbb{C}\bar C \cap \mathbb{C}E_j$ corresponds to a geodesic in $(\mathbb{R}/\pi\mathbb{Z})^2$ in the direction parallel to $\vec n(E_j)$ for a side $E_j \subset \partial\Delta$, cf. [16]. Since $\mathrm{Fr}_2^\Delta(p) \in \mathbb{R}\Delta$ the corresponding geodesic passes through two of the points $(0,0), (\frac{\pi}{2}, 0), (0, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{\pi}{2})$. The image of this circle in P is a geodesic segment connecting the corresponding $\mathbb{Z}_2$-orbifold point of P.

Thus $\bar\beta(\bar S)$ is a 2-cycle in P and it covers a generic point l times (counted algebraically), where l is a number independent on the choice of a generic point. But then $\int\limits_{S^\circ} \beta^* d\,\mathrm{Vol}_P = l\,\mathrm{Area}(P) = l\pi^2/2$. Therefore, $l = 2k(\mathbb{R}C)$. □

Note that this lemma implies Proposition 3.

Proof of Theorem 1. We have $k(\mathbb{R}C) \in \frac{1}{2}\mathbb{Z}$ since $2k(\mathbb{R}C)$ is the degree of β at a generic point of P by Lemma 25. Let $\tilde a \in (\mathbb{R}/\pi\mathbb{Z})^2$ be a generic point and $a \in P$ be the point corresponding to $\tilde a$. The inverse image $\beta^{-1}(a)$ consists of points of S° mapped to $\tilde a$ or $\sigma(\tilde a)$. If $2\,\mathrm{Arg}(p) = -\tilde a$ for $p \in S^\circ$ then $2\,\mathrm{Arg}(\mathrm{conj}(p)) = \tilde a$, where $\mathrm{conj}(p) \in \mathrm{conj}(S^\circ)$. Thus we have a 1-1 correspondence between sets $\beta^{-1}(a)$ and $R = (2\,\mathrm{Arg})^{-1}(\tilde a) \cap \mathbb{C}C^\circ$.

Consider the continuous involution $\mathrm{conj}_{\tilde a} : \mathbb{C}\Delta \to \mathbb{C}\Delta$ extending the involution of $(\mathbb{C}^\times)^2$ defined by $z \mapsto \tilde a\,\mathrm{conj}(\sigma(\tilde a)(z))$. Note that the fixed point locus of this involution in $(\mathbb{C}^\times)^2$ coincides with $(2\,\mathrm{Arg})^{-1}(\tilde a)$, cf. [12]. Note that

$$R \subset \mathbb{C}C^\circ \cap \mathrm{conj}_{\tilde a}(\mathbb{C}C^\circ) \tag{32}$$

while $R \smallsetminus (\mathbb{C}C^\circ \cap \mathrm{conj}_a(\mathbb{C}C^\circ))$ consists of pairs of points interchanged by the involution $\mathrm{conj}_{\tilde a}$. For generic $\tilde a$ the curve $\mathrm{conj}_{\tilde a}(\mathbb{C}C^\circ)$ is transverse to $\mathbb{C}C^\circ$, while $\mathrm{conj}_{\tilde a}(\mathbb{C}C^\circ) \cap \mathbb{C}C^\circ \cap \partial\mathbb{C}\Delta = \emptyset$.

Thus the number of points in R is not greater than $\#(\mathbb{C}C^\circ \cap \mathrm{conj}_{\tilde a}(\mathbb{C}C^\circ))$, while we have $\#(\mathbb{C}C^\circ \cap \mathrm{conj}_{\tilde a}(\mathbb{C}C^\circ)) = 2\,\mathrm{Area}(\Delta)$ by the Kouchnirenko-Bernstein theorem [11]. Thus the degree of β takes values between $-2\,\mathrm{Area}(\Delta)$ and $2\,\mathrm{Area}(\Delta)$. Also $\#(R) = 2\,\mathrm{Area}(\Delta)$. □

Proof of Theorem 2. Let us compute the degree of the map (31) at a generic point $a \in P$ close to the origin $0 \in P$. The set $\bar\beta^{-1}(0) \cap S^\circ$ contributes $2E$ to the degree of $\bar\beta$ as the intersection number gets doubled when we pass from $(\mathbb{R}/\pi\mathbb{Z})^2$ to P.

Note that the set $S_\mathbb{R} = \bar\beta^{-1}(0) \cap \partial\bar S$ can be thought of as the topological closure of $\mathbb{R}C^\circ$ in $\bar S$ by our assumption of transversality to $\partial\mathbb{R}\Delta$. Let $U \supset S_\mathbb{R}$ be a small regular neighborhood of $S_\mathbb{R}$. The image $\bar\beta(U)$ makes $\mathrm{Rot}_{\mathrm{Log}}(\mathbb{R}C)$ rotation around $0 \in P$. Thus $S_\mathbb{R}$ contributes to $-\mathrm{Rot}_{\mathrm{Log}}(\mathbb{R}C)$ to the degree of $\bar\beta$ equal to $2k(\mathbb{R}C)$. We have the

appearance of the negative sign since the basis composed of vectors v_1, v_2, iv_1, iv_2 is negatively-oriented in $\mathbb{C}^2$ whenever vectors v_1, v_2 are linearly independent over $\mathbb{C}$. Thus a positive rotation in $(i\mathbb{R})^2$ (and therefore also in P) corresponds to a negative contribution to the degree of $\bar{\beta}$. □

Proof of Theorem 3. The signed area of $\beta(S^\circ)$ coincides with the signed area of its lift to the universal covering space and thus with $\operatorname{Area}\Sigma(\mathbb{R}C)$ multiplied by π^2. For $(a,b) \in \mathbb{Z}^2$ and $\epsilon \in \tilde{\mathbb{RP}}^1 \smallsetminus \tilde{\mathbb{QP}}^1$ we consider a point p_ϵ obtained by a small translation of (a,b) in the direction of ϵ. A point of S° mapped to p_ϵ by the lift of β must correspond to a point of $\mathbb{R}C^\circ$ of realigned map (a,b) which is either singular or has β as the image of its logarithmic Gauß map. Summing up the contributions of all such points we get (12). □

6.2. **Invariance of the numbers $R_{\Delta,k}(\mathcal{P})$.**

Proof of Theorem 5. First we compute the dimension of the space of rational curves $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ with the Newton polygon Δ. Two coordinate functions on $(\mathbb{C}^\times)^2$ define two meromorphic functions on the Riemann surface $\mathbb{C}\tilde{C}$. The zeroes and poles of these functions are $\partial\mathbb{C}\tilde{C} = \tilde{C} \cap \partial\mathbb{C}\Delta$. The order of these zeroes and poles is determined by the multiplicity of the corresponding intersection points of $\mathbb{C}\bar{C}$ with $\mathbb{C}E_j$ as well as by the slope of $E_j \subset \Delta$. As $\mathbb{C}\tilde{C}$ is rational, we may freely deform $\partial\mathbb{C}\tilde{C}$ in $\mathbb{C}\tilde{C}$ and each such deformation extends to the deformation of the coordinate functions. The group $PSL_2(\mathbb{C})$ of symmetries of $\mathbb{C}\tilde{C}$ is 3-dimensional, so the space of deformation of $\partial\mathbb{C}\tilde{C}$ is $(\#(\partial\mathbb{C}\tilde{C}) - 3)$-dimensional. The resulting curve is well-defined up to the multiplicative translation in $(\mathbb{R}^\times)^2$. Altogether, the dimension of the space of rational curves in $(\mathbb{R}^\times)^2$ is $m-1$, which coincides with the dimension of the space of configurations $\mathcal{P}$.

Note that the deformation of a single point p to p' in $\partial\mathbb{C}\tilde{C}$ corresponds to adding rational functions with zeroes and poles only at p and p' to the corresponding integer multiples of the coordinate functions. A non-immersed point of $\mathbb{C}C^\circ$ corresponds to a common zero of the differentials of the coordinate functions. But the differential of each coordinate function can be perturbed separately by addition of a rational function as above. Thus the set $\mathcal{R}_{\Delta,k}(\mathcal{P})$ of curves in $(\mathbb{R}^\times)^2$ with Newton polygon Δ and quantum index k passing through $\mathcal{P}$ is finite, and each curve in this set is immersed.

It is convenient to consider not only conventional irreducible rational curves, but also the so-called *stable rational curves* (cf. [3]) in $\mathbb{C}\Delta$. These are possibly disconnected Riemann surfaces F enhanced with

a holomorphic map to $\mathbb{C}\Delta$ and with a tree Γ whose vertices are the components of F so that each edge of Γ corresponds to an intersection point of the images of the corresponding components. Conventional rational curves are stable rational curves with the tree Γ consisting of a single vertex. Any sequence of (stable) rational curves in $(\mathbb{R}^\times)^2$ with the Newton polygon Δ converges to a stable rational curve $\mathbb{R}D \subset (\mathbb{R}^\times)^2$ with a Newton polygon $\Delta' \subset \Delta$. Note that if $\mathbb{C}\bar{D} \cap \partial\mathbb{C}\Delta$ is disjoint from the points of intersection of toric divisors $\mathbb{C}E_j$ then $\Delta' = \Delta$. Therefore, the map

$$\mathrm{ev} : \mathcal{M}_\Delta \to \mathcal{M}_{\partial\Delta} \tag{33}$$

is proper. Here the source $\mathcal{M}_\Delta$ is the space of stable oriented real rational curves with the Newton polygon Δ. The target $\mathcal{M}_{\partial\Delta}$ is the space of conj-invariant configurations of m points in $\partial\mathbb{R}\Delta$ with m_j points in $\mathbb{C}E_j$, $m = \sum m_j = \#(\partial\Delta \cap \mathbb{Z}^2$, subject to the Menelaus condition. The map ev sends a curve $\mathbb{R}C^\circ$ to the configuration $\mathrm{Fr}_2^\Delta(\mathbb{C}\bar{C}) \cap \partial\mathbb{C}\Delta \subset \partial\mathbb{C}\Delta$.

We refer to stable rational curves with disconnected F as *reducible rational curves* in $\mathbb{C}\Delta$. Note that the dimension of the space of deformation of such curves is equal to the sum of the space of deformation of their irreducible components, i.e. to m minus the number of components. Thus for a generic choice of $\mathcal{P}$ there are no reducible rational curves of Newton polygon Δ passing through $\mathcal{P}$.

Consider the space $\mathcal{M}_{\partial\Delta,\lambda} \subset \mathcal{M}_{\partial\Delta}$ of real configurations $\mathcal{P} \subset \mathbb{R}\Delta$ with λ_j points in $\mathbb{R}E_j \smallsetminus (\overline{(\mathbb{R}_>0)^2})$. Any curve in $\mathrm{ev}^{-1}(\mathcal{M}_{\partial\Delta,\lambda})$ satisfies to the hypothesis of Theorem 1, thus its quantum index is well-defined. Let $\mathcal{P}, \mathcal{P}' \in \mathcal{M}_{\partial\Delta,\lambda}$ and $\gamma = \{\mathcal{P}_t\}_{t\in[0,1]} \in \mathcal{M}_{\partial\Delta,\lambda}$be a smooth generic path connecting two such configurations $\mathcal{P} = \mathcal{P}_0$ and $\mathcal{P}' = \mathcal{P}_1$.

All but finitely many values of t correspond to $\mathcal{P}_t$ such that $\mathcal{R}_{\Delta,k}$ are irreducible. If this finite set is empty then $\mathrm{ev}^{-1}(\gamma)$ consists of irreducible rational curves. In this case an orientation of a curve uniquely determines the orientations of all curves in its connected component in $\mathrm{ev}^{-1}(\gamma)$. Thus, $\mathrm{ev}^{-1}(\gamma)$ splits to a disjoint union of components according to the quantum index k. Furthermore, we may deduce that $R_{\Delta,k}(\mathcal{P}) = R_{\Delta,k}(\mathcal{P}')$ from the Welschinger theorem [22].

For this deduction we note that the curve $\mathbb{R}\bar{C} \subset \mathbb{R}\Delta$ birationally corresponds to a curve $\mathbb{R}\hat{C} \subset \mathbb{RP}^1 \times \mathbb{RP}^1$. Furthermore, in $\mathbb{RP}^1 \times \mathbb{RP}^1$ we may obtain a deformation $\mathbb{R}\hat{C}'$ of $\mathbb{R}\hat{C}$ as follows. We take the Riemann surface $\mathbb{C}\tilde{C}$ and two coordinate functions with zeroes and poles at $\partial\mathbb{C}\tilde{C}$. Each point $p \in \partial\mathbb{C}\tilde{C}$ is thus associated two numbers, n_x and n_y: the order of zero (or pole) of the first and of the second coordinate function at p. The points are ordered according to the

orientation of $\mathbb{R}\tilde{C}$. We may replace p with n_x+n_y generic points in the neighborhood of p and ask that the first (resp. the second) coordinate function has a simple zero (or pole) at the first n_x point (resp. the last n_y points). The resulting curve $\mathbb{R}\hat{C}' \subset \mathbb{R}\Delta' = \mathbb{RP}^1 \times \mathbb{RP}^1$ has Newton polygon $\Delta' \supset \Delta$ (which is a rectangle). Denote the critical values of the two coordinate functions on $\mathbb{R}\bar{C}$ (resp $\mathbb{R}\hat{C}'$) with Z_x and Z_y (resp. $\hat{Z}'_x$ and $\hat{Z}'_y$). Generically, all these critical values are of order 1, so that $\#(Z'_x) = 2d_x - 2$ and $\#(Z'_y) = 2d_y - 2$. The set Z'_x (resp Z'_y) is obtained from Z_x (resp. Z_y) by a small perturbation (in the circle $\mathbb{R}\tilde{C}$) as well as adding $n_x - 1$ (resp $n_y - 1$) points near the corresponding point of $\partial\mathbb{R}\tilde{C}$ in the prescribed relative position. Thus the number of *hyperbolic nodes*, i.e. the points of self-intersection of $\mathbb{R}\hat{C}'$, is the same as for $\mathbb{R}C^\circ$.

The set $\mathrm{Fr}_2^\Delta(\mathbb{R}\hat{C}') \cap \partial\mathbb{R}\Delta'$ minus one point (determined by the Menelaus condition) can be further perturbed to a generic set $\mathcal{Q} \subset \mathbb{R}\Delta' = \mathbb{RP}^1 \times \mathbb{RP}^1$. Similarly the path $\mathcal{P}_t$ produces a generic path $\mathcal{Q}_t$ so that the curves from $\mathcal{R}_{\Delta,k}(\mathcal{P}_t)$ can be perturbed to real rational curves with the Newton polygon Δ' such that their images under Fr_2^Δ pass through $\mathcal{Q}_t$. The Welschinger signs of the resulting curves in $\mathbb{RP}^1 \times \mathbb{RP}^1$ determine the sign σ for the curves from $\mathcal{R}_{\Delta,k}(\mathcal{P}_t)$ since the Welschinger sign for a given Δ' can be expressed through the number of hyperbolic points and the index k is locally invariant for deformation in the class of irreducible curves. Therefore, by the local invariance of Welschinger count [22], we have $R_{\Delta,k}(\mathcal{P}) = R_{\Delta,k}(\mathcal{P}')$ in this case.

Suppose now that

$$\mathcal{R}_\Delta(\mathcal{P}_t) = \bigcup_{k'=-\mathrm{Area}(\Delta)}^{\mathrm{Area}(\Delta)} \mathcal{R}_{\Delta,k'}(\mathcal{P}_t) \tag{34}$$

contains a reducible curve $\mathbb{R}\bar{D}$ for $t = t_0$.

As the dimension of the space of deformation of each component is equal to the number of points in its intersection with $\partial\mathbb{C}\Delta$ minus 1, for the generic path γ the curve $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{D})$ is the union of two irreducible rational immersed curves $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{D}_j)$, $j = 1, 2$. Also, the presence of multiplicative translations in $(\mathbb{R}^\times)^2$ implies that $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{D}_1)$ and $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{D}_2)$ intersect transversally.

We may assume that all points $\mathrm{Fr}_2^\Delta(\mathcal{P}_t)$ except for two points $p_j(t) \in \mathbb{R}\bar{D}_j$, $j = 1, 2$, remain independent of $t \in [t_0 - \epsilon, t_0 + \epsilon]$ for a small $\epsilon > 0$. For $t \in [t_0 - \epsilon, t_0 + \epsilon]$ the deformation $p_1(t) \in \partial\mathbb{R}\Delta$ determines the deformation $p_2(t)$ by the Menelaus condition. These two points must belong to two different components of $\mathbb{R}\bar{D}$ as there are no reducible curves in $\mathcal{R}_\Delta(\mathcal{P}_t)$ for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ except for $t = t_0$. Denote with

$\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_t)$ for $t_0-\epsilon t \le t_0+\epsilon$ the curves whose images under Fr_2^Δ is close to $\mathrm{Fr}_2^\Delta(\mathbb{R}\bar{D})$.

Let us choose some orientations of $\mathbb{R}\bar{D}_1$ and $\mathbb{R}\bar{D}_2$. Then the intersection points of

$$I = \mathrm{Fr}_2^\Delta(\mathbb{R}\bar{D}_1) \cap \mathrm{Fr}_2^\Delta(\mathbb{R}\bar{D}_2) \tag{35}$$

come with the intersection sign in $\mathbb{R}^2_{>0}$. The set of positive points $I_+ \subset I$ has the same cardinality as the set of negative points $I_- \subset I$, $I = I_+ \cap I_-$ as $\mathbb{R}^2_{>0}$ is contractible.

The curves in $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_{t_0\pm\epsilon})$ are obtained by smoothing a nodal point $q \in I$ in one of the two ways, one that agrees with our choice of orientation and one that does not. Without loss of generality (changing the direction of the path γ if needed) we may assume that the orientation-preserving smoothing in a point $q \in I_+$ corresponds to a curve $\mathbb{R}\bar{D}_{q,+} \in \mathcal{R}_\Delta(\mathcal{P}_{t_0+\epsilon})$ and thus the orientation-reversing smoothing at the same point corresponds to a curve $\mathbb{R}\bar{D}_{q,-} \in \mathcal{R}_\Delta(\mathcal{P}_{t_0-\epsilon})$. The following lemma determines the situation at all the other points of I.

Lemma 26. *A curve obtained by the smoothing of a node from I_+ in the orientation-preserving way, or by the smoothing of nodes from I_- in the orientation-reversing way belongs to $\mathcal{R}_\Delta(\mathcal{P}_{t_0+\epsilon})$, $\epsilon > 0$.*

Accordingly, a curve obtained by the smoothing of a node from I_- in the orientation-preserving way, or by the smoothing of nodes from I_+ in the orientation-reversing way belongs to $\mathcal{R}_\Delta(\mathcal{P}_{t_0-\epsilon})$.

Proof. Let $\mathbb{R}D_{q',s} \in \mathcal{R}_\Delta(\mathcal{P}_{t_0+\epsilon})$, $s = \pm 1$, be the curve obtained by smoothing $\mathbb{R}D$ at a point $q' \in I$ according to the sign s. Note that $\mathrm{Fr}_2^\Delta(\mathbb{R}D_{q,+})$ and $\mathrm{Fr}_2^\Delta(\mathbb{R}D_{q',s})$ are tangent to each other at m points of $\mathcal{P}_{t_0+\epsilon}$ and must intersect each other at pairs of points close to the nodes of $I \smallsetminus \{q, q'\}$. By Pick's formula the total number of intersection is equal to the twice the area of 2Δ, which is the Newton polygon of $\mathrm{Fr}_2^\Delta(\mathbb{R}D_{q,+})$ and $\mathrm{Fr}_2^\Delta(\mathbb{R}D_{q',s})$. By the Kouchnirenko-Bernstein formula, these curves do not have any other intersection points which implies that $s = +1$ if $q' \in I_+$, see Figure 6. □

Note that any curve in $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_{t_0\pm\epsilon})$ is obtained by smoothing $\mathbb{R}D$ at a point $q \in I$. The quantum index of the result is $\pm k(\mathbb{R}D_1) \pm k(\mathbb{R}D_2)$, where the signs are determined by the agreement or disagreement of the orientation of the resulting curve with the chosen orientations of $\mathbb{R}D_j$. Since $\#(I_+) = \#(I_-)$, Lemma 26 implies that $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_{t_0+\epsilon})$ and $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_{t_0-\epsilon})$ have the same number of curves of each quantum index. □

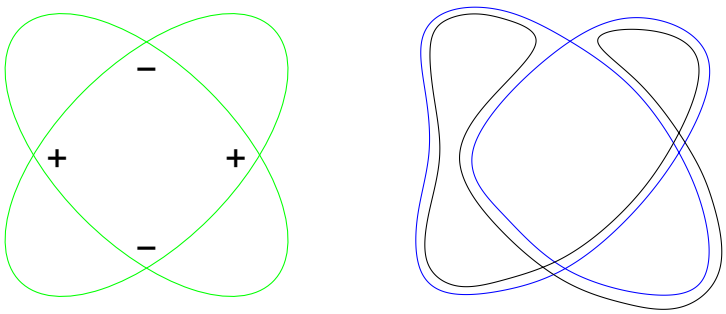

FIGURE 6. The signs of intersection points of two components of $\mathbb{R}D$ and the corresponding direction of smoothing.

Proof of Theorem 6. By the same reason as above we have $\tilde{R}_{\Delta_d,k}(\mathcal{P}) = \tilde{R}_{\Delta_d,k}(\mathcal{P}')$ if there are no reducible curves with the Newton polygon Δ_d that pass through $\mathcal{P}_t$. Also we may assume that if $\mathbb{R}D$ is a reducible curve with the Newton polygon Δ_d passing through $\mathcal{P}_{t_0}$ then it consists of two components $\mathbb{R}D_1$ and $\mathbb{R}D_2$ that intersect transversely at a finite set I. Note that the degree of both components, $\mathbb{R}D_1$ and $\mathbb{R}D_2$, must be even, as a real curve of odd degree must intersect $\partial\mathbb{RP}^2$ in a negative point as the boundary of the positive quadrant is null-homologous.

We have two smoothings of $\mathbb{R}D$ at $q \in I$ that pass through $\mathcal{P}_{t_0-\epsilon}$ and $\mathcal{P}_{t_0+\epsilon}$. One can be oriented in accordance with the orientations of $\mathbb{R}D_1$ and $\mathbb{R}D_2$ and the other in accordance with the orientation of $\mathbb{R}D_1$, but opposite to the orientation of $\mathbb{R}D_2$. The corresponding quantum indices are different by $2k(\mathbb{R}D_2)$. The index $k(\mathbb{R}D_2)$ is integer since the degree of $\mathbb{R}D_2$ is even. □

6.3. **Indices of real phase-tropical curves.** We start by recalling the basic notions of tropical geometry (cf. [14], [15]) specializing to the case of plane curves. Recall that a *metric graph* is a topological space homeomorphic to $\Gamma^\circ = \Gamma \smallsetminus \partial\Gamma$ enhanced with a complete inner metric. Here Γ is a finite graph and $\partial\Gamma$ is the set of its 1-valent vertices. The metric graph is also sometimes called a tropical curve (while in some other instances the term tropical curve is reserved for the equivalence class of metric graphs with respect to tropical modifications). In this paper we require the graph Γ to be connected so that Γ° is irreducible as a tropical curve. We assume that Γ has a vertex of valence at least three, and that Γ does not have 2-valent vertices. The half-open edges of Γ° obtained from the closed edges of Γ adjacent to $\partial\Gamma$ are called *leaves*.

A *plane tropical curve* is a proper continuous map $h : \Gamma^\circ \to \mathbb{R}^2$ such that $h|_E$ is smooth for every edge $E \subset \Gamma$ with $dh(u) \in \mathbb{Z}^2$ for a unit tangent vector u at any point of E. In addition we require the following *balancing condition* at every vertex $v \in \Gamma$

$$\sum_E dh(u(E)) = 0, \tag{36}$$

where $u(E)$ is the unit tangent vector in the outgoing direction with respect to v and the sum is taken over all edges E adjacent to v.

The collection of vectors $\{dh(u_v)\}_{v\in\partial\Gamma}$ where u_v is a unit vector tangent to the leaf adjacent to v (and directed towards v) is called *the (toric) degree* of $h : \Gamma^\circ \to \mathbb{R}^2$. The identity (36) implies that the sum of all vectors in this collection is zero. Therefore this collection is dual to a lattice polygon $\Delta \in \mathbb{Z}^2$ which is well-defined up to translations in $\mathbb{Z}^2$. The polygon Δ is determined by $h(\Gamma^\circ)$. We call Δ *the Newton polygon* of $h : \Gamma^\circ \to \mathbb{R}^2$.

Tropical curves appear as limits of scaled sequences of complex curves in the plane. Let A be any set and $\alpha \to t_\alpha \in \mathbb{R}$ be a function unbounded from above (this function is called *the tropical scaling sequence*). Let $\mathbb{C}C_\alpha \subset (\mathbb{C}^\times)^2$, $\alpha \in A$, be a family of complex curves with the Newton polygon Δ.

Definition 27. We say that a family $\mathbb{C}C_\alpha$ has a *phase-tropical limit* with respect to t_α if for every $p \in \mathbb{R}^2$ we have

$$\lim_{t_\alpha\to+\infty} t_\alpha^{-p}\mathbb{C}C_\alpha = \Phi_p \tag{37}$$

for a (possibly empty) algebraic curve $\Phi(p) \subset (\mathbb{C}^\times)^2$. Here $t_\alpha^{-p}\mathbb{C}C_\alpha$ is the multiplicative translation of the curve $\mathbb{C}C_\alpha$ by $t_\alpha^{-p} \in (\mathbb{C}^\times)^2$. The coefficients of the polynomials defining $t_\alpha^{-p}\mathbb{C}C_\alpha$ represent a point in the projective space of dimension $\#(\Delta \cap \mathbb{Z}^2) - 1$. The limit is understood in the sense of topology of this projective space. The curve $\Phi_p \subset (\mathbb{C}^\times)^2$ may be reducible and even non-reduced.

We say that $h : \Gamma^\circ \to \mathbb{R}^2$ is *the tropical limit* of $\mathbb{C}C_\alpha$ with respect to t_α if for a sufficiently small open convex neighborhood $p \in U \subset \mathbb{R}^2$ the irreducible components $\Psi \subset \Phi(p) \subset (\mathbb{C}^\times)^2$ correspond to the connected components $\psi \subset h^{-1}(U)$ so that the lattice polygon Δ_ψ determined by the ends of the open graph ψ coincides with the Newton polygon Δ_Ψ of the irreducible component Ψ taken with some multiplicity. The same component Ψ may correspond to several components of $h^{-1}(U)$ so that the some of all resulting multiplicities is equal to the multiplicity of Ψ in $\Phi(p)$. Each connected component of $h^{-1}(U)$ corresponds to a unique component of $\Phi(p)$.

If h does not contract any edge of Γ° to a point then the open set $\psi \subset \Gamma^\circ$ may contain at most one vertex. If $v \in \Gamma$ is such a vertex then we call Ψ *the phase* Φ_v *of the vertex* v. If ψ is contained in an edge E then we call Ψ *the phase* Φ_E *of the edge* E. The phases $\Phi(E) \subset (\mathbb{C}^\times)^2$ do not depend on the choice of a point $p \in h(E)$ and are well-defined up to multiplicative translations by $(\mathbb{R}^\times)^2$. The curve $h : \Gamma^\circ \to \mathbb{R}^2$ enhanced with the phases Φ_v and Φ_E for its vertices and edges is called *the phase-tropical limit of* $\mathbb{C}C_\alpha$ *with respect to the scaling sequence* $t_\alpha \to +\infty$.

We consider the phases in $(\mathbb{C}^\times)^2$ that are different by multiplicative translation by vectors from $(\mathbb{R}_{>0})^2$ equivalent.

Note that the Newton polygon of the phase Φ_E of an edge E is an interval. Thus after a suitable change of coordinates in $(\mathbb{C}^\times)^2$ the (irreducible) curve Φ_E is given by a linear equation in one variable. Therefore, Φ_E is a multiplicative translation of a subtorus $S^1 \approx T_E \subset S^1 \times S^1$ in the direction parallel to $h(E)$.

Let us orient E. Then T_E as well as the quotient space $B_E = (S^1 \times S^1)/T_E$ also acquire an orientation. The image $\mathrm{Arg}(\Phi_E)$ coincides with $\pi_E^{-1}(\sigma_E)$ for some $\sigma_E \in B_E$, where $\pi_E : S^1 \times S^1 \to B_E$ is the projection. Since B_E is isomorphic to S^1 and oriented, we have a canonical isomorphism $B_E = \mathbb{Z}/2\pi\mathbb{Z}$. Thus, a phase Φ_E of an oriented edge E of a planar tropical curve is determined by a single argument $\sigma(E) \in \mathbb{Z}/2\pi\mathbb{Z}$. The change of the orientation of E results in the change of sign of $\sigma(E)$.

Let $v \in \Gamma^\circ$ be a vertex and E_j be the edges adjacent to v. Orient E_j outwards from v. The oriented edges E_j can be associated a *momentum* $\mu(E_j)$ with respect to the origin $0 \in \mathbb{R}^2$. This is the wedge product of the vector connecting the origin with a point of E_j and the unit tangent vector $u(E_j)$ coherent with the orientation. Clearly, it does not depend on the choice of the point in E_j.

Recall that the vertex v is dual to the lattice polygon Δ_v determined by the integer vectors $dh(u(E_j))$. The multiplicity is defined as $m(v) = 2\,\mathrm{Area}\,\Delta_v$, cf. [14].

Proposition 28 (tropical Menelaus theorem)**.** *For any tropical curve* $h : \Gamma^\circ \to \mathbb{R}^2$ *and a vertex* $v \in \Gamma^\circ$ *the momenta* $\mu(E_j)$ *of the edges adjacent to* v *and oriented outwards from* v *satisfy to the equality*

$$\sum_j \mu(E_j) = 0. \tag{38}$$

If $\sigma(E_j) \in \mathbb{Z}/2\pi\mathbb{Z}$ are phases of the oriented edges E_j then

$$\sum_j w(E_j)\sigma(E_j) = \pi m(v) \tag{39}$$

(assuming that $\sigma(E_j)$ appear in the phase-tropical limit of a family $\mathbb{C}C_\alpha \subset (\mathbb{C}^\times)^2$ of complex curves).

This statement can be viewed as a generalization of the ancient Menelaus theorem stating that three points D, E, F on the extensions of three sides of a planar triangle ABC are collinear if and only if

$$\frac{|AD|}{|DB|}\frac{|BE|}{|EC|}\frac{|CF|}{|FA|} = -1. \tag{40}$$

Here the length is taken with the minus sign if the direction of an interval (e.g. $|CF|$) is opposite to the orientation of the triangle, see Figure 7.

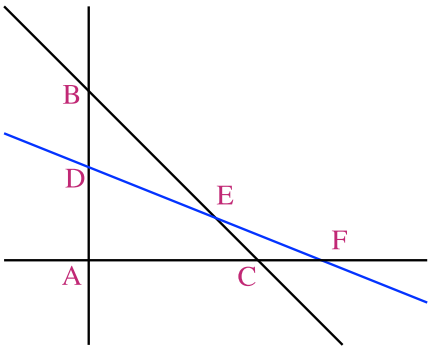

FIGURE 7. The Menelaus theorem.

Proof. The wedge product of the balancing condition (36) with the vector connecting 0 and v gives (38). To deduce (39) we consider the polynomial f_v (whose Newton polygon is Δ_v) defining the phase $\Phi_v \subset (\mathbb{C}^\times)^2$. By Vieta's theorem, the product of the roots cut by f_v on a divisor of $\mathbb{C}\Delta_v$ corresponding to an oriented side $F \subset \Delta_v$ is $(-1)^{\#(F\cap\mathbb{Z}^2)}$ times the ratio of the coefficients at the endpoints of F. Therefore the sum of the phases of the edges of Γ corresponding to F is the argument of this ratio plus $\#(F \cap \mathbb{Z}^2)\pi$. Since by Pick's formula the parity of $\#(\partial\Delta \cap \mathbb{Z}^2)$ coincides with that of $m(v) = 2\,\mathrm{Area}(\Delta_v)$ we recover (39). □

Corollary 29. *We have $\sum_E \mu(E) = 0$, where the sum is taken over all leaves of $h : \Gamma^\circ \to \mathbb{R}^2$ oriented in the outwards direction.*

Proof. Take the sum of the expression (38) over all vertices of Γ°. The momenta of all bounded edges will enter twice with the opposite signs. □

If all curves $\mathbb{C}C_\alpha$ are defined over $\mathbb{R}$ then the phases $\Phi(p)$ must be real for all points $p \in \mathbb{R}^2$. Note, however that in general, the phase Φ_v for a vertex $v \in \Gamma^\circ$ does not have to be real as the involution of complex conjugation may exchange it with $\Phi_{v'}$ for another vertex $v' \in \Gamma$ with $h(v) = h(v')$. We say that a vertex v is *real* if Φ_v is defined over $\mathbb{R}$.

Let $\mathbb{R}C_\alpha$ be a scaled sequence of type I curves enhanced with a complex orientation, so that a component $S_\alpha \subset \mathbb{C}C_\alpha \smallsetminus \mathbb{R}C_\alpha$ is fixed for all α. Suppose that $\mathbb{C}C_\alpha$ has a phase-tropical limit, and the orientations of $\mathbb{R}C_\alpha$ agree with some complex orientations of the real part $\mathbb{R}\Phi(p)$ of the phases $\Phi(p)$. The quantum index of $\mathbb{R}C_\alpha$ is well-defined if it has a purely imaginary or real coordinate intersection. Similarly, the phase $\mathbb{R}\Phi_v$ of a real vertex v of the tropical limit has a well-defined quantum index if $\sigma(E) \equiv 0 \pmod{\pi}$ for any edge E adjacent to v.

Proposition 30. *For large t_α we have*

$$k(\mathbb{R}C_\alpha) = \sum_v k(\mathbb{R}\Phi_v), \tag{41}$$

where the sum is taken over all real vertices whenever all quantum indices in (41) *are well-defined.*

Proof. Additivity of the quantum index with respect to the phases Φ_v follows from Theorem 1 through additivity of the degree of the map $2\operatorname{Arg}$ restricted to $S \cap (\mathbb{C}^\times)^2$. Non-real vertices have zero contribution to $k(\mathbb{R}C_\alpha)$ as the signed area of the amoeba of the whole complex curve is zero. □

Proof of Theorem 7. Recall the definition of the (tropical) Block-Göttsche invariants, see [7], which refine tropical enumerative invariants of [14]. Namely, to any 3-valent (open) tropical immersed curve $h : \Gamma^\circ \to \mathbb{R}^2$ we may associate the Laurent polynomial

$$BG(h(\Gamma^\circ)) = \prod_v \frac{q^{\frac{m(v)}{2}} - q^{-\frac{-m(v)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{42}$$

where v runs over all vertices $v \in \Gamma$ and $m(v)$ is the multiplicity of the vertex v. The genus of a (connected) tropical curve Γ° is the first Betti number of Γ°. In particular, a rational tropical curve is a tree.

Let us fix a collection $\mu = \{m_j\}_{j=1}^m$, $m = \#(\partial\Delta \cap \mathbb{Z}^2)$, of generic real numbers subject to the condition $\sum_{j=1}^m \mu_j = 0$. This means that μ_j,

$j = 1, \dots, m-1$ are chosen generically, and μ_m is determined from our condition.

If $h : \Gamma^\circ \to \mathbb{R}^2$ is a tropical curve with the Newton polygon Δ then we number its leaves so that the first m_1 leaves are dual to the side $E \subset \partial\Delta$, the second m_2 to the side $E_2 \subset \partial\Delta$ and so on with the last m_n leaves dual to E_n. We say that $h : \Gamma \to \mathbb{R}^2$ passes through the $\partial\mathbb{T}\Delta$-points determined by μ if the jth unbounded edge of Γ has the momentum μ_j. Note that a leaf $E \subset \Gamma^\circ$ must have the momentum $\mu(E)$ if it passes through a point p_E on the oriented line parallel to the vector $(dh)u(E)$ with the momentum $\mu(E)$. Thus a generic choice of the momenta ensures that $h : \Gamma^\circ \to \mathbb{R}^2$ passes through a generic collection of $m-1$ points in $\mathbb{R}^2$. Thus we have only finitely many rational tropical curves with the Newton polygon Δ passing through the $\partial\mathbb{T}\Delta$ points determined by μ by Lemma 4.22 of [14] (as the number of combinatorial types of tropical curves with the given Newton polygon Δ is finite). By Proposition 4.11 of [14] all these tropical curves are simple in the sense of Definition 4.2 of [14].

The Block-Göttsche number associated to μ is

$$BG_\Delta = BG_\Delta(\mu) = \sum_{h:\Gamma^\circ\to\mathbb{R}^2} BG(h(\Gamma^\circ)), \tag{43}$$

where the sum is taken over all $h : \Gamma^\circ \to \mathbb{R}^2$ passing through the $\partial\mathbb{T}\Delta$ points determined by μ. Independence of BG_Δ from μ can be proved in the same way as in [7]. Also it follows from Theorem 5 once we prove coincidence of R_Δ and $BG_\Delta(\mu)$.

A toric divisor $\mathbb{C}E_j \subset \mathbb{C}\Delta$ is the compactification of the torus $\mathbb{C}^\times$ obtained by taking the quotient group of $(\mathbb{C}^\times)^2$ by the subgroup defined by the side $E_j \subset \Delta$. Thus a configuration $\mathcal{P} = \{p_j\}_{j=1}^m \subset \partial\mathbb{C}\Delta$ is given by a collection of m nonzero complex numbers as well as an attribution of the points to the toric divisors. This collection is real if the corresponding numbers are real and positive if these number are positive.

We set $\mathcal{P}^t = \{p_1^t, \dots, p_m^t\} \subset \partial\mathbb{R}\Delta$ be the configuration of points with the same toric divisor attribution as $\mathcal{P}$, and given by the positive numbers $\{t^{2\mu_j}\}$, $t > 1$. By Proposition 8.7 of [14] the amoebas of rational complex curves with the Newton polygon Δ passing through $(\mathrm{Fr}_2^\Delta)^{-1}(\mathcal{P}_t)$ converge when $t \to +\infty$ to tropical curves passing through the $\partial\mathbb{T}\Delta$-points determined by μ. Proposition 8.23 of [14] determines the number of complex curves with amoeba in a small neighborhood of a rational tropical curve $h : \Gamma^\circ \to \mathbb{R}^2$ passing through any choice of points $\tilde{p}_j^t \in (\mathrm{Fr}_2^\Delta)^{-1}(p_j^t)$, $j = 1, \dots, m-1$, for large t, while Remark 8.25 of [14] determines the number of the corresponding real curves. E.g. if

the weights of all edges of Γ° are odd we have a single real curve for any choice of $\tilde{\mathcal{P}}^t = \{\tilde{p}_j^t\}$. In general, some choices of $\tilde{\mathcal{P}}_t$ may correspond to no real solutions, while others may correspond to multiple solutions. We claim that nevertheless there are 2^{m-1} different real curves whose amoeba is close to $h : \Gamma^\circ \to \mathbb{R}^2$ with the image under Fr_2^Δ passing through $\mathcal{P}^t$ for large t. Thus we have 2^m different oriented curves. We show this by induction on m as follows.

If Γ° has a single vertex v (so that $m = 3$) then there are 4 different real rational phase Φ_v which differ by the deck transformations of the map Fr_2^Δ. Thus we have 8 different oriented real rational phases in this case. The positive logarithmic rotation number for half of them is positive, for the other half is negative. Adding each new 3-valent vertex v' to the tree Γ doubles the number of oriented real phases as there are two ways to attach the phase for v': so that the logarithmic rotation number of the resulting real curve will increase by one and so that it will decrease by one. Inductively we get 4 real oriented curves for each of the 2^{m-2} possible distribution of signs for the vertices of Γ°.

For each vertex v the real phase $\mathbb{R}\Phi_v$ is the image of a line by a multiplicative-linear map of determinant $m(v)$ by Corollary 8.20 of [14]. Therefore $k(\mathbb{R}\Phi_v) = \pm\frac{m(v)}{2}$, where the sign is determined by the degree of the logarithmic Gauß map. According to our sign convention (18) each oriented real curve comes with the sign equal to the number of negative vertices. Thus by Proposition 30 the contribution of $h : \Gamma \to \mathbb{R}^2$ to $R_\Delta(\mathcal{P}^t)$ for large t is $\prod_v (q^{\frac{m}{2}} - q^{-\frac{m}{2}})$ which coincides with the numerator of the Block-Göttsche multiplicity (42). □

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