

# QUANTUM INDICES OF REAL PLANE CURVES AND REFINED ENUMERATIVE GEOMETRY

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ABSTRACT. We associate a half-integer number, called *the quantum index*, to algebraic curves in the real plane satisfying to certain conditions. The area encompassed by the logarithmic image of such curves is equal to  $\pi^2$  times the quantum index of the curve. We use the quantum index to refine real enumerative geometry in a way consistent with the Block-Göttsche invariants from tropical enumerative geometry.

## 1. INTRODUCTION.

Geometry of real algebraic curves in the plane is one of the most classical subject in Algebraic Geometry. In the first section we review some relevant basic notions needed to formulate the main results of this paper.

**1.1. Real curves of type I and their complex orientation.** A real curve  $\mathbb{R}C \subset \mathbb{R}P^2$  is given by a single homogeneous polynomial equation  $F(z_0, z_1, z_2) = \sum_{j,k,l} a_{k,l} z_0^j z_1^k z_2^l = 0$ ,  $j + k + l = d$ ,  $a_{k,l} \in \mathbb{R}$ . The locus  $\mathbb{C}C \subset \mathbb{C}P^2$  of complex solutions of  $F = 0$  is called the *complexification* of  $\mathbb{R}C$ . The normalization

$$(1) \quad \nu : \mathbb{C}\tilde{C} \rightarrow \mathbb{C}C$$

defines a parameterization of  $\mathbb{C}C$  by a Riemann surface  $\mathbb{C}\tilde{C}$ . The antiholomorphic involution of complex conjugation  $\text{conj}$  acts on  $\mathbb{C}C$  in an orientation-reversing way so that its fixed point locus is  $\mathbb{R}C$ . The restriction of  $\text{conj}$  to the smooth locus of  $\mathbb{C}C$  lifts to an antiholomorphic involution  $\tilde{\text{conj}} : \mathbb{C}\tilde{C} \rightarrow \mathbb{C}\tilde{C}$  on the normalization. We denote with  $\mathbb{R}\tilde{C}$  the fixed point locus of  $\tilde{\text{conj}}$ . Clearly,  $\nu(\mathbb{R}\tilde{C}) \subset \mathbb{R}C$ . In this paper we assume that  $\mathbb{R}C$  is irreducible, i.e. that  $\mathbb{C}\tilde{C}$  is connected.

Following Felix Klein we say that  $\mathbb{R}C$  is *of type I* if  $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$  is disconnected. In such case it consists of two connected components  $S$  and  $S' = \tilde{\text{conj}}(S)$  which are naturally oriented by the complex orientation of the Riemann surface  $\mathbb{C}\tilde{C}$ . We have  $\mathbb{R}\tilde{C} = \partial S = \partial S'$ , so a choice of one of these components, say  $S$ , induces the boundary orientation

on  $\mathbb{R}\tilde{C}$ . The resulting orientation is called *a complex orientation of  $\mathbb{R}\tilde{C}$*  and is subject to Rokhlin's complex orientation formula [13]. If we choose  $S'$  instead of  $S$  then the orientations of all components of  $\mathbb{R}C$  will reverse simultaneously. Thus any orientation of a component of  $\mathbb{R}\tilde{C}$  determines a component of  $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$ .

### 1.2. Toric viewpoint and reality of coordinate intersections.

The projective plane  $\mathbb{C}\mathbb{P}^2$  can be thought of as the toric compactification of the torus  $(\mathbb{C}^\times)^2$ . The curve  $\mathbb{C}C \subset \mathbb{C}\mathbb{P}^2$  is the closure of its toric part  $\mathbb{C}C^\circ = \mathbb{C}C \cap (\mathbb{C}^\times)^2$ . The complement  $\partial\mathbb{C}\mathbb{P}^2 = \mathbb{C}\mathbb{P}^2 \setminus (\mathbb{C}^\times)^2$  is the union of three axes: the  $x$ -axis, the  $y$ -axis and the  $\infty$ -axis. These axes intersect pairwise at the points  $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1) \in \mathbb{R}\mathbb{P}^2$ . If the coefficients  $a_{0,0}, a_{d,0}, a_{0,d}$  are non-zero then  $\mathbb{C}C$  is disjoint from the intersection points of the axes. In the general case it is reasonable to consider other toric surfaces compactifying  $(\mathbb{C}^\times)^2$ , so that the closure of  $\mathbb{C}C^\circ$  there is disjoint from pairwise intersections of toric divisors.

Let us consider the (non-homogeneous) polynomial  $f(x, y) = F(1, x, y)$  and its Newton polygon

$$\Delta = \text{ConvexHull}\{(j, k) \in \mathbb{R}^2 \mid a_{j,k} \neq 0\}.$$

The fan dual to  $\Delta$  defines a toric compactification  $\mathbb{C}\Delta \supset (\mathbb{C}^\times)^2$ . The toric divisors of  $\mathbb{C}\Delta$  correspond to the sides of  $\Delta$ . Their pairwise intersections correspond to the vertices of  $\Delta$  and are disjoint from the compactification  $\mathbb{C}\tilde{C}$  of the curve  $\mathbb{C}C^\circ$ . We denote with  $\partial\mathbb{C}\Delta \subset \mathbb{C}\Delta$  the union of toric divisors. Accordingly, we denote with  $\mathbb{R}\Delta$  (resp.  $\partial\mathbb{R}\Delta, \mathbb{R}C^\circ, \mathbb{R}\tilde{C}$ ) the real part of  $\mathbb{C}\Delta$  (resp.  $\partial\mathbb{C}\Delta, \mathbb{C}C^\circ, \mathbb{C}\tilde{C}$ ). E.g. we have  $\mathbb{R}\mathbb{P}^2 = \mathbb{R}\Delta$  for the triangle  $\Delta = \text{ConvexHull}\{(0, 0), (1, 0), (0, 1)\}$  or any of a positive integer multiple of this triangle.

Let  $\text{Fr}_2 : (\mathbb{C}^\times)^2 \rightarrow (\mathbb{C}^\times)^2$  be the map defined by  $\text{Fr}_2(x, y) = (x^2, y^2)$ . This map extends to a map  $\text{Fr}_2^\Delta : \mathbb{C}\Delta \rightarrow \mathbb{C}\Delta$ .

We call a point  $p \in \mathbb{C}\Delta$  *purely imaginary or real* if  $\text{Fr}_2^\Delta(p) \in \mathbb{R}\Delta$ . We say that a curve  $\mathbb{R}C \subset \mathbb{R}\mathbb{P}^2$  has *pure imaginary or real coordinate intersection* if every point of  $\mathbb{C}C \cap \partial\mathbb{C}\Delta$  is purely imaginary or real.

### 1.3. Logarithmic area and other numbers associated to a real curve of type I.

Let  $\mathbb{R}C$  be a real curve of type I enhanced with a choice of a complex orientation. Consider the image  $\text{Log}(\mathbb{R}C^\circ) \subset \mathbb{R}^2$ , where  $\text{Log} : (\mathbb{C}^\times)^2 \rightarrow \mathbb{R}^2$  the map defined by

$$(2) \quad \text{Log}(x, y) = (\log |x|, \log |y|).$$

For a point  $p \in \mathbb{R}^2 \setminus \text{Log}(\mathbb{R}C^\circ)$  we define  $\text{ind}(p) \in \mathbb{Z}$  as the intersection number of an oriented ray  $R \subset \mathbb{R}^2$  emanating from  $x$  in a generic direction and the oriented curve  $\text{Log}(\mathbb{R}C^\circ)$  (this number can be considered as the linking number of  $p$  and  $\text{Log}(\mathbb{R}C^\circ)$ ).

**Definition 1.** The integral

$$\text{Area}_{\text{Log}}(\mathbb{R}C) = \int_{\mathbb{R}^2} \text{ind}_{\mathbb{R}C}(x) dx$$

is called the *logarithmic area* of  $\mathbb{R}C$ .

This is the signed area encompassed by  $\text{Log}(\mathbb{R}C^\circ)$ , where the area of each region of  $\mathbb{R}^2 \setminus \text{Log}(\mathbb{R}C^\circ)$  is taken with the multiplicity equal to the linking number of  $\text{Log}(\mathbb{R}C^\circ)$ .

Let  $S \subset \mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$  be the component corresponding to the complex orientation of  $\mathbb{R}\tilde{C}$ . The intersection points  $\nu(S) \cap (\mathbb{R}^\times)^2$  are the so-called *solitary real singularities* of  $\mathbb{R}C^\circ$ . The *multiplicity* of a solitary real singularity  $p \in \nu(S) \cap (\mathbb{R}^\times)^2$  is the local intersection number of  $S$  and  $(\mathbb{R}^\times)^2$  at  $p$ . Here the orientation of  $S$  is induced by the inclusion  $S \subset \mathbb{C}\tilde{C}$ , while the orientation of  $(\mathbb{R}^\times)^2$  is induced by the covering  $\text{Log}|_{(\mathbb{R}^\times)^2} : (\mathbb{R}^\times)^2 \rightarrow \mathbb{R}^2$ . In other words, the quadrants  $\mathbb{R}_{>0}^2$  and  $\mathbb{R}_{<0}^2$  are counterclockwise-oriented while the quadrants  $\mathbb{R}_{>0} \times \mathbb{R}_{<0}$  and  $\mathbb{R}_{<0} \times \mathbb{R}_{>0}$  are clockwise-oriented. The *toric solitary singularities number*  $E(\mathbb{R}C) \in \mathbb{Z}$  is the sum of multiplicities over all solitary real singularities of  $\mathbb{R}C^\circ$ , i.e. the total intersection number of  $S$  and  $(\mathbb{R}^\times)^2$  (enhanced with our choice of orientation).

The logarithmic Gauß map sends a smooth point of  $\mathbb{R}C^\circ$  to the tangent direction of the corresponding point on  $\text{Log}(\mathbb{R}C) \subset \mathbb{R}^2$ . This map uniquely extends to a map

$$\gamma : \mathbb{R}\tilde{C} \rightarrow \mathbb{R}P^1,$$

cf. [4], [8]. The *logarithmic rotation number*  $\text{Rot}_{\text{Log}}(\mathbb{R}C) \in \mathbb{Z}$  is the degree of  $\gamma$ .

## 2. QUANTUM INDICES OF REAL CURVES.

**Theorem 1.** *Let  $\mathbb{R}C$  be a real curve of type I enhanced with a complex orientation. If  $\mathbb{R}C$  has pure imaginary or real coordinate intersection then*

$$\text{Area}_{\text{Log}}(\mathbb{R}C) = k\pi^2$$

where  $k \in \frac{1}{2}\mathbb{Z}$ ,

$$-\text{Area}(\Delta) \leq k \leq \text{Area}(\Delta)$$

and  $k \equiv \text{Area}(\Delta) \pmod{1}$ .

Furthermore,  $k(\mathbb{R}C)$  coincides with the degree of the map

$$2 \operatorname{Arg} : \mathbb{C}C^\circ \setminus \mathbb{R}C^\circ \rightarrow (\mathbb{R}/\pi\mathbb{Z})^2,$$

i.e. with the number of inverse images at a generic point of the torus  $(\mathbb{R}/\pi\mathbb{Z})^2$  counted with the sign according to the orientation. Here the orientation of  $\mathbb{C}C^\circ \setminus \mathbb{R}C^\circ$  is defined by the condition that it coincides with the complex orientation of  $\mathbb{C}C$  on the component  $S \subset \mathbb{C}C^\circ \setminus \mathbb{R}C^\circ$  determined by the orientation of  $\mathbb{R}C$  and is opposite to the complex orientation of  $\mathbb{C}C$  on the component  $\operatorname{conj}(S) \subset \mathbb{C}C^\circ \setminus \mathbb{R}C^\circ$ . The map  $2 \operatorname{Arg}$  is defined by  $2 \operatorname{Arg}(x, y) = (2 \arg(x), 2 \arg(y))$ .

Note that as  $\Delta \subset \mathbb{R}^2$  is a lattice polynomial, its area is a half-integer number.

**Definition 2.** We say that  $k(\mathbb{R}C) = \frac{1}{\pi^2} \operatorname{Area}_{\operatorname{Log}}(\mathbb{R}C)$  is the *quantum index* of  $\mathbb{R}C$ .

If  $\mathbb{R}C$  is a real curve of type I with pure real or imaginary coordinate intersection, but the complex orientation of  $\mathbb{R}C$  is not fixed then its quantum index is well-defined up to sign.

The quantum index of  $\mathbb{R}C$  can also be expressed without computing the logarithmic area. We say that  $\mathbb{R}\tilde{C}$  is *transversal to  $\partial\mathbb{R}\Delta$*  if for any  $p \in \mathbb{R}C \cap \partial\mathbb{R}\Delta$  we have  $\nu^{-1}(p) \subset \mathbb{R}\tilde{C}$  and the composition  $\mathbb{R}\tilde{C} \rightarrow \mathbb{R}\tilde{C} \subset \mathbb{R}\Delta$  is an immersion near  $\nu^{-1}(p) \subset \mathbb{R}\tilde{C}$  which is transversal to  $\partial\mathbb{R}\Delta$ .

**Theorem 2.** Let  $\mathbb{R}C$  be a curve of type I with pure real or imaginary coordinate intersections such that  $\mathbb{R}\tilde{C}$  is transversal to  $\partial\mathbb{R}\Delta$ . Then

$$k(\mathbb{R}C) = -\frac{1}{2} \operatorname{Rot}_{\operatorname{Log}}(\mathbb{R}C) + E(\mathbb{R}C).$$

If  $\mathbb{R}\tilde{C}$  is tangent to  $\partial\mathbb{R}\Delta$  then an adjustment of the right-hand side according to the order of tangency and the orientation of  $\mathbb{R}C$  is required.

*Example 3* (Simple Harnack curves). If  $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$  is a simple Harnack curve (see [8]) then  $k(\mathbb{R}C) = \pm \operatorname{Area}(\Delta)$ . Vice versa, if  $k(\mathbb{R}C) = \pm \operatorname{Area}(\Delta)$  then  $\mathbb{R}C^\circ$  is a simple Harnack curve, see [12]. This characterizes real curves of highest and lowest quantum index.

*Example 4* (Quantum indices of real lines). Any real line is a curve of type I and has real coordinate intersections. The quantum index of a real line in  $\mathbb{R}\mathbb{P}^2$  disjoint from the points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ ,  $(0 : 0 : 1)$  is  $\pm \frac{1}{2}$  (depending on its orientation), see Figure 1. The quantum index of a line passing through exactly one of these points is 0.

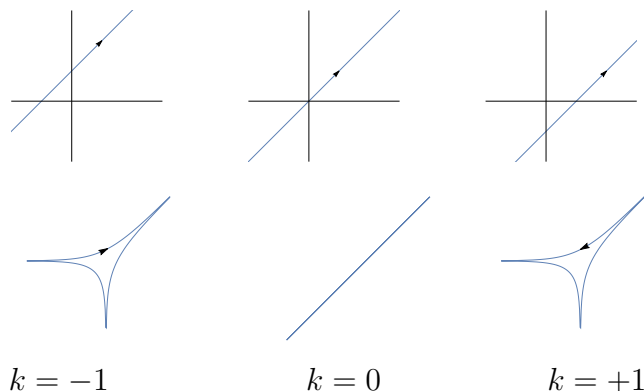


FIGURE 1. Oriented lines, their logarithmic images and quantum indices.

*Example 5* (Quantum indices of real conics). A smooth nonempty real conic is a curve of type I. Figure 2 depicts real conics in  $\mathbb{RP}^2$  that intersect the coordinate axes in 6 real points.

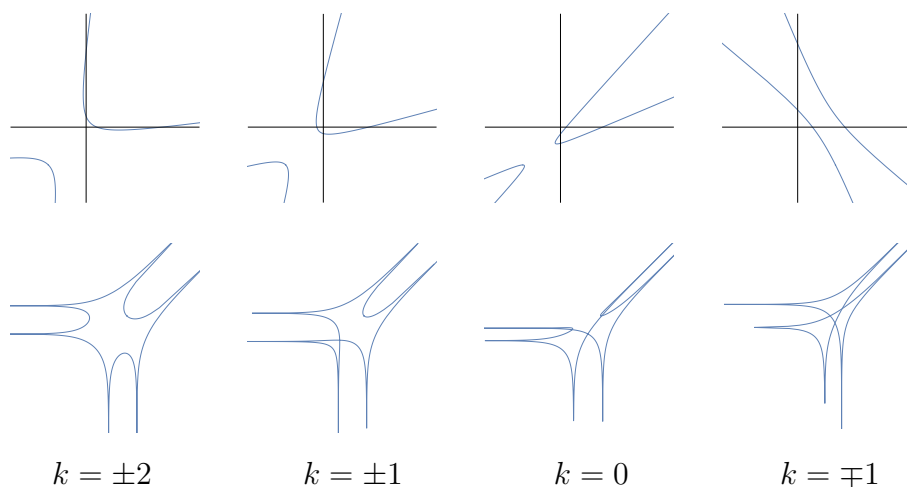


FIGURE 2. Projective hyperbolas, their logarithmic images and quantum indices.

Note that a circle in  $\mathbb{R}^2$  intersects the infinite axis of  $\mathbb{RP}^2$  at the points  $(0 : 1 : \pm i)$ . Thus a circle intersecting the coordinate axes of  $\mathbb{R}^2$  in 4 real points has pure real or imaginary coordinate intersections, see Figure 3. A circle passing through the origin in  $\mathbb{R}^2$  has quantum index  $\pm \frac{1}{2}$ .

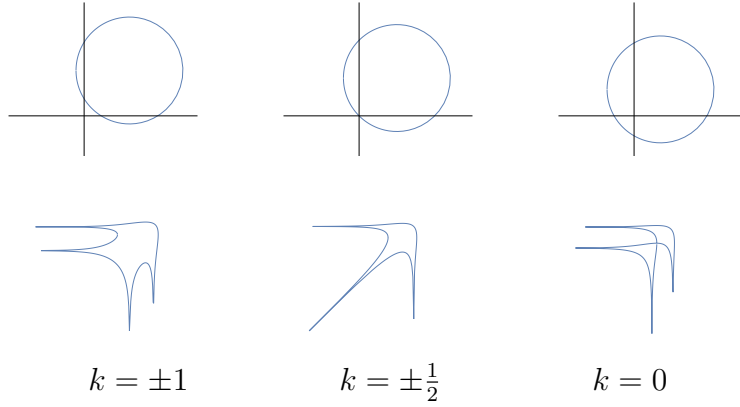


FIGURE 3. Circles, their logarithmic images and quantum indices.

### 3. TORIC TYPE I CURVES: QUANTUM INDICES AND DIAGRAMS

Denote with  $\mathbb{C}\tilde{C}^\circ \subset \mathbb{C}\tilde{C}$  the normalization of an algebraic curve  $\mathbb{C}C^\circ \subset (\mathbb{C}^\times)^2$  and with  $\mathbb{R}\tilde{C}^\circ$  its real part. The composition of the normalization and the inclusion map induces a map  $\mathbb{C}\tilde{C}^\circ \setminus \mathbb{R}\tilde{C}^\circ \rightarrow (\mathbb{C}^\times)^2$ .

**Definition 6.** We say that an irreducible real algebraic curve  $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$  of type I has *toric type I* if the induced homomorphism

$$(3) \quad H_1(\mathbb{C}\tilde{C}^\circ \setminus \mathbb{R}\tilde{C}^\circ) \rightarrow H_1((\mathbb{C}^\times)^2) = \mathbb{Z}^2$$

is trivial.

Each side  $E \subset \Delta$  is dual to a unique primitive integer vector  $\vec{n}(E) \subset \mathbb{Z}^2$  (which sits in the space dual to the vector space containing the Newton polygon  $\Delta$ ) oriented away from  $\Delta$ . We refer to  $\vec{n}(E)$  as *the normal vector to  $E \subset \partial\Delta$* .

**Proposition 7.** *If  $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$  is of toric type I then*

$$\mathbb{C}\bar{C} \cap \partial\mathbb{C}\Delta \subset \mathbb{R}\bar{C} \subset \mathbb{R}\Delta.$$

*In other words  $\mathbb{R}C$  has real coordinate intersection. Thus it has a well-defined quantum index.*

*Proof.* The homology class in  $H_1((\mathbb{C}^\times)^2) = \mathbb{Z}^2$  of a small loop in  $\mathbb{C}C^\circ$  around a point of  $\mathbb{C}\bar{C} \cap \partial\mathbb{C}\Delta$  is a positive multiple of  $\vec{n}(E)$  for a side  $E \subset \Delta$ . Therefore it is non-zero. Thus such loop must intersect  $\mathbb{R}C^\circ$  if  $\mathbb{R}C^\circ$  is of toric type I.  $\square$

Let  $K$  be an oriented component of a real algebraic curve  $\mathbb{R}\tilde{C} \subset \mathbb{R}\Delta$  which has a non-empty intersection with  $\partial\mathbb{R}\Delta$ . Let  $q_1, \dots, q_l \in K \cap$

$\partial\mathbb{R}\Delta$  be the points of intersections with  $\partial\mathbb{R}\Delta$  numbered according to the cyclic order of  $K$ . (Recall that  $K$  is a connected component of the normalized curve  $\mathbb{R}\tilde{C}$ , so the same point of  $\partial\mathbb{R}\Delta$  may lift to more than one of these intersection points.) Let  $E_j$  be the side of  $\Delta$  such that  $q_j$  is contained in the toric divisor corresponding to  $E_j$  (we may have  $E_j = E_{j'}$  for  $j \neq j'$ ) and  $\iota_j$  be the local intersection number of  $\partial\mathbb{C}\Delta$  and  $\mathbb{C}\tilde{C}$  at  $q_j$ .

Let  $\Sigma(K) \subset \mathbb{R}^2$  be the oriented broken line with the vertices to the arcs of  $K \setminus \bigcup_{j=1}^l \{q_j\}$  and edges given by the vectors  $\iota_j \vec{n}(E_j) \in \mathbb{Z}^2$  corresponding to the points  $q_j$  and connecting the vertices corresponding to the adjacent arcs. The starting vertex  $v = (a, b)$  corresponds to the arc  $(q_l, q_1) \subset K$ . We choose  $(a, b)$  so that their parity is compatible with the quadrant  $Q_{(q_l, q_1)} \subset (\mathbb{R}^\times)^2$  containing the arc  $(q_l, q_1)$ , i.e. so that  $Q_{(q_l, q_1)}$  is the  $((-1)^a, (-1)^b)$ -quadrant in  $(\mathbb{R}^\times)^2$ . By induction this implies that the parity of all vertices of  $\Sigma(K)$  are compatible with the quadrants containing the corresponding arcs of  $K \setminus \bigcup_{j=1}^l \{q_j\}$ .

**Proposition 8.** *If  $\mathbb{R}C^\circ$  is a curve of toric type I then the broken line  $\Sigma(K)$  is closed for any connected component  $K$  of  $\mathbb{R}\tilde{C}$ .*

*Conversely, Suppose that  $\mathbb{R}\tilde{C}$  is an  $M$ -curve (i.e. the number of its component is one plus the genus of  $\mathbb{C}\tilde{C}$ ) with  $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C} \cap \partial\mathbb{C}\Delta = \emptyset$  such that  $\Sigma(K)$  is closed for all connected components  $K \subset \mathbb{R}\tilde{C}$ . Then  $\mathbb{R}C^\circ$  has toric type I.*

We prove this proposition together with the proof of the main theorems.

Thus  $\Sigma(K)$  does not depend on the choice of the starting vertex if  $\mathbb{R}C$  is of toric type I and is well-defined up to a translation by  $2\mathbb{Z}^2$ . For such curve we fix one of its complex orientation and denote with  $S$  the corresponding connected component of  $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$ .

**Definition 9.** The closed broken curve  $\Sigma(K) \subset \mathbb{R}^2$  (enhanced with multiplicities as it might pass through the same edge several times) is called *the index diagram* of  $K$ . For the curve  $\mathbb{R}C$  we define its *diagram*  $\Sigma(\mathbb{R}C) \subset \mathbb{R}^2$  as the union of the index diagrams of all the connected components of  $\mathbb{R}\tilde{C}$  that intersect  $\partial\mathbb{R}\Delta$ .

Let

$$(4) \quad \text{Area } \Sigma(\mathbb{R}C) \in \frac{1}{2}\mathbb{Z}$$

be the signed area (with multiplicities) enclosed by  $\Sigma(\mathbb{R}C)$ . It is invariant with respect to translations of the index diagrams of the components by  $2\mathbb{Z}^2$ . Furthermore, once we fix a representative for the diagram  $\Sigma(\mathbb{R}C)$  we may define the linking number  $l_{\Sigma(\mathbb{R}C)}(a, b) \in \mathbb{Z}$  for each point  $(a, b) \in \mathbb{Z}^2$ . Namely,  $l_{\Sigma(\mathbb{R}C)}(a, b) \in \mathbb{Z}$  is the intersection number of the closed oriented curve  $\Sigma(\mathbb{R}C) \subset \mathbb{R}^2$  and a ray in  $\mathbb{R}^2$  emanating from  $(a, b)$  in a generic direction. Here we ignore the intersection at the origin  $(a, b)$  of this ray if it appears. For each quadrant  $Q = ((-1)^a, (-1)^b)\mathbb{R}_{>0}^2$  we define

$$(5) \quad l_{\Sigma(\mathbb{R}C)}(Q) = \sum_{k_a, k_b \in \mathbb{Z}} l_{\Sigma(\mathbb{R}C)}(a + 2k_a, b + 2k_b) \in \mathbb{Z}.$$

Clearly,  $l_{\Sigma(\mathbb{R}C)}(Q)$  does not depend on the choice of a representative for  $\Sigma(\mathbb{R}C)$  as translations by  $2\mathbb{Z}^2$  preserve parity of the lattice points.

Any connected component  $K \subset \mathbb{R}\tilde{C}$  disjoint from  $\partial\mathbb{R}\Delta$  is contained in a single quadrant  $Q$ . Its image  $\text{Log}(K)$  is a closed oriented curve in  $\mathbb{R}^2$ . Let  $\lambda(K) \in \mathbb{Z}$  be the rotation number of  $\text{Log}(K)$ . E.g. if  $K \subset \mathbb{R}^2$  is a positively oriented embedded circle contained in the  $(+, +)$ - or  $(-, -)$ -quadrant (resp. in the  $(+, -)$ - or  $(-, +)$ -quadrant) then  $\lambda(K) = 1$  (resp.  $\lambda(K) = -1$ ). Any point of  $S \cap Q$  is a real isolated singular point  $p \in \mathbb{R}C^\circ$ . We denote with  $\lambda(p) \in \mathbb{Z}$  the intersection number of  $S$  (enhanced with the complex orientation) and  $Q$  (enhanced with the orientation pulled back from the standard orientation of  $\mathbb{R}^2$  by  $\text{Log}|_Q$ ). Set

$$(6) \quad \lambda_{\mathbb{R}C^\circ}(Q) = - \sum_K \lambda(K) + \sum_p \lambda(p),$$

where the sums are taken over all components  $K$  and all isolated singular points of  $\mathbb{R}C^\circ \cap Q$ .

**Theorem 3.** *If  $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$  is a real algebraic curve of toric type I enhanced with a choice of its complex orientation then*

$$(7) \quad k(\mathbb{R}C) = \text{Area } \Sigma(\mathbb{R}C).$$

Furthermore, we have

$$(8) \quad \lambda_{\mathbb{R}C^\circ}(Q) = l_{\Sigma(\mathbb{R}C)}(Q)$$

for each quadrant  $Q \subset (\mathbb{R}^\times)^2$ .

The equality (8) may be viewed as *the toric complex orientation formula* for toric type I curves. The following lemma contains a strengthening of (8). Let  $M$  be a connected compact oriented surface of genus equal to the genus of  $S$  and with the number of boundary components equal to the number of components of the diagram  $\Sigma(\mathbb{R}C)$ .



**Lemma 10.** *If  $\mathbb{R}C^\circ$  is of toric type I then there exists a continuous finite-to-one map*

$$(9) \quad \alpha : M \rightarrow \mathbb{R}^2$$

and a 1-1 correspondence between the points  $q$  of  $\alpha^{-1}(\mathbb{Z}^2) \setminus \partial M$  and the union of the set of closed components  $K$  of  $\mathbb{R}\tilde{C}^\circ$  with the set of points  $p \in S$  mapped to  $(\mathbb{R}^\times)^2$  by the normalization map  $\mathbb{C}\tilde{C} \rightarrow \mathbb{C}\bar{C}$  satisfying to the following properties.

- The map  $\alpha$  is smooth on  $M^\circ = M \setminus \partial M$ .
- The image  $\alpha(\partial M) \subset \mathbb{R}^2$  is the digram  $\Sigma(\mathbb{R}C)$  (well-defined up to translations by  $2\mathbb{Z}^2$ ).
- A point  $q \in M^\circ$  with  $\alpha(q) = (a, b)$  corresponds to a component  $K$  or an isolated real singular point  $p$  contained in the quadrant  $((-1)^a, (-1)^b)\mathbb{R}_{>0}^2$ .
- The local degree of  $\alpha$  at  $q$  coincides with  $\lambda(K)$  or  $\lambda(p)$  for the corresponding component  $K$  or the point  $p$ .

Furthermore, for any linear map  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$  the map  $L \circ \alpha|_{M^\circ}$  does not have local maxima.

**Example 11.** All real rational curves which intersect  $\partial\Delta$  in  $\#(\partial\Delta \cap \mathbb{Z}^2)$  points (counted with multiplicity) have toric type I as  $\mathbb{C}\tilde{C}^\circ \setminus \mathbb{R}\tilde{C}^\circ$  is the disjoint union of two open disks. Therefore we may compute the quantum index of such curves with the help of Theorem 3.

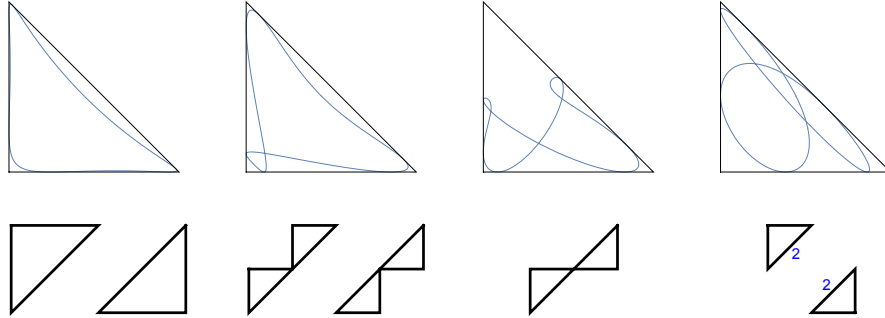


FIGURE 4. Squares of the real conics from Figure 2 and their diagrams  $\Sigma$  for both possible orientations.

Figure 4 depicts the images of the real conics from Figure 2 under  $\text{Fr}_2^\Delta$  (reparameterized with the help of the moment map). Each of these conics may be oriented in two different ways producing two different diagrams. For one of these conics the diagrams for the two opposite orientations coincide. For the other three conics they are different.

Note that in the case of  $\mathbb{C}\Delta = \mathbb{C}\mathbb{P}^2$  the orientation can be uniquely recovered from the diagram as the edges correspond to the normals to  $\Delta$ . E.g. the vertical edges are always directed downwards.

*Remark 12.* The diagram  $\Sigma = \Sigma(\mathbb{R}C)$  may be viewed as a non-commutative version of the polygon  $\Delta$ . Here the set of normal vectors is given an order.

Note that  $-\text{Area } \Delta \leq \text{Area } \Sigma \leq \text{Area } \Delta$  for any collection of closed broken lines  $\Sigma$  whose oriented edges are normal vectors to  $\Delta$  so that each side  $E \subset \Delta$  contribute to  $\#(E \cap \mathbb{Z}^2) - 1$  normal vectors counted with multiplicity. Furthermore, we have  $\text{Area } \Sigma = \pm \text{Area } \Delta$  if and only if  $\Sigma \subset \mathbb{R}^2$  consists of a single broken line and coincides with the polygon  $\Delta$  itself rotated by 90 degrees (as we can represent the primitive normal vector to a vector  $(a, b) \in \mathbb{Z}^2$  by  $(-b, a)$  identifying the vector space  $\mathbb{R}^2$  with its dual).

Recall that the cyclically maximal position of  $\mathbb{R}\bar{C} \subset \mathbb{R}\Delta$  in  $\mathbb{R}\Delta$  from Definition 2 of [8]. It can be rephrased that  $\mathbb{R}\bar{C}$  has a connected component  $K$  intersecting  $\partial\mathbb{R}\Delta$  in  $m = \#(\partial\Delta \cap \mathbb{Z}^2)$  points counted with multiplicity, and the cyclic order of the intersection points on  $K$  agrees with the cyclic order on  $\partial\Delta$ . This condition is equivalent to the equality 10.

The number of points in  $(\Delta \setminus \partial\Delta) \cap \mathbb{Z}^2$  is equal to the arithmetic genus  $g$  of  $\mathbb{R}\bar{C}$ . On the other hand, Lemma 10 implies that the total number of closed connected components of  $\mathbb{R}C^\circ$  and its isolated real singular points is at least  $g$ . Thus all closed components of  $\mathbb{R}C^\circ$  are smooth ovals and all singular points of  $\mathbb{R}C^\circ$  are solitary double points, and the curve  $\mathbb{C}C^\circ$  is a nodal  $M$ -curve. We get the following corollary of Lemma 10.

**Corollary 13.** *If  $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$  is a curve of toric type I with*

$$(10) \quad \text{Area } \Sigma(\mathbb{R}C) = \pm \text{Area } \Delta$$

*then it is an  $M$ -curve whose only singularities are solitary real nodes.*

*Furthermore, the topological type of  $(\mathbb{R}\Delta; \mathbb{R}\bar{C}, \partial\mathbb{R}\Delta)$  is obtained from the unique topological type of a smooth real curve  $\mathbb{R}\bar{A}$  transversal to  $\partial\mathbb{R}\Delta$  by contracting some of the ovals of  $\mathbb{R}A^\circ$  to solitary double points and replacing some  $n$ -tuples of consecutive transversal intersection points of  $\mathbb{R}\bar{A}$  and  $\partial\mathbb{R}\Delta$  (sitting on the same toric divisor) with points of  $n$ th order of tangency.*

*Proof.* The curve  $\text{Fr}_2^\Delta(\mathbb{C}C^\circ)$  also has toric type I. Its diagram is obtained from  $\Sigma(\mathbb{R}C)$  by scaling both coordinates by 2, so the quantum index of  $\text{Fr}_2^\Delta(\mathbb{C}C^\circ)$  is equal to  $\pm \text{Area}(2\Delta)$ . Lemma 10 implies that the only singularities of  $\text{Fr}_2^\Delta(\mathbb{C}C^\circ)$  are solitary real nodes, so that  $\text{Fr}_2^\Delta(\mathbb{R}\bar{C})$

does not have self-intersections. Therefore for each toric divisor  $\mathbb{R}E$  the order of intersection points on  $\mathbb{R}E$  and that on the component  $K \subset \mathbb{R}\tilde{C}$  agree.

Let us look at the compact components of  $\mathbb{R}C^\circ$ . Their number and distribution among the quadrants of  $(\mathbb{R}^\times)^2$  is determined by the lattice points of  $\Sigma(\mathbb{R}C)$  and thus by  $\Delta$ . Furthermore, Lemma 10 implies that in each quadrant of  $(\mathbb{R}^\times)^2$  all ovals and solitary real nodes of  $\mathbb{R}C^\circ$  have the same orientation. The complex orientation formula [13] ensures that these components cannot be nested among themselves and that they are arranged with respect to  $K$  so that their complex orientation is coherent with the complex orientation of  $K$ .  $\square$

*Remark 14.* The proof of Lemma 10 is applicable also for pseudoholomorphic, and even the so-called *flexible* (see [14]) real curves of toxic type I. Thus Corollary 13 may be considered as a further generalization of the topological uniqueness theorem for simple Harnack curves [8] from its version for pseudoholomorphic curves [1] recently found by Erwan Brugallé.

**Example 15.** The curve sketched on Figure 5 is isotopic to a smooth real quartic curve of type I. Namely, it can be obtained as a perturbation of a union of 4 lines. However, it is not an  $M$ -curve while its diagram coincides with the diagram of the simple Harnack curve of the same degree (i.e. the triangle with vertices  $(0, 0)$ ,  $(0, -4)$ ,  $(-4, -4)$  for one of the orientations). By Corollary 13 this curve is not of toric type I. In other words there is a cycle in  $\mathbb{C}C^\circ \setminus \mathbb{R}C^\circ$  that is homologically non-trivial in  $H_1((\mathbb{C}^\times)^2)$ . Also this can be deduced from the toric complex orientation formula (8).

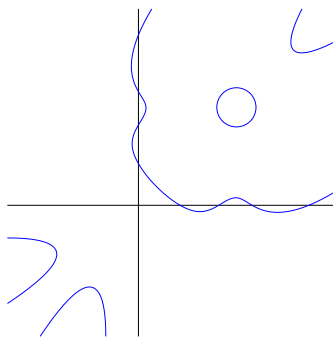


FIGURE 5. A quartic curve of type I, but not of toric type I.

## 4. REFINED REAL ENUMERATIVE GEOMETRY

**4.1. Invariance of real refined enumeration.** Let  $\Delta \subset \mathbb{R}^2$  be a lattice polygon. Let  $E_j \subset \partial\Delta$ ,  $j = 1, \dots, n$ , be its sides of integer length  $m_j = \#(E_j \cap \mathbb{Z}^2) - 1$ . Let  $\mathbb{C}E_j$  be the corresponding toric divisors. Let  $\mathcal{P} = \{p_j\}_{j=1}^m$ ,  $m = \sum_{j=1}^n m_j$ , be a configuration of points on  $\partial\mathbb{R}\Delta$  such that we have exactly  $m_j$  points on the toric divisor  $\mathbb{R}E_j$ .

We require that the first  $m - 1$  points in  $\mathcal{P}$  are chosen on  $\partial\mathbb{R}\Delta$  generically, and that the whole configuration  $\mathcal{P}$  is subject to the condition of the Menelaus theorem (cf. discussion after Proposition 22). This means that the linear system defined by the divisor  $\mathcal{P}$  on the (singular) elliptic curve  $\partial\mathbb{C}\Delta$  is  $\mathcal{O}(\Delta)$ , i.e. any curve with the Newton polygon  $\Delta$  passing through the point  $\{p_j\}_{j=1}^{m-1}$  also passes through  $p_m$ . For any subset of  $m - 1$  points on  $\mathcal{P}$  there exists a unique remaining point with this condition. We say that  $\mathcal{P}$  is a generic configuration of  $m$  points on  $\partial\mathbb{R}\Delta$  subject to *the Menelaus condition*.

The configuration  $(\text{Fr}_2^\Delta)^{-1}(\mathcal{P})$  consists of purely real or imaginary points. Thus any oriented real rational curve  $\mathbb{R}\tilde{C} \subset \mathbb{R}\Delta$  with the Newton polygon  $\Delta$  such that  $\text{Fr}_2^\Delta(\mathbb{R}C)$  passes through  $\mathcal{P}$  has the quantum index  $k(\mathbb{R}C) \in \frac{1}{2}\mathbb{Z}$ . (Note that rationality over the real numbers imply, in particular, that  $\mathbb{R}\tilde{C}$  is non-empty and that  $\mathbb{R}C$  is of type I.)

Define

$$(11) \quad \sigma(\mathbb{R}C) = (-1)^{\frac{m - \text{Rot}_{\text{Log}}(\mathbb{R}C)}{2}}.$$

Since the parities of  $\text{Rot}_{\text{Log}}(\mathbb{R}C)$  and  $m$  coincides we have  $\sigma(\mathbb{R}C) = \pm 1$ .

*Remark 16.* Note that if  $\mathbb{R}C^\circ$  is nodal then its toric elliptic number  $E(\mathbb{R}C)$  has the same parity as the number of solitary nodes of  $\mathbb{R}C^\circ$ . Thus the Welschinger sign  $w(\mathbb{R}C)$  (see [15]) coincides with  $(-1)^{E(\mathbb{R}C)}$ . Since the curve  $\mathbb{R}\tilde{C}$  intersects the union  $\partial\mathbb{R}\Delta$  of toric divisors in  $m$  distinct points and  $\frac{m}{2} \equiv \text{Area}(\Delta) \pmod{1}$  by Pick's formula, we have

$$(12) \quad \sigma(\mathbb{R}C) = (-1)^{\text{Area}(\Delta) - k(\mathbb{R}C)} w(\mathbb{R}C)$$

by Theorem 2.

We define

$$(13) \quad R_{\Delta, k}(\mathcal{P}) = \frac{1}{4} \sum_{\mathbb{R}\tilde{C}} \sigma(\mathbb{R}C),$$

where the sum is taken over all oriented real rational curves  $\mathbb{R}\tilde{C}$  with the Newton polygon  $\Delta$  and  $k(\mathbb{R}\tilde{C}) = k$  such that  $\text{Fr}_2^\Delta(\mathbb{R}\tilde{C}) \supset \mathcal{P}$ . We have the coefficient  $\frac{1}{4}$  in the right-hand side of (13) as the group of

the deck transformations of  $\text{Fr}_2^\Delta : \mathbb{C}\Delta \rightarrow \mathbb{C}\Delta$  is  $\mathbb{Z}_2^2$ , so each curve  $\mathbb{R}\bar{C}$  comes in four copies with the same image  $\text{Fr}_2^\Delta(\mathbb{R}\bar{C})$ . (Alternatively, we can take a sum over different oriented images  $\text{Fr}_2^\Delta(\mathbb{R}\bar{C})$  without the coefficient  $\frac{1}{4}$ .)

We call a point in  $\partial\mathbb{R}\Delta$  *positive* if it is adjacent to the quadrant  $(\mathbb{R}_{>0})^2$  and *negative* otherwise. Note that  $(\text{Fr}_2^\Delta)^{-1}(p)$  consists of real points if  $p$  is positive and of purely imaginary points if it is negative. Let  $\lambda_j$  be the number of negative points in  $\mathcal{P} \cap \mathbb{R}E_j$ . Denote  $\lambda = (\lambda_j)_{j=1}^n$ .

**Theorem 4.** *The number  $R_{\Delta,k}(\lambda) = R_{\Delta,k}(\mathcal{P})$  is invariant of the choice of  $\mathcal{P}$  and depends only on  $\Delta$ ,  $k$  and  $\lambda$ .*

If all points of  $\mathcal{P}$  are negative (i.e.  $m_j = \lambda_j$ ) the number  $R_{\Delta,k}(\lambda)$  is the number of real curves  $\mathbb{R}\bar{C}$  contained in the positive quadrant  $\mathbb{R}_{>0}^2 \subset (\mathbb{R}^\times)^2$  and passing through *all* points of the purely imaginary configuration  $(\text{Fr}_2^\Delta)^{-1}(\mathcal{P})$ . For a positive point  $p \in \mathcal{P}$  a curve  $\mathbb{R}\bar{C}$  should pass through one of the two real points in  $(\text{Fr}_2^\Delta)^{-1}(p)$ .

Define

$$(14) \quad \tilde{R}_{\Delta,k}(\mathcal{P}) = \sum_{\mathbb{R}C} \sigma(\mathbb{R}C)$$

where the sum is taken over all curves  $\mathbb{R}C$  with the Newton polygon  $\Delta$  and  $k(\mathbb{R}C) = k$  passing through  $\mathcal{P}$ . Let

$$(15) \quad \Delta_d = \text{ConvexHull}\{(0,0), (d,0), (0,d)\}.$$

We have  $\mathbb{C}\Delta_d = \mathbb{C}\mathbb{P}^2$ .

**Example 17.** The curves  $\mathbb{R}C$  with the Newton polygon  $\Delta_2$  are projective conics. In this case  $j = 3$ ,  $m_1 = m_2 = m_3 = 2$ ,  $m = 6$ . For any generic  $\mathcal{P} \subset \partial\mathbb{C}\mathbb{P}^2$  subject to the Menelaus condition we have a unique conic through  $\mathcal{P}$ . This gives us two oriented curves through  $\mathcal{P}$  of opposite quantum index.

We may assume (applying the reflections in  $x$  and  $y$  axes if needed) that  $\mathcal{P}$  contains a positive point in the  $x$ -axis and a positive point in the  $y$ -axis. As the positivity of the last point of  $\mathcal{P}$  will be determined by the Menelaus condition we have 3 possibility for the non-decreasing sequence  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ . The possible values of  $k(\mathbb{R}C)$  are listed in the following table, cf. Figure 2. In particular in this case  $\tilde{R}_{\Delta_2,k}(\mathcal{P})$  is different for different configurations with the same  $\lambda$  for two last rows of Table 1. Thus the numbers  $\tilde{R}_{\Delta,k}(\mathcal{P})$  may vary when we deform  $\mathcal{P}$ .

Define

$$(16) \quad \tilde{R}_{\Delta,even}(\mathcal{P}) = \sum_{k \in \frac{m}{2} + 2\mathbb{Z}} \tilde{R}_{\Delta,k}(\mathcal{P}); \quad \tilde{R}_{\Delta,odd}(\mathcal{P}) = \sum_{k \in \frac{m}{2} + 1 + 2\mathbb{Z}} \tilde{R}_{\Delta,k}(\mathcal{P}).$$

$\lambda_1$	$\lambda_2$	$\lambda_3$	$k(\mathbb{R}C)$
0	0	0	$\pm 2$
0	0	2	$\pm 1$ or 0
0	1	1	$\pm 1$ or 1

TABLE 1. Quantum indices for conics

**Theorem 5.** *The numbers  $\tilde{R}_{\Delta_d, \text{even}}(\mathcal{P})$  and  $\tilde{R}_{\Delta_d, \text{odd}}(\mathcal{P})$  do not depend of  $\mathcal{P}$  as long as  $d$  is even and all the points of  $\mathcal{P}$  are positive (i.e.  $\lambda = (0, 0, 0)$ ).*

#### 4.2. Refined real and refined tropical enumerative geometry.

We return to the study of the invariant  $R_{\Delta, k}$  from Theorem 4.

**Definition 18.** The sum

$$(17) \quad R_{\Delta}(\lambda) = \sum_{k=-\text{Area}(\Delta)}^{\text{Area}(\Delta)} R_{\Delta, k}(\lambda) q^k$$

is called *the real refined enumerative invariant* of  $(\mathbb{R}^\times)^2$  in degree  $\Delta$ .

If  $\lambda = 0$  (i.e. all  $\lambda_j = 0$ ) then all points of  $(\text{Fr}_2^\Delta)^{-1}(\mathcal{P})$  are real. In such case we use notations  $R_{\Delta, k} = R_{\Delta, k}(0)$  and  $R_{\Delta} = R_{\Delta}(0)$ .

Recall that the Block-Göttsche invariant is a symmetric (with respect to the substitution  $q \mapsto q^{-1}$ ) Laurent polynomial with positive integer coefficients. This polynomial is responsible for the enumeration of the tropical curves with the Newton polygon  $\Delta$  of genus  $g$ , passing through a generic collection of points in  $\mathbb{R}^2$ , see [3]. The expression  $BG_{\Delta}$  defined by (36) may be viewed as the counterpart for the case when the tropical curves pass through a collection of  $m$  points on the boundary of the toric tropical surface  $\mathbb{T}\Delta$  which are generic among those satisfying to the tropical Menelaus condition (31).

**Theorem 6.**

$$R_{\Delta} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{m-2} BG_{\Delta}.$$

**4.3. Holomorphic disk interpretation.** Recall that an orientation of a real rational curve  $\mathbb{R}C$  defines a connected component  $S \subset \mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$ . Let  $D$  be the topological closure of the image of  $S$  in  $\mathbb{C}\Delta$ . The disk  $D \subset \mathbb{C}\Delta$  is a holomorphic disk whose boundary  $\partial D = \mathbb{R}C$  is contained in the Lagrangian subvariety  $\mathbb{R}\Delta \subset \mathbb{C}\Delta$ .

Let  $L$  be the topological closure of the quadrant  $\mathbb{R}_{>0}^2$  in  $\mathbb{C}\Delta$ . Note that  $L$  is a Lagrangian subvariety of  $\mathbb{C}\Delta$  with boundary. The image  $\text{Fr}_2^\Delta(D)$  is a holomorphic disk whose boundary is contained in  $L$ .

Thus the expression (17) may also be interpreted as a refined enumeration of holomorphic disks with boundary in  $L$ , passing through  $\mathcal{P}$ , and tangent to  $\partial\mathbb{R}\Delta$  at the points of  $\mathcal{P}$ .

These disks are images under  $\text{Fr}_2^\Delta$  of disks  $D$  with boundary in  $\mathbb{R}\Delta$  with

$$(18) \quad D \cap \partial\mathbb{C}\Delta = \mathcal{P} \subset \partial\mathbb{R}\Delta.$$

Let  $\mathbb{C}\hat{\Delta}$  be the result of blowup of the toric variety  $\mathbb{C}\Delta$  at  $\mathcal{P}$ . Let  $\hat{L} = \overline{(\mathbb{R}^\times)^2} \setminus \partial\hat{\mathbb{R}}\Delta$  where  $\overline{(\mathbb{R}^\times)^2}$  is the topological closure of  $(\mathbb{R}^\times)^2$  in  $\mathbb{C}\hat{\Delta}$  and  $\partial\hat{\mathbb{R}}\Delta$  is the proper transform of  $\partial\mathbb{R}\Delta$  in  $\mathbb{C}\hat{\Delta}$ . Then a holomorphic disk  $D$  lifts to a holomorphic disk  $\hat{D}$  with the boundary in the non-compact Lagrangian subvariety  $\hat{L} \subset \mathbb{C}\hat{\Delta}$  without boundary. Furthermore, the Maslov index of  $\hat{D}$  is 0. This construction appears in the Kontsevich-Soibelman wall-crossing approach [5].

## 5. PROOFS

**5.1. Proof of Theorems 1, 2 and 3.** Consider the map  $\text{Arg} : (\mathbb{C}^\times)^2 \rightarrow (\mathbb{R}/2\pi\mathbb{Z})^2$  defined by

$$(19) \quad \text{Arg}(z, w) = (\arg(z), \arg(w))$$

and the map  $2 \text{Arg} : (\mathbb{C}^\times)^2 \rightarrow (\mathbb{R}/\pi\mathbb{Z})^2$  obtained by multiplication of  $\text{Arg}$  by 2, cf. [11]. The involution of complex conjugation in  $(\mathbb{C}^\times)^2$  descends to  $(\mathbb{R}/\pi\mathbb{Z})^2$  as the involution  $\sigma : (a, b) \mapsto (-a, -b)$ . Denote with

$$(20) \quad P = (\mathbb{R}/\pi\mathbb{Z})^2 / \sigma$$

the quotient space. The orbifold  $P$  is the so-called *pillowcase*. The projections of the four points  $(0, 0), (\frac{\pi}{2}, 0), (0, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{\pi}{2})$  form the  $\mathbb{Z}_2$ -orbifold locus of  $P$  (the corners of the pillowcase). All other points are smooth. (The orbifold  $P$  can be viewed as the real part of the Kummer surface.) We denote with  $0 \in P$  the origin of  $P$ , i.e. the projection of  $(0, 0)$ . Note that  $(2 \text{Arg})^{-1}(0, 0) = (\mathbb{R}^\times)^2$ . Since the involution  $\sigma$  preserves the orientation, the standard volume form on  $(\mathbb{R}/\pi\mathbb{Z})^2$  induces the standard volume form  $d\text{Vol}_P$  on the orbifold  $P$ .

Let  $\mathbb{R}C$  be a real curve of type I with purely real or imaginary coordinate intersection. Consider the surface  $S^\circ = S \setminus \nu^{-1}(\partial\mathbb{C}\Delta)$ , where  $S$  is the component of  $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$  corresponding to the orientation of  $\mathbb{R}C$  and  $\nu$  is the normalization map (1). Denote with

$$(21) \quad \beta : S^\circ \rightarrow P$$

the composition of the map  $2 \text{Arg}|_{S^\circ} : S^\circ \rightarrow (\mathbb{R}/\pi\mathbb{Z})^2$  and the projection  $(\mathbb{R}/\pi\mathbb{Z})^2 \rightarrow P$ .

Let  $p \in P$  be a regular point of  $\beta$ . A point  $q \in \beta^{-1}(p)$  is called *positive* (resp. *negative*) if locally near  $q$  the map  $\beta$  is an orientation-preserving (resp. orientation-reversing) open embedding. The difference between the number of positive and negative points in  $\beta^{-1}(p)$  is called *the degree at  $p$* . A priori, since  $\beta$  is a non-proper map, the degree at different points could be different.

**Lemma 19.** *We have*

$$(22) \quad \text{Area}_{\text{Log}}(\mathbb{R}C) = \int_{S^\circ} \beta^* d\text{Vol}_P.$$

Furthermore, the degree of  $\beta$  at a generic point of  $P$  is  $2k(\mathbb{R}C)$ .

*Proof.* Consider the form

$$(23) \quad \frac{dx}{x} \wedge \frac{dy}{y} = (d \log |x| + id \arg(x)) \wedge (d \log |y| + id \arg(y)) = \\ d \log |x| \wedge d \log |y| - d \arg(x) \wedge d \arg(y) + \\ i(d \log |x| \wedge d \arg(y) + d \arg(x) \wedge d \log |y|)$$

on  $(\mathbb{C}^\times)^2$ . As it is a holomorphic 2-form, it must restrict to the zero form on any holomorphic curve in  $(\mathbb{C}^\times)^2$ . In particular, the real part of this form must vanish everywhere on  $S^\circ$ , so that  $d \log |x| \wedge d \log |y| = d \arg(x) \wedge d \arg(y)$  on  $S^\circ$ , and thus (22) holds, cf. [7].

The smooth map  $\beta : S^\circ \rightarrow P$  extends to a continuous map

$$(24) \quad \bar{\beta} : \bar{S} \rightarrow P$$

for a surface with boundary  $\bar{S} \supset S^\circ$  such that  $S^\circ = \bar{S} \setminus \partial \bar{S}$ . Each  $p \in \mathbb{C}\bar{C} \cap \mathbb{C}E_j$  corresponds to a geodesic in  $(\mathbb{R}/\pi\mathbb{Z})^2$  in the direction parallel to  $\vec{n}(E_j)$  for a side  $E_j \subset \partial \Delta$ , cf. [11]. Since  $\text{Fr}_2^\Delta(p) \in \mathbb{R}\Delta$  the corresponding geodesic passes through two of the points  $(0, 0)$ ,  $(\frac{\pi}{2}, 0)$ ,  $(0, \frac{\pi}{2})$ ,  $(\frac{\pi}{2}, \frac{\pi}{2})$ . The image of this circle in  $P$  is a geodesic segment connecting the corresponding  $\mathbb{Z}_2$ -orbifold point of  $P$ .

Thus  $\bar{\beta}(\bar{S})$  is a 2-cycle in  $P$  and it covers a generic point  $l$  times (counted algebraically), where  $l$  is a number independent on the choice of a generic point. But then  $\int_{S^\circ} \beta^* d\text{Vol}_P = l \text{Area}(P) = l\pi^2/2$ . Therefore,  $l = 2k(\mathbb{R}C)$ .  $\square$

*Proof of Theorem 1.* We have  $k(\mathbb{R}C) \in \frac{1}{2}\mathbb{Z}$  since  $2k(\mathbb{R}C)$  is the degree of  $\beta$  at a generic point of  $P$  by Lemma 19. Let  $\tilde{a} \in (\mathbb{R}/\pi\mathbb{Z})^2$  be a generic point and  $a \in P$  be the point corresponding to  $\tilde{a}$ . The inverse image  $\beta^{-1}(a)$  consists of points of  $S^\circ$  mapped to  $\tilde{a}$  or  $\sigma(\tilde{a})$ . If  $2 \text{Arg}(p) = -\tilde{a}$  for  $p \in S^\circ$  then  $2 \text{Arg}(\text{conj}(p)) = \tilde{a}$ , where  $\text{conj}(p) \in$



$\text{conj}(S^\circ)$ . Thus we have a 1-1 correspondence between sets  $\beta^{-1}(a)$  and  $R = (2 \text{Arg})^{-1}(\tilde{a}) \cap \mathbb{C}C^\circ$ .

Consider the continuous involution  $\text{conj}_{\tilde{a}} : \mathbb{C}\Delta \rightarrow \mathbb{C}\Delta$  extending the involution of  $(\mathbb{C}^\times)^2$  defined by  $z \mapsto \tilde{a} \text{conj}(\sigma(\tilde{a})(z))$ . Note that the fixed point locus of this involution in  $(\mathbb{C}^\times)^2$  coincides with  $(2 \text{Arg})^{-1}(\tilde{a})$ , cf. [7]. Note that

$$(25) \quad R \subset \mathbb{C}C^\circ \cap \text{conj}_{\tilde{a}}(\mathbb{C}C^\circ)$$

while  $R \setminus (\mathbb{C}C^\circ \cap \text{conj}_{\tilde{a}}(\mathbb{C}C^\circ))$  consists of pairs of points interchanged by the involution  $\text{conj}_{\tilde{a}}$ . For generic  $\tilde{a}$  the curve  $\text{conj}_{\tilde{a}}(\mathbb{C}C^\circ)$  is transverse to  $\mathbb{C}C^\circ$ , while  $\text{conj}_{\tilde{a}}(\mathbb{C}C^\circ) \cap \mathbb{C}C^\circ \cap \partial\mathbb{C}\Delta = \emptyset$ .

Thus the number of points in  $R$  is not greater than  $\#(\mathbb{C}C^\circ \cap \text{conj}_{\tilde{a}}(\mathbb{C}C^\circ))$ , while we have  $\#(\mathbb{C}C^\circ \cap \text{conj}_{\tilde{a}}(\mathbb{C}C^\circ)) = 2 \text{Area}(\Delta)$  by the Kouchnirenko-Bernstein theorem [6]. Thus the degree of  $\beta$  takes values between  $-2 \text{Area}(\Delta)$  and  $2 \text{Area}(\Delta)$ . Also  $\#(R) = 2 \text{Area}(\Delta)$ .  $\square$

*Proof of Theorem 2.* Let us compute the degree of the map (24) at a generic point  $a \in P$  close to the origin  $0 \in P$ . The set  $\bar{\beta}^{-1}(0) \cap S^\circ$  contributes  $2E$  to the degree of  $\bar{\beta}$  as the intersection number gets doubled when we pass from  $(\mathbb{R}/\pi\mathbb{Z})^2$  to  $P$ .

Note that the set  $S_{\mathbb{R}} = \bar{\beta}^{-1}(0) \cap \partial\bar{S}$  can be thought of as the topological closure of  $\mathbb{R}C^\circ$  in  $\bar{S}$  by our assumption of transversality to  $\partial\mathbb{R}\Delta$ . Let  $U \supset S_{\mathbb{R}}$  be a small regular neighborhood of  $S_{\mathbb{R}}$ . The image  $\bar{\beta}(U)$  makes  $\text{Rot}_{\text{Log}}(\mathbb{R}C)$  rotation around  $0 \in P$ . Thus  $S_{\mathbb{R}}$  contributes to  $-\text{Rot}_{\text{Log}}(\mathbb{R}C)$  to the degree of  $\bar{\beta}$  equal to  $2k(\mathbb{R}C)$ . We have the appearance of the negative sign since the basis composed of vectors  $v_1, v_2, iv_1, iv_2$  is negatively-oriented in  $\mathbb{C}^2$  whenever vectors  $v_1, v_2$  are linearly independent over  $\mathbb{C}$ . Thus a positive rotation in  $(i\mathbb{R})^2$  (and therefore also in  $P$ ) corresponds to a negative contribution to the degree of  $\bar{\beta}$ .  $\square$

*Proof of Proposition 8.* If  $\mathbb{R}C$  has toric type I then the map (21) lifts to the universal covering  $\mathbb{R}^2$  of the torus  $(\mathbb{R}/2\pi\mathbb{Z})^2$ . The boundary of the push-forward of  $S^\circ$  to  $\mathbb{R}^2$  multiplied by  $\frac{1}{\pi}$  is a representative of  $\Sigma(\mathbb{R}C)$ , so the broken line  $\Sigma(K)$  must be closed. Conversely, if  $\mathbb{R}\tilde{C}$  is an  $M$ -curve and  $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$  is disjoint from  $\partial\mathbb{C}\Delta$  then  $H_1(S^\circ)$  is generated by the ends of  $S^\circ$  which in turn corresponds to  $\Sigma(\mathbb{R}C)$  and the compact components of  $\mathbb{R}C^\circ$ . Any compact connected component of  $\mathbb{R}C^\circ$  is null-homologous in  $(\mathbb{C}^\times)^2$  as it is contained in a single quadrant of  $(\mathbb{R}^\times)^2$ .  $\square$

*Proof of Lemma 10.* The surface  $M$  is obtained from  $\bar{S}$  by contracting all components of  $\partial\bar{S}$  corresponding to compact components of  $\mathbb{R}C^\circ$ .

The map (9) is obtained by lifting the map (21) to the universal covering space  $\mathbb{R}^2$  of the orbifold  $P$  followed by multiplication by  $\frac{1}{\pi}$ . We have a correspondence between  $\alpha^{-1}(\mathbb{Z}^2) \cap M^\circ$  and the set of compact components of  $\mathbb{R}C^\circ$ . The map  $L \circ \alpha|_{M^\circ}$  is harmonic and thus cannot have local maxima by the maximum principle.  $\square$

*Proof of Theorem 3.* The signed area of  $\beta(S^\circ)$  coincides with the signed area of its lift to the universal covering space and thus with  $\text{Area } \Sigma(\mathbb{R}C)$  multiplied by  $\pi^2$ . Summing up the contributions of local degrees at  $\alpha^{-1}(M)$  for different parities we get (8).  $\square$

## 5.2. Invariance of the numbers $R_{\Delta,k}(\mathcal{P})$ .

*Proof of Theorem 4.* First we compute the dimension of the space of rational curves  $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$  with the Newton polygon  $\Delta$ . Two coordinate functions on  $(\mathbb{C}^\times)^2$  define two meromorphic functions on the Riemann surface  $\mathbb{C}\tilde{C}$ . The zeroes and poles of these functions are  $\partial\mathbb{C}\tilde{C} = \tilde{C} \cap \partial\mathbb{C}\Delta$ . The order of these zeroes and poles is determined by the multiplicity of the corresponding intersection points of  $\mathbb{C}\tilde{C}$  with  $\mathbb{C}E_j$  as well as by the slope of  $E_j \subset \Delta$ . As  $\mathbb{C}\tilde{C}$  is rational, we may freely deform  $\partial\mathbb{C}\tilde{C}$  in  $\mathbb{C}\tilde{C}$  and each such deformation extends to the deformation of the coordinate functions. The group  $PSL_2(\mathbb{C})$  of symmetries of  $\mathbb{C}\tilde{C}$  is 3-dimensional, so the space of deformation of  $\partial\mathbb{C}\tilde{C}$  is  $(\#\partial\mathbb{C}\tilde{C} - 3)$ -dimensional. The resulting curve is well-defined up to the multiplicative translation in  $(\mathbb{R}^\times)^2$ . Altogether, the dimension of the space of rational curves in  $(\mathbb{R}^\times)^2$  is  $m - 1$ , which coincides with the dimension of the space of configurations  $\mathcal{P}$ .

Note that the deformation of a single point  $p$  to  $p'$  in  $\partial\mathbb{C}\tilde{C}$  corresponds to adding rational functions with zeroes and poles only at  $p$  and  $p'$  to the corresponding integer multiples of the coordinate functions. A non-immersed point of  $\mathbb{C}C^\circ$  corresponds to a common zero of the differentials of the coordinate functions. But the differential of each coordinate function can be perturbed separately by addition of a rational function as above. Thus the set  $\mathcal{R}_{\Delta,k}(\mathcal{P})$  of curves in  $(\mathbb{R}^\times)^2$  with Newton polygon  $\Delta$  and quantum index  $k$  passing through  $\mathcal{P}$  is finite, and each curve in this set is immersed.

It is convenient to consider not only conventional irreducible rational curves, but also the so-called *stable rational curves* (cf. [2]) in  $\mathbb{C}\Delta$ . These are possibly disconnected Riemann surfaces  $F$  enhanced with a holomorphic map to  $\mathbb{C}\Delta$  and with a tree  $\Gamma$  whose vertices are the components of  $F$  so that each edge of  $\Gamma$  corresponds to an intersection point of the images of the corresponding components. Conventional rational curves are stable rational curves with the tree  $\Gamma$  consisting of

a single vertex. Any sequence of (stable) rational curves in  $(\mathbb{R}^\times)^2$  with the Newton polygon  $\Delta$  converges to a stable rational curve  $\mathbb{R}D \subset (\mathbb{R}^\times)^2$  with a Newton polygon  $\Delta' \subset \Delta$ . Note that if  $\mathbb{C}\bar{D} \cap \partial\mathbb{C}\Delta$  is disjoint from the points of intersection of toric divisors  $\mathbb{C}E_j$  then  $\Delta' = \Delta$ . Therefore, the map

$$(26) \quad \text{ev} : \mathcal{M}_\Delta \rightarrow \mathcal{M}_{\partial\Delta}$$

is proper. Here the source  $\mathcal{M}_\Delta$  is the space of stable oriented real rational curves with the Newton polygon  $\Delta$ . The target  $\mathcal{M}_{\partial\Delta}$  is the space of conj-invariant configurations of  $m$  points in  $\partial\mathbb{R}\Delta$  with  $m_j$  points in  $\mathbb{C}E_j$ ,  $m = \sum m_j = \#(\partial\Delta \cap \mathbb{Z}^2)$ , subject to the Menelaus condition. The map  $\text{ev}$  sends a curve  $\mathbb{R}C^\circ$  to the configuration  $\text{Fr}_2^\Delta(\mathbb{C}\bar{C}) \cap \partial\mathbb{C}\Delta \subset \partial\mathbb{C}\Delta$ .

We refer to stable rational curves with disconnected  $F$  as *reducible rational curves* in  $\mathbb{C}\Delta$ . Note that the dimension of the space of deformation of such curves is equal to the sum of the space of deformation of their irreducible components, i.e. to  $m$  minus the number of components. Thus for a generic choice of  $\mathcal{P}$  there are no reducible rational curves of Newton polygon  $\Delta$  passing through  $\mathcal{P}$ .

Consider the space  $\mathcal{M}_{\partial\Delta, \lambda} \subset \mathcal{M}_{\partial\Delta}$  of real configurations  $\mathcal{P} \subset \mathbb{R}\Delta$  with  $\lambda_j$  points in  $\mathbb{R}E_j \setminus ((\mathbb{R}_{>0})^2)$ . Any curve in  $\text{ev}^{-1}(\mathcal{M}_{\partial\Delta, \lambda})$  satisfies to the hypothesis of Theorem 1, thus its quantum index is well-defined. Let  $\mathcal{P}, \mathcal{P}' \in \mathcal{M}_{\partial\Delta, \lambda}$  and  $\gamma = \{\mathcal{P}_t\}_{t \in [0,1]} \in \mathcal{M}_{\partial\Delta, \lambda}$  be a smooth generic path connecting two such configurations  $\mathcal{P} = \mathcal{P}_0$  and  $\mathcal{P}' = \mathcal{P}_1$ .

All but finitely many values of  $t$  correspond to  $\mathcal{P}_t$  such that  $\mathcal{R}_{\Delta, k}$  are irreducible. If this finite set is empty then  $\text{ev}^{-1}(\gamma)$  consists of irreducible rational curves. In this case an orientation of a curve uniquely determines the orientations of all curves in its connected component in  $\text{ev}^{-1}(\gamma)$ . Thus,  $\text{ev}^{-1}(\gamma)$  splits to a disjoint union of components according to the quantum index  $k$ . Furthermore, we may deduce that  $R_{\Delta, k}(\mathcal{P}) = R_{\Delta, k}(\mathcal{P}')$  from the Welschinger theorem [15].

For this deduction we note that the curve  $\mathbb{R}\bar{C} \subset \mathbb{R}\Delta$  birationally corresponds to a curve  $\mathbb{R}\hat{C} \subset \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$ . Furthermore, in  $\mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$  we may obtain a deformation  $\mathbb{R}\hat{C}'$  of  $\mathbb{R}\hat{C}$  as follows. We take the Riemann surface  $\mathbb{C}\tilde{C}$  and two coordinate functions with zeroes and poles at  $\partial\mathbb{C}\tilde{C}$ . Each point  $p \in \partial\mathbb{C}\tilde{C}$  is thus associated two numbers,  $n_x$  and  $n_y$ : the order of zero (or pole) of the first and of the second coordinate function at  $p$ . The points are ordered according to the orientation of  $\mathbb{R}\tilde{C}$ . We may replace  $p$  with  $n_x + n_y$  generic points in the neighborhood of  $p$  and ask that the first (resp. the second) coordinate function has a simple zero (or pole) at the first  $n_x$  point (resp. the last  $n_y$  points). The resulting curve  $\mathbb{R}\hat{C}' \subset \mathbb{R}\Delta' = \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$  has Newton

polygon  $\Delta' \supset \Delta$  (which is a rectangle). Denote the critical values of the two coordinate functions on  $\mathbb{R}\tilde{C}$  (resp  $\mathbb{R}\hat{C}'$ ) with  $Z_x$  and  $Z_y$  (resp.  $\hat{Z}'_x$  and  $\hat{Z}'_y$ ). Generically, all these critical values are of order 1, so that  $\#(Z'_x) = 2d_x - 2$  and  $\#(Z'_y) = 2d_y - 2$ . The set  $Z'_x$  (resp  $Z'_y$ ) is obtained from  $Z_x$  (resp.  $Z_y$ ) by a small perturbation (in the circle  $\mathbb{R}\tilde{C}$ ) as well as adding  $n_x - 1$  (resp  $n_y - 1$ ) points near the corresponding point of  $\partial\mathbb{R}\tilde{C}$  in the prescribed relative position. Thus the number of *hyperbolic nodes*, i.e. the points of self-intersection of  $\mathbb{R}\hat{C}'$ , is the same as for  $\mathbb{R}C^\circ$ .

The set  $\text{Fr}_2^\Delta(\mathbb{R}\hat{C}') \cap \partial\mathbb{R}\Delta'$  minus one point (determined by the Menelaus condition) can be further perturbed to a generic set  $\mathcal{Q} \subset \mathbb{R}\Delta' = \mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$ . Similarly the path  $\mathcal{P}_t$  produces a generic path  $\mathcal{Q}_t$  so that the curves from  $\mathcal{R}_{\Delta,k}(\mathcal{P}_t)$  can be perturbed to real rational curves with the Newton polygon  $\Delta'$  such that their images under  $\text{Fr}_2^\Delta$  pass through  $\mathcal{Q}_t$ . The Welschinger signs of the resulting curves in  $\mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$  determine the sign  $\sigma$  for the curves from  $\mathcal{R}_{\Delta,k}(\mathcal{P}_t)$  since the Welschinger sign for a given  $\Delta'$  can be expressed through the number of hyperbolic points and the index  $k$  is locally invariant for deformation in the class of irreducible curves. Therefore, by the local invariance of Welschinger count [15], we have  $R_{\Delta,k}(\mathcal{P}) = R_{\Delta,k}(\mathcal{P}')$  in this case.

Suppose now that

$$(27) \quad \mathcal{R}_\Delta(\mathcal{P}_t) = \bigcup_{k' = -\text{Area}(\Delta)}^{\text{Area}(\Delta)} \mathcal{R}_{\Delta,k'}(\mathcal{P}_t)$$

contains a reducible curve  $\mathbb{R}\bar{D}$  for  $t = t_0$ .

As the dimension of the space of deformation of each component is equal to the number of points in its intersection with  $\partial\mathbb{C}\Delta$  minus 1, for the generic path  $\gamma$  the curve  $\text{Fr}_2^\Delta(\mathbb{R}\bar{D})$  is the union of two irreducible rational immersed curves  $\text{Fr}_2^\Delta(\mathbb{R}\bar{D}_j)$ ,  $j = 1, 2$ . Also, the presence of multiplicative translations in  $(\mathbb{R}^\times)^2$  implies that  $\text{Fr}_2^\Delta(\mathbb{R}\bar{D}_1)$  and  $\text{Fr}_2^\Delta(\mathbb{R}\bar{D}_2)$  intersect transversally.

We may assume that all points  $\text{Fr}_2^\Delta(\mathcal{P}_t)$  except for two points  $p_j(t) \in \mathbb{R}\bar{D}_j$ ,  $j = 1, 2$ , remain independent of  $t \in [t_0 - \epsilon, t_0 + \epsilon]$  for a small  $\epsilon > 0$ . For  $t \in [t_0 - \epsilon, t_0 + \epsilon]$  the deformation  $p_1(t) \in \partial\mathbb{R}\Delta$  determines the deformation  $p_2(t)$  by the Menelaus condition. These two points must belong to two different components of  $\mathbb{R}\bar{D}$  as there are no reducible curves in  $\mathcal{R}_\Delta(\mathcal{P}_t)$  for  $t \in [t_0 - \epsilon, t_0 + \epsilon]$  except for  $t = t_0$ . Denote with  $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_t)$  for  $t_0 - \epsilon t \leq t_0 + \epsilon$  the curves whose images under  $\text{Fr}_2^\Delta$  is close to  $\text{Fr}_2^\Delta(\mathbb{R}\bar{D})$ .

Let us choose some orientations of  $\mathbb{R}\bar{D}_1$  and  $\mathbb{R}\bar{D}_2$ . Then the intersection points of

$$(28) \quad I = \text{Fr}_2^\Delta(\mathbb{R}\bar{D}_1) \cap \text{Fr}_2^\Delta(\mathbb{R}\bar{D}_2)$$

come with the intersection sign in  $\mathbb{R}_{>0}^2$ . The set of positive points  $I_+ \subset I$  has the same cardinality as the set of negative points  $I_- \subset I$ ,  $I = I_+ \cap I_-$  as  $\mathbb{R}_{>0}^2$  is contractible.

The curves in  $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_{t_0 \pm \epsilon})$  are obtained by smoothing a nodal point  $q \in I$  in one of the two ways, one that agrees with our choice of orientation and one that does not. Without loss of generality (changing the direction of the path  $\gamma$  if needed) we may assume that the orientation-preserving smoothing in a point  $q \in I_+$  corresponds to a curve  $\mathbb{R}\bar{D}_{q,+} \in \mathcal{R}_\Delta(\mathcal{P}_{t_0+\epsilon})$  and thus the orientation-reversing smoothing at the same point corresponds to a curve  $\mathbb{R}\bar{D}_{q,-} \in \mathcal{R}_\Delta(\mathcal{P}_{t_0-\epsilon})$ . The following lemma determines the situation at all the other points of  $I$ .

**Lemma 20.** *A curve obtained by the smoothing of a node from  $I_+$  in the orientation-preserving way, or by the smoothing of nodes from  $I_-$  in the orientation-reversing way belongs to  $\mathcal{R}_\Delta(\mathcal{P}_{t_0+\epsilon})$ ,  $\epsilon > 0$ .*

Accordingly, a curve obtained by the smoothing of a node from  $I_-$  in the orientation-preserving way, or by the smoothing of nodes from  $I_+$  in the orientation-reversing way belongs to  $\mathcal{R}_\Delta(\mathcal{P}_{t_0-\epsilon})$ .

*Proof.* Let  $\mathbb{R}D_{q',s} \in \mathcal{R}_\Delta(\mathcal{P}_{t_0+\epsilon})$ ,  $s = \pm 1$ , be the curve obtained by smoothing  $\mathbb{R}D$  at a point  $q' \in I$  according to the sign  $s$ . Note that  $\text{Fr}_2^\Delta(\mathbb{R}D_{q,+})$  and  $\text{Fr}_2^\Delta(\mathbb{R}D_{q',s})$  are tangent to each other at  $m$  points of  $\mathcal{P}_{t_0+\epsilon}$  and must intersect each other at pairs of points close to the nodes of  $I \setminus \{q, q'\}$ . By Pick's formula the total number of intersection is equal to the twice the area of  $2\Delta$ , which is the Newton polygon of  $\text{Fr}_2^\Delta(\mathbb{R}D_{q,+})$  and  $\text{Fr}_2^\Delta(\mathbb{R}D_{q',s})$ . By the Kouchnirenko-Bernstein formula, these curves do not have any other intersection points which implies that  $s = +1$  if  $q' \in I_+$ , see Figure 6.  $\square$

Note that any curve in  $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_{t_0 \pm \epsilon})$  is obtained by smoothing  $\mathbb{R}D$  at a point  $q \in I$ . The quantum index of the result is  $\pm k(\mathbb{R}D_1) \pm k(\mathbb{R}D_2)$ , where the signs are determined by the agreement or disagreement of the orientation of the resulting curve with the chosen orientations of  $\mathbb{R}D_j$ . Since  $\#(I_+) = \#(I_-)$ , Lemma 20 implies that  $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_{t_0+\epsilon})$  and  $\mathcal{R}_\Delta^{\mathbb{R}D}(\mathcal{P}_{t_0-\epsilon})$  have the same number of curves of each quantum index.  $\square$

*Proof of Theorem 5.* By the same reason as above we have  $\tilde{R}_{\Delta_d,k}(\mathcal{P}) = \tilde{R}_{\Delta_d,k}(\mathcal{P}')$  if there are no reducible curves with the Newton polygon  $\Delta_d$  that pass through  $\mathcal{P}_t$ . Also we may assume that if  $\mathbb{R}D$  is a reducible

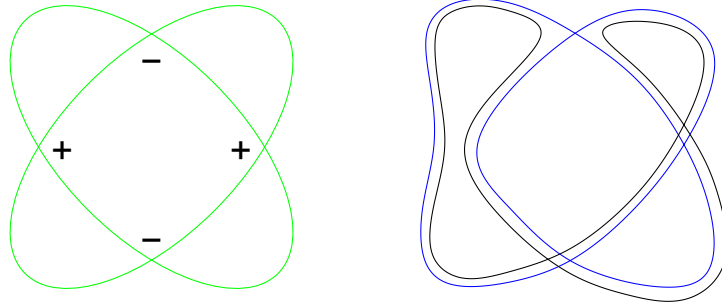


FIGURE 6. The signs of intersection points of two components of  $\mathbb{R}D$  and the corresponding direction of smoothing.

curve with the Newton polygon  $\Delta_d$  passing through  $\mathcal{P}_{t_0}$  then it consists of two components  $\mathbb{R}D_1$  and  $\mathbb{R}D_2$  that intersect transversely at a finite set  $I$ . Note that the degree of both components,  $\mathbb{R}D_1$  and  $\mathbb{R}D_2$ , must be even, as a real curve of odd degree must intersect  $\partial\mathbb{R}P^2$  in a negative point as the boundary of the positive quadrant is null-homologous.

We have two smoothings of  $\mathbb{R}D$  at  $q \in I$  that pass through  $\mathcal{P}_{t_0-\epsilon}$  and  $\mathcal{P}_{t_0+\epsilon}$ . One can be oriented in accordance with the orientations of  $\mathbb{R}D_1$  and  $\mathbb{R}D_2$  and the other in accordance with the orientation of  $\mathbb{R}D_1$ , but opposite to the orientation of  $\mathbb{R}D_2$ . The corresponding quantum indices are different by  $2k(\mathbb{R}D_2)$ . The index  $k(\mathbb{R}D_2)$  is integer since the degree of  $\mathbb{R}D_2$  is even.  $\square$

**5.3. Indices of real phase-tropical curves.** We start by recalling the basic notions of tropical geometry (cf. [9], [10]) specializing to the case of plane curves. Recall that a *metric graph* is a topological space homeomorphic to  $\Gamma^\circ = \Gamma \setminus \partial\Gamma$  enhanced with a complete inner metric. Here  $\Gamma$  is a finite graph and  $\partial\Gamma$  is the set of its 1-valent vertices. The metric graph is also sometimes called a tropical curve (while in some other instances the term tropical curve is reserved for the equivalence class of metric graphs with respect to tropical modifications). In this paper we require the graph  $\Gamma$  to be connected so that  $\Gamma^\circ$  is irreducible as a tropical curve. We assume that  $\Gamma$  has a vertex of valence at least three, and that  $\Gamma$  does not have 2-valent vertices. The half-open edges of  $\Gamma^\circ$  obtained from the closed edges of  $\Gamma$  adjacent to  $\partial\Gamma$  are called *leaves*.

A *plane tropical curve* is a proper continuous map  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  such that  $h|_E$  is smooth for every edge  $E \subset \Gamma$  with  $dh(u) \in \mathbb{Z}^2$  for a unit tangent vector  $u$  at any point of  $E$ . In addition we require the following

*balancing condition* at every vertex  $v \in \Gamma$

$$(29) \quad \sum_E dh(u(E)) = 0,$$

where  $u(E)$  is the unit tangent vector in the outgoing direction with respect to  $v$  and the sum is taken over all edges  $E$  adjacent to  $v$ .

The collection of vectors  $\{dh(u_v)\}_{v \in \partial\Gamma}$  where  $u_v$  is a unit vector tangent to the leaf adjacent to  $v$  (and directed towards  $v$ ) is called *the (toric) degree* of  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$ . The identity (29) implies that the sum of all vectors in this collection is zero. Therefore this collection is dual to a lattice polygon  $\Delta \in \mathbb{Z}^2$  which is well-defined up to translations in  $\mathbb{Z}^2$ . The polygon  $\Delta$  is determined by  $h(\Gamma^\circ)$ . We call  $\Delta$  *the Newton polygon* of  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$ .

Tropical curves appear as limits of scaled sequences of complex curves in the plane. Let  $A$  be any set and  $\alpha \rightarrow t_\alpha \in \mathbb{R}$  be a function unbounded from above (this function is called *the tropical scaling sequence*). Let  $\mathbb{C}C_\alpha \subset (\mathbb{C}^\times)^2$ ,  $\alpha \in A$ , be a family of complex curves with the Newton polygon  $\Delta$ .

**Definition 21.** We say that a family  $\mathbb{C}C_\alpha$  has a *phase-tropical limit* with respect to  $t_\alpha$  if for every  $p \in \mathbb{R}^2$  we have

$$(30) \quad \lim_{t_\alpha \rightarrow +\infty} t_\alpha^{-p} \mathbb{C}C_\alpha = \Phi_p$$

for a (possibly empty) algebraic curve  $\Phi(p) \subset (\mathbb{C}^\times)^2$ . Here  $t_\alpha^{-p} \mathbb{C}C_\alpha$  is the multiplicative translation of the curve  $\mathbb{C}C_\alpha$  by  $t_\alpha^{-p} \in (\mathbb{C}^\times)^2$ . The coefficients of the polynomials defining  $t_\alpha^{-p} \mathbb{C}C_\alpha$  represent a point in the projective space of dimension  $\#(\Delta \cap \mathbb{Z}^2) - 1$ . The limit is understood in the sense of topology of this projective space. The curve  $\Phi_p \subset (\mathbb{C}^\times)^2$  may be reducible and even non-reduced.

We say that  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  is *the tropical limit* of  $\mathbb{C}C_\alpha$  with respect to  $t_\alpha$  if for a sufficiently small open convex neighborhood  $p \in U \subset \mathbb{R}^2$  the irreducible components  $\Psi \subset \Phi(p) \subset (\mathbb{C}^\times)^2$  correspond to the connected components  $\psi \subset h^{-1}(U)$  so that the lattice polygon  $\Delta_\psi$  determined by the ends of the open graph  $\psi$  coincides with the Newton polygon  $\Delta_\Psi$  of the irreducible component  $\Psi$  taken with some multiplicity. The same component  $\Psi$  may correspond to several components of  $h^{-1}(U)$  so that the sum of all resulting multiplicities is equal to the multiplicity of  $\Psi$  in  $\Phi(p)$ . Each connected component of  $h^{-1}(U)$  corresponds to a unique component of  $\Phi(p)$ .

If  $h$  does not contract any edge of  $\Gamma^\circ$  to a point then the open set  $\psi \subset \Gamma^\circ$  may contain at most one vertex. If  $v \in \Gamma$  is such a vertex then we call  $\Psi$  *the phase  $\Phi_v$  of the vertex  $v$* . If  $\psi$  is contained in

an edge  $E$  then we call  $\Psi$  *the phase  $\Phi_E$  of the edge  $E$* . The phases  $\Phi(E) \subset (\mathbb{C}^\times)^2$  do not depend on the choice of a point  $p \in h(E)$  and are well-defined up to multiplicative translations by  $(\mathbb{R}^\times)^2$ . The curve  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  enhanced with the phases  $\Phi_v$  and  $\Phi_E$  for its vertices and edges is called *the phase-tropical limit of  $\mathbb{C}\mathbb{C}_\alpha$  with respect to the scaling sequence  $t_\alpha \rightarrow +\infty$* .

We consider the phases in  $(\mathbb{C}^\times)^2$  that are different by multiplicative translation by vectors from  $(\mathbb{R}_{>0})^2$  equivalent.

Note that the Newton polygon of the phase  $\Phi_E$  of an edge  $E$  is an interval. Thus after a suitable change of coordinates in  $(\mathbb{C}^\times)^2$  the (irreducible) curve  $\Phi_E$  is given by a linear equation in one variable. Therefore,  $\Phi_E$  is a multiplicative translation of a subtorus  $S^1 \approx T_E \subset S^1 \times S^1$  in the direction parallel to  $h(E)$ .

Let us orient  $E$ . Then  $T_E$  as well as the quotient space  $B_E = (S^1 \times S^1)/T_E$  also acquire an orientation. The image  $\text{Arg}(\Phi_E)$  coincides with  $\pi_E^{-1}(\sigma_E)$  for some  $\sigma_E \in B_E$ , where  $\pi_E : S^1 \times S^1 \rightarrow B_E$  is the projection. Since  $B_E$  is isomorphic to  $S^1$  and oriented, we have a canonical isomorphism  $B_E = \mathbb{Z}/2\pi\mathbb{Z}$ . Thus, a phase  $\Phi_E$  of an oriented edge  $E$  of a planar tropical curve is determined by a single argument  $\sigma(E) \in \mathbb{Z}/2\pi\mathbb{Z}$ . The change of the orientation of  $E$  results in the change of sign of  $\sigma(E)$ .

Let  $v \in \Gamma^\circ$  be a vertex and  $E_j$  be the edges adjacent to  $v$ . Orient  $E_j$  outwards from  $v$ . The oriented edges  $E_j$  can be associated a *momentum*  $\mu(E_j)$  with respect to the origin  $0 \in \mathbb{R}^2$ . This is the wedge product of the vector connecting the origin with a point of  $E_j$  and the unit tangent vector  $u(E_j)$  coherent with the orientation. Clearly, it does not depend on the choice of the point in  $E_j$ .

Recall that the vertex  $v$  is dual to the lattice polygon  $\Delta_v$  determined by the integer vectors  $dh(u(E_j))$ . The multiplicity is defined as  $m(v) = 2 \text{Area } \Delta_v$ , cf. [9].

**Proposition 22** (tropical Menelaus theorem). *For any tropical curve  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  and a vertex  $v \in \Gamma^\circ$  the momenta  $\mu(E_j)$  of the edges adjacent to  $v$  and oriented outwards from  $v$  satisfy to the equality*

$$(31) \quad \sum_j \mu(E_j) = 0.$$

*If  $\sigma(E_j) \in \mathbb{Z}/2\pi\mathbb{Z}$  are phases of the oriented edges  $E_j$  then*

$$(32) \quad \sum_j w(E_j)\sigma(E_j) = \pi m(v)$$



(assuming that  $\sigma(E_j)$  appear in the phase-tropical limit of a family  $\mathbb{C}C_\alpha \subset (\mathbb{C}^\times)^2$  of complex curves).

This statement can be viewed as a generalization of the ancient Menelaus theorem stating that three points  $D, E, F$  on the extensions of three sides of a planar triangle  $ABC$  are collinear if and only if

$$(33) \quad \frac{|AD|}{|DB|} \frac{|BE|}{|EC|} \frac{|CF|}{|FA|} = -1.$$

Here the length is taken with the minus sign if the direction of an interval (e.g.  $|CF|$ ) is opposite to the orientation of the triangle, see Figure ??.

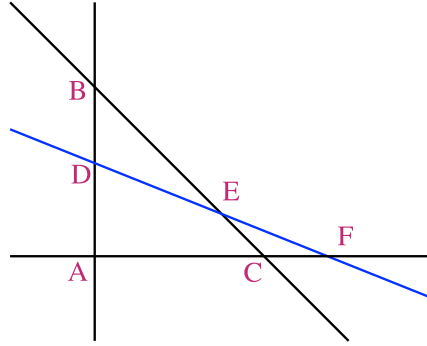


FIGURE 7. The Menelaus theorem.

*Proof.* The wedge product of the balancing condition (29) with the vector connecting 0 and  $v$  gives (31). To deduce (32) we consider the polynomial  $f_v$  (whose Newton polygon is  $\Delta_v$ ) defining the phase  $\Phi_v \subset (\mathbb{C}^\times)^2$ . By Vieta's theorem, the product of the roots cut by  $f_v$  on a divisor of  $\mathbb{C}\Delta_v$  corresponding to an oriented side  $F \subset \Delta_v$  is  $(-1)^{\#(F \cap \mathbb{Z}^2)}$  times the ratio of the coefficients at the endpoints of  $F$ . Therefore the sum of the phases of the edges of  $\Gamma$  corresponding to  $F$  is the argument of this ratio plus  $\#(F \cap \mathbb{Z}^2)\pi$ . Since by Pick's formula the parity of  $\#(\partial\Delta \cap \mathbb{Z}^2)$  coincides with that of  $m(v) = 2 \text{Area}(\Delta_v)$  we recover (32).  $\square$

**Corollary 23.** *We have*

$$\sum_E \mu(E) = 0,$$

where the sum is taken over all leaves of  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  oriented in the outwards direction.

*Proof.* Take the sum of the expression (31) over all vertices of  $\Gamma^\circ$ . The momenta of all bounded edges will enter twice with the opposite signs.  $\square$

If all curves  $\mathbb{C}C_\alpha$  are defined over  $\mathbb{R}$  then the phases  $\Phi(p)$  must be real for all points  $p \in \mathbb{R}^2$ . Note, however that in general, the phase  $\Phi_v$  for a vertex  $v \in \Gamma^\circ$  does not have to be real as the involution of complex conjugation may exchange it with  $\Phi_{v'}$  for another vertex  $v' \in \Gamma$  with  $h(v) = h(v')$ . We say that a vertex  $v$  is *real* if  $\Phi_v$  is defined over  $\mathbb{R}$ .

Let  $\mathbb{R}C_\alpha$  be a scaled sequence of type I curves enhanced with a complex orientation, so that a component  $S_\alpha \subset \mathbb{C}C_\alpha \setminus \mathbb{R}C_\alpha$  is fixed for all  $\alpha$ . Suppose that  $\mathbb{C}C_\alpha$  has a phase-tropical limit, and the orientations of  $\mathbb{R}C_\alpha$  agree with some complex orientations of the real part  $\mathbb{R}\Phi(p)$  of the phases  $\Phi(p)$ . The quantum index of  $\mathbb{R}C_\alpha$  is well-defined if it has a purely imaginary or real coordinate intersection. Similarly, the phase  $\mathbb{R}\Phi_v$  of a real vertex  $v$  of the tropical limit has a well-defined quantum index if  $\sigma(E) \equiv 0 \pmod{\pi}$  for any edge  $E$  adjacent to  $v$ .

**Proposition 24.** *For large  $t_\alpha$  we have*

$$(34) \quad k(\mathbb{R}C_\alpha) = \sum_v k(\mathbb{R}\Phi_v),$$

where the sum is taken over all real vertices whenever all quantum indices in (34) are well-defined.

*Proof.* Additivity of the quantum index with respect to the phases  $\Phi_v$  follows from Theorem 1 through additivity of the degree of the map  $2 \operatorname{Arg}$  restricted to  $S \cap (\mathbb{C}^\times)^2$ . Non-real vertices have zero contribution to  $k(\mathbb{R}C_\alpha)$  as the signed area of the amoeba of the whole complex curve is zero.  $\square$

*Proof of Theorem 6.* Recall the definition of the (tropical) Block-Göttsche invariants, see [3], which refine tropical enumerative invariants of [9]. Namely, to any 3-valent (open) tropical immersed curve  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  we may associate the Laurent polynomial

$$(35) \quad BG(h(\Gamma^\circ)) = \prod_v \frac{q^{\frac{m(v)}{2}} - q^{-\frac{m(v)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

where  $v$  runs over all vertices  $v \in \Gamma$  and  $m(v)$  is the multiplicity of the vertex  $v$ . The genus of a (connected) tropical curve  $\Gamma^\circ$  is the first Betti number of  $\Gamma^\circ$ . In particular, a rational tropical curve is a tree.

Let us fix a collection  $\mu = \{m_j\}_{j=1}^m$ ,  $m = \#(\partial\Delta \cap \mathbb{Z}^2)$ , of generic real numbers subject to the condition  $\sum_{j=1}^m \mu_j = 0$ . This means that  $\mu_j$ ,

$j = 1, \dots, m - 1$  are chosen generically, and  $\mu_m$  is determined from our condition.

If  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  is a tropical curve with the Newton polygon  $\Delta$  then we number its leaves so that the first  $m_1$  leaves are dual to the side  $E \subset \partial\Delta$ , the second  $m_2$  to the side  $E_2 \subset \partial\Delta$  and so on with the last  $m_n$  leaves dual to  $E_n$ . We say that  $h : \Gamma \rightarrow \mathbb{R}^2$  passes through the  $\partial\mathbb{T}\Delta$ -points determined by  $\mu$  if the  $j$ th unbounded edge of  $\Gamma$  has the momentum  $\mu_j$ . Note that a leaf  $E \subset \Gamma^\circ$  must have the momentum  $\mu(E)$  if it passes through a point  $p_E$  on the oriented line parallel to the vector  $(dh)u(E)$  with the momentum  $\mu(E)$ . Thus a generic choice of the momenta ensures that  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  passes through a generic collection of  $m - 1$  points in  $\mathbb{R}^2$ . Thus we have only finitely many rational tropical curves with the Newton polygon  $\Delta$  passing through the  $\partial\mathbb{T}\Delta$  points determined by  $\mu$  by Lemma 4.22 of [9] (as the number of combinatorial types of tropical curves with the given Newton polygon  $\Delta$  is finite). By Proposition 4.11 of [9] all these tropical curves are simple in the sense of Definition 4.2 of [9].

The Block-Göttsche number associated to  $\mu$  is

$$(36) \quad BG_\Delta = BG_\Delta(\mu) = \sum_{h: \Gamma^\circ \rightarrow \mathbb{R}^2} BG(h(\Gamma^\circ)),$$

where the sum is taken over all  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  passing through the  $\partial\mathbb{T}\Delta$  points determined by  $\mu$ . Independence of  $BG_\Delta$  from  $\mu$  can be proved in the same way as in [3]. Also it follows from Theorem 4 once we prove coincidence of  $R_\Delta$  and  $BG_\Delta(\mu)$ .

A toric divisor  $\mathbb{C}E_j \subset \mathbb{C}\Delta$  is the compactification of the torus  $\mathbb{C}^\times$  obtained by taking the quotient group of  $(\mathbb{C}^\times)^2$  by the subgroup defined by the side  $E_j \subset \Delta$ . Thus a configuration  $\mathcal{P} = \{p_j\}_{j=1}^m \subset \partial\mathbb{C}\Delta$  is given by a collection of  $m$  nonzero complex numbers as well as an attribution of the points to the toric divisors. This collection is real if the corresponding numbers are real and positive if these number are positive.

We set  $\mathcal{P}^t = \{p_1^t, \dots, p_m^t\} \subset \partial\mathbb{R}\Delta$  be the configuration of points with the same toric divisor attribution as  $\mathcal{P}$ , and given by the positive numbers  $\{t^{2\mu_j}\}$ ,  $t > 1$ . By Proposition 8.7 of [9] the amoebas of rational complex curves with the Newton polygon  $\Delta$  passing through  $(\text{Fr}_2^\Delta)^{-1}(\mathcal{P}_t)$  converge when  $t \rightarrow +\infty$  to tropical curves passing through the  $\partial\mathbb{T}\Delta$ -points determined by  $\mu$ . Proposition 8.23 of [9] determines the number of complex curves with amoeba in a small neighborhood of a rational tropical curve  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  passing through any choice of points  $\tilde{p}_j^t \in (\text{Fr}_2^\Delta)^{-1}(p_j^t)$ ,  $j = 1, \dots, m - 1$ , for large  $t$ , while Remark 8.25 of [9] determines the number of the corresponding real curves. E.g.

if the weights of all edges of  $\Gamma^\circ$  are odd we have a single real curve for any choice of  $\tilde{\mathcal{P}}^t = \{\tilde{p}_j^t\}$ . In general, some choices of  $\tilde{\mathcal{P}}^t$  may correspond to no real solutions, while others may correspond to multiple solutions. We claim that nevertheless there are  $2^{m-1}$  different real curves whose amoeba is close to  $h : \Gamma^\circ \rightarrow \mathbb{R}^2$  with the image under  $\text{Fr}_2^\Delta$  passing through  $\mathcal{P}^t$  for large  $t$ . Thus we have  $2^m$  different oriented curves. We show this by induction on  $m$  as follows.

If  $\Gamma^\circ$  has a single vertex  $v$  (so that  $m = 3$ ) then there are 4 different real rational phase  $\Phi_v$  which differ by the deck transformations of the map  $\text{Fr}_2^\Delta$ . Thus we have 8 different oriented real rational phases in this case. The positive logarithmic rotation number for half of them is positive, for the other half is negative. Adding each new 3-valent vertex  $v'$  to the tree  $\Gamma$  doubles the number of oriented real phases as there are two ways to attach the phase for  $v'$ : so that the logarithmic rotation number of the resulting real curve will increase by one and so that it will decrease by one. Inductively we get 4 real oriented curves for each of the  $2^{m-2}$  possible distribution of signs for the vertices of  $\Gamma^\circ$ .

For each vertex  $v$  the real phase  $\mathbb{R}\Phi_v$  is the image of a line by a multiplicative-linear map of determinant  $m(v)$  by Corollary 8.20 of [9]. Therefore  $k(\mathbb{R}\Phi_v) = \pm \frac{m(v)}{2}$ , where the sign is determined by the degree of the logarithmic Gauß map. According to our sign convention (11) each oriented real curve comes with the sign equal to the number of negative vertices. Thus by Proposition 24 the contribution of  $h : \Gamma \rightarrow \mathbb{R}^2$  to  $R_\Delta(\mathcal{P}^t)$  for large  $t$  is  $\prod_v (q^{\frac{m}{2}} - q^{-\frac{m}{2}})$  which coincides with the numerator of the Block-Göttsche multiplicity (35).  $\square$

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