Tropical curves as scaling limits of deviation sets in sandpiles on the plane

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Definition

A sandpile is a collection of indistinguishable sand grains distributed among a subset $\Gamma$ of $\mathbb{Z}^2$, that is a function $\phi : \Gamma \to \mathbb{N}_0$. A vertex $v$ is unstable if $\phi(v) \geq 4$. An unstable vertex can topple by sending one grain of sand to each of 4 neighbours.
Sandpile model

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\[
\begin{array}{ccc}
3 & 4 & 3 \\
1 & 0 & 1 \\
\end{array}
\begin{array}{ccc}
\text{toppling} \\
\end{array}
\begin{array}{ccc}
4 & 0 & 4 \\
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\end{array}
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Denote by $\phi^\circ$ the result of the relaxation of $\phi$. The maximal stable state is $\langle 3 \rangle = \phi \equiv 3$. 

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Problem: add several grains to $\langle 3 \rangle$, what is the result of the relaxation?
Formalization

\[ \Omega \subset \mathbb{R}^2, \text{ with non-empty interior } \Omega^\circ, \text{ different points } p_1, p_2, \ldots, p_n \in \Omega^\circ. \]

Consider the grid \( \frac{1}{N} \mathbb{Z}^2 \) with the mesh \( \frac{1}{N} \). Let \( \Gamma = \Omega \cap \frac{1}{N} \mathbb{Z}^2 \).

We consider the state \( \phi = \langle 3 \rangle + \sum \delta_{[p_i]} \) on \( \Gamma \).

Look at the relaxation \( \phi^\circ \) of \( \phi \).

Define \( C_N = \{ x \in \Gamma | \phi^\circ \neq 3 \} \).
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Experiments ([2]) suggest that \( \phi^\circ \equiv 3 \) almost everywhere (as long as \( N \) is big enough). More precisely, \( C_N \) is a “thin graph.”
Simulation, snapshots

Just after beginning... ... in the middle... Et voilà ! Final result!

White: 3 grains, Green: 2 grains, Yellow: 1 grain, Red: no grains, Black: more than 3 grains. Blue points – the points $p_i$. 
Consider the number $F(x, y)$ of topplings at a point $(x, y)$ during the relaxation.

**Proposition**

*If the number of sand grains at $(x, y)$ after relaxation is the same as before the relaxation, then*

$$F(x - 1, y) + F(x + 1, y) + F(x, y - 1) + F(x, y + 1) - 4F(x, y) = 0.$$
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**Proof.**

Indeed, count the number of incoming and outgoing grains. 

That is, $F$ is harmonic at $(x, y)$. In fact, $F(x, y)$ is a “piece-wise” linear function.
Sandpile on a disk: a point in the center

Extra grain at the center.

The grid is 3 times finer. The grid is 9 times finer.

White: 3 grains, Green: 2 grains, Yellow: 1 grain, Red: 0 grains
Scaling limit of toppling function

The circle $\Omega = \{(x, y) | x^2 + y^2 \leq 1\}$ and the corresponding limit of $C_N$. The limit $f(x, y)$ of toppling functions $\frac{1}{N} F_N$ satisfies $f(x, y) = \min_{(i,j) \in \mathbb{Z}^2} (a_{ij} + ix + jy)$ where $a_{ij} = -\min_{\Omega}(ix + jy)$. 
\(\Omega\)-tropical curves

**Definition**

Let \(\mathcal{A}\) be a finite subset of \(\mathbb{Z}^2\). A **tropical polynomial** is a function

\[
f(x, y) = \min_{(i, j) \in \mathcal{A}} (a_{ij} + ix + jy), \quad a_{ij} \in \mathbb{R}.
\]

Note that \(f : \mathbb{R}^2 \to \mathbb{R}\).
Ω-tropical curves

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A **tropical curve** is the set of non-smooth points of a tropical polynomial.

**Definition**

An **Ω-tropical series** on $\Omega$ is a function $f : \Omega^\circ \to \mathbb{R}$, $f|_\Omega \geq 0$, $f|_{\partial \Omega} = 0$, $\mathcal{A} \in \mathbb{Z}^2$ is not necessary finite, and $f(x, y) = \min_{(i, j) \in \mathcal{A}} (a_{ij} + ix + jy)$.

An **Ω-tropical analytic curve** on $\Omega$ is the corner locus (i.e. set of non-smooth points) of an Ω-tropical series on $\Omega$.

**Remark**

An Ω-tropical curve is a (locally finite) graph with weights on the edges. At each vertex we have the balancing condition.
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An Ω-tropical curve is a (locally finite) graph with weights on the edges. At each vertex we have the balancing condition.
Let $\Omega \subset \mathbb{R}^2$ be a convex set, with nonempty interior $\Omega^\circ$ of $\Omega$; let $\Omega$ do not contain a line with irrational slope, and let $p_1, \ldots, p_n \in \Omega^\circ$ be different points. We intersect $\Omega$ with the grid $\frac{1}{N}\mathbb{Z}^2$ of the mesh $\frac{1}{N}$ and consider a sandpile model on the vertices of the grid inside $\Omega$, $\Gamma = \Omega \cap \frac{1}{N}\mathbb{Z}^2$. Define $\phi_N = \langle 3 \rangle + \sum \delta_{[p_i]}$, $C_N = \{x \in \Gamma | \phi_N^\circ(x) \neq 3\}$.
Main Theorem

Let $\Omega \subset \mathbb{R}^2$ be a convex set, with nonempty interior $\Omega^\circ$ of $\Omega$; let $\Omega$ do not contain a line with irrational slope, and let $p_1, \ldots, p_n \in \Omega^\circ$ be different points. We intersect $\Omega$ with the grid $\frac{1}{N} \mathbb{Z}^2$ of the mesh $\frac{1}{N}$ and consider a sandpile model on the vertices of the grid inside $\Omega$, $\Gamma = \Omega \cap \frac{1}{N} \mathbb{Z}^2$. Define $\phi_N = \langle 3 \rangle + \sum \delta_{[p_i]}$, $C_N = \{ x \in \Gamma | \phi_N^\circ(x) \neq 3 \}$.

Theorem (Kalinin, Shkolnikov, [1])

*In the above hypothesis, the sequence $C_N$ converges (by Hausdorff, on compacts) to an $\Omega$-tropical curve $C$. 
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Theorem (Kalinin, Shkolnikov, [1])

*In the above hypothesis, the sequence $C_N$ converges (by Hausdorff, on compacts) to an $\Omega$-tropical curve $C$."

There is a relatively simple algorithm how to find the equation of $C$ for the given $\Omega$ and the set $p_1, \ldots, p_n$ (without running an actual simulation for the corresponding sandpile).
Technicalities, I (relaxation)

We consider only locally finite relaxations, i.e. we perform only finite number of topplings at each vertex. Let $F^N_{\Omega, p_1, \ldots, p_n}$ be the toppling function of the relaxation of $\phi_N$ (it exists if $\Omega$ as in the theorem).
We consider only **locally finite** relaxations, i.e. we perform only finite number of topplings at each vertex. Let $F^N_{\Omega,p_1,\ldots,p_n}$ be the toppling function of the relaxation of $\phi_N$ (it exists if $\Omega$ as in the theorem).

**Main Idea:** The sequence of functions $\frac{1}{N} F^N_{\Omega,p_1,\ldots,p_n}$ converges to an $\Omega$-tropical series $f_{\Omega,p_1,\ldots,p_n}$ (piece-wise linear non-negative function on $\Omega$), this function defines $C$. 
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The function $f_{\Omega,p_1,...,p_n}$ is the pointwise minimal $\Omega$-tropical series on $\Omega$, non-smooth at the points $p_1, \ldots, p_n$. This is an incarnation of the Least Action Principle for sandpiles.
Technicalities, I (relaxation)

We consider only **locally finite** relaxations, i.e. we perform only finite number of topplings at each vertex. Let $F_N^{\Omega,p_1,\ldots,p_n}$ be the toppling function of the relaxation of $\phi_N$ (it exists if $\Omega$ as in the theorem).

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The function $f_{\Omega,p_1,\ldots,p_n}$ is the **pointwise minimal** $\Omega$-tropical series on $\Omega$, non-smooth at the points $p_1, \ldots, p_n$. This is an incarnation of the **Least Action Principle** for sandpiles.

The toppling function $F_{\Omega,p_1,\ldots,p_n}$ of the relaxation of $\phi_N$ is the pointwise minimal function among those $F$, such that $F \geq 0$ on the graph, $-3 \leq \Delta F(x) \leq 0$ and $\Delta F([p_i]) < 0$ for all $p_i$. 
We said that $C$ is the Hausdorff limit of $C_N$ on compacts. This means that $\forall K \subset \Omega^\circ$ compact set, $\text{dist}(C_N \cap K, C \cap K) \to 0$ as $N \to \infty$. 

Here $\text{dist}(X, Y)$ for $X, Y \subset K$ is defined as follows: $\text{dist}(X, Y) = \max(\sup_{x \in X} \inf_{y \in Y} \text{dist}(x, y), \sup_{y \in Y} \inf_{x \in X} \text{dist}(x, y))$. 

Firstly, we prove the theorem for $\mathbb{Q}$-polygons (i.e. $\Omega$ is a finite intersection of half-planes with rational slopes). Then we present $\Omega$ as the limit of $\mathbb{Q}$-polygons $\mathbb{Q}'_i, \mathbb{Q}''_i$ such that $\mathbb{Q}'_i \subset \Omega \subset \mathbb{Q}''_i$. Here the condition that $\Omega$ does not contain a line with irrational slope appears.
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Technicalities, II (limits)

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We prove that there exists $\ast$-weak limit of $N(3 - \phi_N^\circ)$ as $N \to \infty$, where $E$ is the set of the edges of $C$, $w(e)$ is the weight of $e$, and $l(e)$ is the length of the primitive vector of $e$ (i.e., the minimal lattice vector of the direction of $e$).
Technicalities, III (weights on the edges of $C$, reconsidered)

We prove that there exists $\ast$-weak limit of $N(3 - \phi_N^\circ)$ as $N \to \infty$, i.e. for any $f : \Omega \to \mathbb{R}$, with compact $\text{supp}(f) \subset \Omega^\circ$ we have

$$\int_{\Omega} N(3 - \phi_N^\circ) f = \int_{C} f d\mu$$

Remark ([3])
The number $w(e) \cdot l(e)$ can be though as the degeneration of pushforwards of symplectic area of degenerating complex curves.
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$$\int_{\Omega} N(3 - \phi_N^\circ)f = \int_{C} fd\mu = \sum_{e \in E} \int_{e} f \cdot w(e) \cdot l(e),$$

where $E$ is the set of the edges $e$ of $C$, $w(e)$ is the weight of $e$, $l(e)$ is the length of the primitive vector of $e$ (i.e. the minimal lattice vector of the direction of $e$).

Remark ([3])

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$$\min_{(i,j) \in A} (a_{ij} - C + ix + jy) \leq F(x, y) \leq \min_{(i,j) \in A} (a_{ij} + ix + jy).$$

Here, $i, j, C$ depend only on $\Omega$ and $p_1, p_2, \ldots, p_n$ and do not depend on $N$ (clearly, $a_{ij}$ depend on $N$).
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Here, \( i, j, C \) depend only on \( \Omega \) and \( p_1, p_2, \ldots, p_n \) and do not depend on \( N \) (clearly, \( a_{ij} \) depend on \( N \)). Then, \( C_N \) is the set of points where \( \Delta F \neq 0 \), we apply above estimate and find the limit of \( C_N \).
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Then, $C_N$ is the set of points where $\Delta F \neq 0$, we apply above estimate and find the limit of $C_N$.

**Remark**

The estimate $F(x, y) \leq \min_{(i,j)\in A}(a_{ij} + ix + jy)$ is easy. It implies that the amount of sand lost is of order $N$, therefore $C_N$ is also of order $N$. 

Self-reproducing patterns. The patterns on the picture appear as the smoothings of the piecewise linear integer valued functions. We decompose the relaxation into the sequence of waves, and the patterns just move under the action of the waves, without changing the structure.

The laplacians of the smoothings of $\min(x + 4y, 0)$ and $\min(x + 2y, x + y, 0)$. The third picture is an edge of weight 2.


Thank you for your attention!

Oaxaca, 2015