Tropical curves as scaling limits of deviation sets in sandpiles on the plane

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Sandpile model

Definition

A sandpile is a collection of indistinguishable sand grains distributed among a subset Γ of $\mathbb{Z}^2$, that is a function $\phi : \Gamma \to \mathbb{N}_0$. A vertex v is unstable if $\phi(v) \geq 4$. An unstable vertex can topple by sending one grain of sand to each of 4 neighbours.

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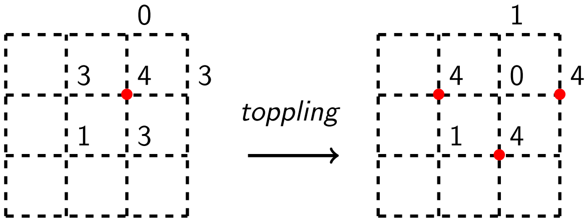

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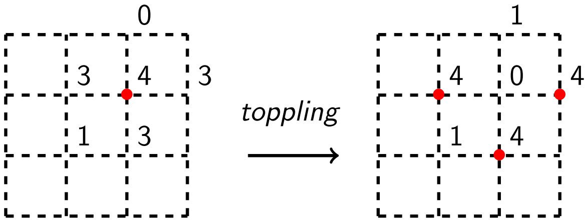

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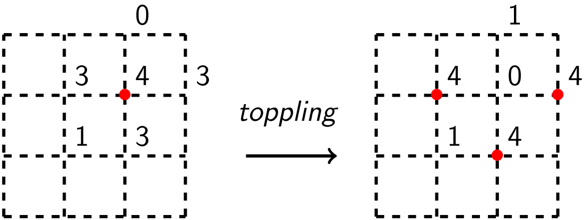

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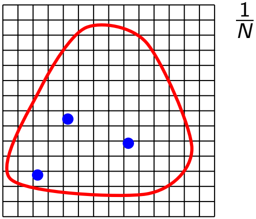
Problem: add several grains to $\langle 3 \rangle$, what is the result of the relaxation?

Formalization

$\Omega \subset \mathbb{R}^2$, with non-empty interior Ω°, different points $p_1, p_2, \ldots, p_n \in \Omega^\circ$. Consider the grid $\frac{1}{N}\mathbb{Z}^2$ with the mesh $\frac{1}{N}$. Let $\Gamma = \Omega \cap \frac{1}{N}\mathbb{Z}^2$.



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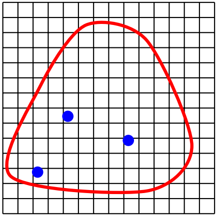

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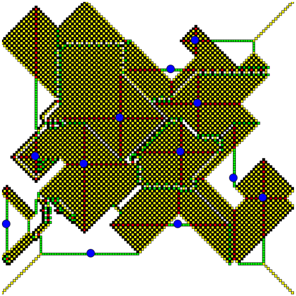
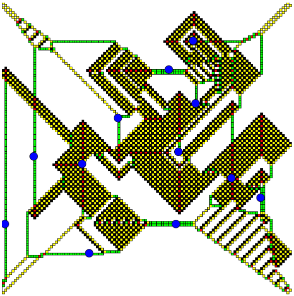
Experiments ([2]) suggest
that $\phi^\circ \equiv 3$ almost everywhere (as long as N is big enough). More precisely, C_N is a "thin graph".

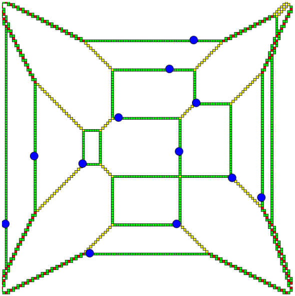
Simulation, snapshots



Just after beginning...



... in the middle...



Et voilà ! Final result!

White: 3 grains, Green: 2 grains, Yellow: 1 grain, Red: no grains, Black: more than 3 grains. Blue points – the points p_i.

Explanation: look at the number of topplings

Consider the number $F(x, y)$ of topplings at a point (x, y) during the relaxation.

Proposition

If the number of sand grains at (x, y) after relaxation is the same as before the relaxation, then
$F(x-1, y) + F(x+1, y) + F(x, y-1) + F(x, y+1) - 4F(x, y) = 0.$

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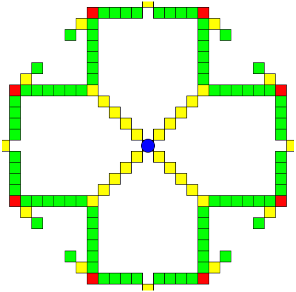
Proof.

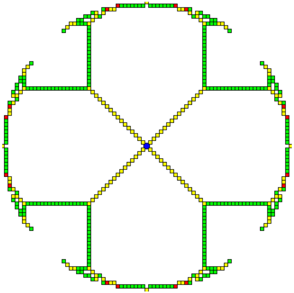
Indeed, count the number of incoming and outgoing grains. □

That is, F is harmonic at (x, y). In fact, $F(x, y)$ is a "piece-wise" linear function.

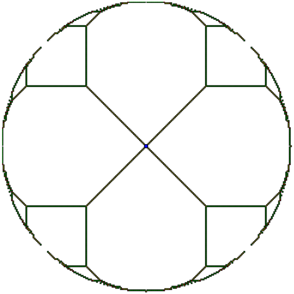
Sandpile on a disk: a point in the center



Extra grain at the center.



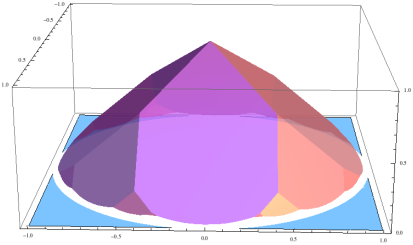
The grid is 3 times finer.

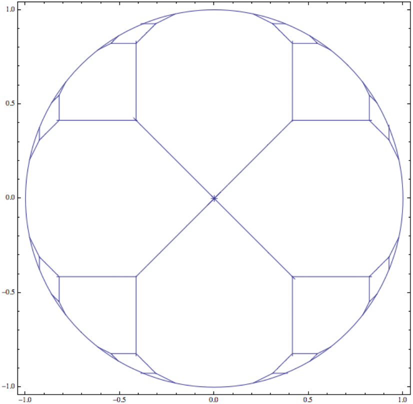


The grid is 9 times finer.

White: 3 grains, Green: 2 grains, Yellow: 1 grain, Red: 0 grains

Scaling limit of toppling function





The circle $\Omega = \{(x, y) | x^2 + y^2 \leq 1\}$ and the corresponding limit of C_N.
The limit $f(x, y)$ of toppling functions $\frac{1}{N} F_N$ satisfies
$f(x, y) = \min_{(i,j)\in\mathbb{Z}^2}(a_{ij} + ix + jy)$ where $a_{ij} = -\min_\Omega(ix + jy)$.

Ω-tropical curves

Definition

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^2$. A tropical polynomial is a function $f(x,y) = \min_{(i,j)\in\mathcal{A}} (a_{ij} + ix + jy), a_{ij} \in \mathbb{R}$. Note that $f : \mathbb{R}^2 \to \mathbb{R}$.

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Definition

An Ω-tropical series on Ω is a function $f : \Omega^\circ \to \mathbb{R}$, $f|_\Omega \geq 0, f|_{\partial\Omega} = 0$, $\mathcal{A} \in \mathbb{Z}^2$ is not necessary finite, and $f(x,y) = \min_{(i,j)\in\mathcal{A}}(a_{ij} + ix + jy)$.

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Remark

An Ω-tropical curve is a (locally finite) graph with weights on the edges. At each vertex we have the balancing condition.

Main Theorem

Let $\Omega \subset \mathbb{R}^2$ be a convex set, with nonempty interior Ω° of Ω; let Ω do not contain a line with irrational slope, and let $p_1, \dots, p_n \in \Omega^\circ$ be different points. We intersect Ω with the grid $\frac{1}{N}\mathbb{Z}^2$ of the mesh $\frac{1}{N}$ and consider a sandpile model on the vertices of the grid inside Ω, $\Gamma = \Omega \cap \frac{1}{N}\mathbb{Z}^2$. Define $\phi_N = \langle 3 \rangle + \sum \delta_{[p_i]}$, $C_N = \{x \in \Gamma | \phi_N^\circ(x) \neq 3\}$.

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There is a relatively simple algorithm how to find the equation of C for the given Ω and the set $p_1, \dots, p_n$ (without running an actual simulation for the corresponding sandpile).

Technicalities, I (relaxation)

We consider only locally finite relaxations, i.e. we perform only finite number of topplings at each vertex. Let $F^N_{\Omega,p_1,\ldots,p_n}$ be the toppling function of the relaxation of ϕ_N (it exists if Ω as in the theorem).

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Main Idea: The sequence of functions $\frac{1}{N}F^N_{\Omega,p_1,\dots,p_n}$ converges to an Ω-tropical series $f_{\Omega,p_1,\dots,p_n}$ (piece-wise linear non-negative function on Ω), this function defines C.

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The function $f_{\Omega,p_1,\ldots,p_n}$ is the pointwise minimal Ω-tropical series on Ω, non-smooth at the points $p_1,\ldots,p_n$. This is an incarnation of the Least Action Principle for sandpiles.

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The toppling function $F^N_{\Omega,p_1,\dots,p_n}$ of the relaxation of ϕ_N is the pointwise minimal function among those F, such that $F \geq 0$ on the graph, $-3 \leq \Delta F(x) \leq 0$ and $\Delta F([p_i]) < 0$ for all p_i.

Technicalities, II (limits)

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This means that $\forall K \subset \Omega^\circ$ compact set, $dist(C_N \cap K, C \cap K) \to 0$ as $N \to \infty$.

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Firstly, we prove the theorem for $\mathbb{Q}$-polygons (i.e. Ω is a finite intersection of half-planes with rational slopes).Then we present Ω as the limit of $\mathbb{Q}$-polygons Q'_i, Q''_i such that $Q'_i \subset \Omega \subset Q''_i$. Here the condition that Ω does not contain a line with irrational slope appears.

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$$\int_\Omega N(3-\phi_N^\circ)f = \int_C fd\mu = \sum_{e\in E}\int_e f\cdot w(e)\cdot l(e),$$

where E is the set of the edges e of C, $w(e)$ is the weight of e, $l(e)$ is the length of the primitive vector of e (i.e. the minimal lattice vector of the direction of e).

Remark ([3])

The number $w(e)\cdot l(e)$ can be though as the degeneration of pushforwards of symplectic area of degenerating complex curves.

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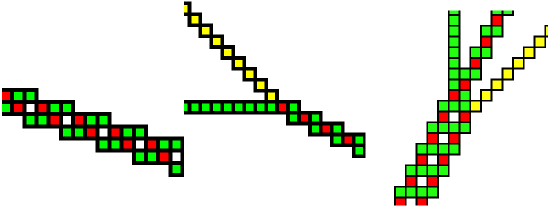
Remark

The estimate $F(x,y) \leq \min_{(i,j)\in\mathcal{A}}(a_{ij} + ix + jy)$ is easy. It implies that the amount of sand lost is of order N, therefore C_N is also of order N.

Proofs, II

Self-reproducing patterns. The patterns on the picture appear as the smoothings of the piecewise linear integer valued functions. We decompose the relaxation into the sequence of waves, and the patterns just move under the action of the waves, without changing the structure.



The laplacians of the smoothings of $\min(x+4y,0)$ and $\min(x+2y,x+y,0)$. The third picture is an edge of weight 2.

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[3] The number of vertices of a tropical curve is bounded by its area, Tony Yue Yu, L'Enseignement Mathématique, 2014.

Thank you for your attention!

Oaxaca, 2015