Tropical curves in Sandpiles II

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Describe ϕ_h° for small h.

Claim

We know that ϕ_h° coincides with the maximal stable state almost everywhere. Therefore, we are interested in describing the shape of the set $E(\phi_h^\circ) = \{v \in \Gamma | \phi_h^\circ(v) < 3\}$.

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There exist an Ω-tropical curve $C = C(\Omega, \{p_i\})$ such that

$$E(\phi_h^\circ) \subset B_r(C \cup \partial\Omega)$$

for $r = O(h)$.

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Lemma

Let ψ be a state such that its toppling function is bounded by $c > 0$. Then $E(\psi^\circ) \subset B_{ch}(E(\psi) \cup \partial\Omega)$.

Finally, let $\psi = \phi_h + \Delta F_-$. Note that $\psi^\circ = \phi_h^\circ$.

Minimization problem

Let H be an Ω-tropical series and $p_1, \dots, p_n$ be a collection of points in Ω°. Consider a space $\mathcal{F}(H, \{p_i\})$ of all Ω-tropical series H' such that $H' \geq H$ and H' is not smooth at all p_i.

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Definition

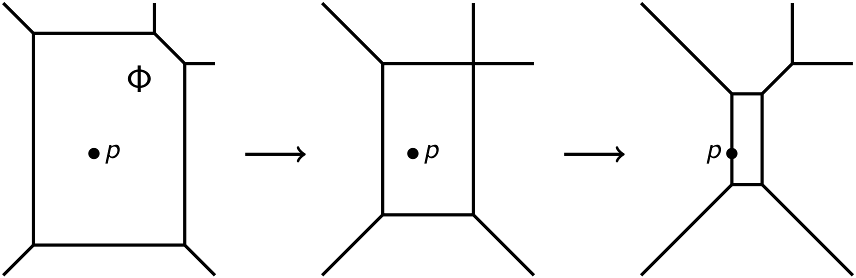
Denote by $G_{p_1,\ldots,p_n}H$ the function on Ω given by

$$G_{p_1,\ldots,p_n}H(v) = \inf_{H' \in \mathcal{F}(H,\{p_i\})} H'(v).$$

Lemma

$G_{p_1,\ldots,p_n}H \in \mathcal{F}(H, \{p_i\})$.

G_p action



Action of G_p by shrinking the face Φ where p belongs to. This corresponds to incrementing the coefficient dual to Φ. Note that combinatorics of the new curve can change.

The curve

Let F_0 be the Ω-tropical series given by

$$F_0 = G_{p_1,\dots,p_n}0$$

and $C = C(\Omega, \{p_i\})$ be the Ω-tropical curve defined by F_0.

We claim that $E(\phi_h^\circ)$ converges to C as h tends to 0.

The upper bound

For a given $h > 0$, consider a non-negative integer-valued function F_+ on Γ given by

$$F_+(v) = [h^{-1} F_0(v)].$$

The function F_+ is superharmonic on Γ and strictly superharmonic at $[p_i]_h$. In particular, $\phi_h + \Delta F_+ \leq 3$ everywhere. Therefore,

$$F \leq F_+$$

by the least action principle.

Lower bounds: reduction to $\mathbb{Q}$-polygons

Definition

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Lemma

For any $\varepsilon > 0$ the set $\Omega_\varepsilon = F_0^{-1}([\varepsilon, \infty)) \subset \Omega$ is a $\mathbb{Q}$-polygon.

This observation allows to reduce the case of general domain to the case of $\mathbb{Q}$-polygon. We can take the lower bound F_- for F on Ω to be the toppling function for the state ϕ restricted to $\Omega_\varepsilon \cap h\mathbb{Z}^2$.

Waves

For each point in $p \in \Gamma$ denote by W_p the **wave operator** acting on the space of stable states on Γ and given by

$$W_p \psi = (T_p \psi)^\circ,$$

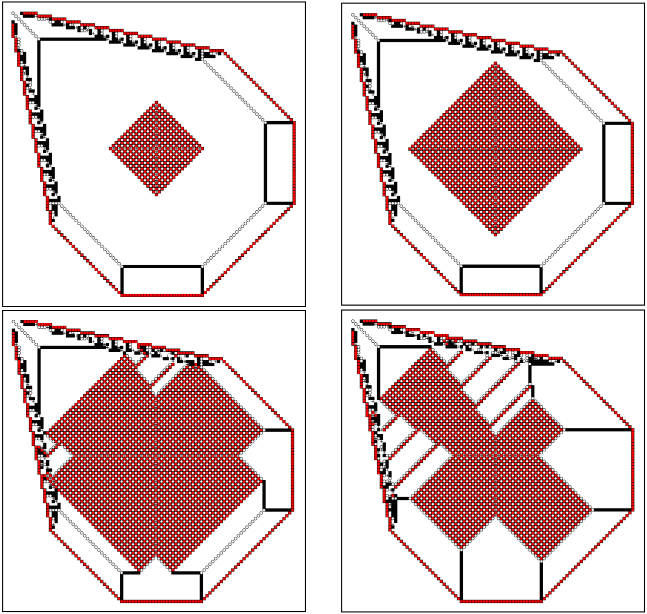
where T_p is the toppling operator.

Lemma

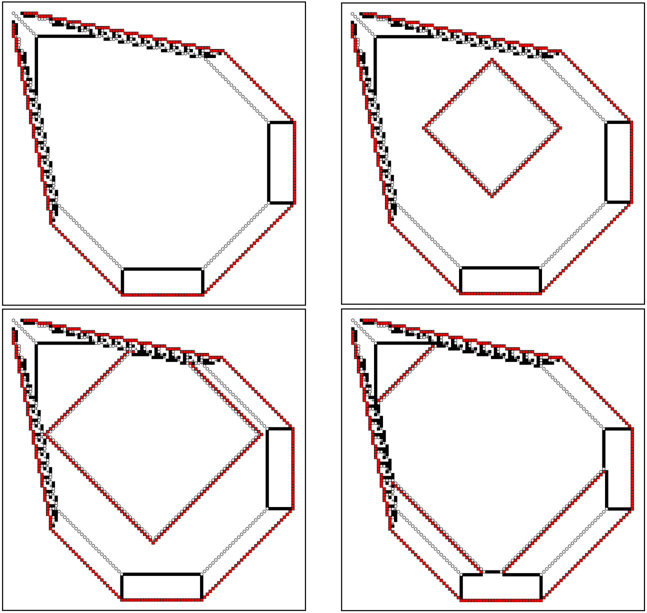
Let ψ be a stable state. Then there exist m such that

$$(\psi + \delta_p)^\circ = W_p^m \psi + \delta_p.$$

One point on a $\mathbb{Q}$-polygon: avalanche



One point on a $\mathbb{Q}$-polygon: waves



Smoothings I

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Consider a tropical polynomial

$$f(x,y) = \min_{(i,j)\in A} (ix + jy + a_{ij})$$

where A is a finite subset of $\mathbb{Z}^2$ and $a_{ij} \in \mathbb{Z}$. Let C be a tropical curve given by f.

Smoothings II

Denote by f_0 the restriction of f to $\mathbb{Z}^2$. Note that f_0 is superharmonic. For any integer $n \geq 0$ consider a space $\mathcal{F}_n$ of all $g\colon \mathbb{Z}^2 \to \mathbb{Z}$ such that

- g is superharmonic
- $f - n \leq g \leq f$
- there exist $r > 0$ such that $g = f$ on $\mathbb{Z}^2 \backslash B_r(C)$.

Denote by f_n the pointwise minimum of all functions in $\mathcal{F}_n$.
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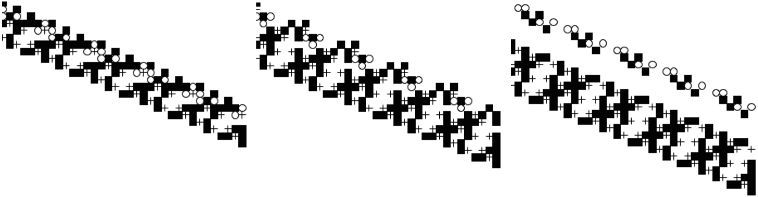
Theorem

The sequence f_n stabilizes if and only if the area of the Newton polygon of C is either

- *a segment of lattice length* 1
- *a triangle with area* $\frac{1}{2}$
- *a parallelogram with area* 1

In these cases C is a local model of an edge, a vertex or a simple node.

Wave action on half-planes



Emergence of a discrete tropical edge of direction $(3, 7)$ under the action by waves. It is an example of a self-reproducing pattern.

Discrete curves

Consider a simple nodal curve $C \subset \Omega$ defined by an Ω-tropical polynomial H. Using the smoothing procedure we can define the state C_h on $\Gamma = \Omega \cap h\mathbb{Z}^2$ such that $E(C_h)$ is close to C.

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Lemma

Let p be a point in $\Omega^\circ \backslash C$. Suppose that $G_p f$ defines a simple nodal curve $\tilde{C}$. Then for h small enough the state $\tilde{C}_h$ coincide with $(C_h + \delta_{[p]_h})^\circ - \delta_{[p]_h}$ outside $B_{O(h)}\partial\Omega$.

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Thus, the operator G_p can be interpreted as a continuous analogue for the operator G_p^h given by

$$\psi \mapsto (\psi + \delta_{[p]_h})^\circ - \delta_{[p]_h}.$$

Finite dynamics

Therefore, in order to find $\phi^\circ = (3 + \sum \delta_{[p_i]_h})^\circ$ we can iteratively apply the operators $G^h_{p_i}$. This gives a process of the type

$$(3) \to G^h_{p_1}(3) \to G^h_{p_2} G^h_{p_1}(3) \to G^h_{p_1} G^h_{p_2} G^h_{p_1}(3) \to G^h_{p_3} G^h_{p_1} G^h_{p_2} G^h_{p_1}(3) \ldots$$

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This motivates to consider a sequence of polynomials $H_0, H_1, \dots$ such that $H_0 = 0$ and $H_{m+1} = G_{p_{k_m}} H_m$ for $k_m = 1, \dots, n$.

Proposition

If each number $i = 1, \dots, n$ appears infinitely many times in the sequence $k_0, k_1 \dots$ then the sequence of functions H_m stabilizes at the function $H_N = G_{p_1,\dots,p_n} 0$.

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The main problem in this approach is that the curve defined by $G_{p_{k_i}} G_{p_{k_{i-1}}} \dots G_{p_{k_1}} 0$ can be very singular and we cannot deal with its discrete analogue. But we still can use this dynamics to get the lower bound for the toppling function of ϕ.

The lower bound

The tropical polynomial $H_{m+1} = G_{p_{k_m}} H_m$ is the result of incrementing by $b_m > 0$ of a certain coefficient of H_m.
Consider a large enough integer $c > 0$. Define a sequence of states $\psi_0, \psi_1 \ldots$ such that $\psi_0 = (3)$ and

$$\psi_{m+1} = W_{p_{k_m}}^{[h^{-1} b_m]-c} \psi_m = \psi_m + \Delta g_m,$$

where g_m is the toppling function of the wave action.

Claim

For "good" Ω the function $F_- = g_0 + \cdots + g_N$ can be taken as the lower bound for the toppling function of $\phi = 3 + \sum \delta_{[p_i]_h}$.

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A general $\mathbb{Q}$-polygon can be deformed to a good one by a sequence of blow-ups. This gives small corrections to the lower bound F_-.

Thanks!