TROPICAL CURVES IN 2-DIMENSIONAL SANDPILE MODEL

NIKITA KALININ, MIKHAIL SHKOLNIKOV

Abstract. We consider the sandpile model on a part of the square lattice, bounded by a polygon. We modify the maximal stable state by adding a grain of sand at each of the n fixed points: the consequent relaxation produces pictures where we can see tropical curves. These curves pass through the same n fixed points and solve a version of the Steiner tree problem: minimization of *tropical symplectic area*. In order to show this, we develop several technics to deal with particular integer-valued solutions of certain Dirichlet problems and to study the continuous version of the considered relaxation which reveals an interesting dynamics on polytopes.

1. Basic definitions and main statements

1.1. **Terminology.** Let Γ be a finite graph with the set of vertices $V(\Gamma)$ and $\partial\Gamma$ be a distinguished subset of $V(\Gamma)$, called *the boundary* of Γ. Vertices in $\Gamma \setminus \partial\Gamma$ are called *internal*. We denote by $n(v) \subset V(\Gamma)$ the set of vertices adjacent to v. A *state* is a non-negative integer function on the set of vertices $V(\Gamma)$, in the following it represents a distribution of sand grains on Γ. A *toppling* T_v at $v \in V(\Gamma) \setminus \partial\Gamma$ is a partially defined operation on the space of states. Namely, for a state ϕ such that $\phi(v) > |n(v)| - 1$ we define a new state $\phi' = T_v(\phi)$ by

$$\phi'(v') = \begin{cases} \phi(v') - |n(v)| & \text{if } \ v = v', \\ \phi(v') + 1 & \text{if } v' \in n(v), \\ \phi(v') & \text{otherwise.} \end{cases}$$

We say that ϕ' is a result of application of *the toppling* at v for a state ϕ. Note that we prohibit to apply toppling at the vertices in the boundary of Γ.

A state ϕ is called *stable* if $\phi(v) < |n(v)|$ for all $v \in \Gamma \setminus \partial\Gamma$. Thus, by definition, topplings can be applied only to *non-stable* states. It is easy to see that if each connected component of Γ contains a vertex from $\partial\Gamma$, then after a finite number of topplings the state becomes stable. Indeed, the total amount of sand in $\Gamma \setminus \partial\Gamma$ is finite and topplings at vertices adjacent to the boundary decrease this amount. The process of applying topplings while it is possible is called *the relaxation.* This version of the abelian sandpile model was defined in [2].

The result of relaxation doesn't depend on the order of topplings [3], that is why it is called an *abelian* model. The *maximal stable state* ϕ_0 is defined by $\phi_0(v) = n(v) - 1$ for every $v \in \Gamma \setminus \partial\Gamma$ and $\phi_0(v) = 0$ for $v \in \partial\Gamma$. For a modern survey about sandpiles we refer the reader to [16].

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Key words and phrases. Tropical curves, sandpile model, tropical dynamics, discrete harmonic functions, Steiner problem.

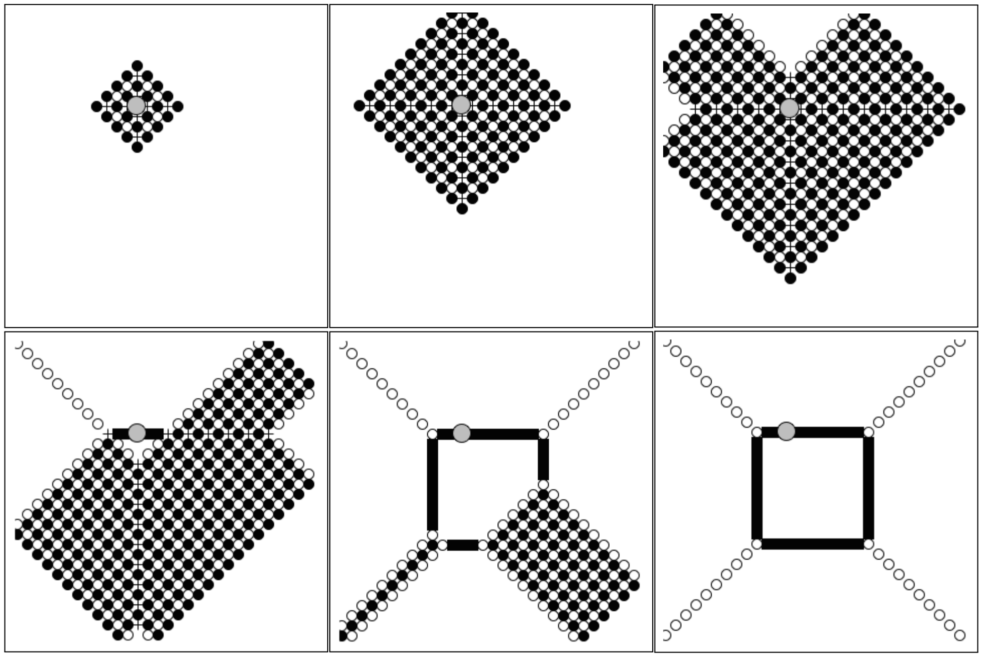

FIGURE 1. Snapshots during the relaxation for the state $\phi \equiv 3$ on a square after adding an extra grain at one point p (the big grey point). Black rounds represent v with $\phi(v) \geq 4$, black squares (which are arranged along the vertical and horizontal edges on the final picture) represent the value of sand equal to 2, white rounds (arranged along diagonals on the final picture) are 1, and whites cells are 3. Rare cells with zero grains are marked as crosses, one can see them during the relaxation on the vertical and horizontal lines through p. The value of the final state at p is 3.

1.2. **Explanation of results.** We are interested in the result of the relaxation for states of a special type on a large graph which is a finite piece of the standard square lattice. Such a state is derived from a maximal stable state by adding an extra grain of sand to vertices $v_1, v_2, \dots, v_n$, i.e. $\phi(v_i) = \phi_0(v_i)+1$ and $\phi(v) = \phi_0(v)$ for all other vertices.

To illustrate the main results, we consider now a very basic example (we learn in from [1]) where $V(\Gamma) = [0,N]^2 \cap \mathbb{Z}^2$, and $\partial\Gamma$ is the intersection of the boundary of the square $[0,N]^2$ with the lattice. Two vertices are connected by an edge iff the distance between them is equal to one. For such a graph the maximal stable state ϕ_0 is equal to 3 at each internal vertex. We define a new state ϕ to be equal ϕ_0 everywhere except for one point p where it is equal to 4. The result of the relaxation for ϕ is shown on Figure 1. We relax ϕ by doing topplings by *generations*, i.e. the first generation is the point p, the second generation is the set of vertices which became unstable after the toppling at p, the third generation is the set of vertices which became unstable after doing topplings at all the vertices in the second generation, etc.

After the relaxation of this configuration we have less than three units of sand on the four segments of the diagonals and the sides of a small square inside, except the point p, where we still have three grains. At all other vertices we have three units of sand. Similar pictures appear in a general situation as explained below.

Now we fix the notation we use throughout this paper. Let Γ be a finite full subgraph of a standard planar lattice graph, bounded by a polygon, $\partial\Gamma$ be the vertices near the boundary of this polygon. Then we take the maximal stable state ϕ_0 and add sand grains to the points $p_1, \dots, p_n$, this gives us a state ϕ^0. We denote result of the relaxation of ϕ^0 by ϕ^{end}. On the other hand, we can drop sand to ϕ_0 into the points $p_1, \dots, p_n$ one by one and relax after each addition. Because of the Abelian property, this gives ϕ^{end} too.

The graphic representation of ϕ^{end}, in general, will be very similar with the above example: vertices with an amount of sand less then three will be arranged along a graph (see Figure 2). Moreover, this graph has a natural structure of a plane tropical curve. This paper, mainly, is devoted to the proof of this result.

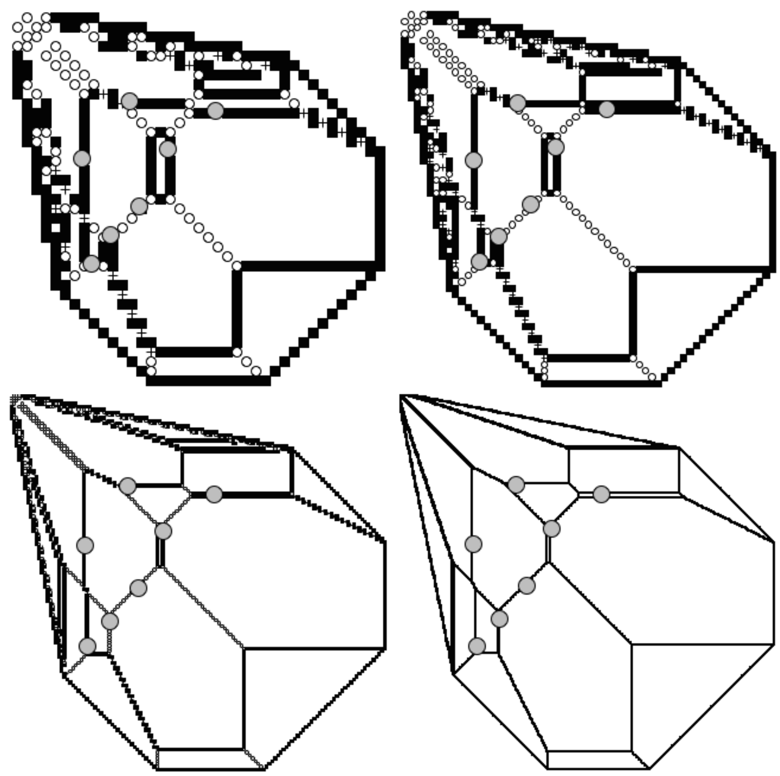

FIGURE 2. Septagonal boundary. The relative position of points and the boundary is remains the same but a scale of the grid is different. In spite of general positions of the points $p_1, ..., p_7$ we have edges with multiplicity bigger than one. The scale of last picture is twenty times bigger than the scale of the first.

In order to recognize a graph in the picture we need to assume that the number n of the points, for which we add extra grains, is small with respect to the size of the graph. In a stable state ϕ, vertices v with $\phi(v) < |n(v)| - 1$ will be called *colored*. We prove that the number of colored vertices in ϕ^{end} is proportional to the product of n and the length of the boundary of Γ. Since the picture appears to be scaling invariant and extremal in a certain sense, in the limit we have a one dimensional object, which reveals to be a tropical curve. This is formalized and summarized in the following subsection.

We learned this model and pictures from [1, 6, 7], where this problem and its properties are approached from the physics perpective. This paper, on the contrary, is devoted to the mathematic aspects of the obtained pictures. We state precise theorems and prove them via studying the limiting behaviour of the relaxation process.

1.3. **Formalization.** In our setup Δ is a lattice polygon, and p_1,...,p_n are distinct points in Δ°, the interior of Δ. Now, let N be a positive integer. Consider a graph Γ_N, a full subgraph of $\mathbb{Z}^2$, with a set of vertices equal to $(N \cdot \Delta) \cap \mathbb{Z}^2$. Let $\partial\Gamma$ be the set of vertices of Γ of degree less than 4. Consider a collection of vertices P_1^N,...,$P_n^N \in V(\Gamma_N)$ defined by taking the coordinate-wise integer part of the points $N \cdot p_1$,...,$N \cdot p_n$ in $N \cdot \Delta$. We define a non-stable state φ_N^0 on Γ_N to be equal 4 at the points P_1^N,...,P_n^N and equal 3 at all other internal points. Consider the result of the relaxation process for ϕ_N^0, a new stable state ϕ_N^{end}. We define an exceptional (colored) set of vertices E_N in the relaxation as a set of all such vertices where ψ_N^{end} is not equal to 3, i.e. $E_N = (\phi_N^{\text{end}})^{-1}\{0, 1, 2\}$.

Theorem 1. The sequence of sets $\frac{1}{N}E_N$ has a limit $\tilde{C}$ in Δ in the Hausdorff sense. Let C be the closure of $\tilde{C}\backslash\partial\Delta$. Then, C is a finite part of a tropical curve. Moreover, C passes through the points $p_1, \dots, p_n$ and the endpoints of C, i.e. $C \cap \partial\Delta$ are exactly the vertices of Δ.

Definition 1. The curve $C = C(\Delta; p_1, \dots, p_n)$ is called the limiting curve for a configuration of points $p_1, \dots, p_n \in \Delta$.

For us, the most exciting thing is that C has a natural structure of a finite part of a plane tropical curve. A *plane tropical curve* is a planar graph, whose edges e are intervals with rational slopes and prescribed positive integer weights $w(e)$. The edges are allowed to be unbounded, the number of edges and vertices is finite and at each vertex the balancing condition is satisfied (refer to Figure 4). The latter means that for each vertex v of a tropical curve the condition $\sum_e w(e)p(e) = 0$ holds, where e runs over all edges incident to v and $p(e)$ is the outgoing primitive vector of e. We need to recall that *the outgoing primitive vector* for an edge ending at the vertex v is a shortest integer vector in the direction of the edge going out of v. For further details on tropical curves see Section 2. A fundamental review of tropical curves can be found in [4].

We mentioned in the theorem that C is a finite part of a tropical curve. We call a planar graph *a finite part of a tropical curve* if it can be represented as an intersection of a planar tropical curve and some polygon P, such that P contains all bounded edges of this curve. Therefore the limiting curve C is a result of a partial cutting of all unbounded edges for a tropical curve. See an example on Figure 1, the colored cells are arranged along a tropical curve; all the weights for the edges in the picture are equal to one.

In general terms, Theorem 1 claims that near the limit the set of colored vertices will be arranged along a well organized finite collection of straight segments with rational slopes. Also it states that large clusters of non-colored vertices are convex in the limit, and for big N all non-convexity is in a small neighborhood of the boundary of a cluster.

It is worth mentioning that the weights for C don't come directly from the limit for the set of colored points. Nevertheless, we can extract the weights with a more delicate analysis of the relaxation process. Namely, instead of looking at the result of the relaxation one can count a number of topplings at each point.

Namely, for any state ϕ on Γ there exist a sequence of states $\phi^0, \dots, \phi^{\text{end}}$ such that $\phi^0 = \phi$, the state ϕ^{end} is stable and ϕ^i is a result of a toppling at a vertex q_i of the state ϕ^{i-1}. As we mentioned before the state ϕ^{end} is a result of relaxation for ϕ^0 and doesn't depend on the sequence of intermediate steps.

Definition 2. Let $Toppl_{\phi^0}(v)$, *the toppling function*, be the number of topplings at v during the relaxation process $\phi^0 \to \cdots \to \phi^{\text{end}}$.

The function $Toppl_{\phi^0}$ depends only on ϕ^0 and doesn't depend on the choice of the sequence $\phi^0, \dots, \phi^{\text{end}}$, and the argument is the same as for the uniqueness of the relaxation. Furthermore, $Toppl_{\phi^0}$ is minimal in certain sense, see Section 5.1.

In our setup we define such functions $Toppl_{\phi_N}$ on $V(\Gamma_N)$ for each N. It appears that $Toppl_{\phi_N}$ behaves well while N grows, namely, $Toppl_{\phi_N}$ is a piecewise linear function almost everywhere. We will see that this linear function provides a natural choice for multiplicities for C. Finally, we will see that C enjoys not only the standard balancing condition at vertices in the interior of Δ but also certain analog of balancing at the boundary. This and the previous observations are made precise in the following theorem.

Theorem 2. A sequence of functions $F_N\colon \Delta \to \mathbb{R}$ given by

$$F_N(x,y) = \frac{1}{N} Toppl_{\phi_N}([Nx],[Ny])$$

uniformly converges to a continuous function $F(\Delta; p_1, \dots, p_n) = F\colon \Delta \to \mathbb{R}$. This function is concave, piecewise linear with integer slopes and vanishes at the boundary of the polygon. Then, C is the locus where $F|_\Delta$ is not smooth, and thus C coincides with a finite part of a tropical curve as a set.

Proof. As explained in the Section 5.1, Theorem 2 implies Theorem 1. □

In other words, F is a tropical polynomial and C is its non-smoothness locus in Δ. Again, see Section 2 for more details on tropical curves and polynomials.

Definition 3. We call the function $F(\Delta; p_1, \dots, p_n)$ *the limiting toppling function* for the configuration $p_1, \dots, p_n$ of points in Δ.

Hence, a collection of points in the polygon gives rise to a curve C, and it is natural to ask what is special about this curve. It turns out that $C(\Delta; p_1, \dots, p_n)$ minimizes the tropical symplectic area in the class of all curves passing through the chosen set of points $\{p_i\}$ and balanced at the boundary of Δ.

Definition 4 (See [8])**.** A tropical symplectic area of a finite segment l with a rational slope is given by $Area(l) = Length(l) \cdot Length(v)$, where $Length(-)$ denotes a Euclidean length and v is a primitive integer vector parallel to l. If C' is a finite part of a tropical curve, then its symplectic area is given by a weighted sum of areas for its edges, i.e.

$$Area(C') = \sum_{e \in E(C')} Area(e) \cdot Weight(e).$$

In Proposition 1 it is proven that the tropical symplectic area comes as a limit for the symplectic area of holomorphic curves and is a deformation invariant. Usually, there is a whole smooth family of area-minimizing curves passing through the given collection of points, see Section 11. Therefore, minimization of area does not specify the curve C. Fortunately, there is another way to do that.

Theorem 3. Let V be a set of all tropical polynomials $\tilde{F}$ that vanish at $\partial\Delta$ and the curve defined by $\tilde{F}$ passes through the points $\{p_i\}$. Then the curve $C(\Delta; p_1, \ldots, p_n)$ has the minimal tropical symplectic area in the class of all curves given by polynomials in V. Additionally, $F(\Delta; p_1, \ldots, p_n)$ is a unique minimum for the functional on V given by

$$\tilde{F} \longmapsto \int_\Delta \tilde{F} dxdy.$$

Definition 5. The curve given by the polynomial minimizing this integral functional will be called *the minimal curve* for the points $p_1, \ldots, p_n$.

Thus, the theorem above states that $C(\Delta; p_1, \ldots, p_n)$ is the minimal curve. The problem of finding such curve can be seen as a tropical analog for the famous Steiner tree problem: given a set of points, find the tree of minimal length containing all these points.

As distinct from mainstream approach started in [2], we do not study the algebraic properties of recurrent configurations, but consider the scaling limit of certain configurations. In some sense, the convergence results of our paper are analogous to the results in [14] and study of the limiting properties can be compared with [17]. But, since the setups and techniques are quite different, this analogy is very limited. For the generalisations and restrictions of hypothesis, see Section 11.

1.4. **Hidden forces in the geometric dynamics and our methods.** Due to complicated logical structure of the proof we recommend the reader to take a glance on the list of methods we use. We sincerely believe that the reader will be able to reproduce all the statements in this article only using these methods and ideas. During the first reading, we recommend to find all the definitions, remarks, propositions, and theorems, understand their statements – usually it is necessary to read a few lines above and below a statement – and only after that start reading the parts of this paper, which seem to be more interesting or suspicious.

Here is the list of principal ideas and methods.

- Looking at the relaxation process after adding a grain to ϕ_0 we observe a sort of *a ripple* this point produces. This suggests the idea of decomposing this ripple into *waves*. A similar decomposition has also appeared in [7].
- Firstly, we want to know what will happen if we drop a grain to the maximal stable state. For this we consider what is going on near the boundary of the rational slope. Since we are interested in the result at the limit $N \to \infty$, during this analysis we can suppose that Γ is just a semi-plane and we send waves at a point distant from the its boundary. We prove that a *discrete tropical edge* will detach from such a boundary. For that, we estimate the amount of sand, which falls to the boundary, and prove that the toppling

function is linear near the boundary. These methods rely on the fact that a toppling function is the minimal solution in certain type of Dirichlet problems.

- Secondly, we carefully studied the process how a discrete tropical edges and vertices appear from *smoothing* of a piecewise linear function. That proves that *discrete tropical edges* move, when we send waves, without changing the structure of their *traceries*. That is why we call such a subset of coloured vertices *self-reproducing*.
- Thirdly, we need to globalize the above consideration. This is done via operators G_p acting on the space of piecewise functions. These G_p are defined in purely tropical terms, but play important role in the proofs about sandpiles. In fact, G_p is an "integral" phenomena while a wave is its "differential" version.
- Fourthly, we need to resolve different problems of non-genericity, which appear even for generic situations and in the proofs. We applied different technics for that. For example, we change the boundary: we cut a relatively small piece of Δ and "freeze" it, i.e. prohibit to perform toppling there. Then, when the rest of the picture is stabilized, we "unfreeze" this part and prove that the topplings we produce now do not change the picture in the limit. Also we have to master states which represents discretization of *all* tropical curves. Since for non-minimal curves we can not obtain their discretization via relaxations, we obtain them via sending waves only.
- Fifthly, we do not need the assumption that the points $p_1, \ldots, p_n$ are in general position, but it is tricky to handle. For that, we study the dynamics of aforementioned operators G_p and prove convergence results about them. Then, the crucial thing is the finiteness of some product of G_p (this product represent the whole schema of relaxation). We prove this type of finiteness for rational $p_1, \ldots, p_n$ and then, we use the convergence results in order to localize picture near each point. Finally, we see that in localized picture the irrationality does not matter.

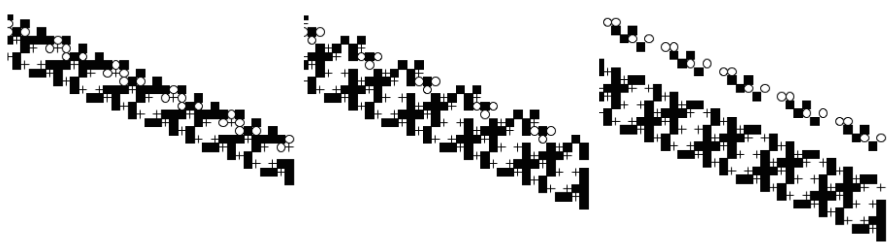

Figure 3. Emergence of a discrete tropical edge of direction $(3, 7)$ under the action by waves. An example of a tracery, which is a self-reproducing pattern.

2. A brief introduction to planar tropical curves and their finite parts

A tropical Laurent polynomial F in two variables is a function on $\mathbb{R}^2$ which can be written as

$$F(x) = \min_{w\in A}(c_w + w\cdot x), \tag{1}$$

where A is a finite subset of $\mathbb{Z}^2$. The numbers $c_w \in \mathbb{R}$ are called *the coefficients* of F and A is called *the support* for the coefficients of F. To emphasize the analogy with usual polynomials, sometimes we write $F =$ "$\sum_{w\in A} c_w x^w$" using the standard tropical addition "$a+b$" $= min(a,b)$ and multiplication "ab" $= a+b$. See [10], [9] or [11] for details and motivation.

The locus of those points where a tropical polynomial F is not smooth is a *tropical curve* (see [9]).

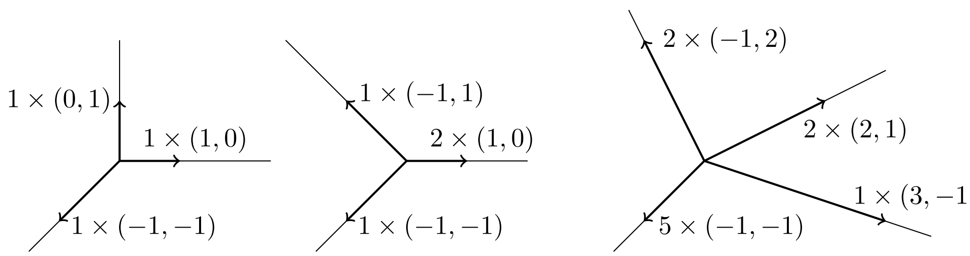

FIGURE 4. Examples of vertices satisfying the balancing condition. We used the balancing condition to define a planar tropical curve (see page 4). The notation $m\times(p,q)$ means that a corresponding edge has a weight m and has a primitive vector (p,q).

A basic object that one associates with a polynomial F is its *Newton polygon* $P(F)$, that is defined as a collection of all integer points in the convex hull of the support for coefficients of F, i.e. $P(F) = \mathbb{Z}^2 \cap Conv(A)$. Note that it is not necessary that all points in $P(F)$ correspond to monomials in F.

Remark 1. We always assume that, given a tropical curve C, we construct a tropical polynomial F which defines C such that $P(F)$ is minimal by inclusion.

The polynomial F gives a subdivision for the polygon $P(F)$ in the following way. The extended Newton polyhedron $\tilde{P}(F) \subset \mathbb{R}^3$ is defined by

$$\tilde{P}(F) = Conv\{(w,t) \in P(F)\times\mathbb{R} | t \geq c_w\}.$$

There is an obvious projection of $\tilde{P}(F)$ to $\mathbb{R}^2$. Taking the image under this projection of one-skeleton of $\tilde{P}(F)$ we get a polyhedral subdivision for $P(F)$.

The tropical curve C defined by the polynomial F is dual to the subdivision of $P(V)$: vertices of the curve correspond to faces in the subdivision, edges of the curve are in one to one correspondence with edges of the subdivision and faces of the curve correspond to the vertices of the subdivision. See [12] for pictures and applications of extended Newton polyhedra.

Note, that not all integer points of the Newton polygon are necessarily the vertices of the subdivision defined by F. If a point $w \in P(F)$ is not a vertex of the subdivision, i.e. it belongs to an interior of a face or an edge, then it means that the coefficient c_w in the expression (1) can be increased without any changes for the function F. In particular, this implies that a function represented by a tropical polynomial doesn't determine the coefficients of the polynomial in general.

Remark 2. So we will always assume that $A = P(F)$ and among all the coefficients $\{c_w\}_{w\in A}$ that represent the same function F the one we used in (1), the definition for F, are the minimal ones.

Note, that from now on, **all** the lattice points in $P(F)$ correspond to monomials in F.

In this paper we deal with a special type of tropical polynomials, the ones that vanish at the boundary of a fixed polygon Δ. Let F be a tropical polynomial such that $F|_{\partial\Delta} = 0$ and C be an intersection of tropical curve defined by F with Δ. So C is a finite part of a tropical curve.

We want to analyze the behavior of F near the boundary. In the neighborhood of any edge e of Δ the function F can be locally written as $x \mapsto c_w + w(e)\cdot x$, where $w(e) \in P(F)$ is orthogonal to e. The integer vector $w(e)$ is a multiple of a certain primitive vector, i.e. $w(e) = m(e)n(e)$, where $n(e)$ is *a primitive normal* vector to e towards the interior of Δ. Thus, for a finite part of a tropical curve C we constructed the function m on the set $E(\Delta)$ of the edges of Δ, $m = m_C \colon E(\Delta) \to \mathbb{Z}_{>0}$.

Definition 6. The aforementioned function m_C is called the *quasi-degree* for the finite part of a tropical curve $C \subset \Delta$.

Remark 3. Note that $m_C(e)n(e) \subset P(F)$ for each $e \in E(\Delta)$. We always assume that the convex hull of the set $\{m_C(e)n(e)\}$ coincides with $P(F)$, since the monomials from the outside of this convex hull don't contribute to $F|_\Delta$.

3. Relation to holomorphic curves

In this section we sketch how one can translate the story about minimal tropical curves (see Definition 5) to the context of classical holomorphic curves. We recommend the reader, whose interest is mostly about sandpiles, to skip this section during the first reading.

Recall that an amoeba of an algebraic curve S in the algebraic torus $(\mathbb{C}^*)^2$ is an image of S in $\mathbb{R}^2$ under the logarithm map Log given by $Log(z_1, z_2) = (log|z_1|, log|z_2|)$. Consider a family of algebraic curves S_t in $(\mathbb{C}^*)^2$ for $t > 0$. We say that the family S_t tropicalizes to the tropical curve C if the family of their rescaled amoebas $Log_t S_t \subset \mathbb{R}^2$ converges to C when t tends to ∞. Here Log_t simply denotes the map $(log\, t)^{-1}Log$. It could seem that the topicalization C is defined only as a set. In fact, the multiplicities for the edges of C can be also canonically restored from the family S_t.

First of all we are going to justify the name " tropical symplectic are" that we extensively used. Suppose a family S_t tropicalizes to a tropical curve C. Let

$$\omega = -id\log(z_1) \wedge d\log(\bar{z}_1) - id\log(z_2) \wedge d\log(\bar{z}_2)$$

be the symplectic form on $(\mathbb{C}^*)^2$.

Proposition 1. Let C be a tropicalization for S_t and B be a convex bounded open subset of $\mathbb{R}^2$. Then

$$\int_{S_t\cap B_t} \omega \underset{t\to\infty}{\sim} 4\pi^2 Area(C \cap B)\log t,$$

where $B_t = Log_t^{-1}(B)$.

Thus the symplectic area for a tropical curve indeed can be interpreted as a main part in the asymptotic for symplectic areas of a family of holomorphic curves.

Proof. The outline is the following. For a large t the rescaled amoeba $Log_t(S_t)$ is in a small neigborhood of the tropical curve C. Moreover S_t itself will be close to a certain lift of C to the torus $(\mathbb{C}^*)^2$. It is performed by lifting each edge with a slope (p, q) to a piece of holomorphic cylinder $\{(z^p, z^q)|z \in \mathbb{C}\}$ translated by the action of the torus. This lift is called a complex tropical curve (see [4] for the details).

Therefore, we can compute the area of S_t near the limit by looking at the areas of the cylinders. There also can be minor corrections coming from the vertices of C but the corrections are small with respect to $log(t)$ and so do not appear in the final statement.

To complete the proof we need to compute the contribution from each edge in $C \cap B$. It is clear that for each segment in $C \cap B$ the area of its lift is proportional to the length of the segment. So if we show that the area of the lift for the interval going from the origin to the integer vector (p, q) is equal to $4\pi^2(p^2 + q^2)\log t$ then we will be done. This computation is given in the following lemma. □

Lemma 1. Let $v = (p, q)$ be a primitive integer vector. Let C_t^{pq} be a lift of an interval $[0, v]$ to the torus $(\mathbb{C}^*)^2$ under Log_t, i.e. $C_t^{pq} = \{(z^p, z^q)|1 \le |z| \le t\}$. Then

$$\int_{C_t^{pq}} d\log(z_1) \wedge d\log(\bar{z}_1) = -4i\pi^2 p^2 \log t.$$

Proof. Let z_1 be $r\exp(i\phi)$, where $r > 0$ and $\phi \in [0, 2\pi]$. Then

$$d\log z_1 = d\log r + i\phi d\phi \text{ and}$$

$$d\log z_1 \wedge d\log \bar{z_1} = -id\log r \wedge d\phi^2$$

Then the left hand side of the equality we are proving is equal to

$$-i\int_1^t \int_0^{2\pi} d\log r \wedge d\phi^2 = -4i\pi^2 p^2 \log t.$$

□

Remark 4. The specific choice for ω is not crucial while it is invariant under the action of $(C^*)^2$. Indeed, if ω' is an arbitrary 2-form then its restriction to any holomorphic curve will not have contributions from pure holomorphic and anti-holomorphic parts of ω'. So we can think that ω' is a $(1, 1)$-form. There is a two dimensional family of torus-invariant $(1, 1)$-forms. Different choices for ω from this family correspond to coordinate dilatations on the level of tropical curves.

Proposition 1 suggests us that symplectic area for tropical curves should be deformation invariant. Indeed, this should follow from the fact that the 2-form ω is closed. And actually, we can prove the deformation invariance directly.

Lemma 2. Consider a continuous family C_s of finite parts of tropical curves with common fixed endpoints. Then $Area(C_s)$ is constant.

Proof. Any deformation C_s locally can be decomposed into the elementary ones. An elementary deformation is a process of moving and shortening two edges while growing the one in the opposite direction (see Figure 5).

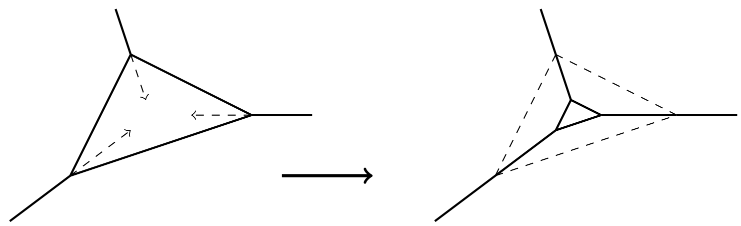

FIGURE 5. In the picture we shrink a triangular cycle. Any deformation of a tropical curve can be decomposed into operations such operations or their inversions.

Globally this corresponds to enlarging a coefficient for a tropical polynomial. For example in Figure 8 we change the coefficient for the central region.

Up to a scaling an elementary deformation simply replaces the union of segments $[0, v_1]$ an $[0, v_2]$ by a single segment $[0, v_1 + v_2]$. Here v_1 and v_2 are the primitive (or appropriate multiples of primitive) vectors for the edges we are moving. Denote by w_i the projection of $v_1 + v_2$ on the line spanned by v_i (see 6) Then after the deformation the two edges together loose

$$|v_1||w_1| + |v_2||w_2| = |v_1|(v_1 + v_2) \cdot \frac{v_1}{|v_1|} + |v_2|(v_1 + v_2) \cdot \frac{v_2}{|v_2|} = |v_1 + v_2|^2$$

of their symplectic area. On the other hand, the growing edge contributes exactly $|v_1 + v_2|^2$ to the symplectic area of the deformed curve. □

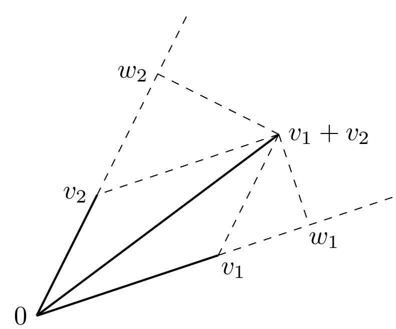

FIGURE 6. Computing contributions for symplectic area.

Lets get back to our specific case. Let the ends for a part of a tropical curve C be the vertices of a lattice polygon Δ. Suppose there is a well-defined quasidegree m_C for this curve. Then we can deform C to the union of all edges e of the polygon taken with the multiplicities $m_C(e)$. This observation together with the deformation-invariance proves the following lemma.

Lemma 3. $Area(C) = \sum_{e \in E(\Delta)} m_C(e) Area(e)$.

4. Dynamics towards minimal tropical curves

In this section we will describe certain operators G_p on the space of piece-wise linear functions, the definition will be done in purely tropical terms. Such an operator G_p is a limiting incarnation for the relaxation after dropping a grain to point p, and this relation is an important part of the proofs. It is hard to imagine that the remarkable properties of these operators could be ever guessed without using a sandpile interpretation. Moreover, with no relation to sandpiles the operators G_p are instruments in the construction of tropical curves with minimal tropical symplectic area.

4.1. **Implicit definition of $G_{p_1,..,p_n}$ and G_p.**

Definition 7. Let $V(\Delta, p_1, \dots, p_n, F)$ be the space of all tropical polynomials $\tilde{F}$ such that

- $\tilde{F}(\partial\Delta) = 0$
- $\tilde{F}(p) \geq F(p)$ for any $p \in \Delta$
- $\tilde{F}$ is not smooth at each of the points p_i.

Lemma 4. Let Δ be a lattice polygon, F be a tropical polynomial vanishing on $\partial\Delta$ and $p_1, \dots, p_n$ be a collection of points in Δ°. Then functional $F' \mapsto \int_\Delta F' dxdy$ defined on the space $V(\Delta, p_1, \dots, p_n, F)$ has a unique minimum $\tilde{F}$. Moreover, $\tilde{F}$ is a pointwise minimum on V, i.e. $\tilde{F} \leq F'$ for all $F' \in V$.

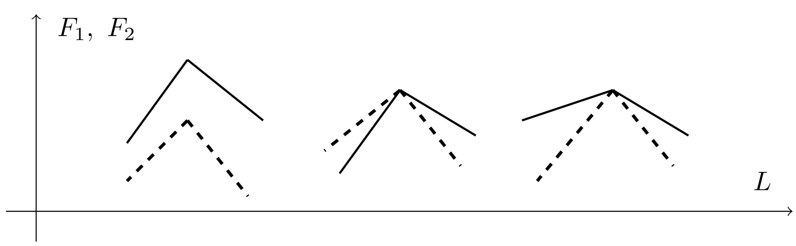

FIGURE 7. Three different relative positions for graphs of tropical polynomials F_1 and F_2 restricted to a generic line L that are not smooth at a given point. A minimum of two convex functions having a kink at the same point also has a kink at this point.

Proof. The only idea of the proof is that V is a semigroup with respect to the operation of taking pointwise minimum. If F_1 and F_2 are in V then $min(F_1, F_2)$ majorettes F and vanishes at the boundary of Δ. To show that $min(F_1, F_2)$ is in V it is enough to show that it is not smooth at all the points p_i. Consider a generic line L through a point p_i. Restrictions of F_1 and F_2 are not smooth at p_i. Then from the convexity (see Figure 7) it follows that the restriction of $min(F_1, F_2)$ to the line is also not smooth at p_i. And thus the minimum itself is not smooth at all the points $p_1, \dots, p_n$.

Since taking minimum does not increase the integral we can conclude that its minimizer $\tilde{F}$ exists, unique and is given by

$$\tilde{F}(p) = \min_{F' \in V} F'(p).$$

□

Definition 8. The function $G_{p_1,..,p_n} F$ is this unique minimum $\tilde{F}$ in lemma above.

The lemma has a very nice consequence. It appears that the curve defined by the minimizer $\tilde{F}$ always minimizes its symplectic area.

Lemma 5. For any curve C' defined by $F' \in V$

$$Area(C') \geq Area(\tilde{C}),$$

where $\tilde{C}$ is given by the polynomial $\tilde{F}$.

Proof. Point-wise minimization implies the point-wise minimization for quasidegree. To see this, one needs to look at the neighbourhood of each edge of Δ. Now the statement follows from the deformation invariance for symplectic area in the form of Lemma 3. □

Now, as an initial data we have a point in the interior of the lattice polygon Δ. We transform a tropical polynomial F vanishing at the boundary of Δ to a tropical polynomial F' also vanishing at the boundary of the polygon. Also, the curve C' given by F' will pass through p, i.e. the function F' will be not smooth at p. We will show that for any F there exist a canonical way to find such a tropical polynomial and we denote it by $F' =: G_pF$.

Definition 9. Let $V(\Delta, p, F)$ be the space of all tropical polynomials $\tilde{F}$ such that

- $\tilde{F}(\partial\Delta) = 0$
- $\tilde{F}(p') \geq F(p')$ for any $p' \in \Delta$
- $\tilde{F}$ is not smooth at p.

Proposition 2. For any point $p \in \Delta^\circ$ and any tropical polynomial F vanishing on $\partial\Delta$ the functional $\tilde{F} \mapsto \int_\Delta \tilde{F}$ defined on the space $V(\Delta, p, F)$ has a unique pointwise minimum F'.

Proof. We proceed analogously to the proof of Lemma 4. □

Definition 10. We denote by G_pF this unique minimum F'.

Essentially, G_p can be seen as a result of squeezing the face Φ where the point p sits, until the curve and p meet (see Figure 8). For more details see Section 4.4.

4.2. **Recursive definition of $G_{p_1,..,p_n}$.** Now we start with a sequence of points $p_1, .., p_n$ and the tropical polynomial F vanishing at the boundary of Δ. We construct a certain polynomial $F' = H_{p_1,..,p_n}F$ such that the curve it defines passes through all the given points. We will define the new function recursively. If F is not smooth at all the points $p_1, .., p_n$ then we take F' to be F. If it is not the case, i.e. there exist p in the set $\{p_i\}$ of the given points such that F is smooth at p, we define $H_{p_1,..,p_n}F$ as $H_{p_1,..,p_n}G_pF$. It is completely unclear from the very beginning that this recursive definition always terminates and the result doesn't depend on the choice of the point $p \in \{p_1, .., p_n\}$ at each step.

Lemma 6. Let F be a restriction of a tropical polynomial to the polygon Δ and $p_1, \ldots, p_n$ be a collection of points in Δ°. Then

$$G_{p_1,\ldots,p_n}F(x) \geq G_{p_{k_m}} \ldots G_{p_{k_1}}F(x),$$

where for every point $x \in \Delta$ and $k_i = 1, \ldots, n$.

Proof. For $m=0$ the inequality is trivial. Suppose we know that the inequality is true for $m-1$, i.e $G_{p_1,\dots,p_n}F \geq G_{p_{k_{m-1}}}\dots G_{p_{k_1}}F$. It is clear that $G_{p_1,\dots,p_n}F$ is not smooth at p_m and so using the implicit definition (Def. 7) of G_{p_m} we finish the proof. □

Therefore, passing to the limit, we have $G_{p_1,\dots,p_n}F(x) \geq H_{p_1,\dots,p_n}F(x)$. From the other hand, $H_{p_1,\dots,p_n}F(x) \in V(\Delta, p_1,\dots,p_n, F)$ and it is not smooth at all p_i. Therefore, by definition $G_{p_1,\dots,p_n}F(x) \leq H_{p_1,\dots,p_n}F(x)$ and we proved the following lemma.

Lemma 7. The operator $H_{p_1,..,p_n}$ is well defined, does not depend on the order of points and coincides with $G_{p_1,..,p_n}$.

We will never use the notation $H_{p_1,\dots,p_n}$, but, depending on the situation, we will treat $G_{p_1,\dots,p_n}$ either as the minimum of the functional, or as the limit of some product of G_{p_i}. In fact, in order to get $G_{p_1,\dots,p_n}$ it is enough to apply only finite number of G_{p_i} that is proved in Theorem 6. Also, it appears that $G_{p_1,..,p_n}$ is a piece-wise linear analog for the relaxation process, and this relation is formalized in the following paragraphs.

4.3. **Back to discrete tropical curves and their limits.** Now we return to the problem of describing the limiting curve C obtained as the limit for relaxations of ϕ_N^0. Suppose that we already know the curve $C(\Delta; p_1,..,p_n)$. Consider another point p_{n+1} in the interior of Δ. It is natural to ask how the new curve $C(\Delta; p_1,..,p_{n+1})$ can be derived from the old one.

There are two very different cases. The limiting curve either passes through p_{n+1} or doesn't pass through it. When the new point is on the curve we claim that a new limiting curve and a new limiting toppling function are the same as the old ones. Recall that F is the limiting toppling function.

Lemma 8. If the point $p_{n+1} \in \Delta^\circ$ is on the curve $C(\Delta; p_1,\dots,p_n)$ then

$$F(\Delta; p_1,..,p_{n+1}) = F(\Delta; p_1,..,p_n).$$

In particular, $C(\Delta; p_1,..,p_{n+1}) = C(\Delta; p_1,..,p_n)$.

Proof. □

In the case when p_{n+1} is not on C the situation is more complicated. It appears that the operator $G_{p_1,..,p_{n+1}}$ is precisely the one transforming the old curve to the new one.

Proposition 3. For any collection of distinct points $p_1,\dots,p_{n+1} \in \Delta^\circ$

$$F(\Delta; p_1,\dots,p_{n+1}) = G_{p_1,..,p_{n+1}}F(\Delta; p_1,\dots,p_n)$$

Proof. See Proposition 6. □

Proposition 3 together with Lemmas 7, 13 and the construction of $G_{p_1,..,p_n}$ give the following remark.

Remark 5. $F(\Delta; p_1,\dots,p_n) = G_{p_1,..,p_n}(0)$, where 0 denotes the function on Δ which gives zero identically.

Theorem 4. The curve given by the function $G_{p_1,..,p_n}(0)$ is minimal for the collection of points $p_1,..,p_n$.

Proof. Remark 5 and Theorem 3 show that the operator $G_{p_1,..,p_n}$ provides an entirely geometric way to construct a minimal curve, for details see Section 10. □

4.4. **Explicit description of G_p.** In the remainder of the section we are going to give an intrinsic description for the operator G_p. This can be seen, perhaps, as a constructive version of the second part of Proposition 12.

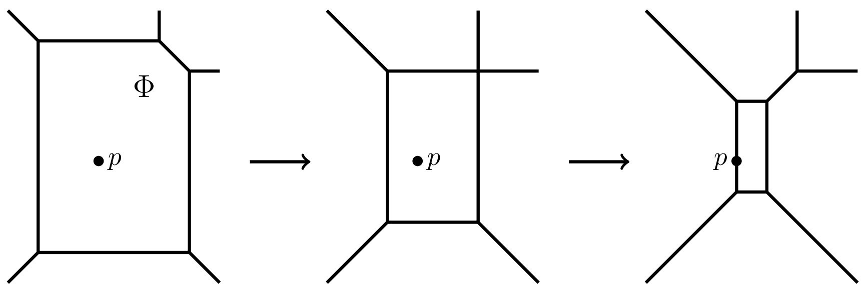

FIGURE 8. Action of G_p by shrinking the face Φ where p belongs to. Firstly, $t = 0$, then $t < t_0$, and finally $t = t_0$. Note that combinatorics of the new curve can change.

Let us take a point p in the interior of Δ. The construction of the new polynomial G_p will depend on position of p with respect to C.

The definition of G_p implies that if the curve C passes through a point p then $G_pF = F$.

Suppose that p is in the complement to C in Δ. Then we have two more cases. Let Φ be the connected component, i.e. the face of $\Delta \backslash C$ to which p belongs. The face can be adjacent to the boundary of Δ or not adjacent to it.

If Φ doesn't touch the edges of Δ then the procedure is quite straightforward. Consider a vector $w \in P(F)$ that corresponds to Φ. If we are increasing the coefficient c_w we squeeze the face Φ (see Figure 8) This produces the family of polynomials F_t such that F_t differs from F only by a coefficient before the monomial "x^w" which is defined as $c_w + t$. We take $G_p = F_{t_0}$ such that $t_0 > 0$ is the smallest number for which the curve given by F_{t_0} passes through p.

In the case when Φ has common edges with the polygon Δ we cannot do exactly the same thing because we will violate the vanishing at the boundary. To avoid this we increase the quasi-degree for the curve C and, if it is required , we extend the Newton polygon of F.

In fact, if $F|_\Delta$ is not identically zero and $C \cap \Delta^\circ$ is not empty, then the set $\Phi \cap \partial\Delta$ consists of at most one edge. So we suppose for now that C is not empty and $\Phi \cap \partial\Delta = e$. We can define a new curve C', which coincides as a set with C incide Δ, and a new polynomial F' by modifying the old ones, i.e $C' = C \cup e$ and

$$F'(x) = min(F(x), (m_C(e) + 1)(n(e) \cdot x + a_e)),$$

where a_e is chosen in such a way that the function $x \mapsto n(e) \cdot x + a_e$ vanishes on the edge e. Note that F' coincides with F inside Δ. So we have essentially reduced the situation to the previous case. We can start increasing the coefficient for $w = m_C(e)n(e)$ and so forth, until a curve crosses the point p, thereby defining $G_pF := G_pF'$.

The only thing that really matters in this approach is the accurate choice of the polynomial representation for F' (for F in the previous case). As we explained in Section 2, we should assume that the coefficients for F' which are not the vertices of the subdivision for the Newton polygon $P(F')$ are chosen to be smallest possible. Afterwards we will give another constructive description for G_p free of these technicalities (see Proposition 12).

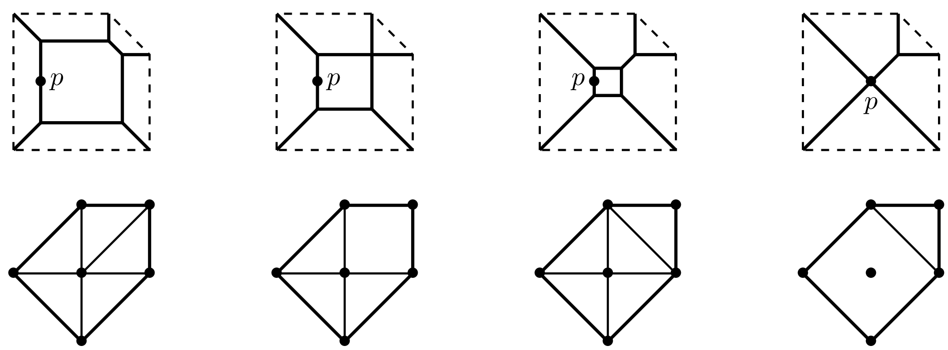

FIGURE 9. First row shows how minimal curves for one point on a pentagon boundary depend on the positions of the point. The second raw shows a dual decomposition for their Newton polygon.

The last case is very special, it is the case when $\Phi^\circ = \Delta^\circ$. The curve C itself is not necessarily empty but it can be the a union of some edges in the boundary of Δ. Following the analogy with the previous case, we define a new curve by completing C to the whole boundary, i.e. $C' = \partial\Delta$, and in the same way we construct a new function F' by taking the minimum of all linear functions $n(e)\cdot x + a_e$ for $e \in E(\Delta)$. Then G_pF is equal to G_pF'. And as before G_pF' is derived from F' by increasing the coefficient c_0. The dependence of the result on the position of point p is shown in Figure 9.

Lemma 9. The above paragraphs give direct construction for G_p from Definition 10.

Proof. Indeed, G_pF is the point-wise minimum and during this construction we increase only one coefficient of F as less as possible to have p in the non-smooth locus. □

5. THE CRITICALITY OF THE TOPPLING FUNCTION AND THE STRUCTURE OF SELF-REPRODUCING WAVES

5.1. **The criticality of the toppling function.** Given a state ϕ^0, we denote by h_v the number of topplings at the vertex v during a relaxation process of ϕ^0, so $h_v = Toppl_{\phi^0}(v)$ in the terminology of the first chapter. The inequality

$$0 \leq \sum_{w\in n(v)} h_w + \phi^0(v) - |n(v)|h_v \leq |n(v)| - 1 = \phi_0(v) \tag{2}$$

follows from counting the number of incoming and outcoming grains at v. Recall that ϕ_0 is the maximal stable state.

Definition 11. Let $\Delta h_v = h_v - \frac{1}{|n(v)|}\sum_{w\in n(v)} h_w$ be the Laplacian of f.

If $\phi^0(v)=\phi_0(v)$ for a vertex v then the inequality (2) implies $1>\Delta h_v\geq 0$. If $\phi^0(v)\geq |n(v)|$ then

$$\frac{\phi^0(v)-\phi_0(v)}{|n(v)|}+1>\Delta h_v\geq \frac{\phi^0(v)-\phi_0(v)}{|n(v)|}>0 \tag{3}$$

Remark 6. Note that $\Delta h_v = 0$ means that $\phi^0(v)=\phi^{end}(v)$. This implies that the main object of our study, i.e. the set of vertices where ϕ^{end} differs from ϕ_0, has other description:

$$\{v|\phi_0(v)\neq\phi^{end}(v)\}=\{v|\Delta h_v>0\}.$$

In the limit, the function h becomes piecewise linear with integer slopes (Theorem 2). Hence the limit of the set where $\Delta h>0$ is exactly a non-smooth set of such a function, i.e. this limit is a tropical curve by definition. Note that we have $\Delta h_v>0$ at a vertex v where we have added sand (i.e. $\phi^0(v)=\phi_0(v)+1$), because of (3), so v belongs to the tropical curve. Then, Theorem 2 implies Theorem 1.

We fix an order of topplings during the relaxation of ϕ^0, i.e. a sequence $\phi^0\to\phi^1\to\dots\to\phi^{end}$ of states. We consider functions $h^1,h^2,\dots,h^{end}$ where $h^i_v=h^i(v)$ is the number of topplings at v among first i topplings during the relaxation. Note that the possibility of applying the toppling to ϕ^i at v is equivalent to $\Delta h^i_v<0$.

We state a new problem. Find a function $H:\Gamma\to\mathbb{Z}_{\geq 0}, H|_{\partial\Gamma}\equiv 0$, which has $\Delta H\geq 0$ at the internal vertices, and $\Delta H>0$ at the fixed different vertices $p_1,p_2,\dots,p_n$. This initial data correspond to the initial state ϕ^0 which differs from ϕ_0 by added sand grains at $p_1,p_2,\dots,p_n$. The sand relaxation of ϕ^0 provides us some solution for the problem of finding such H.

We are going to show that this sand solution (the function $Toppl_{\phi^0}=h^{end}$) is **the** minimal solution. This follows from the fact that each of h^i is smaller or equal at every point than any solution H. For $h^0\equiv 0$ it is clear. Let h^i be less then H at every point. We will show that this implies the same inequality for h^{i+1}. Indeed, let h^{i+1} differs from h^i by the toppling at w. That means $\Delta h^i_w<0$, and the latter implies that

$$|n(w)|\cdot h^i_w+1\leq(-1+\sum_{v\in n(w)}h^i_v)+1\leq\sum_{v\in n(w)}H_v\leq|n(w)|\cdot H_w.$$

Since $h^i_w,H_w\in\mathbb{Z}$, we proved that $h^i_w+1\leq H_w$.

5.2. **Waves.** On the simulation of the relaxation process we see a sort of ripple effect, see Figure 1. So we decompose the relaxation as a sequence of *waves* ([15, 16]). Composed in a different order, the waves constitute the operators G_p, that explains why the relaxation in the limit is a sequence of the operators G_p.

Definition 12. Given a stable state ϕ, consider the full subgraph that contains all the vertices v such that $\phi(v)=|n(v)|-1$. Consider the connected components of this subgraph. Those who contain at least two vertices are called *territories*.

Definition 13. ([15]) Let v be a vertex in some territory in a stable state. The *wave* at v is the following procedure. We add a grain to v and topple v if possible. Then, we "freeze" v, i.e. prohibit to topple it and perform a relaxation. After stabilisation we remove a grain from v.

Remark 7. It is easy to see ([16]) that after "freezing" of v and subsequent relaxation, each vertex can topple no more than one time. So, basically, a wave increase the function $Toppl$ by one in this territory.

Lemma 10. The adding of a grain to v and the relaxation makes the same effect as making waves at v while it change something (i.e. until we have $\phi(v) < |n(v)| - 1$), then adding a grain to v.

Proof. Left to the reader. □

On the video of the relaxation process (Figure 3) we will see that a wave moves the set of colored vertices towards v. We will prove that the set of coloured vertices is made of self-reproducing traceries, and under this action by waves this tracery finally crawls over v, and the relaxations terminates. In the limit this crawling is easily recognized as the operators G_p.

Proposition 4. The results of waves made at two different vertices in one territory coincide.

Proof. Consider a state ϕ and two adjacent vertices v and w in the same territory. It is enough to prove that the results of waves at v and w coincide, that is straightforward. □

5.3. **Relaxation process near the boundary of a slope** (p, q)**.** Now we come to one of the the bottom level arguments of this paper. Let us fix $p, q \in \mathbb{N}$. The only thing we need to demonstrate is why the sequence of waves near the boundary of a slope (p, q) produces a new edge of a tropical curve with the same slope, see Figure 3. The set of colored vertices constituting this edge possesses the property that after waves it moves without changing the pattern, that is why the name *self-reproducing*. Once we prove this property, we can essentially speak about operators G_p, see Proposition 12. All this is done via demonstrating that the toppling function near such a boundary is of type $-qx + py + c$.

Consider the sandpile problem on the square lattice Γ on the plane with a boundary $\partial\Gamma = \{(i,j)|iq > jp\}$. Let us send n waves from the point $(0, N)$ with N very big. Let $h^{(n)}(i,j)$ be the number of topplings at a vertex (i,j) during this process. We denote by $\phi^{(n)}$ the obtained state. Since the boundary is periodic we can model this situation by the following half-infinite cylinder.

To be specific, we construct an auxiliary graph Σ, where

$$V(\Sigma) = [0, p] \times [0, +\infty) \cap \mathbb{Z}^2$$

is its set of vertices. In this graph we connect (i,j) with $(i-1,j)$ for $i > 0$, (i,j) with $(i, j-1)$ for $j > 0$ and $(1, j)$ with $(p, j+q)$. Let $\partial\Sigma$ be a subset of $[0,p] \times [0,q]$, we suppose that $(0,i), (j,0) \in \partial\Sigma$ for all i, j.

Let us try to solve the problem of finding $H : \Sigma \to \mathbb{Z}_{>0}$ with $\Delta H \geq 0$ and $H|_{\partial\Sigma} = 0, H(i,j) = n$, for $i > n+p+q$. Clearly, the resultat h^n of sending n waves produces the minimal solution of this problem. By abuse of notation we suppose that h^i are defined on Σ, which cases no problem since $h^i(i,j) = h^i(i+p, j+q)$.

Definition 14. In what follows by $C_{a,b,\dots}$ we define a function which depends only on the list $a, b, \dots$ of variables. We use this notation for constants numerously appearing below.

Theorem 5. There is a constant $k = C_{p,q,\partial\Sigma} \in \mathbb{Z}$ such that the function $H(i,j,n) = \min(pj - qi + k, n)$ is equal to $h^{(n)}$ almost everywhere. Namely, there is a constant $C_{p,q}$ such that for the defect set

$$Def = \{(i,j)|h^{(n)}(i,j) \neq H(i,j,n)\},$$

we have $|Def| < C_{p,q}$ for each $n \in \mathbb{N}$. Moreover, Def consists of two parts $Def = B \cup E$ where $B \subset [0,p] \times [0, C_{p,q}]$ and $E \subset [0,p] \times [a, a + C_{p,q}]$ for some a. In fact, E represents the tracery of the discrete tropical edge in the direction (p,q) and B is the boundary artifacts – i.e. the colored vertices which inevitably appear near the boundary, but in a negligible amount. See Figure 2 for example of discrete tropical edges and an example of the boundary artifacts.

The proof of this theorem relies on two observations. The first is that the amount of colored vertices does not exceed a constant while n grows. The second observation states that if a non-negative harmonic integer-valued function is less than a linear function on a big territory then this function is also linear. Recall that we call a vertex v *colored* if $\phi(v) \leq 3$ in the currently considered state ϕ. Now we derive some benefits of this theorem, and we postpone its proof till the end of this section.

5.4. **Dynamic and static of discrete tropical edges.** Theorem 5 says that after n waves the number $h^{(n)}(i,j)$ of topplings near the boundary is equal to the function $\min(-qi + pj + k, n)$ almost everywhere. This provides the other point of view on the discrete tropical edges.

Indeed, we start from the function $h(i,j) = \min(-qi + pj + k, n)$. Easy computations show that near the line $-qi + pj + k = n$ we have $\Delta h \geq 1$. This suggests the following procedure. We take a point (v_0) where $(\Delta h)(v_0) \geq 1$ and construct h_1 which satisfies the properties $h_1(v_0) = h(v_0) - 1$ and $h_1(v) = h(v)$ for all others points v. Then, if $(\Delta h_1)(v_1) \geq 1$ at a point v_1 we produce h_2 which is h_1 decreased by one at v_1. Clearly we have the inequalities $h_k \geq h^{(n)}$ while we find points v_k with $(\Delta h_k)(v_k) \geq 1$. Theorem 5 implies that in the half-infinite cylinder Σ we have $h_N = h^{(n)}$ for some $N = C_{p,q,\partial\Sigma}$, since we need to decrease h only in a finite neighbourhood of the boundary and in a finite neighborhood of the line $-qi+pj+k = n$. We call this process *smoothing* of $\min(-qi + pj + k, n)$.

Recall that a tracery is defined by the set of points where Δ of the toppling function is not zero.

Proposition 5. The tracery of smoothing of $\min(-qi+pj+k, n)$ and the smoothing of $\min(-qi + pj + k, n + 1)$ differ by parallel translation.

Proof. These functions are differs by a change of coordinates, which is a translation $(i,j) \to (i + c_1, j + c_2)$. □

This shows that the discrete tropical edges are self-reproducing, i.e. sending a wave does not change their tracery.

Thus, the discrete tropical edges emerge from smoothing of piece-wise functions which are linear on the one side and constant on the other.

In general, if the toppling function is $ai + bj + c$ on one side and $pi + qj + r$ on the other, then we can start again from $h = \min(ai + bj + c, pi + qj + r)$ and smooth it. The following proposition is, then, clear.

Proposition 6. Provided $(a-p, b-q)$ is a primitive vector, this smoothing produce the same tracery as the smoothing of $\min((a - p)i + (b - q)j + c, r)$.

Note, that total integral of $\Delta h^{(i)}$ is preserved, that is why the area of tropical edge in Σ is no more than $(p+q)^2$ (see [6]). Indeed, for $h = min(-qi+pj+k, n)$ the total integral of $\Delta h^{(n)}$ over E can be estimated via the sum of the values of Δh near the line $-qi+pj+k=n$. Recall, that we consider everything in Σ, so we will have about $p+q$ terms of order $p+q$ the maximal.

If we consider two territories with toppling functions $ai+bj+c, pi+qj+r$ and $(a-p, b-q)$ is not a primitive vector, then smoothing can produce a territory in between of these two territories. If this new territory is visible in the limit, that means that by repeating the argument that if a function is harmonic and less then a linear, then it is linear, we conclude that in the new territory the toppling function is again linear.

The places where we have a convergence zone of three territories, we repeat all the same: we start from the function

$$f(i,j) = \min(p_1 i - q_1 j + k_1, p_2 i - q_2 j + k_2, p_3 i - q_3 j + k_3)$$

and smooth it. By the similar method as above we see that if a new territory appeared there, then the toppling function on it is of type $p_4 i - q_4 j + k_4$ where the point (p_4, q_4) is inside the triable with vertices $(p_1, q_1), (p_2, q_2), (p_3, q_3)$.

5.5. **Proof of Theorem 5.**

Remark 8. Denote by $L^{(n)}$ the amount of sand that is lost after n waves, i.e. the sand which have fallen into $\partial\Sigma$. Therefore

$$L^{(n)} = \sum_{v\in\partial\Sigma}\sum_{w\in n(v)} h^{(n)}(w) = \sum_{v}(\phi^0(v) - \phi^{(n)}(v)) \tag{4}$$

Lemma 11. The number $L^{(n)}$ is bounded by a constant $C_{p,q,\partial\Sigma}$ which does not depend on n.

Proof. Consider the function $h(i,j) = \min(-qi+pj+pq, n)$. Clearly, it satisfies $\Delta h \geq 0$ and $h|_{\partial\Sigma} = 0, h(i,j) = n$, for all $i > n+p+q$, i.e. the sand-solution $h^{(n)}$ is less or equal than h everywhere. Denote the set $[1,p]\times[0,2q+1]\cap\mathbb{Z}^2$ by A. Since $n(v)\subset A$ for $v\in\partial G$, we have

$$L^{(n)} < 4\sum_{(i,j)\in A}(-qi+pj+pq) = C_{p,q}$$

for arbitrary n. □

Now we consider how the picture is changing when we increase n, i.e. send more and more waves. Since the amount of lost sand is bounded by L, the number of colored vertices is also bounded by L.

Proposition 7. For n big enough, there will be no topplings near the boundary and the picture will stabilize there.

Proof. Suppose that the number of topplings at a vertex (i,j) tends to infinity, when $n\to\infty$. Then the number of topplings at all neighbors also tends to infinity. Then the number of topplings near the boundary is also tends to infinity, that contradicts to Remark 8. □

So, we can divide the colored vertices into two parts – those who stabilize and those who change their positions. Take the territory T_0 which lies above the stabilizing colored vertices. We rely on the following lemma.

Lemma 12. Let $A \subset \mathbb{Z}^2$ and $f : A \to \mathbb{N}_0$ be a discrete harmonic function (i.e. $\Delta f = 0$). Let $f(x, y) \le px + qy + r$ with $p, q, r \in \mathbb{Z}$. Then, if $A = [0, N]^2$ and N is big enough (i.e. $N > C_{p,q,r}$) then f is linear on $[C_{p,q,r}, N - C_{p,q,r}]^2$.

Proof. This can be proven via direct calculation of the Poisson kernel, as in Section 2.2.3 of [5]. The more elementary approach is the following[1]. We sketch the idea and leave the details for the reader. For a harmonic function f we define $var(f, n, A) = \max_{(i,j)\in A} f(i+n, j) - f(i, j)$. Trivially, $var(f, n, A)$ is monotone by A, and we prove its convexity, i.e. $2 \cdot var(f, n, A) \ge var(f, n-1, A) + var(f, n+1, A)$ by considering terms of Δf at (i, j) and $(i+n, j)$. Since f is integer-valued we have $var(f, 1, A) \ge 1$ and $f(x, y) \le px+qy+r$ implies that $var(f, n, A) < np+1$. Therefore $var(f, n, A)/n$ attains its maximum for some finite $n = C_{p,q,r}$, that finishes the proof. □

Proposition 8. In the territory above the stabilized part the function $h^{(n)}$ is $-qi + pj + c$ with some constant c.

Proof. Indeed, in this territory the function $h^{(n)}$ is discrete harmonic function, and $h^{(n)} < -qi + pj + pq$. Since this territory is growing while n grows, we can use the estimate for discrete harmonic functions: if $h^{(n)}$ is less than a linear function, then it is linear itself. Since our function is periodic and integer, we easily see that $h^{(n)} = k(-qi + pj) + c$ with some integer k. Clearly $k = 1$. □

Proposition 9. The part with moving colored vertices is bounded, i.e. it is contained in the set $[0, p] \times [c_0(n), c_0(n) + c_1]$, where c_1 does not depend on n and depends only on p and q.

Proof. Indeed, imagine that it is possible to have a big territory above the territory where $h^{(n)} = -qi + pj + c$, and $h^{(n)}$ is not everywhere constant there. We repeat the same arguments as above and see that in this case $h^{(n)}$ is a constant or of the type $-qi + pj + c$ there. It is not possible due to the maximal principle. Namely, let h^n is a constant in the middle territory. Take the point where $h^{(n)} = n$ with a neighbor with value of $h^{(n)}$ less than n. Then this neighbor must have a neighbor with lesser value, etc. That indicates that we can not enter the middle territory because there the vertex have no neighbours with lesser values. We arrived to the contradiction. □

So we proved that the function $h^{(n)}$ come as the smoothing of the function $h(i, j) = \min(-qi + pj + pq, n)$. Thus we proved that an edge will detach.

We drop a grain of sand to a point p which belongs to a discrete tropical edge. If p was coloured, then we have nothing to do. If not, then p has been dropped to a point with three grains and we must proceed by relaxation.

Lemma 13. This relaxation has an effect only in the constant neighborhood of p which does not depend on the scale N.

Proof. Indeed, remember the process of smoothing of a piecewise linear function. We start from $h(x, y) = \min(ax + by + c, px + qy + r)$ and then start to decrease h at the points where $\Delta h \ge 1$. If during this process we decreased p, then it is easy to recover the result after the above relaxation. Indeed, let us decrease the function at p one less time than we do normally. Because of that we also will not decrease it at some other points, and that gives the result of the relaxation. It is

[1]We thank Misha Khristoforov for pointing us out these classical ideas.

clear, that this affects the picture only locally. If we didn't change p during the smoothing, then the toppling function at p is still $ax+by+c$ or $px+qy+r$ what is not possible by the following reason. Take the territory where p belong to. Since in all neighboring to this territory points the function h was decreased, we also can decrease h on this territory keeping the condition $\Delta \geq 0$ preserved. □

6. Finiteness of the tropical dynamics

In this section we will prove that we need only a finite number of steps in the recursive definition for $G_{p_1,\dots,p_n}$ (see section 4.2).

Lemma 14. Consider an integer sequence $\{k_i\}_{i=1}^{\infty}$ such that $k_i \in [1,n]$ and for any $l \in [1,n]$ the sequence k_i takes value l infinitely many times. Then the sequence of functions $F_n = G_{p_{k_n}} \dots G_{p_{k_1}} F$ converges to $G_{p_1,\dots,p_n} F$ to point-wisely.

Proof. Each term of the sequence is represented by a tropical polynomial and from Lemma 6 the Newton polygon for each of this polynomials is a subset of the Newton polygon for $G_{p_1,\dots,p_n}F$, which is proven to exist by Lemma 4. So, in fact we need to prove the convergence for a finite collection of bounded coefficients.

Each G_p increases a certain coefficient (see section 4.4). This implies the convergence for each sequence of coefficients and a point-wise convergence for a sequence of functions F_i. Denote the limit by F_∞.

Now we fix $l = 1,\dots,n$ and take a monotone sequence m_i such that $k_{m_i} = l$. Then the idempotency of G_{p_l} implies that $G_{p_l} F_{m_i} = F_{m_i}$ for all i. Since the operator G_{p_l} is continuous (see Remark 11) we conclude that $G_{p_l} F_\infty = F_\infty$.

The last equality is equivalent to the property for F_∞ of being not smooth at the point p_l. Again using Lemma 14 and an implicit description for $G_{p_1,\dots,p_n}$ (see section 4.2) we conclude that

$$G_\infty = G_{p_1,\dots,p_n}.$$

□

To proceed, we need the following intermediate result.

Lemma 15. If $p_1,\dots,p_n$ have rational coordinates and F is restriction to Δ of a tropical polynomial with rational coefficients then to compute $G_{p_1,\dots,p_n}F$ it is required to use each G_{p_k} only a finite number of times.

More precisely, any particular way of computing $G_{p_1,\dots,p_n}F$ via G_{p_k} is finite.

Proof. First of all we can reduce a problem to the case when all the points and coefficient are integer. This can be done by a simple rescaling of Δ. Namely, denote by Q the product of all denominator for coordinates of p_i and all denominators of coefficients of F. Then we consider the points $Q \cdot p_k$ on $Q \cdot \Delta$ and a new function $F : x \mapsto QF(Q^{-1}x)$.

Now, all objects are integer and they remain be integer while applying G_p. So, we consider G_pF, it is clear that $G_{p_1,\dots,p_n}F \geq G_pF$. It is also clear that if G_pF and F are not the same, then G_p have increased some coefficient for F by an integer. Thus each G_p increases the sum of all coefficient at least by 1. So the process cannot be infinite. □

Theorem 6. For any collection of points $p_1,\dots,p_n \subset \Delta$ and a tropical polynomial F, the function $G_{p_1,\dots,p_n}F$ can be obtained as a finite product of G_{p_i}.

Proof. We proceed by induction on number n of points. For $n = 1$ there is nothing to prove. Let us suppose that there is an infinite chain $G_{p_{i_1}} G_{p_{i_2}} G_{p_{i_3}} \dots$ which converges to $G_{p_1,\dots,p_n} F$. Let F_1 be $G_{p_{i_m}} G_{p_{i_{m-1}}} \dots G_{p_{i_1}} F$ such that F_1 is very close to $G_{p_1,\dots,p_n} F$. We can choose such m by Lemma 14. Now we take the point p_n and stop applying G_{p_n}. By induction we can apply G_{p_i} to F_1 with $i < n$ only finite number times, so such a way we obtain F_2. These applications can not change F_1 a lot, since it is close to the limit. Now we remember about p_n. Let ε be the minimal weighted distance from p_n to the tropical edges. Then, G_{p_n} applied to F_2 increases some coefficient of F_2 by ε. We claim that all the G_{p_i} applied after that also increase coefficients by ε, and this implies that this process can not be infinite.

Since F_2 is close to the limit, then during the application of G_{p_i} there will be no changes in the tropical curve except local ones: an edge can move slightly back and forth, a vertex can produce several vertices or even a new face, but in very small neighbourhood. Using the definition of G_p in Section 4.4, we observe only changing of the coefficients of F_2. Let us take any p_i. We can suppose that it is $(0, 0)$ and look how we change coefficients. Whether $(0, 0)$ is a vertex or an edge, all the tropical monomials around it are of the type $ax + by$, without a constant; then by the same reasoning as in Lemma 15 we see that all incrementation are multiples of ε.

Therefore, starting from F_1 we applied only finite number of operations and arrived to the limit. That means, taking N big enough that $G_{p_{i_N}} G_{p_{i_{N-1}}} \dots G_{p_{i_{m+1}}}$ contains all these operations, we also arrive to the limit. □

Remark 9. Theorem 6 implies Lemma 7.

7. One point case

Now we are going to discuss in more details what happens if we add one point to the maximal state. The relaxation for the state of this type serves as a local model for more general relaxations. In particular it provides an easy description for the operator G_p. Another goal of this section is to show how to apply in the most basic situation the general techniques we developed.

For the case when the underlying graph of the model was a square piece of a rectangular lattice the result has been already described in the first section (see Figure 1). Here we continue to work with a subgraph of the same lattice but the boundary now can be a lattice polygon of an arbitrary shape.

Lets fix δ, a large enough polygon with rational slopes. Consider another polygon δ' which is constructed as a union of δ with its two horizontal and two vertical shifts

$$\delta' = \delta \cup \delta + (\pm 1, 0) \cup \delta + (0, \pm 1).$$

This new polygon can be seen as a set of those points in $\mathbb{R}^2$ that can be moved to δ by the action of standard generators of $\mathbb{Z}^2 \subset \mathbb{R}^2$ (or already were in δ). Now we define a graph $\Gamma = \Gamma(\delta)$.

Definition 15. Let $\Gamma(\delta)$ be a full subgraph of $\mathbb{Z}^2$ whose set of vertices is $\delta' \cap \mathbb{Z}^2$. The boundary of this graph is defined as $(\delta' \backslash \delta) \cap \mathbb{Z}^2$.

Consider the maximal state ϕ_0 and a vertex $v \in V(\Gamma) \backslash \partial\Gamma$. Let us add one extra grain of sand to ϕ_0 at v obtaining an unstable state ϕ. Suppose that v is far enough from the boundary of δ. This quite vague assumption together with the assumption that δ is large enough could be made more rigorous in the following way. In the spirit of the second section we fix a lattice polygon Δ and a point p in its interior. Then we can take a positive integer N and define δ to be $N \cdot \Delta$ and v to be the coordinate-wise integer part for $N \cdot p$. Then as we claimed before in the Theorem 1 we expect to see a very regular behavior for the relaxation of ϕ when N is large enough.

7.1. **Corners of Δ are smooth.** Now we suppose for simplicity that all the corners of Δ are smooth, i.e. the primitive vectors for the adjacent edges span a parallelogram of area 1 (see definition 16). The general situation will be described in the end of the section.

Definition 16. A corner of a polygon is called *smooth* if

$$det|w_1, w_2| = \pm 1,$$

where w_1 and w_2 denote the the primitive vectors in the directions of edges adjacent at the angle and $|w_1, w_2|$ denotes a 2×2 matrix made of the vectors w_1 and w_2. A polygon is called smooth if all of its corners are smooth. This terminology is inspired by the theory of toric surfaces.

To understand the relaxation for ϕ we decompose topplings on waves. As it was explained in the Proposition 7, after applying a number of waves the picture will stabilize near the boundary of the polygon δ. At his point the set of colored vertices will consist of boundary artifacts, edges parallel to the boundary and edges going from the corners of the boundary. Each new wave emitting from v doesn't touch boundary artifacts, shorten edges parallel to the boundary and move them towards v (see Lemma 16) and prolongs edges going from the corners. During this process some of the edges can collapse. This creates a vertex outside the territory of v. The collision is shown in the Figure 8 for $v = [N \cdot p]$.

Lemma 16. A periodic pattern in the primitive direction (α, β) after applying M waves moves at the distance $M(\alpha^2 + \beta^2)^{-1/2} + o(N)$ towards the source of a wave.

Proof. Without loss of generality, suppose that $\alpha \geq 0$ and $\beta > 0$ and the pattern is arranged along a line passing through (α, β) and the origin. Lets send M waves from the point $(N, 0)$. Let h be as usual the height function for this process. Then in the territory of the source of wave it is equal to M. Thus the new location for the pattern is along the locus of non-smoothness of the function

$$(x, y) \mapsto min(M, \beta x - \alpha y).$$

This locus is again a line with the same slope. The new line passes through a point $(M/\beta, 0)$. And so the distance between the lines is

$$\langle n, (M/\beta, 0)\rangle = \frac{M}{\sqrt{\alpha^2 + \beta^2}},$$

where n is an orthonormal direction for the pattern. □

We continue sending waves until one of the edges parallel to the boundary will not pass through v. This simply means that at this moment v went outside its former territory and waves can no longer be applied. To complete the description of the relaxation for ϕ we simply add one grain of sand at v to our current state (we can do it safely because v doesn't belong to any territory). From this description it is clear that the dependence on N is quite weak and we can actually find a limit for the picture when N tends to infinity. To do this explicitly we introduce a weighted distance function.

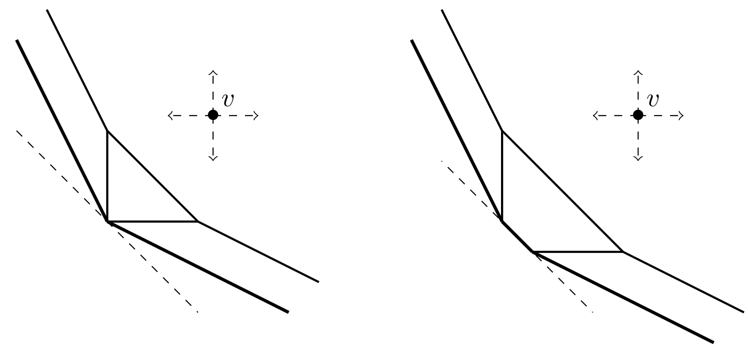

FIGURE 10. A result of sanding the same number of waves from a point v near a non-smooth corner and its perturbation. The non-smooth corner is spanned by the vectors $(-1,2)$ and $(2,-1)$. On the second picture we cut the corner with an edge in the direction $(1,-1)$.

Definition 17. A weighted distance function l_Δ is a function on points p in a lattice polygon Δ. It is defined as a minimum of $dist(p,e)\sqrt{\alpha^2+\beta^2}$ over all edges e in the boundary of Δ, where (α,β) is a primitive vector in the direction of e and $dist(p,e)$ is a distance from p to a line containing e.

The function l_Δ measures how fast a point will meet the set of colored vertices. It also appears that it counts the number of topplings at v.

Proposition 10. Let F_N be the toppling counting function during the relaxation of the maximal state on $\Gamma(N\cdot\Delta)$ after we add an extra gain at $v=[N\cdot p]$. Then

$$\frac{1}{N}F_N([N\cdot p']) \longrightarrow \min(l_\Delta(p), l_\Delta(p'))$$

7.2. **General Δ.** When we apply waves near a non-smooth corner it produces more edges then in a smooth case. An enlightening way to treat a non-smooth corner is to consider it as a collection of degenerate smooth ones (see Figure 10).

Except for this pathology of extra edges coming from bad corners, nothing have really changed. Conceptually, we can thing about a general polygon as a degeneration of a good one, this principle is realized in Proposition 11. After that, we start to emanate waves from the point v as we did before. And we do that until some of the edges parallel to an edge of the *perturbation* of the $\partial\Delta$ passes through v. This procedure dictates us the general definition for the weighted distance function.

Definition 18. Let Ω be a compact subset of $\mathbb{R}^2$. The weighted distance function l_Ω on Ω is defined by

$$l_\Omega(x) = \underset{w\in\mathbb{Z}^2}{inf}\,(w\cdot x - c_w) = \text{“}\sum_{w\in\mathbb{Z}^2} c_w x^w\text{”},$$

where

$$c_w = \underset{x\in\Omega}{min}\, w\cdot x.$$

The function l_Ω serves as an example of a tropical power series. In this terms the interior of the convex hull for Ω should be thought as a *domain of convergence* for l_Ω. We will discuss more of that in [13].

With the current definition of l_Δ Proposition 10 works for a general Δ without any changes. What is more questionable is a consistence of Definitions 17 and 18 and potential infinitude for l_Δ. This two problems are resolved by the following lemma.

Lemma 17. Let $P(\Delta)$ be a set of all primitive vectors in the union of all triangles with vertices $n(e_1),\ n(e_2)$ and 0, where e_1 and e_2 are adjacent edges of Δ. Then

$$l_\Omega(x) = \underset{w\in P(\Delta)}{inf}(w\cdot x - c_w)$$

Here $n(e)$ denotes a primitive normal vector to e towards the interior of Δ. It is clear that $P(\Delta)$ is finite. And for Δ having only smooth corners $P(\Delta)$ coincides with a collection of all primitive normals to the edges of Δ, i.e. $P(\Delta) = n(E(\Delta))$.

Proof. It is clear that non-primitive vectors do not contribute to l_Δ because $kw\cdot x - c_{kw} = k(w\cdot x - c_w)$ and $w\cdot x - c_w$ is nonnegative for all $x \in \Delta$.

Let w be a primitive vector not orthogonal to any of the boundary edges. This implies that $w\cdot x - c_w$ vanishes exactly at one vertex of Δ. We can move the origin to this vertex (which makes c_w to be equal 0) and consider the two edges e_1 and e_2 going from the vertex. So we have $w = a_1 n(e_1) + a_2 n(e_2)$. And a_i are positive because $a_i|n(e_i)|^2 = w\cdot e_i > 0$.

Suppose that $a_1 + a_2 > 1$, i.e. w is outside of a triangle with vertices 0, $n(e_1)$ and $n(e_2)$. We want to prove that w will not contribute to l_Δ. The whole polygon Δ is a subset of the cone spanned by e_1 and e_2. So it is enough to show that $w\cdot$ dominates either $n(e_1)\cdot$ or $n(e_2)\cdot$ in the cone. Let p be a point in this cone, i.e. $n(e_1)\cdot p$ and $n(e_2)\cdot p$ are nonnegative. Then $w\cdot p = a_1 n(e_1)\cdot p + a_2 n(e_2)\cdot p \geq (a_1+a_2) max(n(e_1)\cdot p, n(e_2)\cdot p) > max(n(e_1)\cdot p, n(e_2)\cdot p)$. □

8. Sand approximation for tropical polynomials

In the next two sections we are going to show how statements of Section 4 can be deduced from the results of the section 5. For any tropical curve we will construct its sand model called a *discrete approximation.* This model will be represented by a particularly nice set of colored vertices on a large lattice. In this approach the operators $G_{p_1,\dots,p_n}$ will be interpreted in terms of certain relaxations process. In the previous section we saw the most basic realization of this principle for the case of an empty curve and G_p.

Let Δ be a lattice polygon and F be a non-constant tropical polynomial vanishing at the boundary of Δ. Let C be the intersection of a curve given by F with Δ. Consider a graph $\Gamma = \Gamma(N\cdot\Delta)$ (see Definition 10) for N large enough.

We want to construct a new state ϕ such that its set of colored vertices, i.e. where $\phi < 4$, contains $N\cdot C$ in its tubular neighborhood of radius m, where $m\mathbb{N}$ is fixed for given F and doesn't depend on N. If we have such a state, then any connected component in the complement to a curve C would naturally define a territory for ϕ. As another requirement for ϕ we want this mapping to be injective, i.e. the territories separated by the edges of $N\cdot C$ should be distinct. The territories for ϕ which are in the correspondence with connected components of $\Delta\backslash C$ will be called *large territories.*

Last requirement is necessary to leave the state untouched outside a small neighborhood of a large territory while dropping a grain of sand in the territory and then relaxing. During this process we could have certain artifacts near the boundary of the territory. Clearly, there are many states ϕ satisfying the conditions listed above. And if we chose a generic ϕ the artifacts will inevitably appear.

To avoid this problem we could specifically choose the structure of ϕ along for edge e in C. Namely, we can fill a neigborhoud of $N \cdot e$ with a periodic pattern corresponding to the slope of e (see Section 5.4). This will ensure that the colored set along $N \cdot e$ moves without artifacts during the relaxations. And this is almost it.

It only remains to take care of the vertices. To avoided artifacts near trivalent vertex we can model ϕ by a corner patch arising near the corners of the boundary. In the case when our curve has vertices of higher valence or cycles we have an issue.

The first issue can be easily resolved. We can consider a tropical curve $N \cdot C$ (or rather a function $p \mapsto F(\frac{1}{N}p)$ for $p \in N \cdot \Delta$) and then make it generic using a perturbation of order $\sqrt{N}$. Thus all the vertices will be trivalent and macroscopically picture remains the same.

The second issue is more delicate. We need to show that we can actually glue all the patterns for edges with the corner patches while going around the cycles. Indeed, we will see that this is possible. The idea of the proof is very simple. Instead of making a surgery on patterns we can naturally grow a whole curve by emitting many waves from different places. We formulate it in the following proposition.

Proposition 11. For any N there exist a smooth lattice polygon δ_N, such that all δ_N are dual to the same fan $S(Fan(\Delta))$ (see definition 19), length for each edge tends to infinity and $\frac{1}{N}\delta_N$ converges to δ, and a sequence of stable states

$$\phi_0 \to \cdots \to \phi_N^{end}$$

on $\Gamma(\delta_N)$, where each ϕ_{i+1} is a result of sending a wave for ϕ_i and a sequence of toppling number functions $h_N : \mathbb{Z}^2 \to \mathbb{Z}$ of the processes $\phi_0 \to \phi_N^{end}$ converges to F, i.e.

$$\frac{1}{N} h_N([N\bullet]) \underset{N\to\infty}{\longrightarrow} F(\bullet).$$

The state ϕ_0 is as usual a maximal stable state and as before h_N counts the number of topplings at each point while transforming ϕ_0 to ϕ_N^{end}, outside $V(\Gamma(\delta_N))$ the toppling number function h_N is naturally prolonged by 0. The proposition can be seen as certain inversion for the Theorem 2.

In the statement of the proposition we used a notion of a fan and a concept of duality for fans and polygons that were not introduced yet in this paper, here is a definition for them.

Definition 19. A (complete, lattice) fan Φ is a finite collection of rays with rational slopes starting at the origin such that the angle between two consecutive rays is less than π. To encode a fan we take a set of all primitive vectors in the directions of its rays and say that a fan is generated by this vectors. A polygon Δ is dual to a fan Φ if Φ is generated by all the vectors $n(e)$ for $e \in E(\Delta)$, where $n(e)$ is a primitive normal to vector to an edge e (see the page 9 for the definition of $n(e)$). Let $S(\Phi)$ denote the (minimal) smooth refinement for $S(\Phi)$ that is a fan generated by all the primitive vectors from the triangles spanned by 0, w_1 and w_2, where w_1

and w_2 are the primitive vectors in the directions of two consecutive rays in Φ (see figure 11)

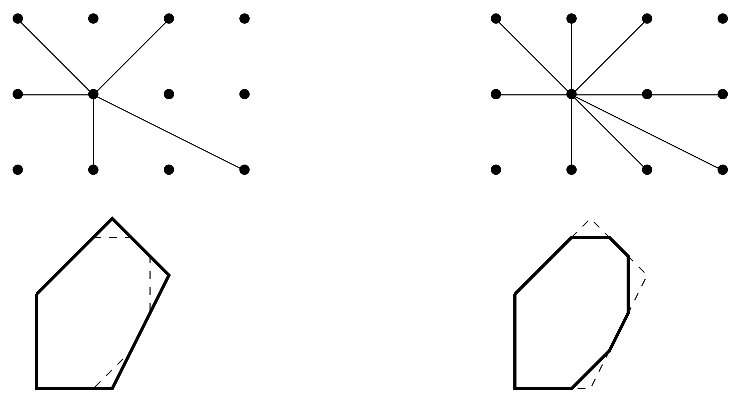

FIGURE 11. Two polygons and their dual fans. The fans are generated by collections of primitive vectors, that are shown on the picture. The right fan is a minimal smooth refinement of the left one, and the right polygon is a result of corner cuts for the left polygon.

Proof. Fix $N \gg 0$. First we need to define a polygon δ_N. Consider a lattice polygon $[\sqrt{N}]\Delta$. We can make each of its vertices smooth (see Definition 16) by doing a sufficient number of small lattice corner cuts (see Figure 10). This gives us a polygon Δ'_N dual to the fan $S(Fan(\Delta))$. We define a polygon δ_N as $[\sqrt{N}]\Delta'_N$ and graph Γ_N as $\Gamma(\delta_N, 1)$. Clearly $\frac{1}{N}\delta_N$ converges to Δ in the Hausdorff sense.

Consider the state ϕ_0. We start deforming it by sending $[\sqrt{N}]$ waves from the middle of $N \cdot \Delta$. This will create one territory of the area proportional to N^2 in the center of the polygon. Also for each edge e of Δ we will have a territory with area proportional to $N^{3/2}$ adjacent to (the neighbourhood) of $N \cdot e$. And finally, for each non-smooth vertex v of Δ we will see a number of territories with area of order $O(N)$ near $N \cdot v$. All other territories will be of the area $O(1)$.

Consider an edge e of $\partial\Delta$. We know that F is linear in the neighbourhood of e and vanishes at e. This means that near e the function F can be written as

$$F(x, y) = m(e)(qx - py + c)$$

for $m(e) \in \mathbb{Z}$, coprime p and q and $c \in \mathbb{R}$.

Let $e_1 \dots e_m$ be the sequence of all edges in Δ. Take a territory near $N \cdot e_1$ with $m(e_1) > 1$. This territory has a shape of trapezoid. More formally we can say that the territory sits in the small neigbourhood (for example of radius $N^{1/4}$) of the trapezoid. The larger base of this trapezoid is $N \cdot e_1$. Consider a point in the territory near a smaller base of the trapezoid. We emit $[\sqrt{N}/m(e_1)]$ waves from this point. This will split the territory on two territories with areas of order $O(N^{3/2})$. Both of this territories are again trapezoids.

If $m(e_1) > 2$ we repeat the procedure for the trapezoid adjacent to the boundary. This will again split the trapezoid into two pieces. Thus we create $m(e_1) - 1$ consecutive trapezoids with bases parallel to e_1.

After we pass to $N \cdot e_2$. We do exactly the same thing but the number of waves emitting from one point is equal to $[\frac{\sqrt{N}}{m(e_1)m(e_2)}]$. In general when creating a trapezoid territory for e_k we use $m(e_k)$ times less waves then for a single territory parallel to e_{k-1}.

At this point all the colored vertices are arranged along the boundary and a small neighbourhood of a certain tropical curve C_N. Note that the territories with the area $O(N^\alpha)$ for $\alpha > 0$ correspond to the connected component of the complement to C_N.

The main benefit of our construction of δ_N is that C_N is smooth. This implies that the connected components of the complement to C_N, are in one to one correspondence with the integer points inside its Newton polygon. This means that by sending waves we can freely deform all the "coefficients" for h_N. Since the Newton polygon for C_N contains the Newton polygon for our initial function F we can deform h_N to those that converge to F. □

Definition 20. A sequence of polygons δ_N, functions h_N and states $\phi_N = \phi_N^{end}$ satisfying the conditions of Proposition 11 will be called a *discrete approximation* for F.

It is clear that h_N determines ϕ_N and F. So abusing the notation we will say that h_N is a discrete approximation for F.

Remark 10. It is clear that a choice of exponents for N in the proof of Proposition 11 is not crucial. In particular, the size of the cuts for $N \cdot \Delta$ and the order of perturbation for a discrete curve could be taken smaller than $\sqrt{N}$.

9. Sand interpretation of G_p

Finally, we are going to debunk the mystery of tropical relaxation. In particular we will show that the curves and functions arising from the sandpile model enjoy certain extreme properties.

Lets fix a discrete approximation h_N for F. Take a point $p \in \Delta$. For each N we modify ϕ_N in the following way. If $[N \cdot p]$ is in the territory for ϕ_N we add a grain of sand at the vertex $[N \cdot p]$, then relax and remove a grain of sand at $[N \cdot p]$, otherwise we leave ϕ_N unchanged. This gives a new state ϕ'_N. We define a function $\tilde{h}_N$ on vertices as a the number of topplings during the relaxation.

Proposition 12. There exist a tropical polynomial F' vanishing at the boundary of Δ such that a sequence of functions $h'_N = h_N + \tilde{h}_N$ and states ϕ'_N is a discrete approximation for F'. Moreover,

- $F'(x) = G_p(F)(x)$ for all $x \in \Delta$;
- if p belongs to the connected component Ω of the complement to a curve defined by F then

$$F'(x) = \begin{cases} F(x) + l_{\overline{\Omega}}(x) & x \in \Omega \\ F(x) & x \in \Delta \backslash \Omega. \end{cases}$$

Recall that $l_{\overline{\Omega}}$ denotes the weighted distance function (see Definition 18). Here $\overline{\Omega}$ of course is a polygon, so we can use lemma 17 to see that $l_{\overline{\Omega}}$ is a tropical polynomial.

Proof. The state ϕ'_N is a result of applying waves to ϕ_0 because we can decompose the relaxation in the construction of ϕ'_N into a sequence of waves applied to ϕ_N. Thus the only thing we need to prove is that ϕ'_N converges.

Consider a vertex $v_N = [N \cdot p]$. If p is on a curve C defined by F then from the convergence of ϕ_N the distance between v_N and the boundary of the boundary of the territory is $o(N)$. This implies that $h'_N - h_N$ is $o(N)$. And thus h'_N converges to $F' = F$.

If p is not on C then v_N is inside of a large territory T. The state ϕ'_N is a result of applying a maximal number of waves at v. Each wave will simply move each edge in the boundary of the territory towards v, as it was described in the previous section. Each edge e' of the territory T will be moved on the distance $M/\sqrt{\alpha^2+\beta^2}$ after M consecutive waves (see Lemma 16). Here (α, β) is a primitive vector parallel to e and $M \gg 0$ but $M \ll N$.

We can actually calculate h' up to $o(n)$ at every vertex v'. If v' is not at the same territory T then $h'(v') = h(v)$. Otherwise, each wave emitted from v will contribute 1 to $h'(v') - h(v')$ until v' or v is not at the boundary of the territory. This means that if $l(v) > l(v')$, i.e. v' reaches the boundary first, then $h'_N(v') = h_N(v') + l(v') + o(N)$. Here $l = l_{N \cdot \Delta'}$ denotes a weighted distance to the boundary of $N \cdot \Delta'$ (see Definition 17), where Δ' is a connected component to the complement of C corresponding to the large territory T. And if $l(v) \leq l(v')$ then $h'(v') = h(v') + l(v) + o_(N)$. It is clear that after rescaling the function l vanishes at the boundary of the territory T and after rescaling converges to a piecewise linear function on the connected component to the complement of C. This implies the convergence for h'_N. □

Remark 11. Proposition 12 implies that G_p is a continuous operator on the space of tropical polynomials.

10. Convergence for the sand relaxation

In previous sections we developed some kind of theory that provides a nice understanding of certain relaxation processes. Unfortunately, we didn't have an occasion to prove several first theorems. We're going to fix this.

Consider a tropical polynomial F vanishing at the boundary of a lattice polygon Δ. We fix a collection of points $p_1, \dots, p_n$ in Δ°.

Consider a discrete approximation h_N for F (see Definition 20). Consider the sequence of stable states ϕ_N defined by h_N on graphs δ_N. We construct a new sequence of possibly non-stables states ϕ^0_N by adding extra grains of sand to ϕ_N at the points $[N \cdot p_i]$, i.e.

(5) $$\phi^0_N(v) = \begin{cases} \phi_N(v) + 1 & v \in \{[Np_1], \dots, [Np_n]\} \\ \phi_N(v) & \text{otherwise.} \end{cases}$$

As we did before we count the number of topplings at each point $v \in V(\delta_N)$ during the relaxation for ϕ^0_N. This gives a function $\tilde{h}_N : V(\delta_N) \to \mathbb{Z}_{\geq 0}$.

Proposition 13. The sequence $\tilde{h}_N$ converges to a function on Δ, i.e.

(6) $$\frac{1}{N}\tilde{h}_N([N\bullet]) \underset{N\to\infty}{\longrightarrow} \tilde{F}(\bullet).$$

Moreover $G_{p_1,\dots,p_n} F = F + \tilde{F}$.

Proof. Following the recursive definition for $G_{p_1,\dots,p_n}$ of Section 4, suppose that the curve given by F passes through all the points p_i. It means that for N large enough each of the points $[N \cdot p_i]$ is in the small neighborhood for same edge in the set of colored vertices. So from the Lemma 13 it follows that $\tilde{h}_N = o(N)$ and so $\tilde{F}$ exists and is identically zero.

Suppose now that some point p_k is not on the curve defined by F. This means that for a large N the vertex $[N \cdot p_k]$ is in a large territory of the state ϕ_N. We modify the state ϕ_N by adding an extra grain at $[N \cdot p_k]$ and then relax it and remove a grain at $[N \cdot p_k]$. Denote the resulting state by ϕ'_N. Proposition 12 says that a sequence of states ϕ'_N together with corresponding toppling counting function h'_N is a discrete approximation for a function F'. And from Proposition 12 it follows that $F' = G_{p_k} F$.

In the last paragraph we mimic the induction step (we will call it the step for the point p_k) in the recursive definition of $G_{p_1,\dots,p_n}$. On each step we contribute h'_N to $\tilde{h}_N$. From Theorem 6 we know that for all $N \gg 0$ the number of steps is finite, doesn't depend on N and the steps can be chosen coherently for different N then we will prove (6) and $G_{p_1,\dots,p_n} F = F + \tilde{F}$.

On the other hand, each step provides a partial relaxation for the state ϕ^0_N defined by (5). Since we know that any relaxation on a graph δ_N should terminate, to complete an honest relaxation of ϕ^0_N for any given N it is enough to do only finite number of the steps and in the end add one grain at each vertex $[Np_i]$. This observation also implies that the steps can be applied in an arbitrary order.

Now we want to analyze how each inductive step for a point p_k terminates on the level of discrete approximations. Recall that the step consist of applying a series of waves at the same large territory where the point $[Np_k]$ sits. We stop sending waves when $[Np_k]$ appears to be at the boundary of the territory.

On the limiting level this correspond to passing of the curve given by $G_{p_k} F$ through a point $[Np_k]$. Moreover, by calculating the speed of each edge (see Lemma 16) under the action of waves we know exactly which edge (or vertex) of the curve will pass through the point, i.e. we index termination of steps with such combinatorial events.

Since the set of colored points is located along the curve, for a large N the cause of termination of the step is the same as for all N. This implies that for $N_1, N_2 \gg 0$ after applying a sequence of steps for the points $p_{k_1}, \dots, p_{k_m} \in \{p_1 \dots, p_n\}$ to the states ϕ_{N_1} and ϕ_{N_2} the combinatorial distributions for the rescaling of points $p_1 \dots, p_n$ on big territories of ϕ_{N_1} and ϕ_{N_2} coincide. Thus, the steps can be applied coherently to different ϕ_N for N large enough. □

If we apply the previous proof to the case when $F = 0$ we get essentially a proof for Theorems 1 and 2.

Proof of Theorem 2. As we explained in the introduction, Theorem 1 is an easy consequence of Theorem 2. So we will concentrate on Theorem 2.

The only difference between the statement of Theorem 2 and the essence of the proof of Lemma 7 is that the theorem was formulated without the involvement of discrete approximations. Comparing with Section 4, in this theorem and in the whole introduction we always start with an empty curve on a lattice polygon Δ and an identically zero polynomial function.

In this case the discrete approximation (see Definition 20) constructed in Proposition 11 is very easy to describe. The sequence of states ϕ_N consists of maximal states and the toppling number function h_N is identically zero. The only nontrivial thing here is the choice of a sequence of polygons δ_N for the underlying graphs of sandpile models.

These polygons δ_N are not very different from the rescaling of Δ, they are essentially derived from $N \cdot \delta_N$ by performing lattice corner cuts for for non-smooth corners. As it is claimed in the Remark 10 we can take smaller exponents for the cuts then in the original proof of Proposition 11.

So we *construct* δ_N by taking the corner cuts of order $N^{1/4}$ for the polygon $N \cdot \Delta$. Together with a sequence of trivial toppling counting functions $h_N = 0$ this defines a discrete approximation for $F = 0$. Applying the proof for Lemma 7 to this discrete approximation we get the convergence for the relaxation of a sequence of states ϕ'_N (see equation (5)), where ϕ_N is a sequence of maximal states.

Now we need to estimate the difference between the relaxations for ϕ'_N on δ_N and the analogous state on $N \cdot \Delta$. Clearly δ_N is a subgraph for $N \cdot \Delta$, so we can think of a relaxation on δ_N as an incomplete relaxation on $N \cdot \Delta$. Other way round, to have a relaxation on δ_N we can run a relaxation on $N \cdot \Delta$ and *freeze* the vertices from $N \cdot \Delta \backslash \delta_N$, i.e. we forbid to do topplings at these vertices.

After the partial relaxation there can be more than 3 units of grain at each of the frozen vertices. If we unfreeze one grain of sand at a vertex v it will produce a certain number of waves until the amount of sand at v will be reduced. Each wave changes the toppling number function at each vertex of $N \cdot \Delta$ at most by 1. A number of waves that will be emitted from v is of order $O(N^{1/4})$, since the distance from v to the boundary of $N \cdot \Delta$ is at most $N^{1/4}$. So the total distortion for the toppling number function coming from unfreezing the grain at v has an order $O(N^{1/4})$.

Lemma 18. There is a constant $C_{n,\Delta} < \infty$ depending only on the number of initial points $p_1, \dots, p_n \in \Delta^\circ$ and the polygon Δ, that bounds an amount of sand at each frozen vertex $v \in (N \cdot \Delta) \backslash \delta_N$ after the relaxation on δ_N.

Proof. The existence of the constant follows from the fact that there is only finite number of different *types* of v. We will define a type of a vertex in $\cdot \Delta \backslash \delta_N$ in such a way that vertices of the same type will have the same amount of sand in them after the partial relaxation.

The simplest and probably the largest type consists of the vertices not adjacent to $N \cdot \Delta$, such vertices need not be relaxed. Some vertices are located around the corners, there number doesn't depend on $N \gg 0$, so we can declare each each such a vertex to be of its own type.

The last type consists of those vertices that are located on the cuts, i.e. the small edges of δ_N. There number tends to infinity with N, but there is only a finite number of locally different vertices. In this case we declare two vertices v_1 and v_2 to be of the same type if they both are on the same edge e of δ_N and $v_1 - v_2 = mw$ for some $m \in \mathbb{Z}$, where w is a primitive vector in the direction of e. □

The total number of frozen points has an order $O(N^{1/4}) \cdot O(N^{1/4})$. And from the computation of the distance to the boundary we deduced that each extra grain at these vertices produces at most $O(N^{1/4})$. By Lemma 18, a number of extra grains at any fixed frozen vertex is at most $C_{n,\Delta}$. Thus at every vertex of the graph on

$N \cdot \Delta$ the difference between the difference between toppling number functions for the relaxation on δ_N and $N \cdot \Delta$ is at most $O(N^{3/4})$. Therefore, the two toppling number functions converge to the same limit in the sense of equation (6). □

Despite the fact that the logic is a bit twisted and dependence of claims slightly counterintuitive, the rest of the proofs will go much easier. For example, what we've done during the proof of Theorem 2 clearly gives the proof for Remark 5, that says that $G_{p_1,..,p_n}(0)$ coincides with a piecewise linear function from Theorem 2.

Lemma 8 follows from Lemma 13, Remark 5 and Proposition 13 in the following way. We consider a descrete approximation (h_N, ϕ_N) for the function $F(x_1, \dots, x_n)$. From Proposition 13 it follows that if we add a gratin at $[Np_{n+1}]$ to ϕ_N and relax it then the resulting state will approximate $F(x_1, \dots, x_{n+1})$. Since the curve given by $F(x_1, \dots, x_n)$ passes throug x_{n+1} we know that the vertex $[Np_{n+1}]$ is at the distance $o(N)$ from some descrete tropical edge in ϕ_N. We what to show that it is required to send at most $o(N)$ waves from $[Np_{n+1}]$ to complete the relaxation. This would imply that the resulting descrte approxiamtion has the same limit as the original one. The case when $[Np_{n+1}]$ is inside a descrete tropical edge is covered by Lemma 13. So we can assume that $[Np_{n+1}]$ outside a tropical edge. Thus, by Lemma 16 we can estimate the number of waves as a constant depending on an edge times $o(N)$.

Theorem 4 that claims the minimality of $G_{p_1,\dots,p_n}F$, actually follows from the correctness of the implicit description for $G_{p_1,\dots,p_n}F$. Together with Theorem 4 and Lemma 5 this implies Theorem 3. Lemma 8 is an easy consequence of Lemma 13. And finally, in this presentation Proposition 3 is a trivial consequence of a definition and correctness for $G_{p_1,\dots,p_n}$ and Remark 5.

11. Discussion

11.1. **Minimal curves and genus.** Sandpile model provides us a solution of the following problem: given a set of points in a lattice polygon Δ we want to find a tropical curve of minimal symplectic tropical area, passing through these points. More precisely, we a looking for tropical curve, which intersect Δ only at vertices of Δ and the intersection of the curve and Δ has the minimal tropical area.

We should mention that there is a whole family of such curves – shrinking or inflating a component of the complement to the curve does not change its tropical symplectic area.

So, the question remains: what is so special about our curve among all the minimal curve. Theorem 3 says that the sand solution minimizes the integral of the toppling function, i.e. of the polynomial which defines the limiting curve.

In fact, this means that the limiting curve has the **maximal** genus among all the minimal curves passing through these points and under above conditions.

11.2. **Generalizations to other graphs.** As the reader can notice, the fact that the dimension of the lattice is two was not heavily used in the proofs. So, all similar results are obtained for higher-dimensional lattices, where we get not tropical curves but tropical hypersurfaces. This approach can be further generalized for any graph with a certain map to $\mathbb{Z}^n$. For example, for the Cayley graph of a group G this map imitates the map to the Cayley graph of the abelianization of G. The changes in the proofs are quite straightforward. The only trick is how to define the boundary. We proceed the following way.

We start from a graph G with a collection of maps $x_1, x_2, \dots, x_n : G \to \mathbb{Z}$. We require that all these function x_i are discrete *linear* harmonic functions on G. Linear means here that for each path $v_1, v_2, \dots, v_n$ of a length n we have a uniform on n estimation $x_i(v_n) - x_i(v_1) \leq cn$. Then, we define a boundary by the inequalities $\partial G = \{v | \sum c_{ij} x_i(v) \leq c_j\}$ for some chosen set of integers $c_{ij}, c_j, i \in 1..m, j \in 1..n$.

11.3. **Sand dynamic on tropical varieties, divisors.** It is easy to see that we can produce the same type of problems for tropical varieties, wince we have a sort of grid there, given by the affine structure and maps $x_1, \dots, x_n$ usually given by projections on the coordinates. The convergence results can be proven in the same way.

What is an interesting aspect of the possible applications is the tropical divisors. Indeed, using relaxation we can understand by the sand dynamic does there exist a divisor linearly equivalent to a given one, passing through prescribed set of points. We that we represent this divisor as a coloured set, using the patterns we already discovered, then we add sand to the points $p_1, p_2, \dots, p_n$ and relax the obtained state. If the relaxation terminates, it produces the divisor with the required properties. If not, that means that such a divisor does not exist.

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