Université de Genève — Faculté des Sciences
Section des Mathématiques — **Professeur Grigory Mikhalkin**

Tropical Curves, Convex Domains, Sandpiles and Amoebas

Thèse

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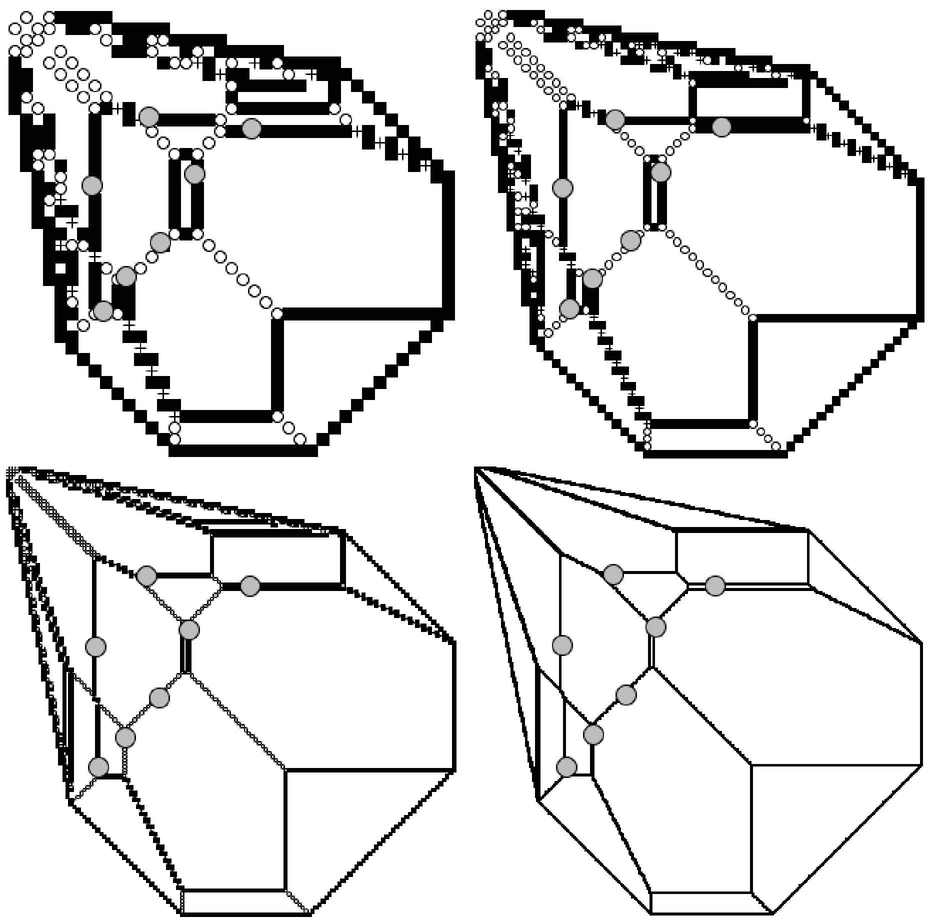

Figure 0.1: Illustration of the scaling limit Theorem 1.1.

Chapter 1

Main results

Consider a lattice polygon Δ. A **state** ϕ of the sandpile model is a non-negative integral valued function on $\Delta \cap \mathbb{Z}^2$. We think of every lattice point as container or cell. It can contain some integer amount of sand, where "sand" is a metaphor for mass-energy. So the state of a system is a function ϕ representing the amount of grains in each cell. If a cell has more than three grains of sand it **topples**, i.e. gives one grain to the each of the four sides (some grain can cross $\partial\Delta$ and leave the system). A state is stable if there is no toppling to be applied, i.e. $\phi \leq 3$. A process of doing topplings while possible is called the **relaxation**, we denote by ϕ° the stable state appearing in the result of the relaxation of a state ϕ. The details about relaxations can be found in [LP10] or in Appendix B of Chapter 9, the joint work with Nikita Kalinin [KS15b].

Caracciolo-Paoletti-Sportiello [CPS10, CPS12] observed that perturbing the maximal stable state produces, what they called "strings", thin graphs made of soliton patterns, the strings are realised as gaps in sand (energy) levels. The maximal stable state is represented by the function equal to 3 at every point. In the example shown on Figure 1.1 we see a perturbation $(3 + \delta_p)^\circ$ made by one point on a square. The result differs from 3 along a **balanced** graph with a single loop, i.e. a

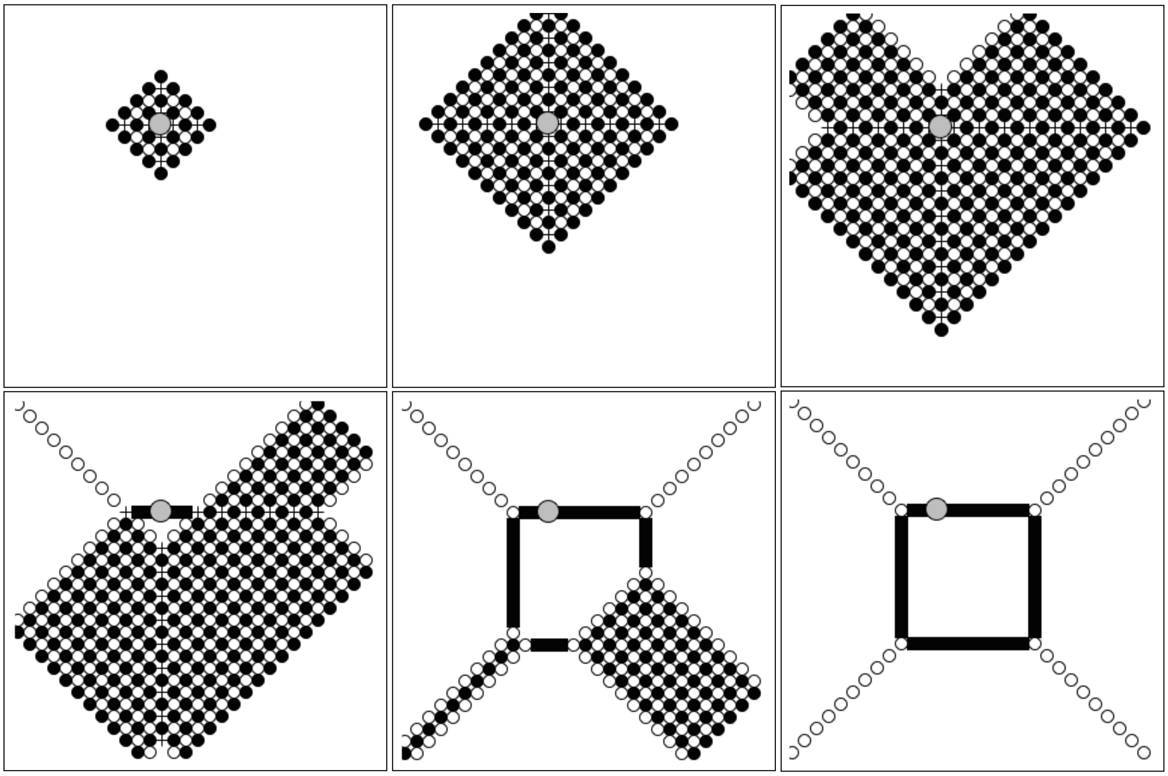

Figure 1.1: Snapshots during the relaxation for the state $\phi \equiv 3$ on a square after adding an extra grain at one point p (the big gray point). Black rounds represent v with $\phi(v) \geq 4$, black squares (which are arranged along the vertical and horizontal edges on the final picture) represent the value of sand equal to 2, white rounds (arranged along diagonals on the final picture) are 1, and white cells are 3. Rare cells with zero grains are marked as crosses, one can see them during the relaxation on the vertical and horizontal lines through p. The value of the final state at p is 3.

tropical elliptic curve passing through the perturbation point.

In Chapter 9, we develop a sufficient mathematical formalism to show that the behaviour of heavy sandpiles is governed by tropical geometry. Inspired by breakthrough results [PS13, LPS16] of Levine-

Pegden-Smart, we've chosen to use scaling limit as a basic tool to give an explicit statement. We show that for big polygon Δ the maximal stable state perturbed in several points approximates the solution of the tropical Steiner problem for these perturbation points (see Figure 0.1 for the demonstration). The tropical Steiner problem is the problem of drawing a **Δ-tropical curve** (given by a tropical polynomial $F = \min(ix + jy + a_{ij})$ vanishing at the boundary $\partial\Delta$ of the polygon) passing through the given collection of points in the interior Δ° and minimising the **action** defined as $\int_\Delta F$. The origin of the minimisation is the least action principle in sandpiles [FLP10] and the action corresponds to the total number of topplings performed during the relaxation.

Theorem 1.1. Consider a lattice polygon Δ and a collection of points

$$p_1, \ldots, p_n \in \Delta^\circ.$$

For any $N \in \mathbb{Z}_{>0}$ consider the perturbation of the maximal stable state on $N\Delta \cap \mathbb{Z}^2$ at $Np_1, \cdots, Np_n$. Denote by $D_N \subset N\Delta$ the locus of point where it is not maximal, i.e.

$$D_N = \{(3 + \delta_{Np_1} + \cdots + \delta_{Np_n})^\circ \neq 3\}.$$

Then $\frac{1}{N}D_N$ converges to (the unique) Δ-tropical curve minimising the action in the class of curves passing through the points $p_1, \ldots, p_n$.

The limit is taken over all compacts contained in the interior Δ°. This extra technicality (becoming crucial when working on general convex domain instead of Δ) is due to irregular behaviour near $\partial\Delta$. Otherwise, the plane limit of $\frac{1}{N}D_N$ on Δ would contain some segments of $\partial\Delta$. In Chapter 5 we show that minimisation of action implies minimisation of tropical symplectic area, corresponding to the total mass lost after the perturbation. Another feature of sandpiles [BTW87, Dha99] is the presence of power laws for the sizes of avalanches and other observables. In [GKL$^+$] we provide the evidence of power laws in the tropical version of the sandpile model defined via the scaling limit.

In Chapter 5 we investigate the origin of symplectic area and state the tropical Steiner problem. We give an iterative algorithm for approximating the solution and conjecture that it actually gives the exact solution after enough iterations. We investigate the role of this algorithm in the proof of the scaling limit theorem.

Chapter 9 (our joint paper with Nikita Kalinin) is dedicated to the proof of the scaling limit theorem for the sandpiles bounded by general convex domains. The main technical discovery for reducing the problem to the case of **$\mathbb{Q}$-polygons** (all sides have rational slopes) and to even simpler case of Delzant polygons is the approximation Theorem 1.2. We use the definition of Delzant polytope as it is given in [dS08]. Applied to the two dimensional case we have.

Definition 1.1. A $\mathbb{Q}$-polygon is called **Delzant** polygon if for every pair of adjacent sides the primitive vectors in their directions give a basis in $\mathbb{Z}^2$.

Theorem 1.2. For any compact convex domain Ω there exists a canonical continuous family Ω_ϵ, $\epsilon > 0$, of $\mathbb{Q}$-polygons such that $\Omega_\epsilon \subset \Omega_\delta$ for $\epsilon > \delta$ and $\cup_\epsilon \Omega_\epsilon = \Omega$. Moreover, if Ω has no corners then Ω_ϵ is Delzant.

The approximation is done by the level sets of the tropical analytic series F_Ω. Its definition is the following.

Definition 1.2. For a compact convex domain $\Omega \subset \mathbb{R}^2$ define a function $F_\Omega \colon \Omega \to \mathbb{R}$
$$F_\Omega(z) = \inf_{v \in \mathbb{Z}^2} (a_v + v \cdot z),$$
where a number α_v for a vector $v \in \mathbb{Z}^2 \backslash 0$ is given by $a_v = -\min_{z \in \Omega} z \cdot v$.

This function represents lattice distance to the boundary. It participates in the simplest case of the scaling limit theorem on a general domain, when we perturb the maximal stable state by a singe point. This interprets the evolution of Ω_ϵ as wave front.

Definition 1.3. Let ϕ be a state of a sandpile model on $\Omega \cap \mathbb{Z}^2$. For any point p the value $F(p)$ of the **toppling** function F, corresponding to the relaxation $\phi \mapsto \phi^\circ$, is the number of topplings performed at p during this relaxation.

The toppling function doesn't depend on the particular order of topplings in a relaxation. The discrete Laplacian of the toppling function F is equal to $\phi^\circ - \phi$ (see Appendix B, Chapter 9 for the details).

Theorem 1.3. Consider a convex compact domain $\Omega \subset \mathbb{R}^2$ and a point p in its interior. For any $h > 0$ consider p^h to be the nearest point to p in $h\mathbb{Z}^2$. Denote by $F_h \colon \Omega \cap \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ the toppling function of the relaxation for the maximal stable state perturbed at p^h. Then hF_h converges point-wise to the tropical series on Ω described explicitly as $z \mapsto \min(F_\Omega(z), F_\Omega(p))$.

The choice of a nearby point $p^h \in h\mathbb{Z}^2$ is irrelevant if we take $|p - p^h| \leq h$. In the case of several perturbation points a special care is needed. The main reason is the instability of a solution for some configurations (see Figure 5.4).

We investigate the properties of F_Ω in the case of Delzant polygon Ω in Chapter 3 via interpreting C_Ω, the tropical curve defined by F_Ω, as a caustic curve (manifestation of a curve with maximal quantum index [IM12, Mik15]). As a result we have the following.

Theorem 1.4. The level sets of F_Δ are Delzant polygons if Δ is Delzant.

Therefore, the **canonical evolution** $\Omega \mapsto \Omega_\epsilon$ on convex domains pushes domains with smooth boundary to Delzant polygons. And after, the evolution stays within Delzant polygons, until the degeneration to the maximal level set.

We interpret the evolution as the symplectic version of the minimal model for toric surfaces in Chapter 4 (the approach might work in higher dimensions as well and symplectic Calabi-Yau manifolds are distinguished as stationary points of the evolution). Recall that $c_1(X) = c_1(T_X)$ the first Chern class of X in algebraic geometry is often called "anti-canonical" class of the surface X and denoted as $-K_X$ and $K_X = -c_1(X)$ is called **the canonical class** of X. For a smooth toric surface X with symplectic form ω we consider the family of forms ω_ϵ such that

$$[\omega_\epsilon] = [\omega] + \epsilon K_X \in H^2(X, \mathbb{R}).$$

There exist $T > 0$ such that for $\epsilon < T$ the form ω_ϵ is non-degenerate and ω_T degenerates over some boundary divisors. Consider X' to be the result of contracting those divisors.

Proposition 1.1. If the limit of volumes $\int_X \omega_\epsilon \wedge \omega_\epsilon$ is not zero as $\epsilon \in (0, T)$ tends to T, then X' is smooth and push-forward of the form ω_T is non-degenerate on X'.

After interpreting Ω_ϵ as the image of the moment map ω_ϵ we realise this proposition as a version of Theorem 1.4. A consequence of the proposition is that one can continue deforming the symplectic form ω_T by the canonical class of X' staying within the class of smooth surfaces. It seems, that it is possible to generalise Proposition 1.1 outside toric geometry, giving rise to the canonical evolution on symplectic varieties. Chapter 3 gives an exhaustive description of events performed during this evolution in the case of smooth toric surface. In Chapter 4 we see that the evolution works equally well in singular case (i.e. for general $\mathbb{Q}$-polygons) and investigate the special role of -2 curves and multiple edges of C_Ω manifesting as A_n singularities of surfaces. Finally, I share some thoughts about how we could define a symplectic toric surface for an arbitrary convex domain.

In any case, this mode of thinking suggests to look at the minimal models for general Ω. The minimal model is defined as the germ of Ω_ϵ near the maximal level set. There are countably many types of such minimal models, we give a preliminary description for them in terms of mutants of del Pezzo polygons in Chapter 6. The minimal models, or equivalently the singularities of C_Ω, define an $SL_2\mathbb{Z}$-invariant stratification of the space of convex domains turning it into the infinite-dimensional tropical manifold. We describe types of strata in Chapter 6 and show examples of working with them.

Theorem 1.5. The space of all compact convex domains $\{\Omega \subset \mathbb{R}^2\}$ on the plane is stratified in terms of the type of singularity of C_Ω. Every stratum is a lattice polyhedral cone in the coordinates given by the moduli of the curve C_Ω.

The "moduli" here means the metric on the tree C_Ω given by lengths of all edges. Therefore, it should be possible to express any invariant of convex domains in terms of this moduli. We give a formula for area and some other invariants in general and show how to use them for the unit disc (see section 6.1 and [KS17]). We hope to refine them to the version of the Hirzebruch-Riemann-Roch formula on the toric space of a disc, and a convex domain in general (we sketch possible definitions of such exotic spaces in Section 4.2). Our interest in the subject is due to the feeling that the plane is very weakly understood. A particular example is the Gauss circle problem which we are attempting to solve via the above-mentioned techniques.

A crucial step in planar geometry could be a complete description of lattice invariants for the embeddings of a loop into the plane. Our research shows that the sandpiles applied as a nice intuitive and empirical basis for observing purely geometric phenomena. One of them is that sandpiles work well in higher dimension and, in dimension two, for non-convex sets, also we need more of symplectic geometry to describe what we see.

Consider the group $\mathrm{PSL}_2\,\mathbb{C} \subset \mathbb{CP}^3$ of orientation preserving automorphisms of $\mathbb{H}^3$ and fix a point $O \in \mathbb{H}^3$. Define a map

$$\varkappa\colon\ \mathrm{PSL}_2\,\mathbb{C} \to \mathbb{H}^3$$

by $\varkappa(A) = A(O)$. This map is an analogue of the logarithm acting from algebraic torus to the Euclidean space. Therefore, for a an algebraic subvariety V of $\mathrm{PSL}_2\,\mathbb{C}$ we consider $\varkappa(V) \subset \mathbb{H}^3$ and call it the **hyperbolic amoeba**.

Theorem 1.6. The hyperbolic amoeba of a line is either a geodesic cylinder or a horosphere. If V is an odd degree surface then $\varkappa(V) = \mathbb{H}^3$. If V is an even degree surface then $\mathbb{H}^3 \backslash \varkappa(V)$ is a convex domain.

This is a result of our joint work with Grigory Mikhalkin and the idea is suggested by Yakov Eliashberg (also it seems we have misunderstood the original aspiration). The difference between hyperbolic and Euclidean amoebas is manifested in our sandpile theorems, joint with Nikita Kalinin [KS16, KS15a]. I gave an overview of geometric ideas in Section 5.3 without spoiling the reading with technicalities. The realistic picture on scaling limits is presented in Chapter 9.

Chapter 2

Résumé en français

Considéron un polygon entier Δ. Un **état** du modèle du tas de sable est $\phi\colon \Delta \cap \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$, est un point p de $\Delta \cap \mathbb{Z}^2$ représente une cellule contenant $\phi(p)$ des grains de sable. Si une cellule a plus de trois grains de sable, elle donne une grain à chacune des quatre côtés. Cet événement est appelé **toppling**, renversement. Un état est stable s'il n'y a pas de renversement (topplings) á appliquer, C'est-á-dire $\phi 3$. Un processus de prise de topplings possible est appelé le **relaxation**, on note ϕ° l'état stable apparaissant dans le résultat de la relaxation d'un état ϕ. Pour des détails sur les relaxations voir [LP10] ou l'appendice B du Chapitre 9, du travail en commun avec Nikita Kalinin [KS15b].

Caracciolo-Paoletti-Sportiello [CPS10, CPS12] ont observé que les perturbations de l'état stable maximal produit, ce qu'ils appelaient "strings", des graphs fins réalisés en motifs solitons, les cordes sont réalisées comme des lacunes dans les niveaux de sable (énergie). L'état stable maximal est représenté par la fonction égale à 3 partout.

Théorème 2.1. Considérons un polygon entier Δ est une collection des points

$$p_1, \ldots, p_n \in \Delta^\circ.$$

Pour tout $N \in \mathbb{Z}_{>0}$ considérons la perturbation de l'état maximal stable sur $N\Delta \cap \mathbb{Z}^2$ à $Np_1, \cdots, Np_n$. Désignons par $D_N \subset N\Delta$ l'ensemble du points où ce n'est pas maximal, cet-à-dire

$$D_N = \{(3 + \delta_{Np_1} + \cdots + \delta_{Np_n})^\circ \neq 3\}.$$

Alors $\frac{1}{N}D_N$ converge vers une courbe Δ-tropicale minimisont l'action dans la classe des courbes passant par les points $p_1, \ldots, p_n$.

Le Chapitre 9 est consacré à la preuve du théoréme de la limite d'échelle pour les tas de sables délimités par des domaines convexes généraux. La technique découverte pour réduire le problème au cas des $\mathbb{Q}$-polygones (tous les côtés ont des pentes rationnelles) et même à un cas plus simple de polygones de Delzant est le Théorème d'approximation 2.2. Nous utilisons la définition de polytope Delzant tel qu'il est donné dans [dS08]. Appliqué en dimension deux nous avons.

Définition 2.1. Un $\mathbb{Q}$-polygone est appellé polygone de **Delzant** si, pour chaque paire de côtés adjacents, les vecteurs primitifs dans leurs directions donnent une base en $\mathbb{Z}^2$.

Théorème 2.2. Pour tout domaine convexe et compact Ω, il existe une famille continue et canonique Ω_ϵ, $\epsilon > 0$, de $\mathbb{Q}$-polygones de telle sorte que $\Omega_\epsilon \subset \Omega_\delta$ pour $\epsilon > \delta$ et $\cup_\epsilon \Omega_\epsilon = \Omega$. En outre, si Ω n'a pas de coins alors Ω_ϵ est de Delzant.

L'approximation est faite par les ensembles de niveaux de la série tropicale F_Ω.

Définition 2.2. Pour un domaine convexe compact $\Omega \subset \mathbb{R}^2$ définissons une fonction $F_\Omega \colon \Omega \to \mathbb{R}$

$$F_\Omega(z) = \inf_{v \in \mathbb{Z}^2} (a_v + v \cdot z),$$

où le nombre α_v pour un vectuer $v \in \mathbb{Z}^2 \backslash 0$ est $a_v = -\min_{z \in \Omega} z \cdot v$.

Cette fonction représente la distance de réseau à la limite. Elle participe au cas le plus simple du théorème de la limite sur un domain convexe général , lorsque nous perturbons l'état stable maximal par un grain.Ça donne une interprétation de l'évolution de Ω_ϵ comme front d'onde. Nous analysons les propriétés de F_Ω dans le cas de polygon de Delzant Ω dans le Chapitre 3 par l'intermédiaire de interpréter C_Ω, la courbe définie par F_Ω, en tant que courbe caustique (manifestation d'une courbe avec indice quantique maximal [IM12, Mik15]).

Théorème 2.3. Les ensembles de niveau de F_Δ sont des polygones Delzant si Δ est de Delzant.

Par consquent, **l'é volution canonique** $\Omega \mapsto \Omega_\epsilon$ sur des domaines convexes pousse des domaines avec $\partial\Omega$ lisse aux polygones Delzant.

Nous interprétons l' évolution comme la version symplectique de la modèle minimale pour des surfaces toriques au Chapitre 4. Pour une surface lisse torique X avec la forme symplectique ω nous considérons la famille de formes ω_ϵ telle que

$$[\omega_\epsilon] = [\omega] + \epsilon K_X \in H^2(X, \mathbb{R}).$$

Il existe $T > 0$ tel que pour $\epsilon < T$ la forme ω_ϵ est non dégénéré and ω_T dégénères sur diviseurs du bord de X. Considérons X' être le résultat de la contraction de ces diviseurs.

Proposition 2.1. Si la limite des volumes $\int_X \omega_\epsilon \wedge \omega_\epsilon$ est différent de zéro lorsque $\epsilon \in (0, T)$ tend à T, alors X' est lisse et le push-forward de la forme ω_T est non dégénéré sur X'.

The minimal model is defined as the germ of Ω_ϵ near the maximal level set. There are countably many types of such minimal models, we give a preliminary description for them in terms of mutants of del Pezzo polygons in Chapter 6. The minimal models, or equivalently the singularities of C_Ω, define an $SL_2\mathbb{Z}$-invariant stratification of the space of convex domains.

Le modéle minimal est défini comme le germe de Ω proche de maximal l'ensemble de niveau. Il existe plusieurs types de modéles minimaux, nous leur donnons une description preliminaire en termes de mutants de polygones del Pezzo au Chapitre 6. Les modéles minimaux, oú de façon équivalent les singularités de C_Ω, définissent une stratification $SL_2\mathbb{Z}$-invariante du l'espace de domaines convexes.

Théorème 2.4. L'espace de tous les domaines convexes compacts $\{\Omega \subset \mathbb{R}^2\}$ sur le plan est stratifié par le type de singularité de C_Ω. Chaque strate est un cône polyédrique entier dans les coordonnées données par les modules de la courbe C_Ω.

Considérons le groupe $\mathrm{PSL}_2\,\mathbb{C} \subset \mathbb{CP}^3$ de l'automorphisme de $\mathbb{H}^3$ fixant l'orientation et considérons le point $O \in \mathbb{H}^3$. Définisons une fonction

$$\varkappa\colon\ \mathrm{PSL}_2\,\mathbb{C} \to \mathbb{H}^3$$

par $\varkappa(A) = A(O)$. Cette fonction est un analogue du logarithme agissant du tore algébrique vers l'espace Euclidien. Par conséquent, pour une sousvariété V du $\mathrm{PSL}_2\,\mathbb{C}$ on considére $\varkappa(V) \subset \mathbb{H}^3$ et appelons **l'amibe hyperbolique**.

Théorème 2.5. L'amibe hyperbolique d'une ligne est soit un cylindre géodésique soit une horosphère. Si V est une surface de degré impair alors $\varkappa(V) = \mathbb{H}^3$. Si V est une surface de degré pair alors $\mathbb{H}^3 \backslash \varkappa(V)$ est un domaine convexe.

Ceci est le résultat de notre travail conjoint avec Grigory Mikhalkin et le l'idée est suggérée par Yakov Eliashberg (il semble également que nous avons mal compris l'aspiration originale). La différence entre les amibes hyperbolique et euclidiennes se manifestent dans nos théorèmes sur les tas de sable [KS15a, KS16].

Chapter 3

The caustic curve

First we give a mechanical description of the tropical curve C_Δ in the case of Delzant polygon Δ. We will construct a wave front Δ_ϵ made by moving the boundary of Δ. It is convinient to think of the vertices of the moving polygons as particles. When a side of a polygon is contracted, a pair of particles collide. The trajectory made by particles is C_Δ. Such point of view on tropical curves is well known in applications to enumerative geometry, where, for example, we can look at the system of particles going from infinity, colliding or not, and then count the possible trajectories passing through prescribed points, the answer corresponds to some Gromov-Witten invariant [BIMS15]. In this chapter collisions are compulsory, i.e. if the particles meet they form a new single particle in such a way that the balancing condition is satisfied (see Figure 3.2). The resulting tropical curve is extremal in many senses. Such curves are the curves of maximal quantum index [Mik15, IM12].

We are going to introduce a finite system of particles in the polygon and, as a set, C_Δ will be the trajectory traced by these particles. In the initial position, we set a particle at every vertex of Δ and send it towards the interior with the velocity vector equal to the sum of the primitive vectors spanning the sides adjacent to the vertex. Every particle moves rectilinearly until some group of particles collides. For example, in case of Δ equal to a square with horizontal and vertical sides all four particles meet at the center. Similarly, if Δ is a Delzant triangle all three particles meet at the same time (see Figure 3.1) and the tropical curve C_Δ has a unique vertex at the point of collision, i.e. the tropical curve is *unbranched*. We note that in both cases the polygons correspond to Fano toric surfaces $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2$ with a multiple of anti-canonical polarisation. This is true in general, C_Δ has a unique vertex if and only if Delzant polygon Δ is proportional to a lattice polygon with a unique lattice point in the interior. Indeed, by putting the origin to the vertex of C_Δ and rescaling Δ by the inverse of the total time τ_Δ of the evolution we get such a lattice polygon. The total time of the rescaled polygon is equal to 1 and the initial positions of particles are opposite to their velocities.

If we perform a blowup of $\mathbb{C}P^2$ at one of the three fixed points of the torus action, this corresponds to cutting a corner of its moment triangle. Note that the size of the cut may vary which corresponds to a different choice of symplectic structure on the blowup. If the integral of the new symplectic form along the exceptional divisor is small enough, the particles corresponding to the new corners will collide first forming a new particle with the velocity equal to the sum of velocities of collided particles. Note that this velocity is the same as of the particle which would be emitted from the blown-up corner. Moreover, after the collision the three remaining particles behave in the same way as if they would be simply emitted from the corners of the triangle without the blowup. In particular, they meet at the same terminal point. In this case the tropical curve has two vertices (see Figure 3.3).

While the cut gets bigger, eventually all the four initial particles will be again meeting at the same time (see Figure 3.1 on the right) and the curve has one vertex which is terminal for the process. Larger cut produces trapezium in which the particles are meeting in pairs. After the two collisions (apparently happening at the same time and

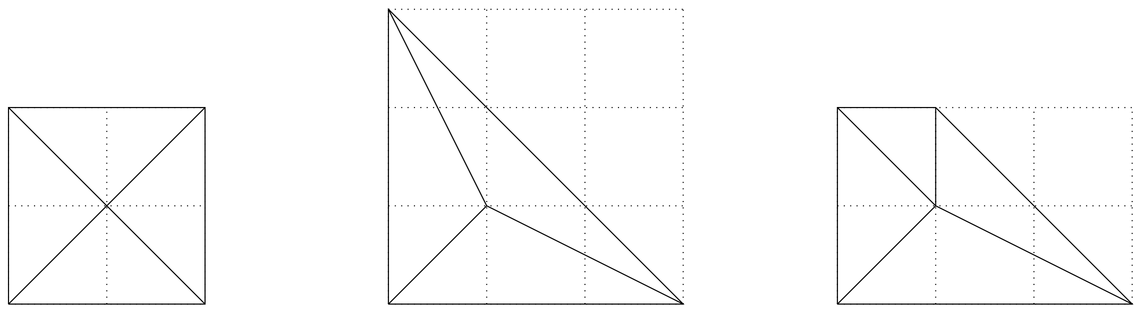

Figure 3.1: Some Delzant polygons with unbranched tropical curve.

at the same altitude) two particles are going horizontally towards each other. They collide at the terminal point (see Figure 3.3).

The rule of particle collision requires a minor clarification. In order to do that, we need to introduce masses of particles which are positive integers. The initial particles (Δ is Delzant) are all taken to be one and set to the vertices of Δ. A momentum of a particle is its mass times velocity and velocity is postulated to be primitive for all existing particles. In the case when a group of particles collide they give a new particle such that the momentum is preserved. For example, in Figure 3.3 on the right we see a pair of mass two particles annihilating each other at the terminal point of the mechanical process.

Lemma 3.1. For any Delzant polygon Δ, after the initial moment all the particles stay inside Δ° and there exists a unique (terminal) time $\tau_\Delta > 0$ at which a group of particles annihilate (without emission of a new particle) each other at the terminal point p_Δ.

In particular, the trajectory of all particles C_Δ is a weighted finite planar graph with rational slopes. The one-valent vertices of C_Δ are the vertices of Δ. The weights (or multiplicities) on edges correspond to the masses of particles tracing them. We note that the moment preservation is equivalent to the balancing condition shown on Figure 3.2. This turns C_Δ into a tropical curve. An edge of a tropical curve

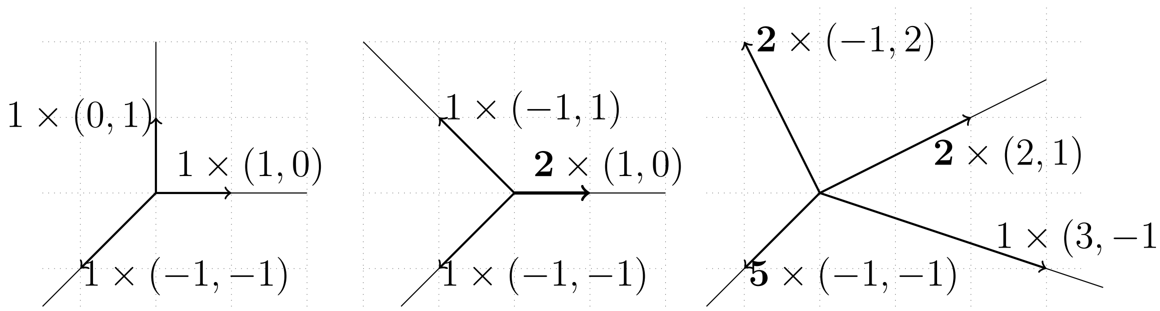

Figure 3.2: Examples of balancing condition in local pictures of tropical curves near vertices. The notation $\mathbf{m} \times (p,q)$ means that the corresponding edge has the weight m and the primitive vector (p,q). The vertex on the left picture is **smooth** (i.e. has multiplicity one), the vertices in the middle and right pictures are not smooth having multiplicities two and forty (see Figure 3.5 for the computation).

is called multiple if its weight is greater than two. One can give a full description of multiple edges in the Delzant case.

Proposition 3.1. If the terminal point p_Δ is a vertex of C_Δ for Delzant polygon Δ then C_Δ has no multiple edges. If the terminal point is not a vertex then it belongs to the middle of the *last* edge l_Δ appearing in the tracing of C_Δ, the last edge l_Δ has multiplicity 2 and it is the only multiple edge of C_Δ. The vertices that are not p_Δ or ends of l_Δ are 3-valent and smooth (see Figure 3.2).

The curve C_Δ contains a lot of geometric information about X the smooth symplectic toric surface whose moment polygon is Δ (we assume Delzant theorem [Del88]). At the most basic level we have.

Proposition 3.2. Consider a compact smooth symplectic toric manifold X and an irreducible boundary divisor D. Then to this divisor one associates a side $s = \mu(D)$ of the Delzant polygon $\Delta = \mu(X)$. Let s_1 and s_2 be the edges of C_Δ coming out of the ends of s. Then

s together with continuations of s_1 and s_2 form a Delzant triangle iff the self-intersection of D in X is -1.

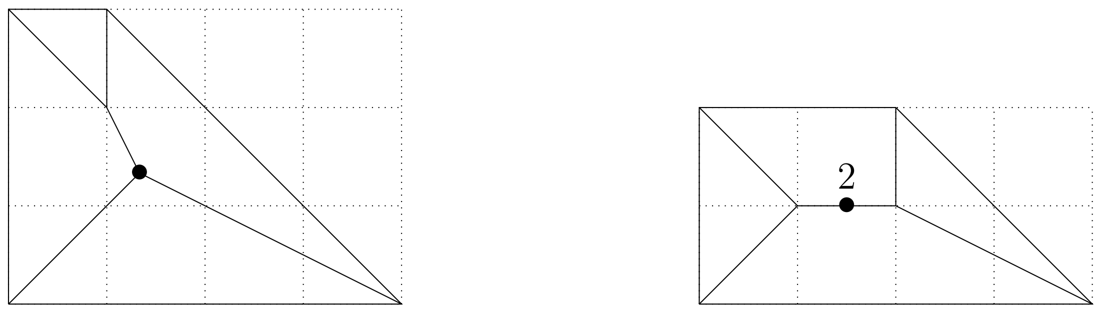

Figure 3.3: Different symplectic structures on a Hirzebruch surface produce different curves on its moment polygon. Note that the terminal point (marked by $\bullet$) of the polygons is not integral (although the polygons are lattice) and the caustic curve on the right has an edge of multiplicity two. The terminal point on the left is the lower vertex and on the right it is the middle of the multiplicity two segment. Note that in both pictures there is a pair of sides of Δ and a pair of edges of C_Ω which are parallel (see Remark 3.1).

Proof. All Delzant triangles are the same up to $SL_2\mathbb{Z}$, rescaling and translations. Therefore, if v_1 and v_2 are the velocities of the particles sent from the ends of $s \subset \partial\Delta$ then this segment is parallel to $v_1 - v_2$ (see Figure 3.4 on the left). If we denote by w_1 and w_2 the primitive vectors in the directions of sides adjacent to s then by the construction $v_1 = w_1 + (v_1 - v_2)$ and $v_2 = w_2 + (v_2 - v_1)$. Thus, $w_1 = v_2$ and $w_2 = v_1$ and the prolongations of sides intersect at a point p_0. Denote by $\tilde{\Delta}$ the convex hull of Δ and p_0. In this case X is realised as a symplectic blow-up of a manifold with the moment polygon $\tilde{\Delta}$ and D is the exceptional divisor. $\square$

If the conditions of Proposition 3.2 are satisfied, we call the side s removable. Removing a side is opposite to cropping a corner.

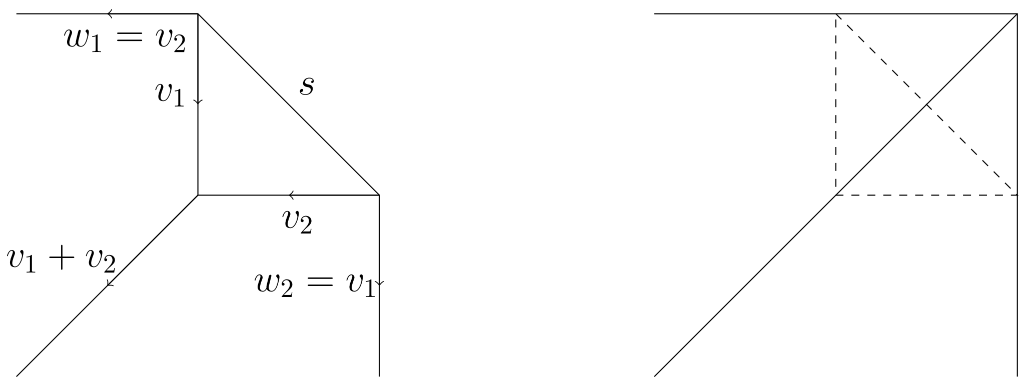

Figure 3.4: Two particles collide, trajectories form a Delzant triangle with the side s, emanate a new one in with velocity $v_1 + v_2$. Note that even if they don't meet geometry implies that two sides of the polygon are parallel to the trajectories of the particles. We can move s towards the intersection of the sides. This corresponds to blowdown on the right. Note that blowing up gives a simple branching of the tropical curve. And the resulting vertex is smooth (non-multiple).

Remark 3.1. A more visual criterium for the removability of D is that the sides adjacent to s are parallel to the two edges of C_Δ adjacent to s in the reversed order (as on Figures 3.3 and 3.4).

By Proposition 3.1, unless C_Δ has only one vertex (which is τ_Δ) or it has only two belonging to the ends of l_Δ (and in this cases we call C_Δ unbranched), there exist a side s such that the particles sent from the ends of this side meet in a non-terminal vertex, such s satisfies conditions of Proposition 3.2. In this case we can easily relate C_Δ and the curve corresponding to Δ with s blown down (see Figure 3.4).

After we recall complex origins of tropical geometry and give an asymptotic description of symplectic area, then we make sense of C_Δ as limiting solution to a Steiner-type problem. In algebraic geometry a curve is defined by a polynomial, tropical curves are logarithmic limits of complex curves. To bring a better global understanding of C_Δ first

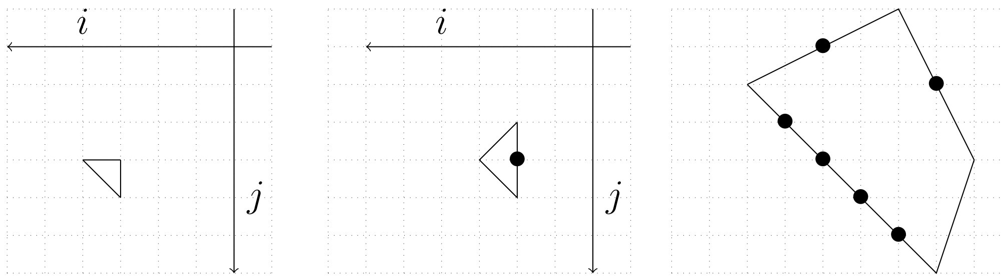

Figure 3.5: Polygons dual to local models of tropical curves on Figure 3.2. A dual polygon is defined up to a translation, its lattice points (i, j) correspond to the monomials $ix + jy + a_{ij}$ (in some polynomial defining a curve) which contribute to the value of the tropical polynomial at the vertex. Note that we need to reverse coordinate axes because of the "inf" (instead of more conventional here "sup") agreement in (6.1), in the definition of the tropical curve. Sides of polygons are orthogonal to the edges of curves. Moreover an integral length of a side, computed as one plus the number of lattice points in its interior, is the weight of the corresponding dual edge. The areas of the polygons are 1/2, 1 and 20, thus the multiplicities of the dual edges are 1, 2 and 40.

we give an intrinsic definition of its defining polynomial.

One may notice that in the cases when Δ is a lattice polygon, C_Δ passes through the vertices of $\Delta' = \mathrm{ConvexHull}(\Delta^\circ \cap \mathbb{Z}^2)$. Iterating the process we get a sequence of polygons $\Delta, \Delta', \Delta'', \ldots$ And we can think of a function F_Δ on Δ whose level sets are $\partial\Delta, \partial\Delta', \partial\Delta'', \ldots$ for the levels $0, 1, 2, \ldots$ This function F_Δ will be piecewise linear and not smooth exactly along C_Δ, and we say that F_Δ defines C_Δ. In general, a formal definition of F_Δ can be given as follows.

Remark 3.2. After the first few moments, a particle sent from a vertex of Δ stays at the same distance from the sides adjacent to that vertex.

The distance is taken in the "integral" (lattice-invariant) normalisation, a length of a lattice vector $v = (x, y)$ is defined to be $|gcd(x, y)|$, this is equal to the quotient of $|v|$ by the length of a primitive vector parallel to v. Therefore, a particle runs a unit of distance through the unit of time in this normalisation. In particular, if we want to compute the distance between a line with a rational slope and a point, one applies an $SL_2\mathbb{Z}$ transformation to make the line horizontal and the integral normalised distance between the new point and the new line (is the same as between the old ones) is equal to the usual distance in this case. Anyway, for a side s of Δ denote by $\lambda_s \colon \Delta \to \mathbb{R}_{\geq 0}$ the distance function from the line prolonging s. Note that this extends to a linear function with integral gradient on $\mathbb{R}^2$ supporting Δ. Define $F_\Delta \colon \Delta \to \mathbb{R}_{\geq 0}$ by

$$F_\Delta(p) = \min_s \lambda_s(p) \tag{3.1}$$

In this formula the minimum is taken over all sides s of Δ, and we may think of $\lambda_s(x, y) = a + ix + jy$ as of monomial "ax^iy^j" and of min as of summation. Therefore, F_Δ is a formal analogue of a polynomial. This mode of thinking is casual for tropical geometry where such piece-wise linear polynomials are seen as the limits of the usual ones. Note that each λ_s is a $\mathbb{Z}$-normalised distance to the supporting line, therefore we call F_Δ **the tropical distance** function from the boundary of Δ.

Theorem 3.1. The distance function F_Δ defines the caustic curve C_Δ.

The verb "defines" means that C_Δ is the corner locus of F_Δ. Remark 3.2 justifies the theorem in the neighbourhood of $\partial\Delta$. To extend this, we note that F_Δ is a continuos concave piecewise-linear function. Consider the maximum $\mathbf{m}_\Delta$ of F_Δ and for $0 < \varepsilon < \mathbf{m}_\Delta$ there exist a polygon Δ_ε obeying $\partial\Delta_\varepsilon = F_\Delta^{-1}(\varepsilon)$. The vertices of Δ_ε are the positions of particles at time ε.

Theorem 3.2. The polygon Δ_ε is Delzant for $\varepsilon \in [0, \mathbf{m}_\Delta)$.

Note that ConvexHull $F_\Delta^{-1}(\mathbf{m}_\Delta)$ has empty interior anyway and cannot be Delzant polygon.

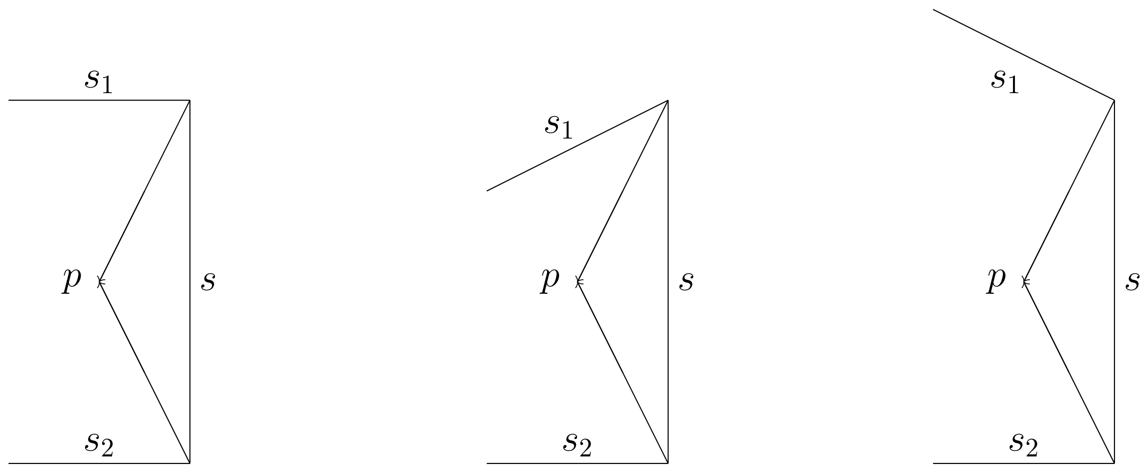

Figure 3.6: The possible configurations of sides with respect to a pair of collided particles. The collision happens at p to the left from the vertical side s. In the first case the adjacent sides s_1 and s_2 are parallel, in the second case their prolongations intersect to the left from s or to the right in the third case.

Proof. Consider the smallest time $\varepsilon > 0$ such that a pair of particles emitted from the ends of a side s collide at point p. If s is removable then we can consider $\tilde{\Delta}$ the moment polygon of the blowdown (see Figure 3.4). In this case either $\tilde{\Delta}_\varepsilon = \Delta_\varepsilon$ if $\varepsilon \geq e_0$ or Δ_ε is the blowup of $\tilde{\Delta}_\varepsilon$ otherwise. In both cases Δ_ε is Delzant whenever $\tilde{\Delta}_\varepsilon$ is Delzant, so we can use induction on the number of sides.

We are going to prove that if s is not removable then $F_\Delta(p) = \varepsilon$ is equal to the maximum $\mathbf{m}_\Delta = \max_\Delta F_\Delta$. In this case for $0 \leq \delta < \mathbf{m}_\Delta$ the polygons Δ_Δ all have the same dual fan.

Lemma 3.2. Consider a point $p \in \Delta^\circ$ such that $F_\Delta(p) = \lambda_{s_1}(p) = \lambda_{s_2}(p)$ for a pair of parallel sides s_1 and s_2. Then $F_\Delta(p) = \mathbf{m}_\Delta$.

Proof. We note that $\Delta_\delta \subset \Delta$ is got squished (i.e. diameter is bounded and the area goes to zero) as $\delta \to_- F_\Delta(p)$ since the distance between a pair of parallel sides tends to zero. This is not possible unless $F_\Delta(p)$ is the maximal value. □

Consider the sides s_1 and s_2 of Δ adjacent to s. There are three different situations (see Figure 3.6). In the first situation the sides s_1 and s_2 are parallel and we are done here by Lemma 3.2.

In the second case, the point of intersection of the prolongations of s_1 and s_2 is on the same side with respect to s as the point of collision p. If we think of F_Δ, its gradient ∇F_Δ is well defined on the compliment to the tropical curve of F_Δ. Consider a side $s_3 \neq s_1, s_2, s$, the gradient of it support function λ_{s_3} is the primitive vector orthogonal to a side s_3 and therefore is a positive combination of gradients for λ_{s_1} and λ_{s_1}. In particular, p is a global maximum of F_Δ.

In the third case, we see that $\nabla\lambda_s$ is a positive linear combination of $\nabla\lambda_{s_1}$ and $\nabla\lambda_{s_2}$. Also, we know that $\nabla\lambda_{s_1}, \nabla\lambda_s$ and $\nabla\lambda_{s_2}, \nabla\lambda_s$ give a basis in $\mathbb{Z}^2$. Now, note that the convex hull of the points $\nabla\lambda_{s_1}, \nabla\lambda_{s_2}, 0$ and $\nabla\lambda_s$ is a four-gone with area 1. It contains a triangle with vertices $\nabla\lambda_{s_1}, \nabla\lambda_{s_2}$ and 0, since s is getting shrinked, see Figure 3.7

Therefore, this lattice triangle has the minimal possible area $\frac{1}{2}$ and s must be removable since

$$\nabla\lambda_s = \nabla\lambda_{s_2} + \nabla\lambda_{s_2}. \tag{3.2}$$

□

Summarising the proof above, we saw that only the right configuration on Figure 3.6 can give a collision at a non-terminal point. In fact, exactly two particles can meet at a non-maximal point.

Lemma 3.3. If when passing from Δ to Δ_ε two sides collapse at the same vertex p of C_Δ then $F_\Delta(p) = \mathbf{m}_\Delta$.

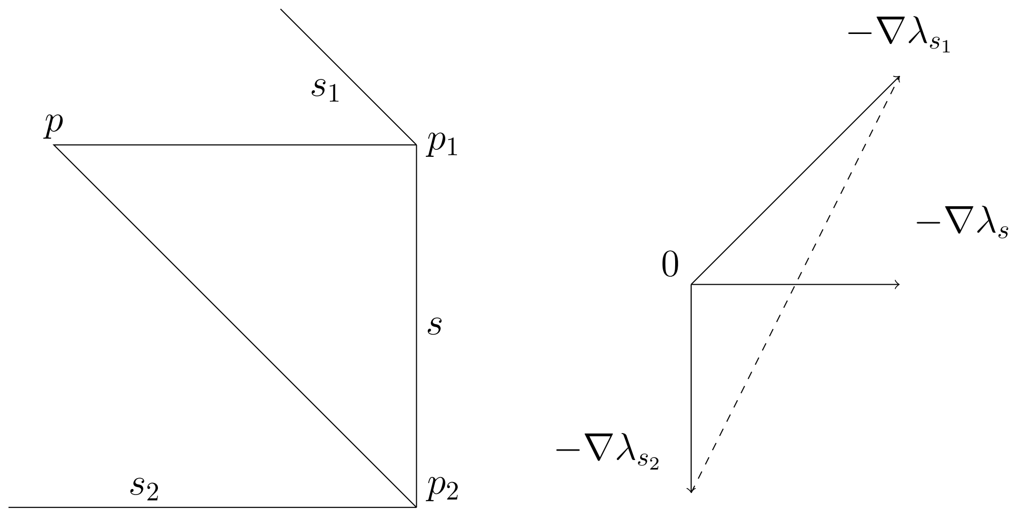

Figure 3.7: The third type of collision and its local dual picture. Note that $\nabla\lambda_s$ doesn't belong to the triangle spanned by $\nabla\lambda_{s_1}$, $\nabla\lambda_{s_2}$ and 0 because a segment $[p, p_i]$, $i = 1, 2$, is orthogonal to the vector $\nabla\lambda_{s_i} - \nabla\lambda_{s_i}$ and $p \in \Delta$.

Proof. Supposing p is not maximal, note that for both contracting sides we have a situation described on Figure 3.7. Also, note that we can assume that these sides are adjacent. Now we apply $SL_2\mathbb{Z}$ transformation to make the contracting sides horizontal and vertical. In terms of Figure 3.7 we have fixed the slopes of s and s_2 which are contracting now. There is only one position (on the dual picture) where we can add $-\nabla\lambda_{s_3}$ for s_3 adjacent to s_2, i.e. (as in the end of the proof of Theorem 3.2) $\nabla\lambda_{s_2} = \nabla\lambda_s + \nabla\lambda_{s_3}$. Therefore, s_1 is parallel to s_3 and we arrive to a contradiction with Lemma 3.2. $\square$

proof of Theorem 3.1. We use induction on time. Consider again the first time of a collision ε. By Remark 3.2, C_Δ coincides with the curve given by F_Δ in $\Delta\backslash\Delta_\varepsilon$. And this is enough for the proof if $\varepsilon = \mathbf{m}_\Delta$. Otherwise, by Theorem 3.2 we can run the particle process for C_{Δ_ε}. On the other hand, by Lemma 3.3 and equation (3.2) there is only one local model for a non-maximal collision shown on Figure 3.4 which guaranties that the processes for Δ after the time ε and the process for Δ_ε are the same. $\square$

The representation of C_Δ as a curve of F_Δ makes Proposition 3.1 and Lemma 3.1 evident. Indeed, the last edge l_Δ becomes just $F_\Delta^{-1}(\mathbf{m}_\Delta)$ and for a point $p \in C_\Delta$ the value $F_\Delta(p)$ is the time at which p is visited by a particle (or a group of particles if p is a vertex). There are no multiple edges except for the maximal edge l_Δ since a non-maximal collision has a unique model shown on Figure 3.4. Finally, note that the maximal edge has multiplicity two since it appears as a limit of a collapsing polygon Δ_ε with a pair of parallel sides and the gradients of the support functions of these sides are primitive and opposite. The multiplicity of an edge is one less the number of monomials participating in F_Δ along the edge. In the case of the maximal edge the monomial are the supporting functions for a pair of parallel supporting lines and the constant monomial, i.e. three monomials along the maximal edge. Experts may think of degenerate elliptic curve.

Remark 3.3. For a Delzant polygon Δ, a non-maximal vertex of C_Δ cannot be multiple because there is a unique model for a non-maximal vertex, see Figure 3.4, where it is smooth.

3.1 $\mathbb{Q}$-polygons and Steiner problem

In the case if Δ has a non-smooth vertex, the construction of the mechanical system needs adjustments. We use the canonical resolution of toric singularities (also related to chain fractions) to approximate Δ by Delzant polygons, see figure 4.2. In general, a non-smooth corner emits more than one particle and those particles can have multiple masses (see Figures 4.1 and 3.9). In any case, the definition of the function F_Δ (3.1) with the corner locus on the trajectory C_Δ of these particles works well. We should take care of invisible sides (corresponding to contracted divisors in a non-smooth case, see Figure 4.2).

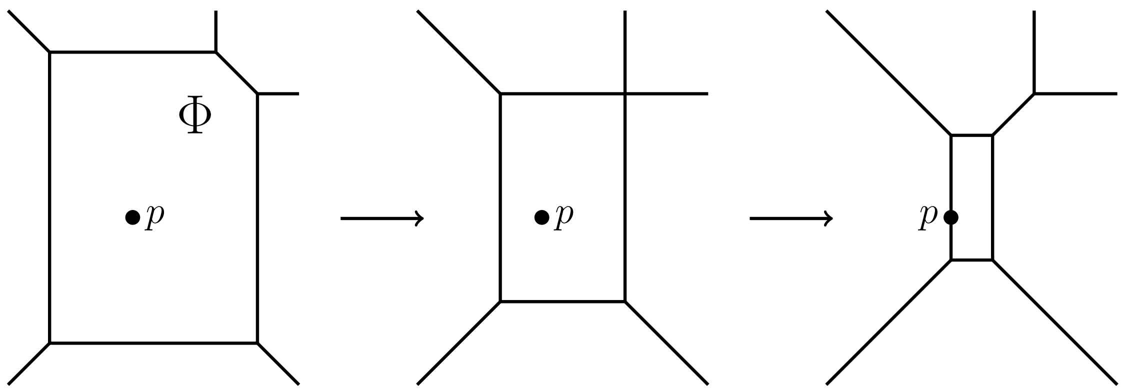

Figure 3.8: The operator G_p shrinks the face Φ where p belongs to. We see the same type of wave front applyed to the face. This is done by increasing the coefficient of the monomial contribution on Φ in the tropical polynomial defining the curve.

In general, we want to look at generalisations of F_Δ, i.e. tropical polynomials non-negative on Δ and vanishing on its boundary. We call such functions **Δ-polynomials** and F_Δ is the point wise minimum of all Δ-polynomials without a constant term (i.e. cannot have zero gradient on a set with non-empty interior). Tropical polynomial F_Δ represents the limit of the toppling function in sandpiles, i.e. the point-wise action of the system. We consider a highly agitated, maximal stable sate on a sandpile bounded by Δ. Perturbing it at one point produces some action (i.e. the total number of topplings), we chose such a perturbation by one point which produces the maximal action. Then the resulting state deviates from the maximal stable state along C_Δ and the toppling function of the perturbation is approximated by a multiple of F_Δ.

In purely tropical level, we may consider a $\mathbb{Q}$-polygon Δ. Then for any Δ-tropical curve C, the corner locus of a restriction of a tropical polynomial F to Δ vanishing on the boundary, we associate the action given by $\int_\Delta F$ and mass, the tropical symplectic area, the limit of the symplectic area for holomorphic curves. Then we ask what is the Δ-tropical curve passing through $p \in \Delta^\circ$ and minimising action or mass. It appears that both problems are solved by the same curve $G_p\varnothing_\Delta = C_\Delta/\partial\Delta \cup \Delta_{F_\Delta(p)}$.

We modified this one-point-perturbation by several points perturbation, dropping additional grains of sand to the maximal stable state at several perturbation points forming a set P. The resulting state deviates from the maximum along a curve $G_P\varnothing$ passing through the points in P. Moreover, its curve has the minimal symplectic area in the class of all curves passing through P, thus solving a version of the Steiner tree problem for the given configuration of points in the polygon Δ. This property is, in fact, implied by the action minimisation.

In "$G_P\varnothing$" the empty set denotes the vacuum state, where no deviation curves are present, we extend the action to the action to G_PF for F an arbitrary tropical Δ-polynomial. The polynomial G_PF is defined as the minimal Δ-polynomial with $G_PF \geq F$ and the curve given by G_PF passes through P.

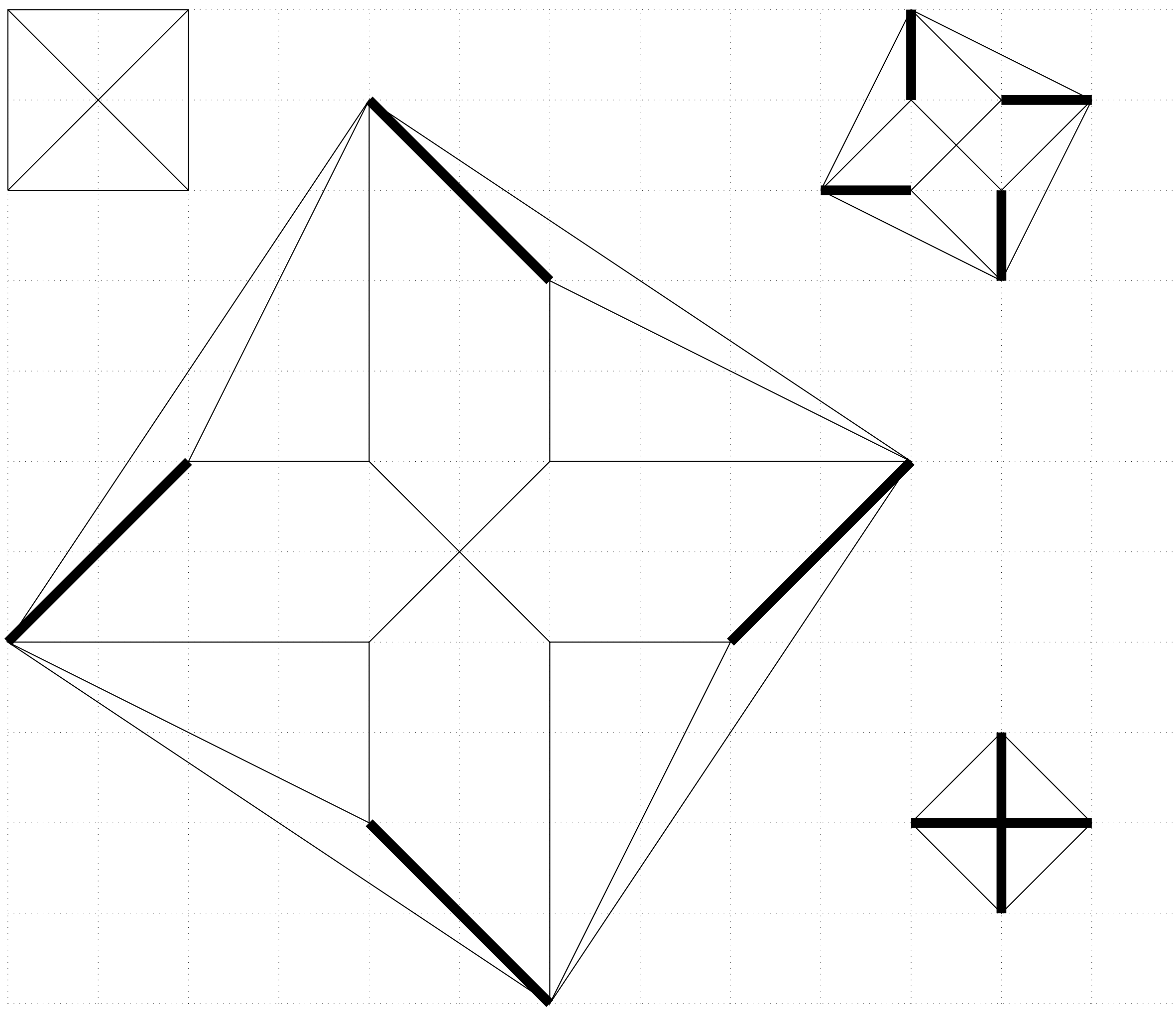

Figure 3.9: Different lattice squares (not Delzant except for the one above on the left) and their caustics, thick edges are of multiplicity two. Be aware that the whole construction is $SL_2\mathbb{Z}$ invariant but not rotation invariant. The curves are the solutions to the simplest version of Steiner type problem where we are trying to connect the corners of squares with its center using only tropical curves.

Chapter 4

The canonical evolution

Consider a Delzant polygon Δ and its tropical function F_Δ defined by (3.1). For any $t \in [0, \mathbf{m}_\Delta)$ we have a symplectic manifold X_{Δ_t} corresponding to $\Delta_t = F_\Delta^{-1}[t, \infty)$ with $2-$form ω_t. Note that X_Δ dominates all these manifolds via blow-downs.

Theorem 4.1.

$$\frac{d}{dt}[\omega_t] = K(X_{\Delta_t}), \tag{4.1}$$

where $K(X_{\Delta_t})$ is the canonical class of X_{Δ_t}.

The equation can be taken either locally (on time) in the cohomology of X_{Δ_t} or globally after the pull-back to the cohomology of X_Δ. Between the critical values of F_Δ the topology of the manifold is unchanged and its canonical class $K(X_{\Delta_t})$ is constant. The pull-back of the form ω_t, as a function of time, is continuous and piece-wise linear.

Proof. It is well known that Δ_1, the convex hull of $\Delta^\circ \cap \mathbb{Z}^2$, is obtained by twisting with the canonical line bundle in the case when Δ is lattice polygon (and so corresponds to an ample line bundle). Applying rescaling and continuity of the correspondence $\Delta \to F_\Delta$, we are able to integrate the observation using the lemma below. $\square$

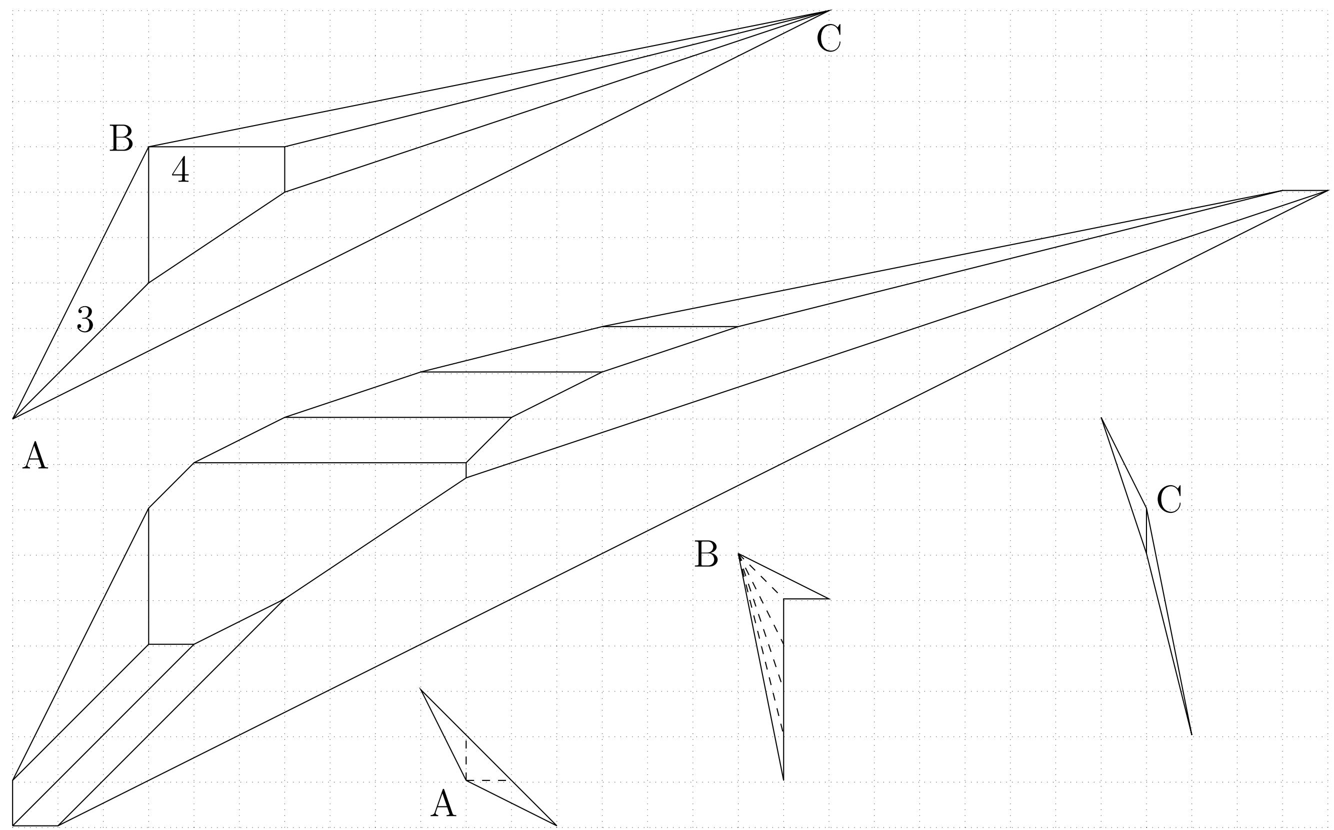

Figure 4.1: Instead of every vertex of a triangle (the same as on Figure 4.3) we've inserted the resolution of singularities according to Figure 4.2. This explains the appearance of multiple edges on the caustic via the contraction of -2 curves. Note that we can use dual corners A, B and C to construct the tropical curve and to resolve the singularities.

Lemma 4.1. Let Ω be a convex compact domain. Define F_Ω as the minimum of all lattice supporting functions (with integral gradients, see (6.1)). Then the non-zero level sets Ω_ϵ of F_Ω are polygons with sides having rational slopes (aka **$\mathbb{Q}$-polygons**). Moreover,

$$F_{\Omega_\epsilon} = F_\Omega|_{\Omega_\epsilon} - \epsilon$$

for $0 < \epsilon < \mathbf{m}_\Omega = \max F_\Omega$.

The hardest part is to prove that F_Ω is well defined and vanishes along the boundary. This and the rest we totally omit here, the proof is given in [KS15b] Appendix C, the paper attached to the thesis. The idea is that only finitely many of tropical monomials give contribution to F_Ω on a compact subset of Δ°. What we can imagine is that the equation (4.1) is the vector field for the canonical evolution on the space of all convex domains. A life of a convex domain Ω begins with the maximal segment/point $F_\Omega^{-1}(m_\Omega)$, its future is totally fixed and runs through Ω_t at the time $t > 0$. The past is undetermined, denote the maximal possible live by M_Ω. It has no past, i.e. $M_\Omega = 0.$, unless Ω is not a $\mathbb{Q}$-polygon with mild singularities (A_n type). The past is finite unless X_Ω is del Pezzo.

Theorem 4.1 suggests how to read the information about the intersection pairing on X_Δ from C_Δ or Δ_ϵ. Recall that the anti-canonical class $-K$ is dual to the sum of all boundary divisors. In particular, for any boundary divisor D its self-intersection is equal to $-D \cdot K - 2$. On the other hand, applying (4.1) we see that the change of length for every side under the canonical evolution is related to its self intersection, see Proposition 4.1. This generalises Remark 3.1, where the length of a blown-up side (i.e. of self-intersection -1) decreases by ϵ after the time ϵ. For sides with self-intersection 0, we see them decreasing by 2ϵ. The only type of sides whose length doesn't change with time are the sides of self-intersection -2, in this case we see parallel edges of the caustic. Contracting a chain of -2 curves, getting A_n singularity corresponds to a collapse of parallel edges and formation of a single multiple edge. On Figure 4.3, there are two multiple edges. By performing the resolution of singularities, shown in Figure 4.1 we see that multiple edges split into a collection of parallel edges.

Proposition 4.1. Let s be a side in a Delzant polygon Δ. Consider Δ_ϵ, for $\epsilon > 0$ it has a side s_ϵ nearby and parallel to s. Then

$$\frac{d}{d\epsilon}\operatorname{Length}_{\mathbb{Z}} s_\epsilon = -D_s^2 - 2,$$

where D_s is a boundary divisor in X_Δ over s.

Proof. If we multiply (4.1) by D_s we get

$$\frac{d}{d\epsilon}\operatorname{Length}_{\mathbb{Z}} s_\epsilon = \frac{d}{d\epsilon}\int_{D_s}\omega_\epsilon = D_s\cdot K = -D_s^2 - 2.$$

□

4.1 The canonical Cauchy problem

.

The construction of C_Ω shows that the canonical evolution defined by (4.1) restricts well to the space of smooth toric surfaces. We consider a smooth symplectic toric manifold X_0 with symplectic form ω_0 and moment polygon Ω. Denote by $K(X_0) = -c_1(X_0)$ the canonical class of X_0. Then for small enough $\varepsilon > 0$ the form $\omega_\varepsilon = \omega + \epsilon K(X_\epsilon)$, where $K(X_\epsilon) = K(X_0)$ because the topology is unchanged. At some moment $t_1 > 0$ the form ω_{t_1} becomes degenerate. It may happen that the curve C_Ω is unbranched and t_1 is the terminal moment of the Cauchy problem, i.e. the Liouville volume of X_t tends to zero as $t \to t_1+$, and so $t_1 = \mathbf{m}_\Omega$.

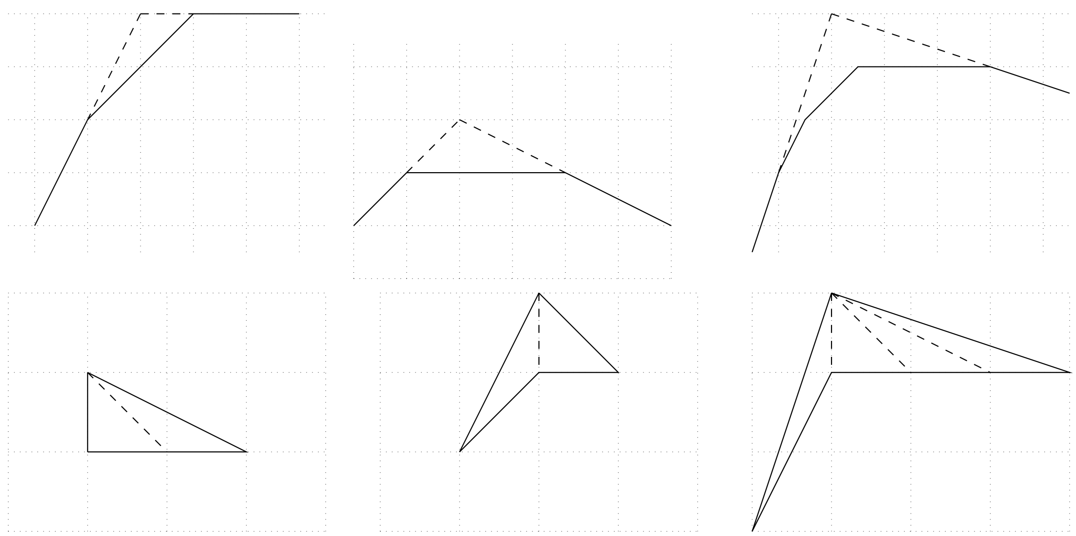

Figure 4.2: The canonical resolution of toric singularities can be extracted either from the dual cone or from the chain fraction for the slope of the corner.

Otherwise, we need to blow-down some boundary divisors to proceed. Denote the blow-down X_{t_1}. Theorem 3.2 guaranties that X_{t_1} is smooth. Passing through the critical level reduces the Picard rank, so we will come to the terminal point moving by the piecewise segments parallel to the canonical class at the moment (the canonical class is considered as the piece-wise constant function of time). The terminal manifolds are rather interesting, they generalise del Pezzo surfaces, and correspond to the stratification of the space of all convex domains. As we see, if a polygon of a symplectic manifold degenerates to a point then the anti-canonical class is ample (or a suitable replacement in singular case, just saying that, up to a rescaling, the moment polygon has exactly one point in its interior). If we see a degeneration of the polygon to a segment of length λ_Ω, it means that $[\omega_0] = \lambda_\Omega c_1(L) - K(X_0)t_1$, where L is a line bundle dual of the divisor D dual to a side parallel to a segment (there are two sides parallel to the segment, but the

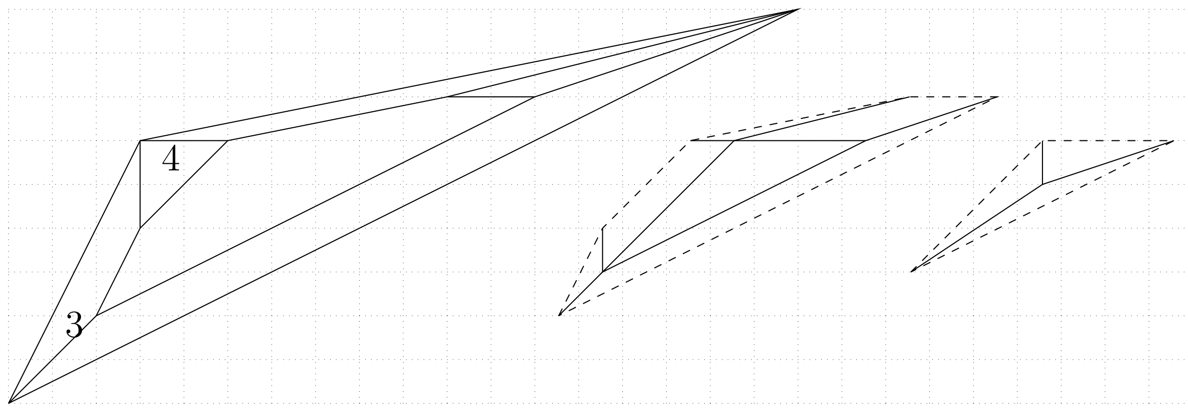

Figure 4.3: The evolution of triangle with no past (only the down left corner has exactly one adjacent edge of the curve) decomposed into three periods. The pentagon in the middle has three smooth corners and two singular, where the multiple edges go. Note that the level set of F_ω near its maximum could have infinite past unless we didn't do so many blow-ups. The big triangle Ω contains the whole information about the future — C_Ω is the trajectory traced by the vertices of shrinking polygons.

difference of their boundary divisors is linearly equivalent to a sum of other boundary divisors and we define $[D] = c_1(L)$ as intersecting by one with all divisors parallel to the maximal segment and by zero with all the rest). We call λ_Ω **the elliptic parameter** of Ω. The reason for the name is that $2\lambda_\Omega$ compute the modulus of the tropical elliptic curve appearing as the normalisation of C_Ω, i.e. the length of the circle parametrising the terminal segment of multiplicity two via the folding.

Now we are going to study the local conditions for Ω_0 to have a past. We can restrict ourselves to the case of $\mathbb{Q}$-polygons by Lemma 4.1, because only them can appear as a result of the evolution, i.e. a non-zero level set. The only event that can prevent a polygon Ω_0 from being $(\Omega_{-\epsilon})_\epsilon$, $\epsilon > 0$ for some $\Omega_{-\epsilon}$ (note that there can be many!) is that in the limit Ω_ϵ, $\epsilon \to 0+$ some sides have collapsed. Therefore, we've almost deduced the following.

Theorem 4.2. A $\mathbb{Q}$-polygon $\Omega = \Omega_0$ has a past iff X_Ω has only A_n-singularities.

Proof. The condition that no sides collapsed just before the present moment is the same as the condition that exactly one side of C_Ω is adjacent to the each corner of Ω. It means that the triangle spanned by the gradients of support functions of a pair of adjacent sides has no lattice points except for vertices and, in the case of the multiple edge of C_Ω, the triangle has lattice points on the side orthogonal to that edge. (see Figure 4.3) □

As an easy consequence we have the following quite useful result.

Theorem 4.3. Let Ω be a compact convex domain with C^1 smooth boundary. Then the polygons in the approximation $\Omega_\epsilon \subset \Omega$ are Delzant.

Proof. Suppose that for some $\epsilon > 0$ the polygon Ω_ϵ is not Delzant. Then, by Theorem 4.2 there is a (non-maximal) multiple edge of C_Ω adjacent to some singular corner of Ω_ϵ. It means that there are at least three supporting lines whose linear functions coincide along this multiple edge of C_Ω. In particular, the supporting lines intersect at one point and this point has to be a corner of Ω. □

4.2 Three topological constructions of X_Ω

In the case when Δ is a polygon, there are several known constructions giving toric spaces X_Δ (see [Bat12, KLMV13]). All constructions coincide for Delzant polygons and provide equivalences of corresponding categories. We suggest some other extensions, but have very little understanding what are relations between the constructions in general.

The main difference betwen our suggestions, and what we understand about other known models is that we have the usual algeberaic torus as an open dense subspace of our models and in the models presented in [Bat12, KLMV13] it can be already replaced with some of its non-commuatative quotients.

Consider compact convex Ω. In the first construction of the toric space X_Ω we use the approximation by polygons $\Omega_\epsilon \subset \Omega$, this approximation is used in sandpiles. The boundary of each polygon is the level set of F_Ω. For $\delta \geq \epsilon$ the projection of X_{Ω_ϵ} to X_{Ω_δ}. Define $X_\Omega = X_\Omega^{pro}$ as $lim_{\epsilon \to 0} X_{\Omega_\epsilon}$, i.e. the projective limit of all these toric spaces. Note that at least on the level of torus the symplectic form survives the limit.

This is seen more directly in the second construction (or rather a version of the first), but now we approximate from outside. This can be viewed as a symplectic version of the minimal model, totally conducted by the symplectic geometry. Note that if one polygon is a blow-up of another, then the second can have a longer history than the first. If we go back in time, we can go until some side collapses, and bowing up decreases self-intersection of adjacent sides. Therefore, if we go first almost to the maximal level, i.e. consider $\Omega_{\mathbf{m}_\Omega - \epsilon}$, for $\epsilon > 0$ small enough, then we can go back in time by $\mathbf{m}_\Omega - \epsilon$ without any branchings to the polygon $\hat{\Omega}$ circumscribing Ω. Consider Ω' to be the result of blowing-up all corners of $\hat{\Omega}$ which doesn't belong Ω, and Ω' circumscribes Ω. Applying cuts to new corners we get a sequence of $Q-$polygons $\hat{\Omega}, \Omega', \Omega'', \ldots$, the limit of the corresponding sequence of symplectic blowups is denoted by X_Ω.

It is interesting to apply the constructions to the case when Ω has no corners. The toric space X_Ω has no fixed points with respect to the action of the torus and, strikingly, it is the same space for all such Ω.

The third construction is different in a spirit, we use irrational foliations. The result $X_\Omega = X_\Omega^{nc}$ feats to the category of non-commutative spaces. Consider the double quotient

$$\mathbb{Z}^2 \backslash (\Omega \times \mathbb{R}^2)/\{(p, L)\},$$

where p runs through all boundary points of Ω and $L+p$ is a supporting line for Ω. If p is a corner of Ω then the fibre in X_Ω over p is a fixed point of the torus action. If Ω is a polygon, we have (possibly non-commutative) boundary divisor over every side. If there are no corners, we see again that the space X_Ω has no fixed points of the torus action. This suggests that X_Ω^{pro} and X_Ω^{nc} might be indeed two models of the same space. This resonates with Kapranov's idea of comparing non-commutative and usual (but big) spaces, see [Kap09].

Another clue is that from the topological perspective the quotients are all the same if Ω has a smooth boundary. It seems that, the space X_Ω can be constructed within the formalism of Shimura varieties. One more step in this direction can be the following formal similarity. The universal elliptic curve over $\overline{\mathbb{H}^2}$ (with non-commutative tori over the irrational points of the real line) is the same topologically as the universal toric surface of a smooth domain. Such non-commutative modular curves are considered in relation to the highly obscured theory of real multiplication and chain-fraction expansions (see [MM02, Man04]).

Chapter 5

Tropical Steiner problem and the limit of sandpiles

Tropical geometry is related to complex geometry in a similar way as particles are related to strings. Simplifying as usual, one can replace a particle with a loop. The world sheet of a string then is a holomorphic curve which retracts to a tropical curve. We can measure the size of a string as a symplectic area of the underlying topological surface. The principle features of the symplectic area are that it is invariant under deformations of the string and survives under tropicalization giving the *symplectic area* of a tropical curve. The idea of its minimisation comes from sandpiles, where this corresponds to the total lost mast of sand. But since the action can be produced (at the level of solitons) without any mass-loss, the problem of action-minimisation dominates the problem of mass-minimisation. This realised through the quasi-degree, measuring the number of emerging solitons (and thus the amount of sand lost) coming from each sink-wall of a polygon Δ, so the mass is the same in each deformation class. The origin for that is the deformation invariance of symplectic area of holomorphic curves.

It is very important to note that we are using the logarithmic picture now, so the whole Euclidean plane is in the image, corresponding to the whole algebraic torus. Elsewhere in the text we use its compactifications, and therefore compact moment polygons. In this section, the moment polygon is thought as a collection of Lagrangian tori in the preimage of its vertices under the logarithm. We need to think about tropical curves with ends on the vertices as open holomorphic curves with boundaries on this tori, the **degree** of the curve is encoded by the total homology classes of boundaries on each torus.

5.1 Symplectic area and complex curves

In this section we sketch how one can translate the story about minimal tropical curves to the context of classical holomorphic curves. Recall that an amoeba (as defined in [GKZ08]) of an algebraic curve S in the algebraic torus $(\mathbb{C}^*)^2$ is an image of S in $\mathbb{R}^2$ under the logarithm map Log given by

$$Log(z_1, z_2) = (\log|z_1|, \log|z_2|).$$

Consider a family of algebraic curves S_t in $(\mathbb{C}^*)^2$ for $t > 0$. We say that the family S_t tropicalizes to the tropical curve C if the family of their rescaled amoebas $Log_t S_t \subset \mathbb{R}^2$ converges to C when t tends to ∞. Here Log_t simply denotes the map $(\log t)^{-1} Log$. It could seem that the tropicalization C is defined only as a set. In fact, the multiplicities for the edges of C can be also canonically restored from the family S_t.

First of all we are going to justify the name "tropical symplectic are" that we extensively used. Suppose a family S_t tropicalizes to a tropical curve C. Let

$$\omega = -id\log(z_1) \wedge d\log(\bar{z}_1) - id\log(z_2) \wedge d\log(\bar{z}_2)$$

be the symplectic form on $(\mathbb{C}^*)^2$.

Proposition 5.1. Let C be a tropicalization for S_t and B be a convex bounded open subset of $\mathbb{R}^2$. Then

$$\int_{S_t \cap B_t} \omega \underset{t \to \infty}{\sim} 4\pi^2 Area(C \cap B) \log t,$$

where $B_t = Log_t^{-1}(B)$.

Thus the symplectic area for a tropical curve indeed can be interpreted as a main part in the asymptotic for symplectic areas of a family of holomorphic curves.

Proof. The outline is the following. For a large t the rescaled amoeba $Log_t(S_t)$ is in a small neighbourhood of the tropical curve C. Moreover S_t itself will be close to a certain lift of C to the torus $(\mathbb{C}^*)^2$. It is performed by lifting each edge with a slope (p, q) to a piece of holomorphic cylinder $\{(z^p, z^q) | z \in \mathbb{C}\}$ translated by the action of the torus.

Therefore, we can compute the area of S_t near the limit by looking at the areas of the cylinders. There also can be minor corrections coming from the vertices of C but the corrections are small with respect to $log(t)$ and so do not appear in the final statement.

To complete the proof we need to compute the contribution from each edge in $C \cap B$. It is clear that for each segment in $C \cap B$ the area of its lift is proportional to the length of the segment. So if we show that the area of the lift for the interval going from the origin to the integer vector (p, q) is equal to $4\pi^2(p^2 + q^2) \log t$ then we will be done. This computation is given in the following lemma. □

Lemma 5.1. Let $v = (p, q)$ be a primitive integer vector. Let C_t^{pq} be a lift of an interval $[0, v]$ to the torus $(\mathbb{C}^*)^2$ under Log_t, i.e. $C_t^{pq} = \{(z^p, z^q) | 1 \leq |z| \leq t\}$. Then

$$\int_{C_t^{pq}} d\log(z_1) \wedge d\log(\bar{z}_1) = -4i\pi^2 p^2 \log t.$$

Proof. Let z_1 be $r\exp(i\phi)$, where $r > 0$ and $\phi \in [0, 2\pi]$. Then

$$d\log z_1 = d\log\, r + i\phi d\phi \text{ and}$$

$$d\log z_1 \wedge d\log \bar{z_1} = -id\log r \wedge d\phi^2$$

Then the left hand side of the equality we are proving is equal to

$$-i\int_1^t \int_0^{2\pi} d\log r \wedge d\phi^2 = -4i\pi^2 p^2 \log t.$$

□

Remark 5.1. The specific choice for ω is not crucial while it is invariant under the action of $(C^*)^2$. Indeed, if ω' is an arbitrary 2-form then its restriction to any holomorphic curve will not have contributions from pure holomorphic and anti-holomorphic parts of ω'. So we can think that ω' is a $(1,1)$-form. There is a two dimensional family of torus-invariant $(1,1)$-forms. Different choices for ω from this family correspond to coordinate dilatations on the level of tropical curves.

Proposition 5.1 suggests us that symplectic area for tropical curves should be deformation invariant. Indeed, this should follow from the fact that the 2-form ω is closed. And actually, we can prove the deformation invariance directly.

Lemma 5.2. Consider a continuous family C_s of bounded parts of tropical curves with common fixed endpoints. Then $Area(C_s)$ is constant.

Proof. Any deformation C_s locally can be decomposed into the elementary ones. An elementary deformation is a process of moving and shortening two edges while growing the one in the opposite direction (see Figure 5.1).

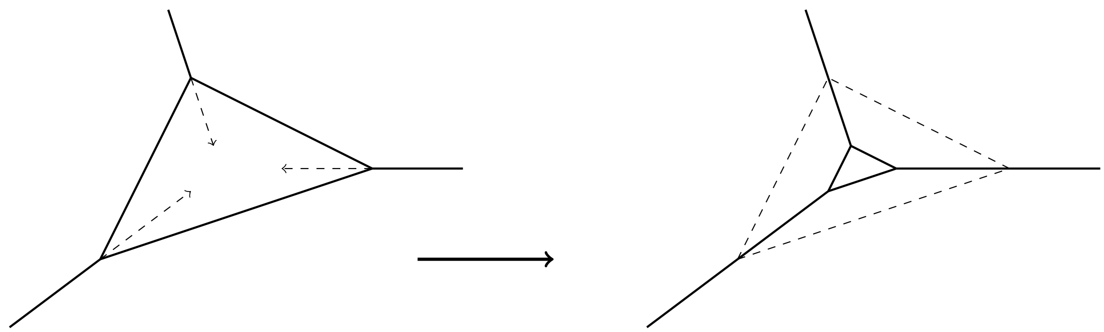

Figure 5.1: In the picture we shrink a triangular cycle. Any deformation of a tropical curve can be decomposed into such operations or their inversions. This can be seen as an application of G_p operator from the next section.

Globally this corresponds to enlarging a coefficient for a tropical polynomial. For example on Figure 3.8 we change the coefficient for the central region.

Up to a scaling an elementary deformation simply replaces the union of segments $[0, v_1]$ an $[0, v_2]$ by a single segment $[0, v_1+v_2]$. Here v_1 and v_2 are the primitive (or appropriate multiples of primitive) vectors for the edges we are moving. Denote by w_i the projection of v_1+v_2 on the line spanned by v_i (see 5.2) Then after the deformation the two edges together loose

$$|v_1||w_1| + |v_2||w_2| = |v_1|(v_1+v_2) \cdot \frac{v_1}{|v_1|} + |v_2|(v_1+v_2) \cdot \frac{v_2}{|v_2|} = |v_1+v_2|^2$$

of their symplectic area. On the other hand, the growing edge contributes exactly $|v_1+v_2|^2$ to the symplectic area of the deformed curve. □

Lets get back to our specific case. Let the ends for a part of a tropical curve C be the vertices of a lattice polygon Δ. Suppose that there exist a tropical polynomial F defining C and vanishing at the

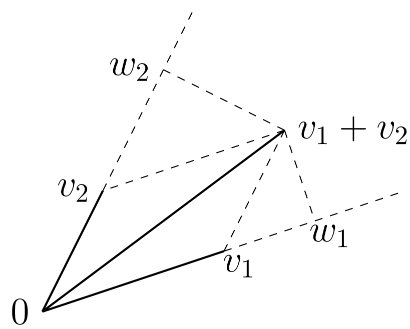

Figure 5.2: Computing contributions for symplectic area.

boundary of Δ. Denote by $d_C(s)$ the **quasi-degree** for this curve along the side s, i.e. the lattice length of the gradient of F near s. Then we can deform C to the union of all edges e of the polygon taken with the multiplicities $\mathrm{d}_C(s)$, by means of decreasing the coefficients of F to zero. This observation together with the deformation-invariance proves the following lemma.

Lemma 5.3. $Area(C) = \sum_{s \in \mathrm{Sides}(\Delta)} d_C(s) Area(s)$.

5.2 Steiner problem

Consider a $\mathbb{Q}$-polygon Δ, that is a convex polygon whose sides have rational slopes. A **Δ-tropical curve** C is an intersection of Δ with the corner locus of some tropical polynomial $F(x, y) = min(ix + jy + a_{ij})$ vanishing at the boundary of Δ. Note that C intersects $\partial\Delta$ only at the vertices. The restriction of F to Δ vanishing on the is called Δ-tropical polynomial. The *action* of C is $\int_\Delta F$. Suppose that we are allowed only to increase F and we want to find such Δ-tropical polynomial $H \geq F$ such that it is not smooth in a given collection of points $p_1, \ldots, p_n \in \Delta^\circ$, i.e. the curve defined by H passes through this points. Such H certainly exist, so we take their point-wise minimum and denote it by $G_{p_1,\ldots,p_n} F$, it defines a Δ-tropical curve $G_{p_1,\ldots,p_n} C$

Proposition 5.2. The curve $G_{p_1,\dots,p_n}C$ minimises the action and the symplectic area in the class of all Δ-tropical curves passing through $p_1, \dots, p_n$ with defining polynomial greater than F.

$G_{p_1,\dots,p_n}C$ is determined by minimisation of action, but there can be many solutions with greater actions but same symplectic area in special cases if the curve has relative deformations.

Proof. By its definition, $G_{p_1,\dots,p_n}F$ is the point-wise minimum – so it minimises the integral, i.e. the action of $G_{p_1,\dots,p_n}C$. It also minimises the quasi-degree and, by Lemma 5.3, it minimises the symplectic area. □

Corollary 5.1. $G_{p_1,\dots,p_n}\varnothing$ is the solution to the Steiner problem, i.e. minimises symplectic action (and therefore the symplectic area) in the class of all Δ-tropical curves passing through $p_1, \dots, p_n$.

Until now, we said nothing about how this solution can be found for the case of more than one point. The idea suggested by sandpiles is to decompose $G_{p_1,\dots,p_n}$ into an a sequence of G_{p_j} applied in arbitrary order and as much as needed (see Figure 3.8). In the case of two points we would simply iterate $G_{p_2}G_{p_1}$. The key statement (crucial for the scaling theorems in sandpiles) is the following.

Proposition 5.3. Let F be a Δ-tropical polynomial and

$$p_1, \dots, p_n \in \Delta^\circ.$$

Consider a sequence of numbers $k_1, k_2 \dots \in \{1, \dots, n\}$ such that every of n possible values is taken infinitely many times. Then the sequence of functions $F_m = G_{p_{k_m}} \dots G_{p_{k_1}}F$ converges to $G_{p_1,\dots,p_n}F$ as $m \to \infty$.

Proof. Note that $C \to G_pC$ is a projector on the space of curves passing through p. It is counties on p just because it is given by an appropriate shrinking of face (see Figure 3.8).

On the other hand, the sequence of Δ-tropical polynomials F_m converges because the sequence of its coefficients for every monomial is non-decreasing, every G_p increases exactly one coefficient, and bounded, because

$$G_{p_1,\dots p_n}F_m = G_{p_1,\dots p_n}F_{m-1} = \dots = G_{p_1,\dots p_n}F$$

by Lemma 5.4 and so $G_{p_1,\dots p_n}F \geq F_m$.

Combining everything, for each j the exist infinitely many $F_m = G_{p_j}F_{m_1}$, and the curve defined by F_m passes through p_j. In particular, the polynomials $G_{p_j}\lim F_m$ and $\lim F_m$ are the same and define the curve passing through all the points $p_1, \dots, p_n$. Therefore, $G_{p_1,\dots p_n}F = F_m$ since $G_{p_1,\dots p_n}F \geq F_m$ and $G_{p_1,\dots p_n}F$ is subject to the minimality. $\square$

Lemma 5.4. For any Δ-polynomial F and $p_j \in \Delta^\circ$

$$G_{p_1,\dots,p_n}G_{p_1}F = G_{p_1,\dots,p_n}F$$

Proof. It is clear that $G_{p_1,\dots,p_n}G_{p_1}F \geq G_{p_1,\dots,p_n}F$ because $G_{p_1,\dots,p_n}G_{p_1}F \geq G_{p_1}F \geq F$ and $G_{p_1,\dots,p_n}G_{p_1}F$ is not smooth at the points $p_1, \dots, p_n$.

On the other hand, $G_{p_1,\dots,p_n}F \geq F$ is not smooth at p_1 and therefore $G_{p_1,\dots,p_n}F \geq G_{p_1}F$. Again by minimality, $G_{p_1,\dots,p_n}F \geq G_{p_1,\dots,p_n}G_{p_1}F$. $\square$

In fact, we know that if the points p_j and the vertices of the polygon Δ belong to the lattice and F has integral coefficients, the sequence F_n from Proposition 5.3 will stabilise (just because we increment coefficients by integers at every step). Even in this case the number of steps required for the stabilisation is very much unpredictable.

Conjecture 5.1. A sequence F_n stabilises for all F, Δ and $\{p_j\}$.

We believe that this conjecture can be proven via the lift of G_p to the algebraic curves. We managed to do such lift only in characteristic two. Let k be a field, $char(k) = 2$, on which Frobenius (taking the square) acts as isomorphism. Let

$$S_pf(z) = f(p)f(z) + f^2(\sqrt{zp})$$

be a $\mathbb{Z}_2$-linear function sending all polynomials over this filed with a given Newton polygon to the polynomials with the same Newton polygon and passing through p. We claim that if k is a non-archimedean field then S_p is a lift of $G_{val(p)}$, where $val\colon k^2 \to \mathbb{R}$ is the coordinate-wise valuation. Therefore, if a lift of the sequence F_n converges in non-archimedean topology, then it would imply the stabilisation at the tropical (only higher asymptotes) level. We mention another relevant operator which is idempotent $f(z) \mapsto f(z) + \sqrt{f(z^2p^{-1})f(p)}$, where p belongs to the two-dimensional algebraic torus over the field k.

5.3 Scaling limit theorem

Consider a big lattice polygon Ω. A sandpile ϕ on Ω is an integral non-negative function on its lattice points. If $\phi(p) > 3$ at some point $p \in \Omega$ then we say that ϕ is unstable and perform a toppling T_p resulting with the state $T_p\phi$ equal to $\phi(p) - 4$ at p, $\phi(p') + 1$ for a neighbour $p' \in \Omega \cap \mathbb{Z}^2$ of p and the same as ϕ elsewhere. We perform such topplings until the we get a stable state ϕ°. This state doesn't depend on the order of topplings and called the relaxation of ϕ. In fact, we can define a function F computing the number of toppling at every point, then

$$\phi^\circ = \phi + \Delta F,$$

where $\Delta F(p) = -4F(p) + \sum_{|p'-p|=1} F(p')$ is the discrete Laplace operator. For a more elaborate exposition consult with [LP10] or our joint work [KS15b] with Nikita Kalinin attached to this thesis (see Appendix B therein).

The sandpiles satisfy the least action principle (see [FLP10]) , i.e. if F is the toppling function of the relaxation for ϕ then it is the point-wise minimum of all functions H such that $\phi + \Delta H < 3$.

If $\phi \equiv 3$ is the maximal stable sate on Δ and we perturb by adding one grain at several points, we get a state which is maximal everywhere

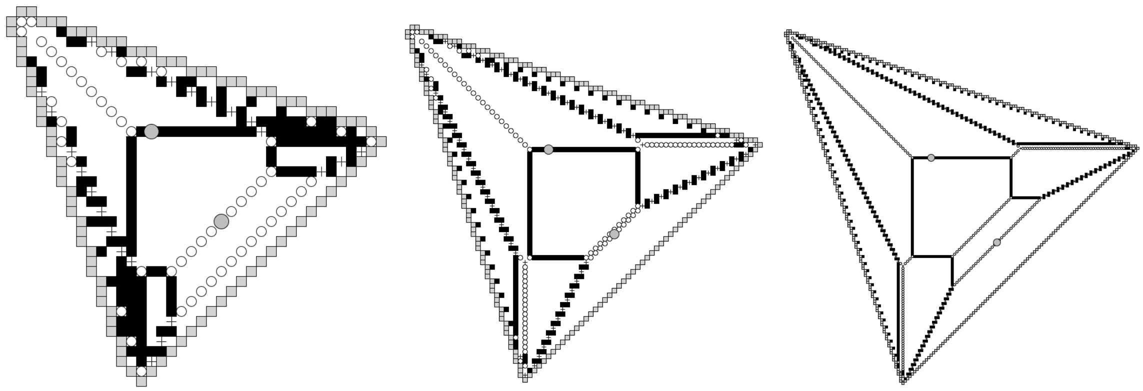

Figure 5.3: The evidence for a thin balanced graph as a deviation set of a sandpile. White corresponds to three grains, black to one, circles for two, crosses to zero, skew lines are the boundary vertices. Gray rounds represent the positions of added grains. The procedure gives a solution of the tropical Steiner problem in the limit. The edges are made of string-solitons.

deviating from it along the thin graph passing through the perturbation points. In [KS15b] we show that as the grid gets finer, the deviation converges to the tropical curve given by $G_P 0$ and solving the Steiner problem for the perturbation points, the limiting version of the least action principle.

In physics, the sandpile model appeared in [BTW87], for the purpose of modelling the self-organised criticality. In the original paper, the sandpile model was a stochastic process of perturbing the sandpile by adding more and more sand at random points. The main observations are the self-organised criticality, the ability to form attractors without tuning parameters, and power-laws for avalanches. A particular power law is that the size of the avalanche satisfies the exponential distribution:

$$P = O(\exp(-cS)), c > 0$$

where P is the probability to have an avalanche of size S. We show empirically that this is also true after the tropical limit, at the level of Δ-curves and operators G_p.

The idea of the operator G_p was certainly known to the experts in sandpiles. We show that this is the limit of the idempotent perturbation at p

$$\phi \mapsto (\phi + \delta_p)^\circ - \delta_p$$

considered in [CPS12]. A comprehensive physical exposition is done in [Pao12]. Clearly, this perturbation does nothing if $\phi(p) < 3$. They considered the states (or interfaces) which are charged enough to see an interesting dynamics. Our research in this direction started with contemplating the idempotent perturbations of the maximal stable state bounded by some region, as we've seen it in [CPS10, CPS12]. These states were maximal almost everywhere except for a locus along a thin graph. Those graphs resembled tropical curves. We shown that this is indeed the case near the thermodynamical limit. The key ingredient is the following.

Theorem 5.1. Let C be a tropical curve defined by a tropical polynomial F with Newton polygon Δ. Consider $F_0\colon \mathbb{Z}^2 \to \mathbb{Z}$ the restriction of F. Consider a sequence of integral super harmonic functions $F_0, F_1, \dots$ on the lattice,$\Delta F_j \leq 0$, such that F_{k+1} is the minimal function with $F_k \geq F_{k+1} \geq F_k + 1$ and F_k coincides with F_{k+1} outside of some finite radius neighbourhood of C. The sequence stabilises if and only if Δ has no lattice points in the interior.

The proof (where we call this process "smoothing") mainly belongs to Nikita Kalinin and is given in [KS15b], Chapter 9. This theorem relies on old estimates from [Duf53]. Denote by F^h the function (on the lattice $h\mathbb{Z}^2$ with the mesh h) on which the sequence has stabilised. There are three types of polygons with no lattice points strictly inside. This can be a segment and so C is a straight line. Denote by C^h the state $3+\Delta_h F$. Note that such a state behaves like a soliton with respect to the wave operator (the front of the avalanche, see [KS15b]) with a source at infinity. Neglecting the width of C^h we could write

$$(C^h + \delta_p)^\circ(z) = C^h(z - p) + \delta_p(z),$$

so the soliton moves towards the perturbation point because the point emits waves that act on the soliton by a translation. This is realised just by changing the coefficients in the tropical polynomial F.

We give an idea of pattern formation in case of a single infinite wall going in the primitive direction. We assume that we send waves from infinity. When this walls crush agains the wall some sand goes out of the system and we see some deviation from the maximum along that wall. More waves we send, more sand we loose. At some point a strip of deviation separates into two. One stays near the wall, the waves doesn't rich this region anymore, and other strip starts moving without leaving any trace. Theorem 5.1 implies that there is a unique strip with such properties for any direction.

The second type of lattice polygons without points is formed by triangle with area 1/2. In this case C^h gives a discrete model of a smooth tropical vertex. This is also a soliton. The third type consists of parallelograms of area 1. The curve C^h gives a model for a simple nodal point. Under the wave action, the four-valent cross of C^h gets decomposed to a pair of three-valent vertices connected by a finite segment. In the dual language this corresponds to the triangulation of the parallelogram, this can be done in to ways corresponding to two combinatorial types of the tropical curve.

This observations enables to construct a discrete model C^h for any nodal tropical curve C. The main feature allowing to use tropical geometry is the following approximate expression (the size of error is o(h) due to the rounding)

$$(C^h + \delta_p)^\circ = (G_p C)^h + \delta_p.$$

Therefore, the addition of a point shrinks the face of C in such a way that the resulting curve, i.e. G_pC, passes through p. Therefore, we can mimic the sandpile relaxation at the tropical level, decomposing it into the sequence of G_p that converges (by Proposition 5.3) to the expected solution of the tropical Steiner problem. Combining this with the idea of canonical approximation by polygons Ω_ϵ we were able to prove the following (see [KS15b]).

Theorem 5.2. Let Ω be compact convex set, $p_1, \dots, p_n \in \Omega^\circ$ be a collection of points. Consider the perturbation of the maximal stable state on $\Omega \cap h\mathbb{Z}^2$ by adding a grain at the roundings $p_1^h, \dots p_n^h \in h\mathbb{Z}^2$. The deviation locus (from 3) of its relaxation converges to $G_{p_1,\dots,p_n}\varnothing_\Omega$.

It is a miracle, that the corresponding tropical model restores the full $SL_2\mathbb{Z}$ symmetry, in sandpiles a choice of the basis in the lattice is essential for the toppling rule. The subtle problem here is the choice of roundings $p_j^h \in h\mathbb{Z}^2$ near $p_j \in \Omega^\circ$. One way to resolve the problem is to consider only lattice points $p_1, \dots, p_n$ and polygon Δ and take the mesh $N = h^{-1}0$ be an integer. This choice is made in [KS16] for simplicity of the first exposition, i.e. the roundings can be taken simply as $p_j^h = p_j \in N^{-1}\mathbb{Z}^2$. In general, this has to be considered as an extra structure. For example, in the cone of all convex domains on $\mathbb{R}^2$ we can define a rounding structure given by

$$\Omega^h = \text{ConvexHull}(\Omega \cap h\mathbb{Z}^2).$$

Similarly, we define a rounding structure on points. The subtlety comes here, since the theorem would work for arbitrary roundings. The stupid example is the following. If we add just one point, the roundings are negligible, but the operator $G_{p_1,\dots,t_n}\varnothing$ is not continuous with respect to the variation of several points. For example, if we take $G_{p_1,p_2}\varnothing$ (see Figure 5.4) on a rectangle with the points in the special position, the curve is unstable with respect to the perturbations of p_j. In particular, small changes in rounding may result in a tremendous change of the relaxation process, just because some point fell off a soliton which is anchored to another point, so it produces lots of waves until some soliton from far away doesn't reach it. So the roundings, in fact, have to be chosen in a consistent way. But this is essential only in the case when the curve is somehow superabundant, i.e. small inner fluctuations produce big jumps of energy, geometrically this means that the configuration is special and the symplectic volume of this configuration is isolated in terms of the symplectic volumes (or even better — actions) of generic nearby configurations.

The scaling limits has appeared before in the context of sandpiles in the case when we are relaxing a big sandpile supported at the origin [PS13] and studied in [LPS16]. Note that tropical curves also appear in their pictures.

There is another relation of sandpiles and tropical curves via the identification of graph Jacobians and sandpile groups, see [BN07]. In this theory the graphs are normally thought to be small and represent intersections of irreducible components of a degenerate curve. In our approach the graphs are arbitrary big, in fact it approximates a surface. We show that the states close to the maximal stable state are governed be tropical structures. This states are recursive (see [Dha99, BTW87]) and belong to the sandpile group, giving the tropical structure to the infinitesimal neighbourhood of the maximal stable state.

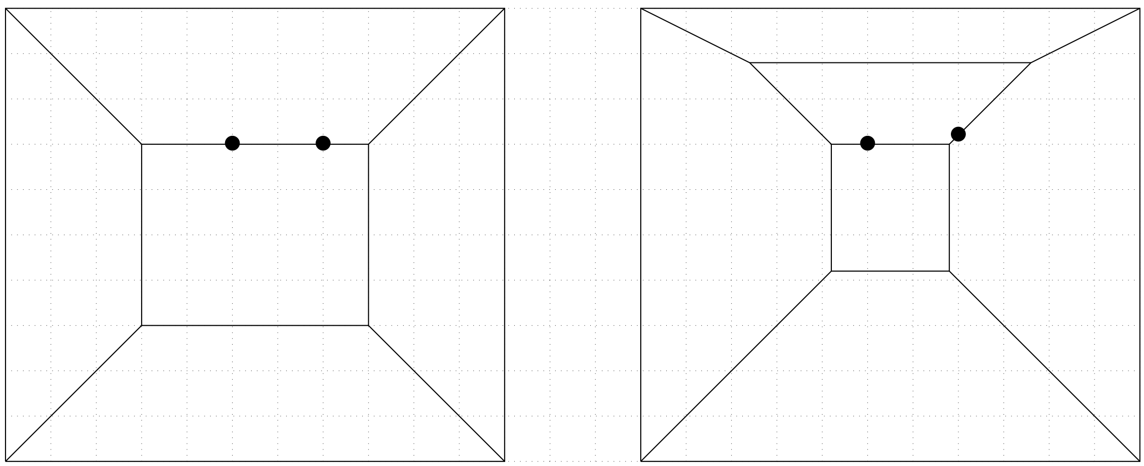

Figure 5.4: An unstable configuration and its perturbation (by a light shift of the right point) together with action minimising curves passing through them (solutions of Steiner problem). A wrong rounding at the sandpile approximation of the problem in similar cases produces globally observable, surviving in the thermodynamical limit, jumps in mass and action comparing to the expected tropical solution of the Steiner problem. We can modify the whole setup, making it sort of more realistic, using a concept of stable tropical intersection taking as the solution the limit of local perturbations, dissolving unstable energy in terms of sandpiles. Also the number of iterations of $G_{p_1}G_{p_2}$ needed to arrive to the picture on the right actually gets tiger when the perturbation is small.

Chapter 6

Integral affine invariants of convex domains

For a compact convex domain $\Omega \subset \mathbb{R}^2$ define a function $F_\Omega \colon \Omega \to \mathbb{R}$

$$F_\Omega(z) = \inf_{v \in \mathbb{Z}^2} (a_v + v \cdot z), \tag{6.1}$$

where a number α_v for a vector $v \in \mathbb{Z}^2 \backslash 0$ is given by

$$a_v = -\min_{z \in \Omega} z \cdot v. \tag{6.2}$$

We call F_Ω the tropical series of Ω. It is discussed a lot in the the joint work with Nikita Kalinin Chapter 9 (see Appendix C where F_Ω is denoted by l_Ω). The terminology comes from the fact that if in (6.1) we replace inf with $\sum$, + with multiplication and the scalar product with an exponentiation we would get something of the shape $\sum a_v z^v$ which is a Laurent power series in two variables. Tropical series and analytic curves have already appeared in several places, for example in [MZ08] in a form of θ-divisor. Following the analogy with analysis, we can say that Ω is the domain of convergence for the series representing the function F_Ω. We describe certain numerical characteristics of Ω in terms of this function. The easiest of them is $\mathbf{m}_\Omega$, the maximal value of F_Ω. Clearly it is homogenous with respect to rescaling and has degree one. It measures the future with respect to the canonical evolution, i.e. $\mathbf{m}_\Omega$ is the time until degeneration to a maximal segment. Its length λ_Ω, the elliptic parameter of Ω, corresponds to the lattice length of the cohomology class of the symplectic form at the terminal point of the canonical evolution starting at ω_Ω, the symplectic form of X_Ω.

Recall that a vector in $\mathbb{Z}^n \otimes \mathbb{R}$ is called *primitive* if it is integral and not proportional to any other integral vector. For a given segment s having a rational slope there exist a primitive vector v in its direction and the length of the segment is proportional to the length of the vector with the coefficient $\operatorname{Length}_{\mathbb{Z}} s$, the integral norm. We define the invariant λ_Ω as $\operatorname{Length}_{\mathbb{Z}} F_\Omega^{-1}(\mathbf{m}_\Omega)$, clearly it has degree one with respect to rescaling.

Remark 6.1. If Ω is a polygon with rational slopes, then only a finite number of monomials contribute to F_Ω, i.e. it can be represented by a tropical polynomial $\min_{v\in A}(a_v + p\cdot v)$ for a finite subset A in $\mathbb{Z}^2\backslash 0$.

In fact, one could give an affine invariant (independent of metric) definition of F_Ω using the integral normalised length. First of all, we define an integral-normalised distance from a point p_0 on the plane to a line l with a rational slope. Taking a projection of the plane along l we have the line projected to a point q and p_0 is projected to p in the quotient $\mathbb{R}^2/l$. This $\mathbb{R} = \mathbb{R}^2/l$ is naturally endowed with an integral affine structure and we define the integral normalised distance from p_0 to l simply as $\operatorname{Length}_{\mathbb{Z}}[p, q]$. For any point $p \in \Omega$ we define $F_\Omega(p)$ as an infimum of distances from p to all lines with rational slopes supporting Ω. Explicitly, the invariance and homogeneity of F_Ω can be written in the following way.

Proposition 6.1. Let Ω_0 be a convex domain. For $A \in SL_2\mathbb{Z}$, $v \in \mathbb{R}^2$

and $r > 0$ consider $\Omega_1 = \{rAp + v | p \in \Omega\}$. Then

$$rF_{\Omega_0}(p) = F_{\Omega_1}(rAp + v).$$

To visualise F_Ω we can look on **the caustic curve** $C_\Omega \subset \Omega$, the locus where F_Ω is not locally smooth. Note that F_Ω is linear on every face, a connected component of $\Omega \backslash C_\Omega$, and every edge of C_Ω has a prescribed multiplicity identified with the integral normalised length of the vector connecting v_1 and v_2, where $v_j \cdot p + a_j$ represent the restrictions of $F_\Omega(p)$ to the faces adjacent to the edge. With this multiplicities we have the balancing condition on slopes satisfied at every vertex (see Figure 3.2). To every vertex p we associate a lattice polygon spanned by the monomials contributing at the vertex and $\mu(p) \in \mathbb{Z}$, *the multiplicity of the vertex* p is defined to be twice the area of this dual polygon. See Figure 3.5 for a more detailed explanation. By definition, a vertex (or an edge) is called *smooth* if it has multiplicity one. Theorem 4.3 asserts that there are no multiple vertices and edges in C_Ω except for the maximal ones if Ω has no corners. In this case C_Ω is an infinite tree (we think of it as of singular infinetly punctured ellipric curve).

Remark 6.2. Ω is $\mathbb{Q}$-polygon if and only if C_Ω has a finite number of segments. Two vertices (or one vertex if $\lambda_\Omega = 0$) where F_Ω attains its maximum are never smooth. The conceptual reason is that C_Ω can be viewed as an elliptic curve and its genus is contracted to the maximal segment (or vertex). In particular, $C_\Omega \cap \Omega^\circ$ has no loops except for this hidden one, and thus is a tree.

6.1 Computations for the disk

This section is based on our joint paper with Nikita Kalinin [KS17]. Some aspects are reviewed better there. Here, the purpose of giving an example, is to show that the universal lattice invariants are fairly easy to compute.

Let Ω be the unit disk $\{x^2+y^2 \le 1\}$. Applying equations (6.2) and (6.1), F is given by

$$F(z) = \min_{v \in \mathbb{Z}^2}(v \cdot z + |v|).$$

The graph of F and its corner locus C are shown on the Figure 6.1. Clearly, that only four *tropical monomials* contribute to F at the origin. These are $1+x, 1-x, 1+y, 1-y$ and the maximum of F is 1. The monomials give an X part of C near the origin where the maximum is taken.

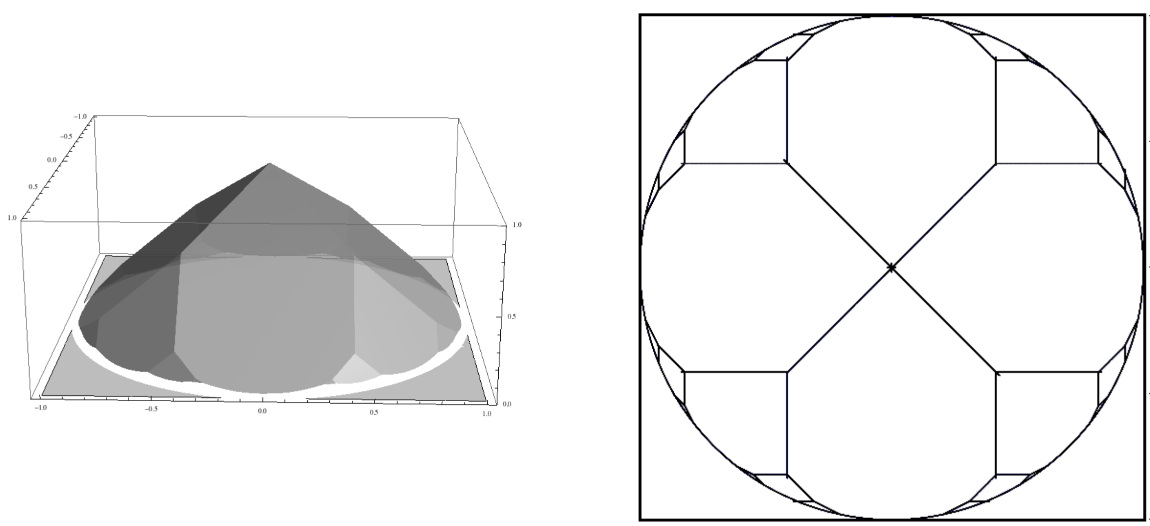

Figure 6.1: A plot of the series F_Ω and the caustic C_Ω for Ω a disk.

Clearly, C is a tree and, apart from the origin, all the vertices of C are trivalent and smooth by Theorem 4.3 and Remark 3.3. Starting from the origin, the tree grows in four directions and branches infinitely many times when gets close to the boundary. A branching gives a vertex and the third monomial adjacent to the new vertex is chosen in a

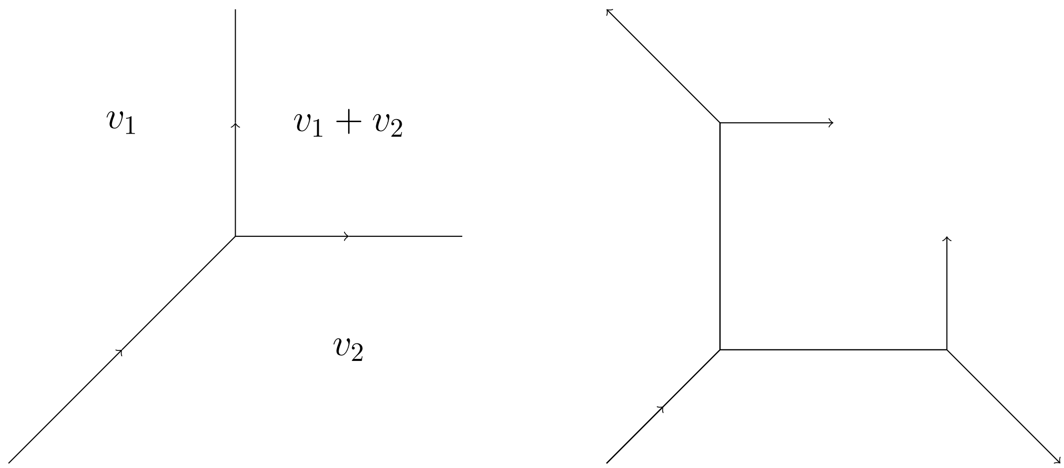

Figure 6.2: On the left: branching pattern for the tropical curve at the vertex $z(v_1, v_2)$, the arrow denotes the direction towards the boundary. The vectors $v_1, v_2, v_1 + v_2$ are the values of the gradient of F in the complement to C, its edges are orthogonal to $v_1 - v_2, v_1$ and v_2. On the right: a schematic picture for two more branchings of C, the vertices are $z(v_1, v_2)$, $z(v_1, v_1 + v_2)$ and $z(v_1 + v_2, v_2)$. The parallel segments of the drawing represent parallel edges of C. Compare this with Figure 6.1.

systematic manner shown in the Figure 6.2. Each such vertex $z(v_1, v_2)$ is adjacent to the three regions in the complement to C and on these regions F is represented by the monomials $|v_1| + z \cdot v_1$, ($|v_2| + z \cdot v_2$ and $|v_1 + v_2| + z \cdot (v_1 + v_2)$, where v_1 and v_2 are vectors from the same quadrant in $\mathbb{R}^2$ forming the basis of $\mathbb{Z}^2$. The value of F at $z(v_1, v_2)$ is given by

$$F(z(v_1, v_2)) = |v_1| + |v_2| - |v_1 + v_2| \tag{6.3}$$

which is exactly the error to the triangular inequality for the primitive triangle formed by vectors $v_1, v_2, v_1 + v_2$. The only multiple vertex of C_Ω is the origin, where the maximum $\mathbf{m} = 1$ is attained and the multiplicity $\mu(0)$ is equal to 4 since δ_Ω is a union of 4 basic triangles with area 1/2.

The symplectic minimal model $\hat{\Omega}$ for the unit disk Ω is given by the square of diameter 2. Recall how it is constructed, we go first to the polygon $\Omega\mathbf{m}-\epsilon$ at the sub-maximal level. After we go back in time by $\mathbf{m}-\epsilon$ without any branchings, extending the open edges. In the case of the disk it just means that we extend the X at the origin until its ends form a square circumscribing the disk. Each branching of the tropical curve C corresponds to the blow-up of the next level and the size of this blow-up is exactly the value of F at the branching (see Figure 3.4). Therefore, the values of F at every vertex of C hold the whole information about Ω if $\hat{\Omega}$ is already fixed.

We define the sequence of polygons starting with $\hat{\Omega}$, then blowing up all its corners in such a way that every corner cut is done by a line tangent to a disk. The area of these polygons converges to the area of disks. The size of every such cut coincides with a value of F at a non-maximal vertex of C, given by (6.3). Therefore, using the fact that we cut only Delzant triangles we get the formulas from [KS17]. The first term correspond to degeneration of lattice perimeter, the last term corresponds to the maximal action of one point in sandpiles, the cubic invariant of Ω, and is computed in terms of cutting tetrahedra from the graph of $F_{\hat{\Omega}}$.

Proposition 6.2.

$$\sum_{(v_1,v_2)\in SL_2^+\mathbb{Z}} (|v_0|+|v_1|-|v_0+v_1|) = 2$$

$$\sum_{(v_1,v_2)\in SL_2^+\mathbb{Z}} (|v_0|+|v_1|-|v_0+v_1|)^2 = 2-\pi/2$$

$$\sum_{(v_1,v_2)\in SL_2^+\mathbb{Z}} (|v_0|+|v_1|-|v_0+v_1|)^3 = 2-\frac{3}{2}\int_\Omega F_\Omega$$

The matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

generate a free monoid inside $SL_2\mathbb{Z}$. This monoid $SL_2^+\mathbb{Z}$ consists of all matrices with determinant one and non-negative integral entries. A matrix

$$M = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

can be viewed as a planar triangle with vertices $v_1 = (x_1, x_2)$, $v_2 = (x_1, x_4)$ and $v_1 + v_2$. We compute the error in the triangular inequality as

$$f(M) = |v_1| + |v_2| - |v_1 + v_2|.$$

For what values of s the series

$$\sum_{M \in Sl_2^+\mathbb{Z}} f^s(M) \tag{6.4}$$

converges? We know that it converges at least for $s \geq 1$. Can it possibly be improved?

We could use the idea originating in the study of Dirichlet series, where we estimate a series of $\frac{1}{n^s}$ over $\mathbb{Z}_{\geq 1}$ by a corresponding integral over $\mathbb{R}_{\geq 1}$. Therefore, we introduce $Sl_2^+\mathbb{R}$, the space of all matrices with non-negative entries and unit determinant. We would like to prove that the convergence of the series (6.4) follows from the convergence of the integral

$$\int_{Sl_2^+\mathbb{R}} f^s(M)\sigma(M),$$

where σ is an invariant measure on $SL_2\mathbb{R}$. It is enough to show that there exist a collection of disjoint domains

$$\{B_M \subset SL_2^+\mathbb{R}\}_{M \in Sl_2^+\mathbb{Z}}$$

of the same (with respect to σ) non-zero volume such that the restriction of f to B_M attains its minimum at M.

6.2 The invariant stratification

Of course, the numbers a_v defined by (6.2) determine Ω. Every invariant of Ω is contained in them. On the other hand, there are non-trivial relations among a_v. We show a particular way of resolving those relations. Every Ω has its minimal model $\hat{\Omega}$, the polygon formed by the supporting lines of Ω whose linear functions contribute to the maximum of F_Ω (in the case of disk $\hat{\Omega}$ is a square). The minimal model has two continuous degree one parameters — the maximum $\mathbf{m}$ and the elliptic parameter λ, their values are the same as for Ω. After that we record the history of symplectic blow-ups $\Omega \to \hat{\Omega}$. The blow-ups correspond to branchings of C_Ω. The size of a blow-up corresponding to a non-maximal vertex z of C_Ω is equal to $F_\Omega(z)$. We can encode this numbers either more geometrically or more algebraically.

In the first case, we observe that if z_1 and z_2 are adjacent vertices of C_Ω then

$$\text{Length}_{\mathbb{Z}}[z_1, z_2] = |F_\Omega(z_1) - F_\Omega(z_2)|,$$

where $[z_1, z_2]$ denotes the segment of C_Ω connecting the vertices. Therefore, we can take the moduli of C_Ω to be the coordinates on the space of convex domains. This already includes the elliptic parameter, the maximum and all the sizes of blow-ups. We can argue, weather C_Ω contains the information about the germ of Ω_ϵ for $\epsilon > 0$. The information about the germ up to rescaling is captured by the dual subdivision for C_Ω in the neighbourhood of the maximum, see Figure 6.6. If there is no maximal segment, then the subdivision is just a convex lattice polygon with exactly one lattice point inside, there is a finite number of their classes modulo the change of basis. Therefore, the abstract curve C_Ω (without a particular embedding to $\mathbb{R}^2$ but non-normalized) or its moduli together with the type of singularity determines Ω up to translations and $Sl_2\mathbb{Z}$. In particular, we've just proved Theorem 1.5.

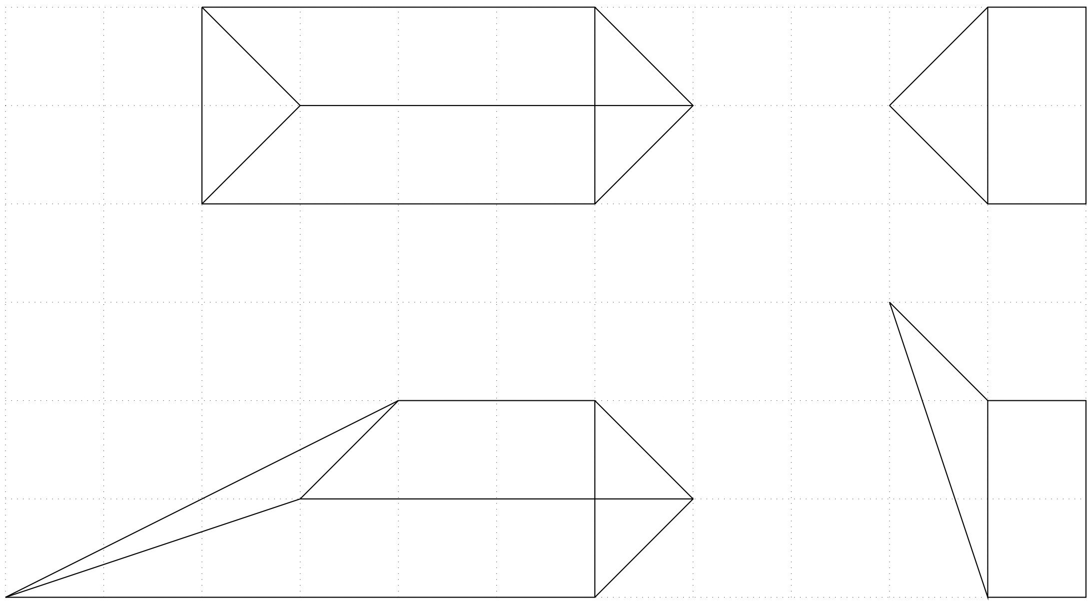

Figure 6.3: The caustic curves whose singularity types are double mutants of each other. A mutation is performed by crossing a codimension one wall while moving from one stable stratum to another. The unions of dual faces to the end-points of the maximal segments are on the right. Note that a polygon Ω with an unbranched curve C_Ω has an infinite past iff $\lambda_\Omega = 0$ or the union of two adjacent faces dual to the ends of the segment is convex. Note that in the non-convex case λ_Ω cannot be zero.

If there is a maximal segment, its dual is a segment of length 2. Two dual polygons of the ends of the maximal segment are glued along its dual (the length two segment). If the union is convex, we again in the case of del Pezzo polygon (there are sixteen types of convex polygons with exactly one lattice point in the interior). If it is not convex, there are infinitely many classes of such non-convex unions. This happens mostly due to the mutations. The stabiliser the length two segment in $SL_2\mathbb{Z}$ is an infinite cyclic group, we can act by its generator on one of the polygons of the union keeping the other one untouched (see Figure 6.3). If we take the combinatorial types of *stabl*e sub-maximal germs of polygons up to $SL_2\mathbb{Z}$ and mutations, there will be again a finite number of their kinds. The combinatorial types of germs, or the types of singularities of C_Ω, give a canonical stratification of the space of all convex domains. Within each stratum the moduli of C_Ω give the complete coordinates. In the case of polygons, every blow-up of a corner reduces adjacent sides by its own size. Therefore, we can compute the length of a side orthogonal to a primitive vector v as a $\mathbb{Z}$-linear combination of the moduli corresponding to the edges of C_Ω belonging to the boundary of the face on which the gradient is equal to v. Thus, each stratum is a lattice polyhedral cone in the coordinates given by the moduli of C_Ω, the inequalities are just that all rational sides of Ω have non-zero length (this has to be controlled when gradually doing blowups starting with $\hat{\Omega}$) and these lengths are expressible as $\mathbb{Z}-$linear combinations of the moduli of C_Ω. In particular, the space of all compact convex domains inherits the structure of infinite dimensional tropical manifold. The linear structure on the strata is compatible with the one given by the Minkowski sum (see Figure 6.4).

A more algebraic way to encode the moduli is the following. Note that a non-maximal multiple edge doesn't branch (as in the proof of Theorem 4.3), because it goes directly to the corner of Ω. Consider a non-multiple edge. Then, there exist two support functions with gradients v_1 and v_2, participating in F_Ω along a segment. Denote by $z(v_1, v_2)$ the end of the segment where the restriction of F_Ω takes its minimal value (these are the analogues of the classical Steiner points).

Proposition 6.3. $F(z(v_1, v_2)) = a_{v_1} + a_{v_2} - a_{v_1+v_2}$.

Proof. Consider a simple pattern of the blow-up given on Figure 6.1.

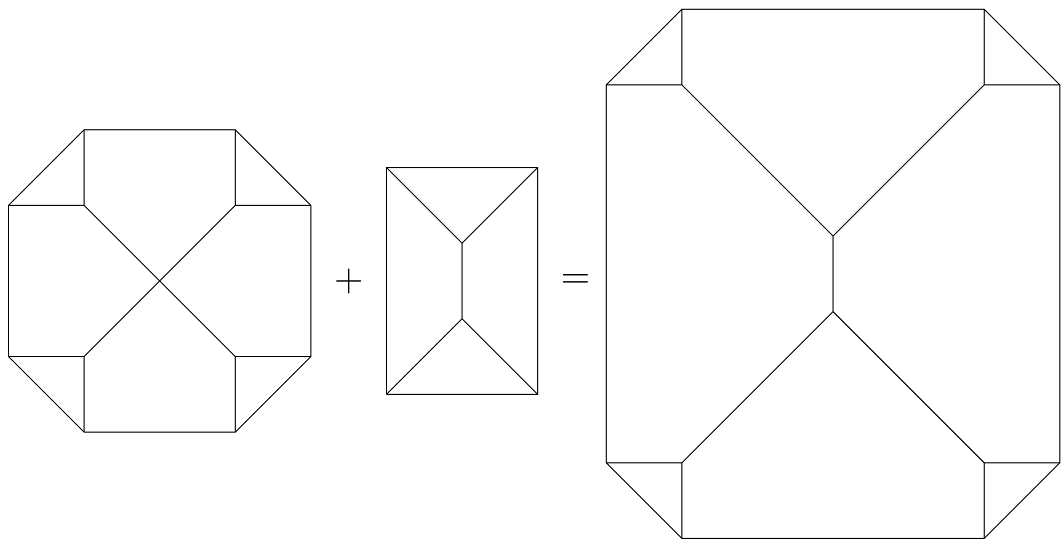

Figure 6.4: The moduli of the curve are additive with respect to the Minkowski sum within the closure of the stratum fixed by the type of singularity of C_Ω.

Even if we have a degenerate branching, see Figure 6.5, the three values

$$z(v_1,v_2)\cdot v_1 + a_{v_1};\quad z(v_1,v_2)\cdot v_2 + a_{v_2};\quad z(v_1,v_2)\cdot(v_1+v_2) + a_{v_1+v_2}$$

of support functions are equal at the vertex $z(v_1,v_2)$. Therefore, we can compute the value by taking the sum of the first two minus the third. $\square$

Therefore, we define a function $f_\Omega\colon \Omega \to \mathbb{R}_{\geq 0}$ by

$$f_\Omega(v_1,v_2) = a_{v_1} + a_{v_2} - a_{v_1+v_2}$$

if there is a side of C_Ω on which the support functions with gradients v_1 and v_2 are equal, otherwise $f_\Omega(v_1,v_2) = 0$. Note that a side can be degenerate if it gets shrinked in the formation of a multiple edge (see Figure 6.5) and we count its contribution to f_Ω. The support of $f = f_\Omega$ is decomposed into a union of free-semigroups, one for each

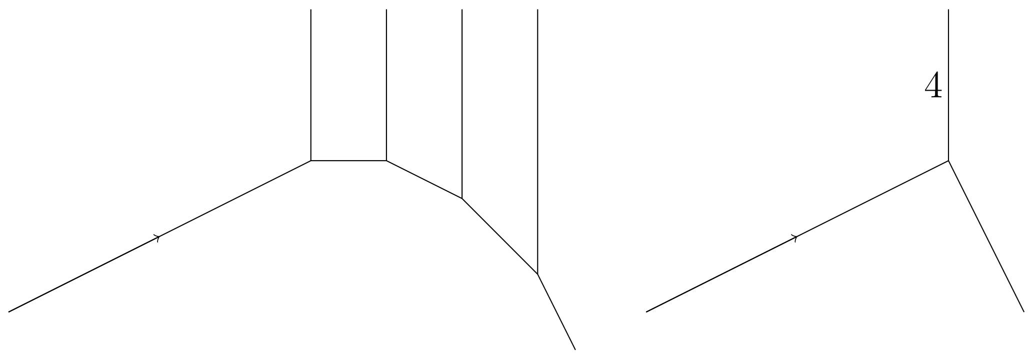

Figure 6.5: We go from the maximum (somewhere down left), accumulating degenerate branchings, when we do a branching right after a previous one, in the limit results in formation of a multiple edge. According to Figure 6.2, the multiplicity of the branching is equal to the multiplicity of the multiple edge stemming from it. Also compare to Figure 4.1, where the resolution is done in a specific example.

non-multiple edge in the curve of a sub maximal germ. Note that the map $(v_1, v_2) \mapsto v_1 + v_2$, acting from the support of f to the primitive vectors, is injective. Knowing the function f is the same as knowing the moduli of the caustic curve. Repeating the same trick with cutting out simplexes, as in the case of the disk, we get the following.

Theorem 6.1.

$$\mathrm{Perimeter}_{\mathbb{Z}}(\Omega) = \mathrm{Perimeter}_{\mathbb{Z}}(\hat{\Omega}) - \sum_{M \in SL_2\mathbb{Z}} f_\Omega(M)$$

$$\mathrm{Area}(\Omega) = \mathrm{Area}(\hat{\Omega}) - \frac{1}{2} \sum_{M \in SL_2\mathbb{Z}} f_\Omega^2(M)$$

$$\mathrm{Action}(\Omega) = \mathrm{Action}(\hat{\Omega}) - \frac{1}{6} \sum_{M \in SL_2\mathbb{Z}} f_\Omega^3(M),$$

where $\mathrm{Action}(\Omega) = \int_\Omega F_\Omega$.

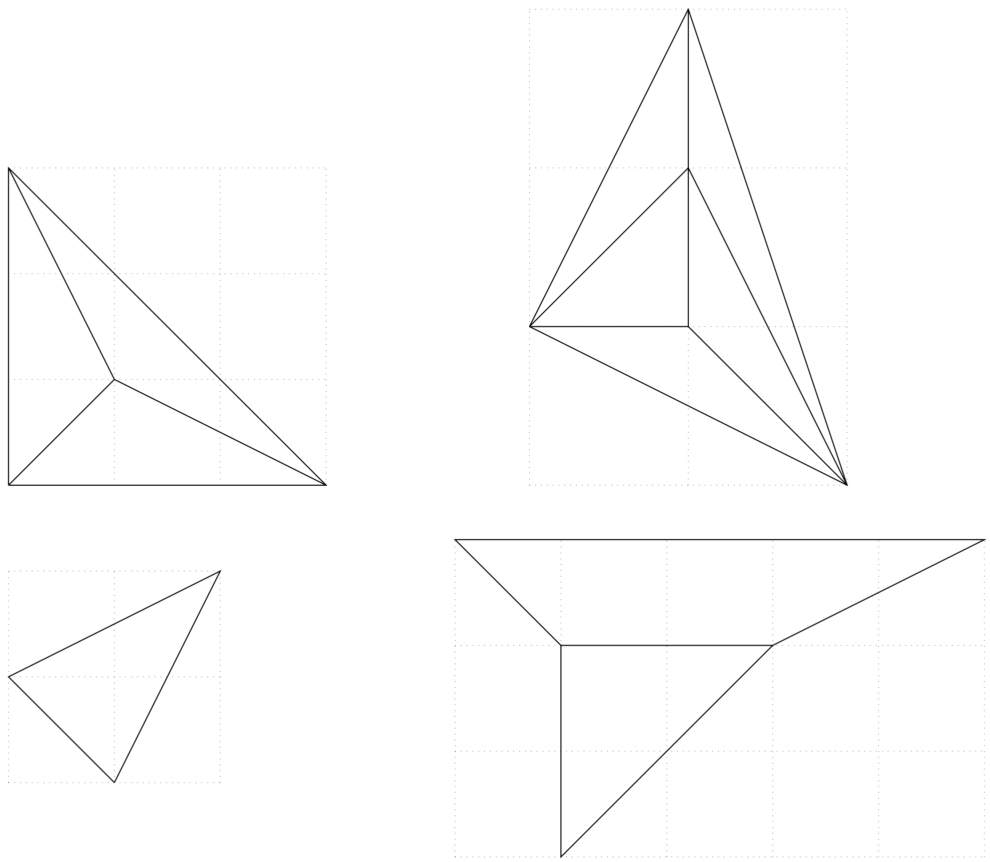

Figure 6.6: Examples of Ω with tropical curves C_Ω inside. The corresponding dual subdivisions, encoding the combinatorial information about the sub-maximal germs of Ω_ϵ, are shown in the second row. Note that the right one is not convex and doesn't get convex even after shift-mutations (see Figure 6.3). The left one belongs to the stable stratum, the neighbourhood of the maximum is unchanged under small perturbations, the right triangle belongs to the unstable stratum because the dual picture can be subdivide further keeping the segment of length two untouched.

The lattice-normalised perimeter is equal to the number of lattice points in the case of the lattice polygon. Just because any small piece of the supporting line will be cut away, it appears, that $\mathrm{Perimeter}_{\mathbb{Z}}$ extends by zero to domains with C^1 smooth boundary. If we think in terms of the sum over the vertices, then they have to be taken with corresponding multiplicities. By Theorem 4.3, all the multiplicities of non-maximal vertices are equal to one.

We see that some sort of localisation is present here. We could compute the area as an integral of 1 over Ω, but we reduce it to the integration of F^2 over the vertices of C_Ω. By Stocks theorem we could went through the integration of F along the edges of C_Ω. Integration of F is reduced to the integration of F^3 along the vertices, etc.

Chapter 7

Panoramic view

I believe that with a fair amount of contemplation, luck and training it is possible to see a relation between two entities, no matter what is the current convention about how remote they are. My research was meant to be a humble attempt to illustrate this principle. The central objects here are tropical curves, convex domains, sandpiles, toric surfaces and unimodular two-by-two matrices. The choice of those entities is somewhat arbitrary and mostly induced by the incredible charisma of several excellent teachers I was lucky to meet and my personal willing to keep things basic but diverse. Contemplation of the vast space of ideas and images in relation to the various geometries (of which to my fault I still know very little) resulted in the discovery of several landmarks, their description can be encapsulated to correspondences described bellow. It seems that the landmarks doesn't stand too close to each other and the main question suggested for the future, more subtle research, is wether we can come directly from one to another, or perhaps to even draw a map. This thesis presents a first sight on a particular location in low-dimensional geometry, although the picture looks to me quite shattered now.

My advisor Grigory Mikhalkin suggested to look at hyperbolic amoebas. Consider a hyperbolic 3-space with a fixed reference point O.

The group of orientation-preserving isometries of $\mathbb{H}^3$ is identified with $\mathrm{PSL}_2\,\mathbb{C}$ (acting conformally on the boundary sphere) and we define a map

$$\varkappa\colon\ \mathrm{PSL}_2\,\mathbb{C} \to \mathbb{H}^3$$

by $\varkappa(A) = A(O)$. For a subvariety V of $\mathrm{PSL}_2\,\mathbb{C}$ its hyperbolic amoeba is $\varkappa(V) \subset \mathbb{H}^3$. This accords with the usual definition of amoeba in the following sense: $\varkappa$ can be seen as factoring out $SO(3)$, which is the maximal compact subgroup of $\mathrm{PSL}_2\,\mathbb{C}$, and $Log\colon (\mathbb{C}^*)^n \to \mathbb{R}^n$ as forgetting the argument subgroup which is also maximal compact. For a hypersurface V in the algebraic torus $(\mathbb{C}^*)^n$, the complement to its logarithmic-amoeba $Log(V)$ consists of some finite union of convex domains and to each of them one injectively associates a lattice point of the Newton polytope of V (Forsberg-Passare-Tsikh theorem [FPT00]), therefore the number of connected components of $\mathbb{R}^n \backslash Log(V)$ is bounded by the number of lattice points in the Newton polyhedron of V and it is easy to show that the bound is sharp for any given polyhedron.

In the case of hyperbolic amoebas, if V is a hypersurface in $\mathrm{PSL}_2\,\mathbb{C}$ then the complement $\mathbb{H}^3 \backslash \varkappa(V)$ is always bounded and connected. In fact, if we compactify $\mathrm{PSL}_2\,\mathbb{C}$ to $\mathbb{C}P^3$ and consider the closure $\bar{V}$ as a surface of degree d then $\varkappa(V) = \mathbb{H}^3$ if d is odd. If d is even we associate to $\bar{V} \subset \mathbb{C}P^3$ a compact convex set $\mathbb{H}^3 \backslash \varkappa(V) \subset \mathbb{H}^3$.

As their different incarnation, the group of unimodular two-by-two matrices $SL_2\mathbb{Z}$ is the group of orientation preserving automorphisms of the square lattice. We investigate the structure of the space of all compact convex domains on the plane modulo $SL_2\mathbb{Z}$, translations and rescaling. The space appears to be nicely stratified into a countable number of contractible infinite-dimensional cells and we give geometric coordinates on each stratum in which it appears to be an open lattice polyhedron. The set of coordinates consist of a function

$$f_\Omega\colon\ SL_2\mathbb{Z} \to \mathbb{R}_{\geq 0}$$

and an "elliptic" parameter $\lambda_\Omega > 0$ corresponding to a domain Ω of a given stratum. We describe a evolution on the space of convex domains giving rise to the canonical approximation of any domain with smooth boundary by smooth (or Delzant) polygons. We describe the stratification of the space of convex domains into infinite-dimensional polyhedral cones revealing the structure of infinite-dimensional tropical manifold. The same structure is induced by the embedding

$$\Omega \to \{a_v(\Omega) = -\inf_{z\in\Omega} z \cdot v\}_{v\in Prim},$$

to the space $\mathbb{R}^{Prim}$, where $Prim \subset \mathbb{Z}^2$ consists of all primitive vectors.

We call a polygon Delzant iff it is as an image of a smooth symplectic toric surface under the moment map or equivalently if primitive vectors directing a pair of adjacent sides for a basis in the lattice (see [Del88]). We use the approximation to give a formalistic construction of a compact symplectic toric pro-space X_Ω^{pro} having a compact convex Ω as its moment domain. Interestingly enough, the resulting topological substratum (forgetting the symplectic structure ω_Ω) appears to be the same for any Ω with C^1 smooth boundary and is isomorphic to the projective plane blown up infinitely many times such that all boundary divisors do not intersect.

This observation is supported by another another construction for a space of Ω – the non-commutative toric surface X_Ω^{nc}. We identify the bare space with the universal elliptic curve over compactified upper half plane. The non-commutative part is located over the boundary and consists of all the foliations on a 2-torus. Very similar spaces appear in Manin-Marcolli [Man04, MM02] in the context of real multiplication and chain fractions in relation to class field theory. The space X_Ω^{pro} has infinitely many independent commuting coordinates and the ring of functions of X_Ω^{nc} is finitely generated but non-commutative. The possibility of such similarities was first noted by Kapranov [Kap09].

The approximation by rational polygons helped a lot in our investigation (joint with Nikita Kalinin) of the behaviour of small perturbations of the maximal sandpile on large convex domains. Grigory Mikhalkin attracted our attention to the series of papers by Carcciolo-Paoletti-Sportiello, a group of Italian physicist which did a number of computer experiments with abelian sandpile model. Among many of their contributions was an empiric discovery of discrete soliton strings propagating in a given primitive direction and moving rectilinearly in the complimentary primitive direction under the wave action. We gave a rigorous proof of this theorem, this enables to make a discrete approximation for every planar tropical curve. Such discrete curves appear when one perturbs the maximal stables state of the sandpile by adding some extra sand grains in few points. Moreover, if we rescale the supporting convex portion Ω of the lattice and keep the positions of perturbation points, the curve remains essentially the same.

This observation was formalised in terms of the scaling limit for sandpiles. This contributes to the study of sandpiles at large initiated by Levine-Pegden-Smart [LPS16, PS13] in the case of a relaxation of a big sandpile concentrated at one point. The operation of adding a grain can be decomposed into the sequence of waves forming the avalanche. The waves propagate through the maximally charged interface interacting with soliton-edges of discrete tropical curves moving them in the direction of a source of the wave. At the level of usual tropical curves this integrates to a deformation of a coefficient of the monomial corresponding to the face to which the source of the wave belongs. Therefore, for any point $p \in \Omega^\circ$ we have a well defined operator G_p on the space of tropical curves in Ω corresponding to the operator of perturbing at p.

If we perturb the maximal stable state on $\Omega \cap \mathbb{Z}^2$ at some collection of points we get the curve passing through the points of perturbation minimising the action, i.e. the total number of topplings during the relaxation. In particular, if Ω is a rational polygon the resulting curve is finite and minimises the symplectic area (this corresponds to the mass of the string and to the symplectic area of holomorphic curve near the tropical limit). In particular, sandpile gives an approximation to the tropical analog of the Steiner problem. If the vertices of the polygon Ω and the points of perturbation $p_1, \ldots, p_n \in \Omega^\circ$ belong to the lattice one can construct the exact solution iterating consequently the idempotent deformation operators G_{p_k} (starting with the empty curve or vacuum state) in an arbitrary order such that all k appear enough many times. The curves minimising the symplectic area and action are closely related to the curves of maximal quantum index [IM12, Mik15].

We consider a stochastic version of the above idempotent process when the point $p \in \Omega^\circ$ in G_p is taken at random after each step. This can be seen as a continuous analogue of the idempotent stochastic process considered by Carcciolo-Paoletti-Sportiello [CPS10, CPS12] where they observed the soliton-patterns. However, their process is a modification of the original Bak-Tang-Wiesenfeld [BTW87] models demonstrating the power law distributions of the sizes of avalanches without any tuning parameters. By means of the supercomputer (cite [GKL$^+$]), we demonstrate empirically that the tropical idempotent dynamics also obeys power laws. Also, it appears that in a suitable regime the resulting random curve has a rather coherent structure.

Another feature of the original sandpile model is the self-organised criticality, i.e. the presence of an attractor of recurrent states having a group structure called the Sandpile group of Dhar (see [Dha99]). The sandpile group on a finite graph (which can be thought as a tropical curve) was interpreted by Baker-Norine in [BN07] as its Jacobian variety, providing yet different relation of sandpiles and tropical curves. Considered together with our sandpile convergence the picture gets rather enigmatic. Indeed, in this interpretation the substratum for the sandpile is a punctured (punctures correspond to sinks on the boundary of the domain) tropical curve supported on a big portion $\Omega \cap \mathbb{Z}^2$ of the lattice, it has a genus depending quadratically on the rescaling of the domain Ω. We see that the "heavy" layer of its sandpile group inherits the tropical sturcture near the scaling limit, since every small perturbation of the maximal stable state is recurrent and described by a discretisation of a tropical curve. The sandpile group considered at large is rather obscure object. For example, one can observe that the unit recurrent state on a square weakly converges when the size of the square goes to infinity. Also geometric and fractal-like patterns are clearly visible on such unit states, it doesn't seem that we have enough technique for the rigorous proof of their presence. In this context, our results shed some light on just the opposite end of the sandpile group, near the maximal stable state.

In general, performing an iterative process of applying G_{p_k} at fixed

$$p_1, \dots, p_n \in \Omega^\circ$$

points we get closer and closer to the solution of the Steiner problem for these points. We couldn't manage to proof that the the process actually stabilises as in the lattice case, but several heuristics speak in favour of this. One of them is that for an arbitrary fine grid we need just a finite number of G_p's to perform a relaxation of the perturbed sandpile and we can estimate the acton of each deformation as a linear function of the rescaling parameter. Another one is that we can extend G_p to an operator S_p acting on algebraic curves over a non-archimedean filed of characteristic 2 such that S_p becomes G_p at the level of non-archimedean amoebas (being tropical curves). We know that iterating $G_{p_1} \dots G_{p_n}$ converges if we would knew the same for S_p's then the iteration of G_p would stabilise. It is an interesting question if we can give similar lifts of G_p to other geometries.

For example, we can view a tropical curve as a limit of logarithmic amoebas. In this case, complex curve degenerates to a union of holomorphic cylinders and Lagrangian pairs of pants connecting them. All cylinders go along edges and pairs of pants belong to the argument tori over vertices of the tropical curve. The boundary of a tropical curve coming from a sandpile on a polygon Ω belongs to the Lagrangian tori over the vertices of Ω. Thinking of that as of topological string we compute its mass as symplectic area. It appear to be deformation invariant and survives under the topicalisation giving the symplectic area computed in terms of the lengths of edges.

The most fundamental perturbation is the perturbation by a single point which produces the maximal action. We call this the central perturbation. This provides a correspondence: to every compact convex domain Ω we associate the tropical analytic curve C_Ω which appears to be the limit of the locus of deviation of the maximal stable state from its central perturbation when the mesh size goes to zero. The moduli of this curve give the invariant system of coordinates on the set of all compact convex Ω's mentioned before.

In the case when Ω is a $\mathbb{Q}$-polygon the curve C_Ω is supported on a finite graph such that $C_\Omega \cap \partial\Omega$ consists of all vertices of Ω. We can modify the logarithmic point of view and consider $\mu\colon X_\Omega \to \Omega$. As before, we can lift $C_\Delta \cap \Omega^\circ$ to the union of holomorphic cylinders and Lagrangian pairs of pants. Over the vertices of Ω we have to glue those cylinders by intersection points of the boundary divisors. This gives a topological cycle $\hat{C}_\Omega \subset X_\omega$ which is dual to the anti-canonical class of X_Ω. Therefore, C_Ω can be seen as a degenerate elliptic curve. This construction suggests a geometric definition of the anti-canonical class of X_Ω for a general Ω.

We give a mechanical description of the caustic curve C_Ω for a rational polygon Ω. We think of the vertices of Ω as of D-branes emitting strings towards the interior. Each string-particle moves with the primitive velocity and collision always results in one new particle respecting the conservation of momentum. The caustic curve C_Ω is the trajectory of the evolution, i.e. the world-sheet of the string. Conservation of momentum corresponds to the balancing condition. We prove that the curve is the corner locus of the weighted distance function F_Ω computing the minimum of lattice distances from all supporting lines of

Ω with rational slopes. Using this intuition we show that the non-maximal level sets of F_Ω are Delzant whenever Ω is Delzant. This defines the canonical evolution on the space of Delzant polygons which extends to convex Ω's with smooth boundary. The refinement of this result is that for a general compact convex Ω non-extremal level sets of F_Ω are rational polygons having only A_n singularities at their vertices. Abusing the notation, we say that a vertex of rational polygon has a singularity of a given type if the toric surface with this moment polygon has such singularity at the corresponding intersection of boundary divisors. Developing the theory in arbitrary dimensions may be worthwhile in approaching towards Mahler conjecture. One of the major obstacles is that certain bodies must be distinguished in terms of a evolution or flow (it might be a version of the one described in Section 4) converging to them:

"The main reason why this conjecture is so difficult is that unlike the upper bound, in which there is essentially only one extremiser up to affine transformations (namely the ball), there are many distinct extremisers for the lower bound - not only the cube and the octahedron, but also products of cubes and octahedra, polar bodies of products of cubes and octahedra, products of polar bodies of well, you get the idea. It is really difficult to conceive of any sort of flow or optimisation procedure which would converge to exactly these bodies and no others; a radically different type of argument might be needed." [Tao08]

We use similar canonical approximations in the spirit of the infinite sequence of symplectic blowups to deduce localisation type formulas for perimeter and area of Ω in terms of the moduli of C_Ω. In case of the unit disc the formula expresses the number π in terms of the total gap in triangular inequalities for all unimodular triangles belonging to a positive quadrant. Pushing this formula to the enumeration of lattice points rather then area suggest a natural approach to the Gauss circle problem. In our initial attempt to realise this program we reduce the general problem of computing the lower order term in the enumeration of lattice points to the accumulation of oscillations in a basic gauge theory on C_Ω. We see that in terms of this theory the disc has additional structure symmetries which could be a partial reason for the better known estimates comparing to the general convex Ω. Another natural approach to the enumeration of lattice points might be routed via K-theory, and particularly the analogue of Riemann-Roch theorem, for the space X_Ω. This is one of the subjects of our current research.

The Gauss circle problem has certain formal similarities with the Riemann hypothesis. Strangely enough, we can associate to every Ω its $\zeta_\Omega(s)$ function equal to the sum of f_Ω^s over $SL_2\mathbb{Z}$. The values at $1, 2, 3$ correspond to the perimeter, area, and action of Ω.

The group $SL_2\mathbb{R}$ acts on $\mathbb{H}^2$ via the automorphisms. This action is a slice of the action by $\mathrm{PSL}_2\,\mathbb{C}$ on $\mathbb{H}^3$. Therefore, the structure of degenerate hyperbolic amoebas might be relevant in the study of sandpiles on hyperbolic tillings in the same way as tropical curves affect sandpiles on square lattice, hexagonal, or indeed any other periodic Euclidean tilling. Our preliminary study of sandpiles on heptagonal tiling gave a conformation of this guess. In [KS15a] we proof that a family of hyperbolic amoebas of hypersurfaces always converges to a complement of a ball centred at the reference point O. In accord with that, a small perturbation of the maximal stable state on the big portion of a tilling deviates from the maximal stable state in the complement of a disc.

Chapter 8

Hyperbolic amoebas

The usual amoebas appear as images of subvarieties in algebraic torus under the coordinate-wise logarithm of absolute values. Topologicaly this map is a trivial fibration over the Eucledean space with topological torus over as a fiber. Algebraicaly, the logarithm is a projection onto the quotient of the algebraic torus by topological torus, its maximal compact subgroup. Here we are going to look what happens if we replace algebraic torus by $PSL_2\mathbb{C}$. We are preparing a separate paper [MS] which will contain some of the presented material and more recent developments. Here I'll review the most basic features of this version of non-commutativity.

$$\begin{tikzcd}
\mathrm{SL}_2\,\mathbb{C} \arrow[r, hook] \arrow[d] \arrow[rr, hook, bend left] \arrow[dd, bend right, "\mathbf{p}r"'] & \mathbb{P}_{\mathbb{C}}(M^{2\times2}_{\mathbb{C}}\times\mathbb{C}) \arrow[dd, "\mathbf{P}r"] & \overline{SL}_2\mathbb{C} \arrow[l, hook'] \arrow[d] \arrow[dd, bend left] & \partial\mathrm{SL}_2\mathbb{C} \arrow[l, hook'] \arrow[d, equal] \arrow[dd, bend left] \\
\mathrm{PSL}_2\,\mathbb{C} \arrow[rr, hook] \arrow[d, "\varkappa"] & & \mathbb{P}_{\mathbb{C}}(M^{2\times2}_{\mathbb{C}}) \arrow[d] \arrow[ul, hook, dashed, "\mathbf{In}"'] & \mathbb{CP}^1\times\mathbb{CP}^1 \arrow[l, hook', "S"'] \arrow[d, "p_1"] \\
\mathbb{H}^3 \arrow[r, hook] \arrow[rr, hook, bend right] & \mathbb{P}_{\mathbb{R}}(\mathbb{R}^{3,1}\times\mathbb{R}) & \overline{\mathbb{H}}^3 \arrow[l, hook'] & \mathbb{CP}^1\cong\partial_\infty\mathbb{H}^3 \arrow[l, hook']
\end{tikzcd}$$

The group $\mathbb{I} := \mathrm{PSL}_2\,\mathbb{C}_2\mathbb{C}$ is the group of all conformal automorphisms of $\mathbb{CP}^1$. This action extends to the action on $\mathbb{H}^3$, where the Riemann sphere becomes the boundary at infinity (see [Thu97]). Take a reference point $O \in \mathbb{H}^3$ and let $SO(3)$ be its stabilizer in $\mathbb{I}$, it is the maximal compact subgroup. Denote by $\varkappa\colon \mathbb{I} \to \mathbb{H}^3$ the map given by $\varkappa(A) = A(O)$. It is the hyperbolic analogue of the logarithm map. The group $\mathbb{I}$ is a complex algebraic group, a complement of $\mathbb{CP}^3$ to the quadric $Q = \{\det = 0\}$. The group $\mathbb{I}$ is a double cover of the affine quadric threefold $\mathrm{SL}_2\,\mathbb{C}$. We can compactify it to the projective quadric threefold $\overline{\mathrm{SL}_2\,\mathbb{C}}$ by comliting it with the projective quadric surface $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$.

This representation is quite enough for the rest of the exposition. For computanional purposes we can take a specific model for $\mathbb{H}^3$. That is, consider the space of Hermitian two-by-two matrices. The determinant is a quadratic form of signature $(3, 1)$. We take $\mathbb{H}^3$ to be the postive part of the hyperboloid $\mathbb{H}^3$. The map $\varkappa$ the is given by $A \mapsto AA^*$, for $\det A = 1$. The big diagram explains how the map is related to $\mathrm{SL}_2\,\mathbb{C}_2$ and how it gets compactified to act from $\mathbb{CP}^3 = \mathbb{P}_{\mathbb{C}}(M^{2\times 2}_{\mathbb{C}} \times \mathbb{C})$ to $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cap \mathbb{CP}^1$.

We are going to start with some general remarks on symmetries of the amoeba map and a priori topological properties of amoebas. Consider a subvariety V in $\mathbb{I}$. The image of V under $\varkappa\colon \mathbb{I} \to \mathbb{H}^3$ is the hyperbolic amoeba of V. The left action of a certain element $A \in \mathbb{I}$ on V translates its amoeba by the isometry A, i.e. $\varkappa(A \cdot V) = A(\varkappa(V))$. We will see that the similar right action in general greatly changes the geometry of an amoeba. There is an obvious exception: the right action on a subvariety by a rotation around O doesn't change its amoeba. If V is irreducible then its amoeba is connected. The existence of $\overline{\varkappa}$ implies that all amoebas are closed subsets in $\mathbb{H}^3$.

8.1 Amoebas of lines

A shape of an amoeba $\mathcal{A} = \varkappa(l \cap \mathbb{I})$ of a line $l \subset \mathbb{CP}^3 = \mathbb{I} \cup Q$ depends on the position of l with respect to the quadric Q: either l can lie on Q, be tangent to Q or intersect transversally in two points.

In the case when the line lies on the quadric the amoeba and the intersection of l with $\mathbb{I}$ are both empty. To describe the image of l under the compactified projection $\overline{\varkappa}$ consider the same identification of Q with $\mathbb{CP}^1 \times \mathbb{CP}^1$ as we had before. There are exactly two families of lines in Q appearing as fibers of two projections from Q to $\mathbb{CP}^1$. If l is of the form $\{p\} \times \mathbb{CP}^1$ for some $p \in \mathbb{CP}^1$ then its image is single a point and moreover if we consider an appropriate identification of $\mathbb{CP}^1$ and $S^2 = \partial_\infty \mathbb{H}^3$ the image $\overline{\varkappa}(l)$ is nothing but a point $\{p\} \in \partial_\infty \mathbb{H}^3$. In another case l must be equal to some $\mathbb{CP}^1 \times \{q\} \subset Q$ and so $\overline{\varkappa}$ projects l identically on $\partial_\infty \mathbb{H}^3$.

If Q doesn't contain l then they intersect either at one or two points. This cases give amoebas of quite different shape. Lets consider first a line l which meets the quadric exactly at one point, in particular l is tangent to Q. This immediately implies that the amoeba of l is non-empty and touches $\partial_\infty \mathbb{H}^3$ exactly at one point. In fact, it is possible to state much more accurate result.

Proposition 8.1. If a line is tangent to Q then its hyperbolic amoeba is a horosphere in $\mathbb{H}^3$.

Recall that a horosphere is a surface in $\mathbb{H}^3$ such that it is orthogonal to any geodesic starting at some fixed point at the infinity.

One can give the following equivalent statement for this proposition. A hyperbolic amoeba of a line in $\tilde{\mathbb{I}}$ is a horosphere. Indeed, a curve in $\tilde{\mathbb{I}}$ is a line if and only if the restriction of the two-fold covering $\tilde{\mathbb{I}} \to \mathbb{I}$ on the curve is injective and its image is a line tangent to Q.

We will prove the proposition after work out necessary technics. Here are some comments on the statement. Actually, any horosphere can be realized as an amoeba of some line and the restriction of $\varkappa$ to such a line is always one-to-one. Two families of lines on Q mentioned above give amoebas, which can be interpreted as infinitely small and infinitely large horosphere.

The case of a line, which is not tangent to the quadric, is generic and so there is no primary reason for its amoeba to have a nice shape. It is clear that an amoeba of such line must be nonempty and its closure must contain two infinite points. Surprisingly, we can give a much stronger description.

Proposition 8.2. If a line is not tangent to Q then its hyperbolic amoeba is a cylinder in $\mathbb{H}^3$.

A cylinder of radius $r \geq 0$ in $\mathbb{H}^3$ is defined to be a locus of points that are at the same distance r from a given geodesic.

We will show that any cylinder can be realized as an amoeba of some line. In particular, any geodesic, which is a degenerate cylinder, is an amoeba of a line. Except for this case all amoebas of lines are smooth surfaces and the restriction of $\varkappa$ on such a line is an embedding. We will use this fact to show that the lines projecting by $\varkappa$ on geodesics are the only examples of algebraic curves in $\mathbb{I}$ with one dimensional hyperbolic amoebas.

After we have announced some of the results lets move on to a more systematic narrative.

Both left and right actions of $\mathbb{I}$ on itself can be uniquely extended to $\mathbb{I} \subset \mathbb{CP}^3$. Clearly, Q is invariant under these actions. And moreover in the identification of Q with $\mathbb{CP}^1 \times \mathbb{CP}^1$ the group $\mathbb{I}$ acts separately on the factors of Q, i.e. the left(right) action on Q acts by automorphisms on the left(right) $\mathbb{CP}^1$ and preserves the right(left) one.

Consider the space of all lines in $\mathbb{CP}^3$ tangent to Q. We have an action of $\mathbb{I} \times SO(3)$ on this space. Lines in the same orbit of this action have congruent hyperbolic amoebas. We claim that there are only three different orbits for the action.

Lemma 8.1. Two families of lines lying on Q and a set of all other lines tangent to Q are the only orbits for the action of $\mathbb{I} \times SO(3)$ on Q.

Thus, if we show that an amoeba of some particular properly tangent line to Q is a horosphere then amoebas of all other lines of this kind will be horosphere.

Proof. Let x be a point in Q. Take a stabilizer subgroup for x under the action of $\mathbb{I} \times SO(3)$ and consider its action on the tangent space to Q at the point x. It is clear that the stabilizer has a subgroup isomorphic to $\mathbb{C}^* \times U(1)$ and acts separately on each multiplier in $T_x Q = \mathbb{C} \times \mathbb{C}$. The action of this subgroup on the projectivization $\mathbb{P}(T_x Q)$ for the tangent space can be obviously reduced to the standard action of $\mathbb{C}^*$ on $\mathbb{CP}^1$. The last is stratified on three orbits: two points and a torus. The points correspond to the lines in the intersection of Q and a plane tangent to Q at x. The torus parametrizes the space of all lines properly tangent to Q at x.

To finish the prove note that $\mathbb{I} \times SO(3)$ acts transitively on Q and evidently preserves the stratification for projectivization of a tangent space to Q at each point. $\square$

Our goal now is to show that there exist a line tangent to Q with an amoeba equal to a horosphere in $\mathbb{H}^3$. The main idea here is to use interactions of some specific subgroups of $\mathbb{I}$ to produce extra symmetries for their amoebas.

As we mentioned before, the group $\mathbb{I}$ can be interpreted as the group of automorphisms for $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. A group B of affine transformations on the complex line $\mathbb{C}$ is a 2-dimensional subgroup of $\mathbb{I}$. It can be also defined to be a stabilizer of ∞. In fact B can be described up to conjugation as a Borel subgroup of $\mathbb{I}$. Each element of

$\mathbb{I}$ can be seen a Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}, \qquad ad-bc \neq 0.$$

Such transformation is affine if and only if $c = 0$ and is given by $z \mapsto az+b$. So the closure of B is a plane in $\mathbb{CP}^3$.

We introduce two natural subgroups in B: a subgroup $l_1 = \{z \mapsto z+b\}$ of translations in $\mathbb{C}$ and a subgroup $l_2 = \{z \mapsto az\}$ generated by homotheties and rotations around $0 \in \mathbb{C}$. In the above notations l_1 is given by the equation $a = 1$ and l_2 is given by $b = 0$. So the closures for both subgroups are lines in $\mathbb{CP}^3$. It is also clear that l_1 is a normal subgroup of B and so l_2 acts on l_1 by the conjugation, l_1 is a maximal affine subgroup and l_2 is a maximal torus in $\mathbb{I}$. Since l_1 and l_2 intersect by a unity and generate the whole group of affine transformations, B is a semi-direct product of l_1 and l_2.

Now we are going to describe amoebas for these groups. First we have to note that an amoeba of any subgroup in $\mathbb{I}$ is smooth at each point because the group acts on its amoeba transitively by isometries.

We start with l_2. The point 0 and ∞ in $\partial_\infty \mathbb{H}^3$ are the only points stabilized by l_2. Denote by γ a geodesic through these points.

Lemma 8.2. An amoeba of l_2 is the geodesic γ.

Proof. The points 0 and ∞ are antipodal on $\partial_\infty \mathbb{H}^3$ with resect to the chosen metric. This means that the fixed point O is inside of a geodesic γ through 0 and ∞. The action of l_2 by isometries preserves the geodesic. Rotations around 0 in $\mathbb{C}$ correspond to rotations around γ and homotheties correspond to translations along γ. Because $1 \in l_2$, the amoeba contains O and since it is closed under translations along the geodesic, the whole γ is contained in l_2.

Now we will show that the amoeba coincides with γ. Suppose a point $p \in \mathbb{H}^3$ in the amoeba is outside of γ. Since the amoeba is connected we can find the path inside of it starting at p and with endpoint on γ. This imply that the amoeba contains a non-degenerate solid cylinder (which is an image of the action by l_2 on γ) and thus 3-dimensional. The amoeba cannot contain any boundary points because it is homogenous. Thus it coincides with $\mathbb{H}^3$, so the amoeba and l_2 have infinitely many boundary points. This is a contradiction. $\square$

The only point in $\partial_\infty \mathbb{H}^3$ which is preserved by l_1 is the point ∞. This automatically implies that the amoeba of l_1 has only one boundary point and l_1 itself is tangent to Q. Denote by η a horosphere through O and touching $\partial_\infty \mathbb{H}^3$ at the point ∞. The lemma below with the Lemma 8.1 completes the proof of the Proposition 8.2.

Lemma 8.3. An amoeba of l_1 is the horosphere η.

Proof. The first observation is that compactified amoebas of l_1 and l_2 share two points: the fixed point O and the point ∞ at infinity. The group l_2 acts on l_1 by conjugation. The action by conjugation of $l_2 \cap SO(3) \cong U(1)$ on $\mathbb{I}$ descends to the level of amoebas. And so the amoeba of l_1 is invariant under the rotations around γ. The amoeba cannot sit on γ because it must not have boundary points and is isolated from $0 \in \partial_\infty \mathbb{H}^3$. This implies that it is not one dimensional, i.e. it is a surface. In particular this means that a tangent space to O is two dimensional and is invariant under the rotations around γ. So the amoeba of l_1 is orthogonal to γ.

Now we apply elements of l_1 to $\mathbb{H}^3$, this doesn't change its amoeba. Since the action is isometric on $\mathbb{H}^3$ and transitive on the amoeba it can move γ to any geodesic starting at the point ∞. Moreover it preservers orthogonality of a geodesic and the amoeba. This forces the amoeba to be equal to the horosphere η. $\square$

Consider a line l which is not tangent to Q. The line meets the quadric at two points (p_1, q_1), $(p_2, q_2) \in Q$. The following short argument shows that an amoeba of this line is a cylinder in $\mathbb{H}^3$ and a geodesic through p_1 and p_2 is an axis of symmetry for this cylinder. In particular, this implies Proposition 8.2.

The group $\mathbb{I}$ acts from the left on itself and on the space of lines not tangent to Q. In terms of intersection points of lines with the quadric the action by $A \in \mathbb{I}$ can be obviously written as

$$A\{(p_1, q_1), (p_2, q_2)\} \mapsto \{(Ap_1, q_1), (Ap_2, q_2)\}.$$

Note that p_1 and p_2 are distinct since the line doesn't lie on the quadric. So the subgroup preserving both p_1 and p_2 is a subgroup of helicial motions around the geodesic through this points. The subgroup also acts from the left on the line l and, similarly as before, it acts by isometries on the amoeba of l.

The question of precise geometric position of an amoeba is more delicate. A cylinder in $\mathbb{H}^3$ has 5-dimensional deformation space. Four of the dimensions correspond to the position of its endpoints at infinity and the rest is the diameter. It is clear that the points p_1 and p_2 (the left components of the intersection of the line with Q) are exactly the endpoints for the amoeba. So there must be a way to extract the diameter from the "invisible" points q_1 and q_2. As before we can move the line using the right action of $SO(3)$ and this will not change its amoeba. In particular, this implies that the diameter d must be a function of an angle between q_1 and q_2 in S^2. It is clear from the discussion of the degenerate cases that the function surjectively maps $(0, \pi]$ on $[0, +\infty)$. It is actually monotone and so one-to-one. Of course, it is possible to find an exact expression:

Remark 8.1. The diameter of the amoeba can be expressed as

$$\text{diameter} = -\log \sin \frac{\alpha}{2}$$

where α is the distance between the points q_1 and q_2 on the sphere.

Proof of this fact is left to the reader.

This gives us an explicit description of the correspondence for lines in general position with Q and their amoebas. A similar thing can be done for lines tangent to the quadric Q. Consider a line l tangent to Q at a point (p, q). We have already proved that an amoeba of this line is a horosphere in $\mathbb{H}^3$ tangent to the absolute at the point p. There is a one dimensional family of such horosphere. A parameter $r \in \mathbb{R}$ can be defined to be an oriented distance from the point O to a horosphere. More precisely, denote by x an intersection point of a horosphere with the geodesic line through the points O and p. Then r is defined to be the distance $|Ox|$ if x is between O and p. In the opposite case $r = -|Ox|$.

A family of lines tangent to Q at a fixed point can be identified with the projectivization of the tangent space to Q at this point. Take a tangent line l to the quadric and consider the correspondent one dimensional tangent subspace to $Q = \mathbb{CP}^1 \times \mathbb{CP}^1$ and its generating vector (u, v). The boundary version of the map $\varkappa$ identifies $\mathbb{CP}^1$ with S^2. In particular it gives a canonical choice of metric on $\mathbb{CP}^1$. Now we are able to describe a precise geometric position for the amoeba of l:

Remark 8.2. The oriented distance from O to the amoeba is given by

$$r = \log \frac{\|u\|}{\|v\|}$$

The proof is provided by a quite straightforward computation and is left to the reader.

8.2 Amoebas of higher degree curves

Consider a curve C of degree d in $\mathbb{CP}$. Since $\overline{\varkappa}$ is continuos and C is compact, it is clear that its compactified amoeba is a closed subset in $\overline{\mathbb{H}^3}$ and thus its amoeba is a closed subset in $\mathbb{H}^3$. It has nonempty complement because it has only finite number of points at infinity. This points come as images of intersection points for the curve C with the quadric Q. So there are at most $2d$ infinitely far points for the amoeba and for a generic generic curve there are exactly $2d$ points. Let us give a slightly dipper result.

Proposition 8.3. Let Ω be a domain in $\mathbb{C}$ and $\phi\colon \Omega \to \mathbb{I}$ be a holomorphic embedding. If $\varkappa \circ \phi$ is critical at every point then ϕ parametrizes a part of a line in $\mathbb{I}$ projecting by $\varkappa$ on a geodesic.

In particular, a hyperbolic amoeba for an irreducible curve of degree greater than one is smooth and two dimensional at its generic point.

Proof. Consider a map Ψ from Ω to the space $G_{\mathbb{C}}(2,4)$ of all lines in $\mathbb{CP}^3$ given by $x \mapsto T_{\phi(x)}\phi(\Omega)$. A point $x \in \Omega$ is critical for $\varkappa \circ \phi$ if and only if a point $\phi(x)$ is critical for $\varkappa|_{\Psi(x)}$. The set of all lines in $\mathbb{I}$ for which a restriction of $\varkappa$ has critical points is precisely the set of those lines which are projected on geodesics. And so, $\Psi(x)$ is not tangent to Q for all $x \in \Omega$. By taking intersection points with Q, the set set of such lines can be identified with a symmetric power for Q without the diagonal. We can locally choose some lift $\widetilde{\Psi}(x) = (p_1(x), q_1(x), p_2(x), q_2(x))$ of Ψ to $Q \times Q = (\mathbb{CP}^1)^{\times 4}$. Since $\varkappa(\Psi(x))$ is a cylinder of zero diameter the angle between $q_1(x)$ and $q_2(x)$ is equal to π and the points are antipodal. Thus $q_2(x) = -q_1(x)$, where "$-$" denotes antiholomorphic involution on $\mathbb{CP}^1$ given by central symmetrizing on $\partial_\infty \mathbb{H}^3$. This implies that both q_1 and q_2 are constant. The points also distinct since all $\Psi(x)$ do not lie on Q.

Now take two non-intersecting lines in Q given by $l_k = \{(p, q_k)\}_{p \in \mathbb{CP}^1}$. There is a unique line through a generic point in $\mathbb{CP}$ and the lines l_k. Consequently, the choice of two points $P_k \in l_k$ and a choice of a point on a line trough P_1 and P_2 together give a local coordinates for $\mathbb{CP}$. From the construction of $\widetilde{\Psi}$ we already know that $P_k(\phi(x)) = (p_k(x), q_k)$ and that the line $\Psi(x)$ is the line trough $P_1(\phi(x))$ and $P_2(\phi(x))$. The line $\Psi(x)$ is tangent to $\phi(\Omega)$ this can be written as

$$\frac{\partial}{\partial x} P_k \circ \phi(x) = 0.$$

Thus, both p_1 and p_2 are constant. In particular, ϕ parametrizes a part of the line trough (p_1, q_1) and (p_2, q_2) which is mapped by $\varkappa$ on a geodesic.

□

8.3 Amoebas of Surfaces

It is clear that a hyperbolic amoeba of a surface is closed. The standard argument for this is that the map $\varkappa$ can be extended to the compactifications of $\mathbb{I}$ and $\mathbb{H}^3$. It is a classical result that a complement for an ordinary (logarithmic) amoeba is always nonempty. In the hyperbolic setup this statement fails: an amoeba of any Borel subgroup always spans $\mathbb{H}^3$.

Indeed, as we sow, a Borel subgroup is a semi-direct product of two lines. An amoeba of the first line is a horosphere and an amoeba of the second is a geodesic – the axis of symmetry for the horosphere. The second line acts on $\mathbb{H}^3$ by translations along its amoeba. Thus, the amoeba of the Borel subgroup contains all the translations of the horosphere along the geodesic, and so coincides with the whole $\mathbb{H}^3$.

From this reasoning we can also conclude that a restriction of $\varkappa$ on a Borel subgroup is equivalent to a quotiening out the rotations around a geodesic, so it is an honest circle fibration. In particular it is an evidence of certain stability for the property of amoeba to span the whole ambient space. We will show that this observation is absolutely true in degree one.

In general an amoeba of a surface fills almost all ambient space. This is verified by the following simple lemma.

Lemma 8.4. Let S be a nonempty surface in $\mathbb{I}$ and $\mathcal{A}$ be its hyperbolic amoeba. Then $\partial_\infty \mathbb{H}^3 \subset \mathcal{A}$ and $\mathbb{H}^3 \backslash \mathcal{A}$ is bounded.

Proof. Consider an intersection of the closure for S in $\mathbb{CP}^3$ with Q. This gives a curve of nonzero symmetric bi-degree on the quadric. The boundary part of $\varkappa$ is holomorphic. Thus we have the first assertion. The second assertion is an obvious consequence of the first one. $\square$

Proposition 8.4. A hyperbolic amoeba of a plane in $\mathbb{I}$ is $\mathbb{H}^3$.

Proof. A plane can be given by a homogenous equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ on $[x_1 : x_2 : x_3 : x_4] \in \mathbb{CP}^3$. This equation can be rewritten as an equation on $X \in \mathbb{I}$:

$$\operatorname{tr}(A_0X) = 0, \text{ where } A_0 = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \text{ and } X_0 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

As usual, multiplication on the right of the plane by unitary elements doesn't change the amoeba, i.e

$$\varkappa(\{X \in \mathbb{I} | \operatorname{tr}(A_0X) = 0\}) = \varkappa(\{X \in \mathbb{I} | \operatorname{tr}(A_0XU) = 0\}),$$

where $U \in U(2)$. But $\operatorname{tr}(A_0XU) = \operatorname{tr}(UA_0X)$. So the geometry of the restriction of $\varkappa$ to the plane and the shape of the amoeba depend only on $A_0 \in \mathbb{C}^4 \backslash \{0\}$ modulo the right action of $U(2)$ and simple rescaling. Thus,

$$[A_0] \in \overline{\mathbb{H}^3} = (\mathbb{C}^4 \backslash \{0\})/(\mathbb{C}^* \times U(2))$$

parametrizes different types of plane projections onto their amoebas.

Using the left action by $\mathbb{I}$ we isometricaly move the amoeba in $\mathbb{H}^3$. This correspond to the analogous action on the space of parameters. We see that the equivalence classes in $\overline{\mathbb{H}^3}/\mathbb{I}$ correspond to congruent amoebas. But the quotient consist only of two elements, i.e $\overline{\mathbb{H}^3}/\mathbb{I} = \{\mathbb{H}^3, \partial_\infty \mathbb{H}^3\}$.

We already know that in the case $[A_0] \in \partial_\infty \mathbb{H}^3$ the amoeba fills the whole space. That is because

$$[A_0] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \operatorname{tr}(A_0 X) = x_3 = 0$$

correspond to the considered example of a Borel subgroup.

The case of $[A_0] \in \mathbb{H}^3$ haven't appeared before. For example, take A_0 to be the identity

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \operatorname{tr}(X) = 0.$$

Clearly, this plane is invariant under conjugation by $\mathbb{I}$. Restricting this action to the action by $SO(3)$, we get that the amoeba is invariant under all rotations around the fixed point O in $\mathbb{H}^3$. The point

$$X_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

obviously sits on the plane. But X_0 is a unitary matrix and so $\varkappa(X_0) = O$. This implies that the amoeba is $\mathbb{H}^3$, because it is connected, contains the absolute and the point O, and invariant under all rotations around the fixed point O. $\square$

From the proof we see that $[A_0] \in \mathbb{H}^3$ actually parametrizes the image for the only singular point of the projection from the plane onto $\mathbb{H}^3$. The fiber over a generic point is a circle and the fiber over $[A_0]$ is a 2-sphere. The planes of this type are not tangent to Q.

The case $[A_0] \in \partial_\infty \mathbb{H}^3$ is a case of a plane tangent to the quadric Q. The restriction of $\varkappa$ to such a plane is nonsingular circle fibration. We can think that a singular fiber has been moved to infinity. And it is actually so. The plane in this case intersects the quadric in two lines.

And one of this lines is mapped to a point $[A_0]$ by the restriction for the boundary part of $\varkappa$ to the plane.

In fact such an enormous behavior is not presented uniquely in degree one. In any degree a space of surfaces with a hyperbolic amoeba spanning $\mathbb{H}^3$ is nonempty and moreover has a nonempty interior. Indeed, if we take a generic collection of planes then any of its small deformations still fills $\mathbb{H}^3$.

The first example of an amoeba with a nonempty complement appears at degree two. Consider a family of quadric surfaces

$$S_\lambda = \{X \in \mathbb{CP} | \operatorname{tr}^2 X = \lambda \det X\}, \quad \lambda \in \mathbb{C}.$$

We know that $\varkappa(S_0) = \mathbb{H}^3$. In fact $\varkappa(S_\lambda) = \mathbb{H}^3$ if and only if $0 \leq \lambda \leq 4$. For all other values of λ an amoeba is a complement to an open nonempty ball with a center at O. This family is canonical in that sense that it interpolates between doubble of the involution plane $\{tr = 0\}$, the quadric Q, and the space of parabolic elements, those that act with only one fixed point on $\mathbb{CP}^1$.

Lets turn to the problem of describing the topology for the complement of a hyperbolic amoeba for a general surface. We already know that it should consist of a disjoint collection of open convex sets. To solve the problem completely it would be enough to describe the exact upper bound for the number of components for an amoeba when the degree of a hypersurface is fixed and show that all the lower values can be obtained on some open family. Surprisingly, it appears that this program can be implemented in a rather simple way and the result from the very beginning seems to by quite unexpected.

Theorem 8.1. A complement for a hyperbolic amoeba of an odd degree surface is empty.

We already proved this result in the case of planes. The case of even degree is less degenerate.

Theorem 8.2. A complement to a hyperbolic amoeba of an even degree surface is either empty or connected.

Moreover, we will see that for each fixed even degree, the locus of surfaces with an amoeba having a nontrivial complement is nonempty open, the locus of surfaces with a trivial amoeba has a nonempty interior.

The following easy argument essentially proves both theorems. Consider a topological $4-$dimensional cycle S in $\overline{\mathbb{I}}$ and an amoeba, i.e. its image under $\overline{\varkappa}$. Suppose its complement is nonempty. Fix one hole (a connected component of the complement) of the amoeba. It is automatically open but not necessarily convex. Take a point p inside the hole. Consider an oriented geodesic passing through the point p. There exists a lift of this geodesic to a line. The line intersects the cycle in a certain number of points which are splinted in to two parts by the position of there images on the geodesic with respect to the point p. Counted with multiplicities these points give us two numbers, say k_1 and k_2 for the points before and after p.

Lemma 8.5. The numbers k_1 and k_2 doesn't depend on the choice of the lift for the geodesic.

Proof. Clearly the sum $k_1 + k_2$ is constant and equals two the degree of the cycle of S. Now we note that the family of lines projecting down two the geodesic is continuos and is actually diffeomorphic to $\mathbb{CP}^1$. We complete the proof by recalling that p is not in the amoeba of S and thus the intersection points cannot jump through $\varkappa^{-1}(p)$. □

This lemma works in the same way in the case of classical logarithmic amoebas. The following phenomena is new.

Lemma 8.6. The numbers k_1 and k_2 are equal.

This implies immediately that the hole can exist only if the degree of S is even. This completes the prove for the first theorem.

Proof. We use the same argument as in the previous lemma, but we are going to vary the geodesic through the point p. This variation will go smoothly (in contrast with the logarithmic setup) since locally the family of lines projecting to the geodesic is parametrized by three points in a sphere. Thus we can choose a path in the space of lines, such that each line is projected by $\varkappa$ to the geodesic passing through distinguish p and these geodesics perform a half-twist around p returning to the initial geodesic, but changing the orientation. This interchanges the two types of the intersection points for S and the lines. $\square$

Now we turn to the case when S is an algebraic even degree surface, and so connected components of its amoeba are convex. Suppose, there are two holes. Consider a geodesic γ through these two holes. Take a point p in the intersection of the geodesic and the amoeba lying between the holes. The intersection of S and $\varkappa^{-1}(p)$ is nonempty. Take a point A in this intersection. There exists a line l through A projecting down on γ.

In analogy with what we were doing before it is possible to distinguish three segments in γ :before the first hole, between the holes, after the second hole; and prescribe some numbers k_1, k_2, k_3 to the segments, given by the intersection points of l and S over the segments. From the construction of l and holomorphisity of S we have k_2 strictly positive. Looking on the first hole individually we have $k_1 = k_2 + k_3$, and on the second $k_1 + k_2 = k_3$. Thus k_2 should be zero. This contradicts to the existence of two distinct points in the amoeba and completes the proof of the second theorem.

Now it becomes not completely evident that the situation when the hole exists is common and that it could be arbitrary large. Clearly, the condition of existence of a hole is open. We are going to show how to deform an arbitrary even degree hypersurfaces to create a hole in its amoeba containing an arbitrary compact subset of $\mathbb{H}^3$.

Consider a compact set $C \subset \mathbb{H}^3$ and a degree $2d$ hypersurface

$S \subset \mathbb{CP}^3$ given by $f = 0$. A preimage $\varkappa^{-1}(C)$ is again compact and doesn't intersect Q. Thus a function $det^{-n} f$ is well defined on $\varkappa^{-1}(C)$ and its absolute value has a finite maximum m. Now take any $\lambda \in \mathbb{C}$ such that $|\lambda| > m$ then an amoeba of $S_\lambda = \{f + \lambda det = 0\}$ doesn't contain C.

8.4 Other

Theorem 8.2 partially explains why there is a combinatorially unique non-deviation locus in heptagonal sandpiles [KS15a]. Taking into account the possibility to lift G_p operators from the Steiner problem to the algebraic S_p, also suggests that there might exist a general framework unifying sandpiles, algebraic groups and amoebas.

We've attempted to perform an analogue of tropicalization in the hyperbolic framework. Clearly, there is nothing interesting at the level of amoebas of surfaces. I would conjecture, a compactness theorem of the form: in every sequence of non-trivial hyperbolic amoebas of surfaces there exist a subsequence such that, combined with rescaling, it converges to the complement of a metric ball (topologically the statemnt is trivial). For curves the naive hyperbolic tropicalization gives floor diagrams. The floors, similarly to [BM07], are the slices of the complementary dimensions, concetric spheres with center at the reference point. They are connected by a finite number of segments, the elevators.

At the first glance, degeneration of hyperbolic amoebas of surfaces doesn't give much information. On the other hand, if we look at the conjugate projection $\varkappa'(A) = A^*A$ together with the standard one, it gives the projection $(\varkappa, \varkappa')(A) = (A^*A, AA^*) \subset (\mathbb{H}^3)^2$ of $\mathrm{PSL}_2\,\mathbb{C}$ to the real cone over the smooth complex quadric surface Q. Degenerating the images of surfaces under this projection we get sort of floor diagram in the cone, where the floor ocupies the whole level of the cone and an elevator between two floors is an algebraic curve of symmetric bi-degree in Q. See [BM07] for the floor diagrams in classic contex.

We have observed that one can define a dual to $\varkappa$

$$\varkappa^*\colon \mathbb{CP}^3 \to \overline{\mathbb{H}^3}$$

acting on the space of planes in the projective space. it is easy to check that the restriction of $\varkappa$ to a plane H has a unique singular value wich we denote by $\varkappa^*(H) \in \mathbb{H}^3$. The fiber over this point is a sphere if the point is on the boundary and real projective space othervise. The key ingredient is to recognize the projectivisation of traceless matrices as the closure of all involutions in $\mathrm{PSL}_2\,\mathbb{C}$.

Finally, we note that despite the fact there are two different hyperbolic amoeba maps (corresponding to the right or left factorization of $\mathrm{PSL}_2\,\mathbb{C}$ by $SO(3)$), there is a unique spherical coamoeba map $\sigma(A) = U \in SO(3)$, where $P = P^*$, $P \geq 0$ and $A = PU$ is the polar decomposition. Note that $A \mapsto (A^{-1})^*$ extends to the real structurcture on the compactification $\mathbb{C}P^3$ of $\mathbb{I} = \mathrm{PSL}_2\,\mathbb{C}$ and $SO(3) = \mathbb{R}P^3$ is the real locus. If we consider a real algebraic knot K of degree d the spherical coamoeba of its complexification $\sigma(\mathbb{C}K) \subset \mathbb{R}P^3$ is a cobordism joining K with a union of d lines. We do not elaborate more on this matter since this goes beyond the scope of the project.

The rest of the section will be devoted to the concept closely related to amoebas hypersurfaces, namely the Gauss map, [Kap91, Mik00].

Definition 8.1. Consider a smooth surface $V \subset G$. Let

$$\gamma_l\colon V \to G_{\mathbb{C}}(2, T_I G)$$

a left Gauss map be given by $\gamma_l(A) = A^{-1} T_A S \subset T_I G$.

Here $G_{\mathbb{C}}(2, T_I G)$ means a Grassman manifold of 2-dimensional complex linear subspaces of $T_I G$. One can define a right Gauss map γ_r in the similar way.

We give a basic fact on this map. A subspace V of $T_I G$ is said to be real if it is invariant under the Hermitian involution. A collection of all 2-dimensional real subspaces in $T_I G$ is denoted by $\mathbb{R}G_{\mathbb{C}}(2, T_I G)$.

Proposition 8.5. A set $Crit(V)$ of all critical points for the restriction of $\mathbf{p}r$ on V coincides with $\gamma_l^{-1}(\mathbb{R}G_{\mathbb{C}}(2, T_I G))$.

Proof. For any point $A \in V$ there are precisely two possible different relative position for a tangent space to a fiber of the projection and a tangent space to the surface. Their intersection is either real one dimensional, if A is not a critical point of $\varkappa$, or two dimensional in a case when A is critical. In the second case we have that $A^{-1}T_A V$ is generated by v_1 and v_2 from the $T_I\,\mathrm{SU}_2$. Then iv_1 and iv_2 are real. Hence $A^{-1}T_A V = \langle iv_1; iv_2\rangle$ is real.

$\square$

Another usual property for the Gauss map relates it with the set of real points of a surface. A surface is said to be real if it is invariant under Hermitian conjugation. A point is called real if it is self-adjoint. A set of all real points of V is denoted by $\mathbb{R}V$. The property that should be satisfied is that an image of a real point of a real surface under the Gauss map is real. But suddenly there is a problem here arises because of non-commutativity. Let us examine this more closely.

Take a real point A in a real surface V. Then $T_A V$ is closed under the conjugation. Consider a conjugate to its translation by A^{-1}:

$$(\gamma_l(A))^* = (A^{-1}T_A V) = T_A V(A^*)^{-1} = (T_A V)A^{-1} = \gamma_r(A).$$

We see that here a right Gauss map appears. Trying to satisfy the property we can construct a variation of Gauss map in this way. Define two-sided Gauss map by $\gamma_{lr}(A) = (\gamma_l A, \gamma_r A)$. Also we introduce a natural real structure on $G_{\mathbb{C}}(2, T_I G)^2$ by $(l_1, l_2)^* = (l_2^*, l_1^*)$.

Proposition 8.6. $\mathbb{R}V \subset \gamma_{lr}^{-1}(\mathbb{R}G_{\mathbb{C}}(2, T_I G)^2)$.

Proof. Take a real point $A \in V$. Then

$$(\gamma_{lr}(A))^* = (\gamma_l(A), \gamma_r(A))^* = ((\gamma_r(A))^*, (\gamma_l(A))^*) = \gamma_{lr}(A).$$

$\square$

If we introduce a symmetric alternative for the product we mentioned at the very beginning then the analog of proposition 8.5 will be true for γ_{lr}. Let a two-sided projection be $\mathbf{p}r_{lr}(A) = (pr_l(A), pr_r(A))$. A set of its critical points on a surface will evidently coincide with $\gamma_{lr}^{-1}(\mathbb{R}G_{\mathbb{C}}(2, T_I G)^2)$ by the same argument as in proposition 8.5. But this map is also not perfect. In this context, it is meaningless to talk about the degree since $(2, T_I G)^2$ is too big.

There is another more complicated, but very interesting way to symmetrize the Gauss map. We can try to translate a tangent space from the both sides simultaneously. By this we mean something of the form

$$A \longmapsto \sqrt{A^{-1}}(T_A V)\sqrt{A^{-1}}.$$

If such a map exists then the analogues of propositions **??** and 8.6 are true for that. In this case, we would have seen the miraculous splitting of the natural properties of the Gauss map [Mik00].

Immediately arises the question of whether you can do something meaningful with that thing (like in [Mik00, Kap91]). We saw many times in this paper that there is nothing special with taking a square root of a Hermitian matrix. One can nicely generalize this for normal matrices. But for a general non-singular matrix it is not possible to tell something relevant about its square root. It happens that for a fixed matrix there can be an infinite family of roots and no way to choose one canonically (or better to say there some good non-global choices).

Chapter 9

Tropical sandpiles

The remaining part of this thesis is occupied by our joint work with Nikita Kalinin [KS15b], it is dedicated to the proof of the scaling limit theorems. Be aware that the terminology doesn't always match with the first part of the thesis. A big effort was made to do things formally and develop a precise mathematical foundation for the intuitive proof explained in Section 5.3. Things like relaxations, waves and sandpiles on infinite grafs are explained in Appendix B. Tropical analytic curves and series are build in Appendix C.

Before we were refering only to compact convex domains. The most general statement of the scaling limit theorem presented here drops compactness exluding only the three cases: when Ω is the whole plane, when it is a half plane with irrational slope or a stripe between two parallel lines with irrational slope. So, in principle we could deal with sandpiles (C_Ω and all other lattice invariants) on an unbounded component of the complemnet to a logarithmic amoeba of an algebraic curve.

TROPICAL CURVES IN SANDPILE MODELS

NIKITA KALININ, MIKHAIL SHKOLNIKOV

ABSTRACT. A state of a sandpile is a configuration of grains at the vertices of a subgraph Γ of $\mathbb{Z}^2$. A vertex with at least four grains is called unstable and can topple by sending one grain to each of its four neighbors; sand falling outside Γ disappears. A relaxation is doing topplings while it is possible.

It was experimentally observed by S. Caracciolo, G. Paoletti, and A. Sportiello that the result of the relaxation of a small perturbation of the maximal stable state on Γ contains a clearly visible thin balanced graph in its deviation set.

In this paper we rigorously formulate (using a certain scaling limit procedure) this fact and prove it. For that, a theory of tropical analytic series was developed. To understand the relaxation dynamic we prove that sending waves in the sandpile model corresponds in the limit to an operator on the set of tropical series. To study the local questions we developed a theory of smoothings of discrete superharmonic functions.

1. INTRODUCTION

1.1. **Sandpiles.** Let h be a small positive real number, $\mathrm{h}\mathbb{Z}^2 = \{(i\mathrm{h}, j\mathrm{h}) | i, j \in \mathbb{Z}\}$. Each two points $z, z' \in \mathrm{h}\mathbb{Z}^2$ with $|z - z'| = \mathrm{h}$ are connected by edge (we write it as $z \sim z'$), thus $\mathrm{h}\mathbb{Z}^2$ is a graph whose all vertices have valency four. Let Γ be a finite subgraph of $\mathrm{h}\mathbb{Z}^2$ and $\partial\Gamma$ be the set of the vertices of Γ, that have neighbors in $\mathrm{h}\mathbb{Z}^2 \setminus \Gamma$.

A *state* ϕ of the sandpile model on Γ is a function $\phi : \Gamma \setminus \partial\Gamma \to \mathbb{Z}_{\geq 0}$ on vertices of Γ. We interpret $\phi(z)$ as the number of grains of sand in $z \in \mathrm{h}\mathbb{Z}^2$. A vertex $z_0 \in \Gamma \setminus \partial\Gamma$ is called *unstable* if $\phi(z_0) \geq 4$, and in this case z_0 can *topple*, producing a new state $\phi' : \Gamma \to \mathbb{Z}_{\geq 0}$ by the following local rule changing only z_0 and its neighbors:

$$\phi'(z) = \begin{cases} \phi(z) - 4 & \text{if } z = z_0; \\ \phi(z) + 1 & \text{if } z \sim z_0; \\ \phi(z) & \text{otherwise.} \end{cases}$$

A *relaxation* is doing topplings at unstable vertices while it is possible. Note that it is not allowed to topple the vertices in $\partial\Gamma$ by definition, this guarantees that any relaxation eventually terminates. We denote by ϕ° the result of a relaxation of ϕ. It is a classical fact that ϕ° does not depend on the relaxation and is uniquely determined by ϕ.

1.2. **A perturbation of the maximal stable state.** Denote by $\langle 3 \rangle$ the *maximal stable state*, the state which has exactly three grains at every vertex of $\Gamma \setminus \partial\Gamma$. Let $\mathbf{p}_1, \ldots \mathbf{p}_k \in \Gamma \setminus \partial\Gamma$.

S. Caracciolo, G. Paoletti, and A. Sportiello proposed to look at the result of the relaxation of a small perturbation of $\langle 3 \rangle$, namely, of the state ϕ_{h} obtained from $\langle 3 \rangle$ by adding one grain to each of $\mathbf{p}_i$. They also made the following observation, which we illustrate on a small concrete example, see Figure 1.

Let Γ_{h} contains all the vertices of $\mathrm{h}\mathbb{Z}^2$ inside the triangle Ω given by three lines

$$x - y = 0, 4x + y = 30, x + 4y = 120.$$

Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ be the points $(7, 22), (12, 20), (12, 16), k = 3$. Figure 1 shows the results of the relaxation of $\phi_{\mathrm{h}} = \langle 3 \rangle + \sum_{i=1}^{k} \delta_{\mathbf{p}_i}$ for each $\mathrm{h} = 1/N$ where $N = 1, 2, 4, 8$.

Such pictures firstly appeared in [3], and the striking fact that ϕ_{h}° is equal to three at most of vertices of Γ_{h} motivates Definition 3.16: the *deviation locus* of a stable state ψ is the set $\{\psi \neq 3\}$. It was

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observed by the authors of [3] that the deviation locus in Figure 1 looks *balanced*: at every vertex of this graph the sum of outgoing primitive vectors in the directions of the edges is zero (see Figure 11).

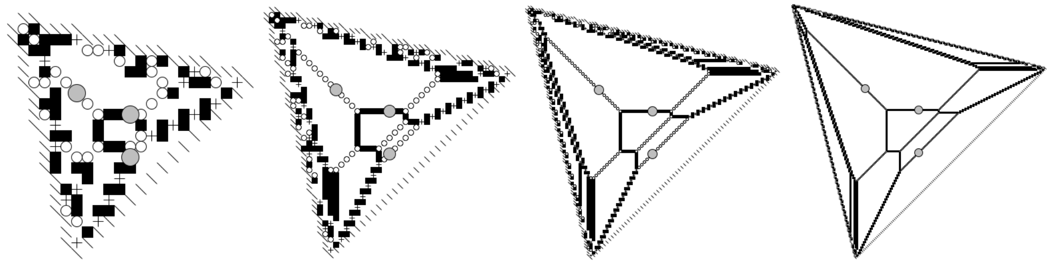

FIGURE 1. The evidence for a thin balanced graph as a deviation set of a sandpile. White corresponds to three grains, black to one, circles for two, crosses to zero, skew lines are the boundary vertices. Grey rounds represent the positions of added grains.

1.3. **Our main results.** Following a suggestion of A. Sportiello, we prove the following: if graphs Γ_{h} are obtained as the intersection of a convex figure Ω with $\mathrm{h}\mathbb{Z}^2$ and we consider the states ϕ_{h} as above, then the deviation locus of ϕ_{h}° tends to a balanced graph as $\mathrm{h} \to 0$. Such graphs are known as *tropical curves*, [18].

Precise formulations can be found in Section 3. In particular, Ω can be unbounded (but must be admissible, see Definition 3.1). The corresponding theory for sandpiles on infinite graphs is presented in Appendix B, though it is absolutely parallel to the theory on finite graphs. For another short statement of our results the reader may read the announce, [10].

1.4. **Application of our methods: fractals and patterns in sandpile.** The sandpile on $\mathbb{Z}^2$ exhibits a fractal structure; see, for example, the pictures of the identity element in the critical group [17]. As far as we know, only a few cases have a rigorous explanation. It was first observed in [21] that if we rescale by $\sqrt{n}$ the result of the relaxation of the state with n grains at $(0,0)$ and zero grains elsewhere, it weakly converges as $n \to \infty$. Than this was studied in [16] and was finally proven in [23]. However the fractal-like pieces of the limit found their explanation later, in [14, 15] and happen to be curiously related to Apollonian circle packing.

Periodic patterns in sandpiles were discovered by S. Caracciolo, G. Paoletti, and A. Sportiello in a pioneer work [3], see also Section 4.3 of [4] and Figure 3.1 in [22]. Experimental evidence suggests that these patterns carry a number of remarkable properties: in particular, they are self-reproducing under the action of waves. That is why we call these patterns *solitons*. Solitons naturally appear during relaxations on convex domains. In Figure 1 on the first two pictures we see these patterns for the directions $(1,0),(1,1),(1,2)$. A pattern for directions $(-1,3),(3,-1)$ can be seen on the third picture (it is represented by two edges at the top left corner). Solitons on pictures correspond to edges of the limiting tropical curve.

The fact that the solitons appear as "smoothings" of piece-wise linear functions was predicted in [26]. We provide a definition of the *smoothing* procedure. We will prove the existence (and uniqueness modulo translation) of solitons for all rational slopes and provide certain estimates on their shape.

We construct *triads* — three solitons meeting at a point — and triads satisfy similar properties. We show that the deviation locus of ϕ_{h}° is essentially glued out of solitons and triads along the graph, which can be constructed via certain dynamic on the set of tropical series.

The papers in experimental physics often lack rigorous proofs and concentrate more on discovering new models and effects. So, we contribute to this area by providing a rigorous formulation and a proof of a particular case of the above mysteriously regular behavior.

In particular, we prove well known (experimentally) fact that these patterns move changeless under the action of waves. A lot of work is to be done in future. Another thesis, [25](see also [26]), contains a lot of pictures and examples with apparent piece-wise linear behavior. We expect that the methods of this article will be used to study the fractal structure in those cases.

1.5. **About proofs.** We developed two completely new technics which independently are also interesting: $G_{\mathbf{p}}$-*dynamics* on the space of tropical series and *smoothing* of discrete superharmonic functions.

Here we present the main ingredients of the proof, avoiding all the technicalities. We propose the reader to concentrate on what seems to be right in the following, because most of claims will be formally wrong. Otherwise we recommend to skip this section. **Boldfaced** terms are supposed to reveal the parallels in the words of sandpiles and tropical series (certain piece-wise linear functions).

The *toppling function* $H_\phi(v)$ of $\phi = 3 + \sum \delta_{\mathbf{p}}$ measures how many topplings at a vertex v happened during a relaxation of ϕ. Knowing H_ϕ means knowing the result of the relaxation, since $\phi^\circ = \phi + \Delta H_\phi$. Here Δ is the discrete laplacian operator and $\Delta H_\phi(v)$ measures how many grains come to v minus how many grains leave v during the relaxation process. Sandpiles are "lazy" in the sense that H_ϕ is the **pointwise minimal** such function H that $\phi + \Delta H \leq 3$. We prove that in our setup the toppling function is **concave**.

Consider a **concave piece-wise linear** function F, which looks like $ix + jy + a_{ij}$ on the linear patches, such that P (the set of points $\mathbf{p}$) belongs to the corner locus of F. Note that ΔF is zero almost everywhere, $\Delta F < 0$ at P and the **deviation** locus $\{3 + \sum \delta_{\mathbf{p}} + \Delta F \neq 3\}$ is the **corner** locus of F, which, in turn, is a balanced graph passing through P.

Since $3 + \sum \delta_{\mathbf{p}} + \Delta F < 0$ at P, we understood that the **pointwise minimum** F_P of all such functions F is an **upper bound** for H_ϕ. Note that $ix + jy + a_{ij}$ should have an **integer slope** (i.e. $i, j \in \mathbb{Z}$) because the toppling function is integer-valued.

Let ψ be a stable state. Adding a grain to a point $\mathbf{p}$ and a relaxation may be **decomposed** as follows: we add a grain to $\mathbf{p}$, perform a toppling at $\mathbf{p}$, prohibit doing topplings at $\mathbf{p}$ for a while, and relax what can be relaxed (this is called *a wave*). One can prove that we will have at most one toppling at every vertex during this process. Then we unfreeze $\mathbf{p}$, topple it, freeze $\mathbf{p}$ again, and relax what can be relaxed (this gives the second wave). We repeat this until $\mathbf{p}$ does not want to topple after unfreezing, i.e. until the deviation locus **reaches** the point $\mathbf{p}$.

A relaxation of $\phi = 3 + \sum \delta_{\mathbf{p}}$ may be decomposed into *waves* from P. Let $\psi = 3 + \Delta G$ where G is a piece-wise linear function with integer slopes. Let $\mathbf{p}$ belongs to a region of linearity where $G = i_0 x + j_0 y + a_{i_0 j_0}$. We prove that sending a wave from $\mathbf{p}$ in ψ produces the state $3 + \Delta G'$ where G' differs from G just by $a'_{i_0 j_0} = a_{i_0 j_0} + 1$.

This motivated us to introduce the operator $G_{\mathbf{p}}$ (Figure 7): it acts on a piecewise linear function, increasing one of its coefficients a_{ij} until the corner locus of the function **reaches** the point $\mathbf{p}$. Finally, the fact that F_P can be **decomposed** as $F_P = \lim_{n\to\infty} \left(\prod_{\mathbf{p}\in P} G_{\mathbf{p}}\right)^n G$ for $G \equiv 0$ implies that F_P is a **lower bound** for H_ϕ.

1.6. **Where to find proofs of previously announced results.** In [10] we announced several theorems which are proven in this paper. Here we list where to look for the proofs. Theorem 1, in [10], is Theorem 1 here. Theorem 2 in [10] is proven in Section 7. Theorem 3 in [10] easily follows from Theorem 1, and will be proven in [9] to keep this paper shorter. Theorem 4 in [10] follows from Theorem 1 here just because of the definition of the function $f_{\Omega,P}$ (see Definition 3.7). We prove Theorem 5 in [10] on the way of the proof of Theorem 1, see Remark 3.24.

1.7. **Strategies of reading and the structure of this paper.** For the convenience of the reader, for many statements we mention where they are cited inside this paper. We encourage to follow these links. We moved more or less standard fact to Appendices. The reader can also choose different strategies of reading, depending on goals.

- For readers, not familiar with tropical geometry, we propose to read Appendix C.1 (basic facts about tropical geometry) first, and follow the references therein in case of interest for algebraic geometry aspects of tropical geometry.
- For the reader, somehow familiar with tropical geometry, we recommend to read entire Section 3 (main definitions, the main theorm), then entire Section 5 (proof of the lower estimate). Then we recommend to read the definitions in Appendices, and Section 4.2 (the dynamic of operators $G_{\mathbf{p}}$ on tropical series). We spear about patterns in Section 6. Then follows the discussion in Section 8.

- For the readers interested in proofs mainly. The combinatorial core of the paper is Section 6 where we study superharmonic "smoothings" of piecewise linear integer-valued functions. Then it is natural to reread Sections 4,5, which provide the lower bound. The upper bound is easier, see Section 3.

1.8. **Acknowledgments.** We thank Andrea Sportiello for sharing his insights on perturbative regimes of the Abelian sandpile model which was the starting point of our work. We also thank Grigory Mikhalkin, who encouraged us and gave a lot of advice about this paper.

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Contents

2. Glossary of symbols

We need considerable notation in this paper. For easy reference we list most of it here.

Domains.

$\Omega \subset \mathbb{R}^2$ is any admissible (see Definition 3.1) closed convex domain, Ω° is its interior.
(x, y) is a point of Ω.
P is a finite subset of Ω°, $P = \bigcup_{i=1}^n \{\mathbf{p}_i\}, \mathbf{p}_i \in \Omega^\circ$.
U, W are open subsets of $\mathbb{R}^2$.
Δ is a $\mathbb{Q}$-polygon (Definition 3.19).
V denotes a vertex of the polygon Δ.
S denotes a side of Δ.
$S(\Delta)$ is the set of all sides of Δ.
$B_r(A)$ is the r-neighborhood of a set $A \subset \mathbb{R}^2$, i.e. the closure of the union of the balls with radius r and center in A.
O stands for the point $(0, 0) \in \mathbb{R}^2$.

Tropical objects.
$f_{\Omega,P}$ is a certain Ω-tropical series associated with Ω and a finite collection $P \subset \Omega$ of points (Definition 3.7).
f, g are tropical series or tropical polynomials.
$C(f)$ is the tropical curve defined by a tropical series (or polynomial) f.
0_Ω is the function $f \equiv 0$ on Ω.
$(i, j) \in \mathbb{Z}^2$ parametrize monomials $ix + jy + a_{ij}$ and occasionally serve for indices.
Φ is a face of a tropical curve (see also Proposition C.3).
E is an edge of a tropical curve.
$d(\cdot)$ is the duality between faces (edges, vertices) of a tropical curve and vertices (resp., edges, faces) of the dual subdivision of the Newton polygon (Proposition C.3).
$m_f : S(\Delta) \to \mathbb{Z}_{>0}$ is the quasi-degree (Definition 4.18) of a Δ-tropical polynomial f.
$\text{Add}_{kl}^c f$ increase by c the coefficient a_{kl} in the (3.5), see Definition 4.33.
$l_\Omega(x, y) : \Omega \to \mathbb{R}_{\geq 0}$ is the tropical weighted distance (from the boundary) function, Definition 4.3.
G_P is an operator on tropical series, Definition 4.8. If P is just one point $\mathbf{p}$ we write $G_\mathbf{p}$.

Lattice graphs.
h is the mesh of the lattice, a positive number small enough.
$\Gamma_\mathrm{h} = \Omega^\circ \cap \mathrm{h}\mathbb{Z}^2$ is the underlying graph for sandpiles.
z stands for a vertex of Γ_h or for a point in Ω.
$z' \sim z$ means that z' is a neighbor of $z \in \Gamma_\mathrm{h}$, $\gamma(z)$ is the set of neighbors of $z \in \mathrm{h}\mathbb{Z}^2$.
$\partial\Gamma_\mathrm{h}$ is the boundary of Γ, i.e $\partial\Gamma_\mathrm{h} = \{z \in \Gamma_\mathrm{h} || \gamma(z) \cap \Gamma_\mathrm{h}| < 4\}$. We never do topplings at the vertices of $\partial\Gamma_\mathrm{h}$, in other words, $\partial\Gamma_\mathrm{h}$ is the set of sinks of our sandpile.
F, G, H are integer-valued functions on graphs Γ_h or $\mathbb{Z}^2$.
$S_n(F)$ is the n-smoothing of F (Definition 5.6).
$\Delta F(z)$ is the discrete Laplacian of a function $F : \Gamma_\mathrm{h} \to \mathbb{Z}$ at a point z, $\Delta F(z) = -4F(z) + \sum_{z' \sim z} F(z')$. Note that ΔF is defined on $\Gamma_\mathrm{h} \setminus \partial\Gamma_\mathrm{h}$, and for $z \in \Gamma_\mathrm{h} \setminus \partial\Gamma_\mathrm{h}$ we always have $|\gamma(z)| = 4$.
$D(F) = \{z \in \Gamma | \Delta F(z) \neq 0\}$, the non-harmonicity locus of F.
v, w are points of $\mathbb{Z}^2$.

Sand objects.
$H_\phi(z)$ is the value of the toppling function of a state ϕ at a point z.
$H_\phi^v(z)$ the value of the toppling function of one wave from v for ϕ at a point z.
$P^\mathrm{h} = \{\mathbf{p}^\mathrm{h}\}_{\mathbf{p} \in P}$ is the set of roundings of the points $\mathbf{p} \in P$ with respect to Γ_h.
$[a]$ is the maximal integer number, such that $[a] \leq a$.
$\langle 3 \rangle$ is the state with 3 grains at every vertex.
$D(\psi)$ is the deviation locus (Definition 3.16) of a state ψ, i.e. the set of vertices where $\psi \neq 3$.

$\mathrm{Smooth}_{\mathrm{h}}(f)$ (Definition 5.14) is the smoothing of a nice (Definition 4.30) Δ-tropical Laurent polynomial f on Γ_{h}.

Other.

δ_v is the function which is equal to one at v and zero otherwise.

$\{X\}$ for some property X is the set where X holds. For example, if f, g are two functions, then $\{f \neq g\}$ is the set of points x where $f(x) \neq g(x)$. Similarly, $\chi(X)$ is the function which is equal to 1 at the points where X holds.

$p_1, p_2, q_1, q_2 \in \mathbb{Z}$ are such numbers that $p_1 q_2 - p_2 q_1 = 1$.

A sand pattern on a plane, corresponding to a tropical edge, is *a soliton.*

A pattern corresponding to a trivalent tropical vertex is a *triad.*

Constants are denoted by $\mathrm{C}, \mathrm{C}_1, \mathrm{C}_2$, etc.

3. The main theorem

Definition 3.1. (Used on pages [2,6,9,10,10]) A convex closed subset $\Omega \subset \mathbb{R}^2$ is said to be *not admissible* if one of the following cases takes place:

- Ω has empty interior Ω° (i.e. Ω is a subset of a line),
- Ω is $\mathbb{R}^2$,
- Ω is a half-plane with the boundary of irrational slope,
- Ω is a strip between two parallel lines of irrational slope.

From now on we always suppose that Ω is an **admissible** convex closed subset of $\mathbb{R}^2$.

Definition 3.2. (Used on pages [41,41,41]) An Ω-*tropical series* is a function $f : \Omega \to \mathbb{R}_{\geq 0}$, $f|_{\partial\Omega} = 0$, such that

$$f(x, y) = \inf_{(i,j)\in\mathcal{A}} (a_{ij} + ix + jy), a_{ij} \in \mathbb{R},$$

and $\mathcal{A} \subset \mathbb{Z}^2$ is not necessary finite. An Ω-*tropical analytic curve* $C(f)$ on Ω° is the corner locus (i.e. the set of non-smooth points) of an Ω-tropical series on Ω°.

The reason to consider only admissible sets is Proposition 4.2: there is no Ω-tropical analytic curves on non-admissible Ω.

Question. An Ω-tropical series can be thought of an analog of a series $w_t(x, y) = \sum_{(i,j)\in\mathcal{A}_\Omega} t^{a_{ij}} x^i y^j$ with $t \in \mathbb{R}_{>0}$ very small. Is is true that Ω is the limit of the images of the region of convergence of w_t under the map $\log : (x, y) \to (\log_t |x|, \log_t |y|)$, and the corresponding Ω-tropical analytic curve is the limit of the images of $\{w_t(x, y) = 0\}$ under $\log_t |\cdot|$ when $t \to 0$?

Lemma 3.3 (The proof is on page 41)**.** (Used on pages [41]) In the definition of an Ω-tropical series f we can replace "inf" by "min", i.e. at every point $(x, y) \in \Omega^\circ$ we have

$$\inf_{(i,j)\in\mathcal{A}} (a_{ij} + ix + jy) = \min_{(i,j)\in\mathcal{A}} (a_{ij} + ix + jy).\textbf{verified}$$

At a point on $\partial\Omega$ where there is no tangent line with a rational slope we actually have to take the infimum, cf. Lemma 4.5.

Note that an Ω-tropical series $f : \Omega \to \mathbb{R}$ always has different presentations as the minimum of linear functions. For example, if Ω is the square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, then $\min(x, 1 - x, y, 1 - y, 1/3)$ equals at every point of Ω to $\min(x, 1 - x, y, 1 - y, 1/3, 2x, 5 - 2x)$.

Definition 3.4. (Used on pages [16,39]) To resolve this ambiguity, we suppose that, in Ω°, a tropical series f is always (if the opposite is not stated explicitly) given by

$$\text{(Used on pages [6,7,16,17,37])}\; f(x, y) = \min_{(i,j)\in\mathcal{A}} (a_{ij} + ix + jy) \tag{3.5}$$

with maximal by inclusion $\mathcal{A}$ and with as minimal as possible coefficients a_{ij}. We call this presentation *the canonical form* of a tropical series. For each Ω-tropical series there exists a unique canonical form.

Example 3.6. The canonical form of $\min(x, 1 - x, y, 1 - y, 1/3)$ on $\Omega = [0, 1] \times [0, 1]$ is $f(x, y)$ as in (3.5) with $\mathcal{A} = \mathbb{Z}^2$, $a_{00} = 1/3$ and $a_{ij} = -\min_{(x,y)\in\Omega}(ix + jy)$ for $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

Proof. It is easy to check that $f(x,y) = \min(x, 1-x, y, 1-y, 1/3)$ on Ω. The set of monomials $\mathcal{A} = \mathbb{Z}^2$ is maximal by inclusion. All the coefficients $a_{ij}, (i,j) \neq (0,0)$ are chosen as minimal with the condition that $ix + jy + a_{ij}$ is non-negative on Ω. Finally, in the canonical form of $\min(x, 1-x, y, 1-y, 1/3)$ the coefficient a_{00} can not be less than $1/3$. $\square$

Definition 3.7. (Used on pages [3,6,11]) Let $\mathbf{p}_1, \dots \mathbf{p}_n \in \Omega^\circ$ be different points, $P = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$. We denote by $f_{\Omega,P}$ the pointwise minimum among all Ω-tropical series non-smooth at all the points $\mathbf{p}_1, \dots, \mathbf{p}_n$.

Lemma 3.8 (The proof is on page 12)**.** (Used on pages [11,13]) The function $f_{\Omega,P}$ is an Ω-tropical series.**verified**

Definition 3.9. Consider the lattice $\mathrm{h}\mathbb{Z}^2$ with the mesh $\mathrm{h} > 0$ and define $\Gamma_\mathrm{h} = \mathrm{h}\mathbb{Z}^2 \cap \Omega^\circ$, Let $\gamma(z)$ denote the set of all four neighbors of z in $\mathrm{h}\mathbb{Z}^2$ and let $\partial\Gamma_\mathrm{h}$ be the set of vertices of Γ_h which have a neighbor vertex outside Ω°. We prohibit making topplings at the vertices in $\partial\Gamma$ (or, equivalently, we think of them as sinks).

Definition 3.10. We say that $\mathbf{p}^\mathrm{h} \in \mathrm{h}\mathbb{Z}^2$ is **a** rounding of a point $\mathbf{p} \in \mathbb{R}^2$ with respect to $\mathrm{h}\mathbb{Z}^2$ if the distance between $\mathbf{p}$ and $\mathbf{p}^\mathrm{h}$ is less than h.

Definition 3.11. (Used on pages [36]) The set of roundings $P^\mathrm{h} = \{\mathbf{p}^\mathrm{h} | \mathbf{p} \in P\}$ for the set of points P is called *proper* if the function

$$F : \mathrm{h}\mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}, F(z) = [\mathrm{h}^{-1} f_{\Omega,P}(z)] \in \mathbb{Z}, z \in \Gamma_\mathrm{h}$$

has negative discrete Laplacian at all points $\mathbf{p}^\mathrm{h}$. Here $[\cdot]$ stands for the usual integer part of a positive number.

Proposition 3.12 (The proof is on page 37)**.** (Used on pages [36,37,37]) For each finite subset P of Ω° there exists a set P^h of proper roundings. **verified**

In fact, the set $P^\mathrm{h} = \{\mathbf{p}^\mathrm{h}\}$ of proper roundings depends on Ω and P, so we should write $\mathbf{p}^\mathrm{h}_{\Omega,P}$ for each point $\mathbf{p}$. Nevertheless, for a fixed h small enough this rounding $\mathbf{p}^\mathrm{h}$ of $\mathbf{p} \in P$ depends only on the behavior of $C(f_{\Omega,P})$ in a small neighborhood of $\mathbf{p}$.

Proposition 3.13 (The proof is on page 37)**.** (Used on pages [9,9,37]) If $\mathbf{p} \in P \cap P', \mathbf{p} \in \Omega \cap \Omega'$ and $C(f_{\Omega,P})$ coincides with $C(f_{\Omega',P'})$ in a neighborhood of $\mathbf{p}$, then $\mathbf{p}^\mathrm{h}_{\Omega,P} = \mathbf{p}^\mathrm{h}_{\Omega',P'}$. **verified**

Let $P^\mathrm{h} = \{\mathbf{p}^\mathrm{h} | \mathbf{p} \in P\}$ be a set of proper roundings of points in P with respect to the lattice $\mathrm{h}\mathbb{Z}^2$. Consider the state ϕ_h of a sandpile on Γ_h defined as

$$\text{(Used on pages [9,37])}\ \phi_\mathrm{h} = \langle 3 \rangle + \sum_{\mathbf{p} \in P} \delta(\mathbf{p}^\mathrm{h}). \tag{3.14}$$

Proposition 3.15 (The proof is on page 37)**.** (Used on pages [9,37]) The function

$$F(z) = [\mathrm{h}^{-1} f_{\Omega,P}(z)], z \in \Gamma_\mathrm{h}$$

bounds from above the toppling function H_{ϕ_h}(see Definitions B.5, B.15) of ϕ_h. **verified**

In particular, this implies that ϕ_h is relaxable (Definition B.11).

Definition 3.16. (Used on pages [1,6]) For a state ψ on Γ_h, its *deviation locus* $D(\psi)$ is

$$D(\psi) = \{z \in \Gamma_\mathrm{h} | \psi(z) \neq 3\}.$$

Our main result is the following theorem.

Theorem 1. (Used on pages [3,3,3,3,8,9,23,30,31,38]) [The proof is on page 9] The family of deviation sets $D(\phi^\circ_\mathrm{h})$ converges (by Hausdorff, on compact sets in Ω°) to the tropical curve $C(f_{\Omega,P})$ as $\mathrm{h} \to 0$. **verified**

The ambiguity with roundings is justified as follows. The corresponding $(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n) \to f_{\Omega,P}$ is continuous for a generic set P of points, and, in this case, Theorem 1 holds for any roundings of points in P. But if P belongs to the discriminant set of configurations, then ϕ_{h}° does not depend in any sense continuously on the points where we drop additional sand; the susceptibility of a sandpile is very big. For different roundings of $\mathbf{p}_1, \dots, \mathbf{p}_n$ we **can** obtain drastically different pictures of ϕ_{h}°.

On the other hand, in a certain setting a rounding is not necessary at all, as it was [10], Section 1.2.

Proposition 3.17 (The proof is on page 37)**.** If $P \subset \mathbb{Z}^2$, Ω is lattice polygon, and $\mathrm{h}^{-1} \in \mathbb{N}$, then we can take the proper roundings $\mathbf{p}^{\mathrm{h}} = \mathbf{p}$ for each $\mathbf{p} \in P$.

3.1. **The scheme of the proof of the main theorem.**

Lemma 3.18 (The proof is on page 37)**.** (Used on pages [10,12,31,37,37]) If the toppling function H_ψ of a state ψ on Γ_{h} is bounded by a constant $\mathrm{C} > 0$, then

$$D(\psi^\circ) \subset B_{\mathrm{Ch}}(D(\psi) \cup \partial\Omega).\textbf{verified}$$

Definition 3.19. (Used on pages [6]) Let $\Delta \subset \mathbb{R}^2$ be a finite intersection of half-planes (at least one) with rational slopes. We call Δ a $\mathbb{Q}$-*polygon* if it is a closed set with non-empty interior.

Note that a $\mathbb{Q}$-polygon is not necessary compact. It is easy to verify that a $\mathbb{Q}$-polygon is admissible (Definition 3.1). The next lemma provides us with compact $\mathbb{Q}$-polygons which exhaust Ω.

Lemma 3.20 (The proof is on page 14)**.** (Used on pages [4,4,9,9,9,12,14,14]) For any compact set $K \subset \Omega^\circ$ such that $P \subset K$ and for any $\varepsilon > 0$ small enough there exists a $\mathbb{Q}$-polygon $\Omega_{\varepsilon,K} \subset \Omega$ such that $B_{3\varepsilon}(K) \subset \Omega_{\varepsilon,K}$ and the following holds:

$$f_{\Omega,P} = f_{\Omega_{\varepsilon,K},P} + \varepsilon \text{ on } B_{3\varepsilon}(K).$$

Note that $f_{\Omega_\varepsilon,P} \le f_{\Omega,P}$ automatically. If Ω is a compact set, then $\Omega_{\varepsilon,K}$ is $\{f_{\Omega,P} \ge \varepsilon\}$ for ε small enough. For non-compact Ω we additionally cut the set $\{f_{\Omega,P} \ge \varepsilon\}$ far enough from K.

Corollary 3.21. (Used on pages [9,9]) Lemma 3.20 implies that for $\varepsilon > 0$ small enough the tropical curves defined by $f_{\Omega,P}$ and $f_{\Omega_{\varepsilon,K},P}$ coincide on K, i.e.

$$C(f_{\Omega,P}) \cap K = C(f_{\Omega_{\varepsilon,K},P}) \cap K$$

therefore by Proposition 3.13 the proper roundings P^{h} of P for Ω and $\Omega_{\varepsilon,K}$ coincide for any $\mathrm{h}, \varepsilon > 0$ small enough.

Proposition 3.22. (The proof is on page 22) (Used on pages [4,4,9,20,22,22]) Suppose that $\Delta \subset \mathbb{R}^2$ is a $\mathbb{Q}$-polygon. Choose any $\varepsilon > 0$. Then, the toppling functions H of the states ϕ_{h} (see (3.14)) satisfy

$$\mathrm{h}H(z) > f_{\Delta,P}(z) - \varepsilon \text{ for all } z \in \Gamma_{\mathrm{h}} \cap \Delta$$

for all $\mathrm{h} > 0$ small enough. Furthermore, the sets $D(\phi_{\mathrm{h}}^\circ) \setminus B_\varepsilon(\partial\Delta)$ are ε-close to $C(f_{\Delta,P})$.**verified**

In other words, the deviation sets converge to the tropical curve everywhere outside an arbitrary small neighborhood of the boundary of Δ. This is the core statement in this paper, the proof of the main theorem heavily relies on it.

Proof of Theorem 1. Consider a compact set $K \subset \Omega^\circ$, such that $P \subset K$, and $\varepsilon > 0$ small enough. Choose $\Delta = \Omega_{\varepsilon,K}$ by Lemma 3.20. Denote by $\Gamma_{\mathrm{h}}^\varepsilon$ the set given by

$$\Gamma_{\mathrm{h}}^\varepsilon = \{z \in \Omega \cap \mathrm{h}\mathbb{Z}^2 | \gamma(z) \subset \Delta^\circ\}.$$

We consider the state $\phi_{\mathrm{h}}^\varepsilon = \langle 3 \rangle + \sum_{\mathbf{p} \in P} \delta(\mathbf{p}^{\mathrm{h}})$ on $\Gamma_{\mathrm{h}}^\varepsilon$. Note that the roundings $\mathbf{p}^{\mathrm{h}}$ of points $\mathbf{p} \in P$ in Δ are the same as for Ω by Corollary 3.21 and Proposition 3.13.

Note that $(\phi_{\mathrm{h}}^\varepsilon)^\circ$ can be thought of a partial relaxation (Definition B.26) of ϕ_{h}. Denote by $H_1 = H_{\phi_{\mathrm{h}}}$ the toppling function of ϕ_{h} (on Γ_{h}) and by $H_2 = H_{\phi_{\mathrm{h}}^\varepsilon}$ the toppling function of $\phi_{\mathrm{h}}^\varepsilon$ (on $\Gamma_{\mathrm{h}}^\varepsilon$). Therefore, $\mathrm{h}H_1 \ge \mathrm{h}H_2$ (Lemma B.27). Since Δ is a $\mathbb{Q}$-polygon, then, using Proposition 3.22 we can choose h small enough, such that $D((\phi_{\mathrm{h}}^\varepsilon)^\circ) \cap K$ is ε-close to $C(f_{\Delta,P}) \cap K = C(f_{\Omega,P}) \cap K$ (Corollary 3.21) and $\mathrm{h}H_2$ is ε-close to $f_{\Delta,P}$.

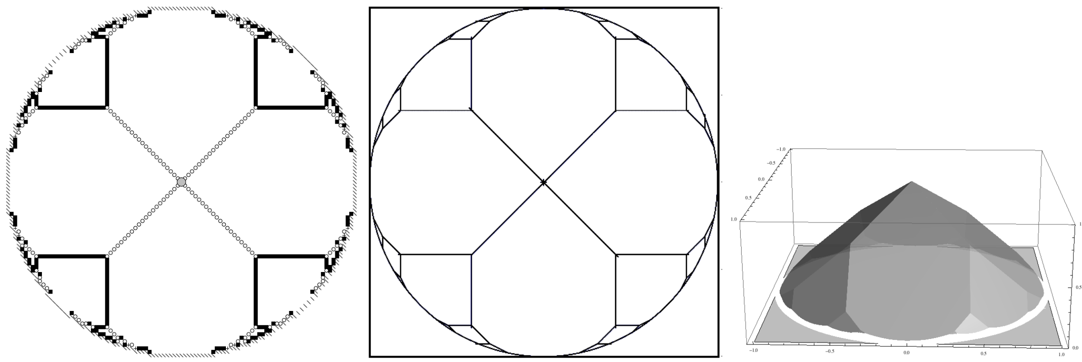


FIGURE 2. Additional grain of sand thrown to the center of $\Omega = \{x^2 + y^2 \le 1\}$. The sand picture for $\mathrm{h} = 1/100$ is on the left and the limit as $\mathrm{h} \to 0$ is in the center. On the right we see the limit of the toppling function. Note that the central picture shows the corner locus of the right picture which is l_Ω (Definition 4.3) for $\Omega = \{x^2 + y^2 \le 1\}$.

On $B_{3\varepsilon}$, combining the above arguments with Proposition 3.15 and Lemma 3.20 we obtain

$$f_{\Delta,P} - \varepsilon < \mathrm{h} H_2 \le \mathrm{h} H_1 \le \mathrm{h}[\mathrm{h}^{-1} f_{\Omega,P}] \le f_{\Omega,P} = f_{\Delta,P} + \varepsilon. \tag{3.23}$$

Therefore, by Lemma 3.18 we see that $D(\phi_{\mathrm{h}}^\circ)$ is 2ε-close to $D((\phi_{\mathrm{h}}^\varepsilon)^\circ) \cup \partial B_{3\varepsilon}(K)$ on $B_{3\varepsilon}(K)$. Therefore $D(\phi_{\mathrm{h}}^\circ) \cap K$ is 3ε-close to $C(f_{\Omega,P}) \cap K$. We proved that for each $\varepsilon, \mathrm{h} > 0$ small enough, $D(\phi_{\mathrm{h}}^\circ) \cap K$ is 3ε-close to $C(f_{\Omega,P}) \cap K$, which finishes the proof of the theorem. □

Remark 3.24. Note that (3.23) implies that $f_{\Omega,P} = \lim_{\mathrm{h}\to 0} \mathrm{h} H_{\phi_{\mathrm{h}}}$ on compact sets $K \subset \Omega^\circ$. This implies the assertion in previously announced Theorem 5 in [10].

4. Tropical tools

4.1. Tropical weighted distance function.

Definition 4.1. (Used on pages [10,14,15,41]) For $(i,j) \in \mathbb{Z}^2$ denote by c_{ij} the infimum of $ix + jy$ over $(x,y) \in \Omega$. Let $\mathcal{A}_\Omega$ be the set of pairs (i,j) with $c_{ij} \neq -\infty$. Note that if Ω is bounded, then $\mathcal{A}_\Omega = \mathbb{Z}^2$. For each $(i,j) \in \mathcal{A}_\Omega$ we define

$$l_\Omega^{ij}(x,y) = ix + jy - c_{ij}.$$

Note that l_Ω^{ij} is positive on Ω°. Also, $\mathcal{A}_\Omega$ always contains $(0,0)$.

Proposition 4.2 (The proof is on page 42)**.** (Used on pages [7,42]) A convex closed set Ω is admissible (Definition 3.1) if and only if $\mathcal{A}_\Omega \neq \{(0,0)\}$ and $\Omega^\circ \neq \varnothing$. **verified**

Definition 4.3. We use the notation of Definition 4.1. The *weighted distance function* l_Ω on Ω is defined by

$$l_\Omega(x,y) = \inf \left\{ l_\Omega^{ij}(x,y) | (i,j) \in \mathcal{A}_\Omega \setminus \{(0,0)\} \right\}.$$

An example of a tropical analytical curve defined by l_Ω is drawn on the right hand side of Figure 2.

Remark 4.4. (Used on pages [11]) If $f(x,y) = ix + jy + a_{ij}, (i,j) \in \mathcal{A} = \mathbb{Z}^2 \setminus \{(0,0)\}, a_{ij} \in \mathbb{R}$, $f|_\Omega \ge 0$, then $f \ge l_\Omega$ on Ω.

Lemma 4.5 (The proof is on page 42)**.** (Used on pages [7,42,42]) The function l_Ω is an Ω-tropical series.**verified**

4.2. **Operators $G_{\mathbf{p}}$.** Recall that Ω is admissible (Definition 3.1). Let $P = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ be a finite collection of points in Ω°. Let g be an Ω-tropical series.

Definition 4.6. (Used on pages) Denote by $V(\Omega, P, g)$ the set of Ω-tropical series f such that $f|_\Omega \geq g$ and f is not smooth at each of the points $\mathbf{p} \in P$.

Lemma 4.7. (Used on pages) The set $V(\Omega, P, g)$ is not empty. **verified**

Proof. Since Ω is admissible, l_Ω is well defined, and the function

$$g'(z) = g(z) + \sum_{\mathbf{p} \in P} \min(l_\Omega(z), l_\Omega(\mathbf{p}))$$

belongs to $V(\Omega, P, g)$. □

Clearly, if $f \geq g$ then $V(\Omega, P, f) \subset V(\Omega, P, g)$.

Definition 4.8. (Used on pages [6]) For a finite subset P of Ω° and an Ω-tropical series f we define an operator G_P, given by

$$G_P f(z) = \inf\{g(z) | g \in V(\Omega, P, f)\}.$$

If P contains only one point $\mathbf{p}$ we write $G_{\mathbf{p}}$ instead of $G_{\{\mathbf{p}\}}$.

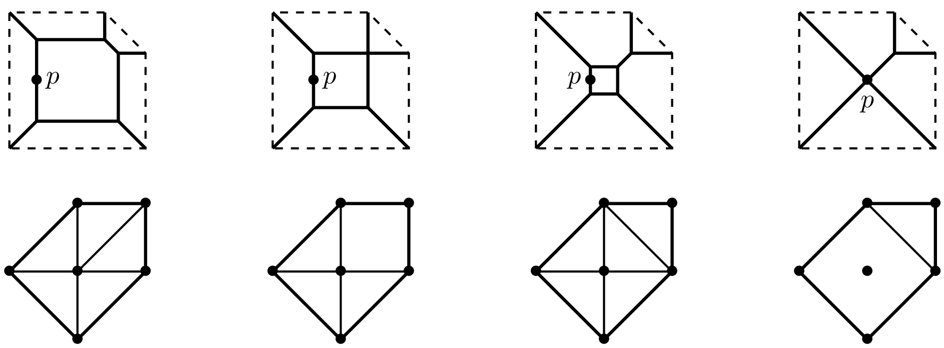


FIGURE 3. First row shows how curves given by $G_p 0_\Omega$ depend on the position of the point in the pentagon Ω. The second row shows a dual decomposition for their Newton polygon. Note that the coordinate axes of the second row are actually reversed as on Figure 12.

In Proposition 4.37 we will prove that G_P can be obtained as the limit of repetitive applications $G_{\mathbf{p}}$ for $\mathbf{p} \in P$ and that each individual $G_{\mathbf{p}}$ simply contracts a face of a tropical curve $C(g)$ such that $C(G_{\mathbf{p}} g)$ passes through $\mathbf{p}$, see Figure 7.

Lemma 4.9. For $\mathbf{p} \in \Omega^\circ$ we have $G_{\mathbf{p}} 0_\Omega(z) = \min(l_\Omega(z), l_\Omega(\mathbf{p}))$.

Proof. Indeed, all the coefficients, except a_{00}, in the canonical form of $G_{\mathbf{p}} 0_\Omega$ can not be less than in l_Ω by Remark 4.4, and if a_{00} were less than $l_\Omega(\mathbf{p})$, then the function would be smooth at $\mathbf{p}$. □

Proposition 4.10. (Used on pages [12,17,42]) For any $z \in \Omega$ and $P = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ the following inequality holds

$$G_P 0_\Omega \leq n \cdot l_\Omega(z).\textbf{verified}$$

Proof. For each point $\mathbf{p} \in P$ we consider the function $(G_{\mathbf{p}} 0_\Omega)(z) = \min(l_\Omega(z), l_\Omega(\mathbf{p}))$, which is not smooth at $\mathbf{p}$ and $(G_{\mathbf{p}} 0_\Omega)|_{\partial\Omega} = 0$. Finally,

$$f_{\Omega,P} \leq \sum_{\mathbf{p} \in P} G_{\mathbf{p}} 0_\Omega \leq n \cdot l_\Omega.$$

□

Lemma 4.11 (The proof is on page 42)**.** (Used on pages [11,42]) The function $G_P f$ is an Ω-tropical series. **verified**

Proof of Lemma 3.8. Note that $G_P 0_\Omega = f_{\Omega,P}(x)$ by the definition (Definition 3.7) of the latter. Applying Lemma 4.11 concludes the proof. □

Lemma 4.12. (Used on pages [17]) Let f_1 and f_2 be two tropical series on Ω° such that $f_1 \leq f_2$ and $P \subset \Omega^\circ$. Then $G_P f_1 \leq G_P f_2$.**verified**

Proof. Indeed, $G_P f_2 \geq f_2 \geq f_1$ and $G_P f_2$ is not smooth at P. Therefore, $G_P f_1 \leq G_P f_2$ by definition of $G_P f_1$. □

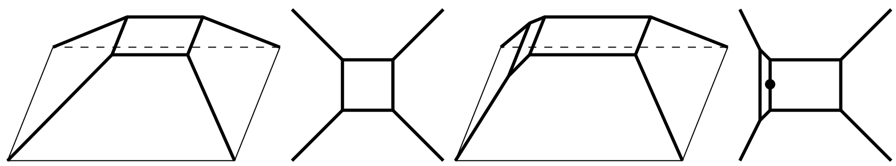


FIGURE 4. On the left: Ω-tropical series $\min(x, y, 1-x, 1-y, 1/3)$ and the corresponding tropical curve. On the right: the result of applying $G_{(\frac{1}{5},\frac{1}{2})}$ to the left picture. The new Ω-tropical series is $\min(x + \frac{2}{15}, y, 1 - x, 1 - y, \frac{1}{3})$ and the corresponding tropical curve is presented on the right. The fat point is $(\frac{1}{5}, \frac{1}{2})$.

Definition 4.13. (Used on pages [12,13,14,15,15]) We say that a tropical series f on Ω is presented in the *small canonical form* if f is written as

$$f(x, y) = \min_{(i,j)\in\mathcal{B}_f} (ix + jy + a_{ij}) \tag{4.14}$$

where all a_{ij} are taken from the canonical form and $\mathcal{B}_f \subset \mathcal{A}_\Omega$ consists of monomials $ix+jy+a_{ij}$ which are equal to f at at least one point in Ω°.

Example 4.15. The small canonical form for Example 3.6 is $\min(x, y, 1 - x, 1 - y, 1/3)$.

Remark 4.16. (Used on pages) Note that for a $\mathbb{Q}$-polygon Δ, the small canonical form of the function l_Δ is a Δ-tropical **polynomial**, i.e. it has only finite number of monomials. It follows from Proposition 4.10 that the small canonical form of $G_P f$ is a Δ-tropical polynomial too, for all Δ-tropical polynomials f.

Lemma 4.17 (Cf. Lemma 3.18)**.** (Used on pages [18,37,37]) Let $\varepsilon > 0, \mathcal{B} \subset \mathbb{Z}^2$ and f, g be two tropical series in Ω° written as

$$f(x, y) = \min_{(i,j)\in\mathcal{B}} (ix + jy + a_{ij}), g(x, y) = \min_{(i,j)\in\mathcal{B}} (ix + jy + a_{ij} + \delta_{ij}).$$

If $|\delta_{ij}| < \varepsilon$ for each $(i, j) \in \mathcal{B}$, then $C(f)$ is 2ε-close to $C(g)$.**verified**

Proof. Suppose that an edge E of $C(f)$ is given by $ix+jy+a_{ij} = i'x+j'y+a'_{ij}$ for some $(i, j), (i', j') \in \mathcal{B}$. The local equation of E is therefore $(i - i')x + (j - j')y + (a_{ij} - a'_{ij}) = 0$. For $C(g)$ this becomes $(i - i')x + (j - j')y + (a_{ij} - a'_{ij}) = (\delta'_{ij} - \delta_{ij})$. Since $i, i'j, j' \in \mathbb{Z}$, the zero set of new local equation of this edge is 2ε-close to E. In other words, when we change a coefficient a_{ij}, the edges of the face F, where $ix + jy + a_{ij}$ is the minimal monomial, move by at most ε. When we change coefficients in the neighboring faces, the edges of F move again by at most ε. Therefore ∂F moves by at most 2ε. If there are vertices of valency bigger than three on ∂F, their perturbations can produce new edges, but again in the ε-neighborhood of ∂F. Some new faces may appear, but, again, in ε-neighborhood of old vertices and edges. □

4.3. **Finite parts of tropical curves and a proof of Lemma 3.20.** Let us fix a $\mathbb{Q}$-polygon Δ. Consider a Δ-tropical polynomial f in the small canonical form (Definition 4.13). Let us analyze the behavior of f near the boundary.

In the neighborhood of each side S of Δ the function f can be locally written as $(x,y)\mapsto ix+jy+a_{ij}$, where the point $(i,j)\in \mathrm{Newt}(f)$ in the Newton polygon $\mathrm{Newt}(f)$ (Definition C.2) of f and the vector (i,j) is orthogonal to S. This integer vector (i,j) is a multiple of a certain primitive vector, i.e. $(i,j)=m(S)n(S)$, where $n(S)$ is *an inward primitive normal* vector to S of Δ and $m(S)\in\mathbb{Z}_{>0}$ is a number. Thus, we constructed the function m on the set $S(\Delta)$ of the sides of Δ, $m=m_f\colon S(\Delta)\to\mathbb{Z}_{>0}$.

Definition 4.18. (Used on pages [6,15]) The aforementioned function m_f is called the *quasi-degree* for the Δ-tropical curve C.

Remark 4.19. Note that $m_f(S)n(S)\in\mathrm{Newt}(f)$ for each $S\in S(\Delta)$. The convex hull of the set

$$\{m_f(S)n(S)\}_{S\in S(\Delta)}$$

coincides with $\mathrm{Newt}(f)$, since the monomials from the outside of this convex hull can not contribute to $f|_\Delta$.

Definition 4.20. (Used on pages [13,15,22]) A quasi-degree m_f is called *nice* if for each side $S\in S(\Delta)$ with $m_f(S)>1$ we have $m_f(S_1)=m_f(S_2)=1$ for the neighboring sides S_1,S_2 of S.

Lemma 4.21. (Used on pages [13,13,22]) Let Δ be a smooth $\mathbb{Q}$-polygon (Definition C.5). Suppose that a quasi-degree d on Δ is nice (Definition 4.20). Then for any $\varepsilon>0$ there exists a tropical Δ polynomial g such that $m_g=d$, the curve $C(g)$ is smooth (Definition C.6) and is contained in the ε-neighborhood of $\partial\Delta$.**verified**

Proof. Let $\{S_k\}_{k=1}^n$ be the sides of Δ. Suppose that the side $S_k, k=1,2$ is given by $i_kx+j_ky+a_k=0$ and these linear function are non-negative on Δ. Choose small $\delta>0$. For each $k=1,\dots,n$ we consider the following tropical polynomial:

$$f_k(x,y)=\min_{l=1,\dots,d(S_k)}\left(l(i_kx+j_ky+a_k)-\frac{l(l+1)}{2d(S_k)}\delta\right).$$

The tropical curve defined by f_k is the collection of $d(S_k)-1$ lines parallel to S_k. Define g as

$$g(x,y)=\min\left(\varepsilon/2,\min_{k=1,\dots,n}f_k(x,y)\right).$$

Clearly, $C(g)\cap\Delta$ is contained in the ε-neighborhood of $\partial\Delta$. It is, as follows, a local calculation near each corner that $C(g)\subset\mathbb{R}^2$ is a smooth tropical curve.

Since the quasi-degree is nice, so near a corner of Δ, $C(g)$ is given locally by

$$\min(\varepsilon/2,x,y,2y-\frac{1}{n},3y-\frac{3}{n},4y-\frac{6}{n},\dots)$$

where $n=\frac{d(S_k)}{\delta}$. Such a curve has an edge locally given by $x=\varepsilon/2$ and, if δ is small enough, $d(S_k)$ edges locally given by $y=\frac{k}{n},1\le k\le d(S_k)$, and these edges meet in smooth position, see Figure 5 for an illustration.

∎

Lemma 4.22. (Used on pages [14]) If Ω is bounded, then for any $\varepsilon>0$ the set $\Omega_\varepsilon=\{x\in\Omega|f_{\Omega,P}\ge\varepsilon\}$ is a $\mathbb{Q}$-polygon and $f_{\Omega,P}|_{\Omega_\varepsilon}$ is a tropical polynomial.**verified**

Proof. By Lemma 3.8, $f_{\Omega,P}$ is continuous and vanishes at $\partial\Omega$. Since Ω is bounded, the set $f_{\Omega,P}=\varepsilon$ is a curve disjoint from $\partial\Omega$. We claim that the intersection of Ω_ε with $C(f_{\Omega,P})$ is a graph with a finite number of vertices. Suppose the contrary. Then a sequence of vertices of this graph converges to a point $z\in\Omega^\circ$. Thus, there is no neighborhood of z where the series $f_{\Omega,P}$ can be represented by a tropical polynomial, which is a contradiction with Definition C.8. The finiteness of the number of vertices implies that there is only a finite number of monomials participating in the restriction of $f_{\Omega,P}$ to the domain Ω_ε, therefore the restriction is a tropical polynomial. ∎

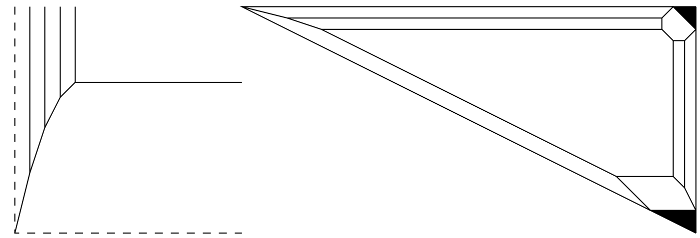


FIGURE 5. Left: the curve corresponding to the function g from Lemma 4.21, near a corner. Each vertex of the curve is smooth because it is dual to a triangle of the type $(0,1),(k,0),(k+1,0)$. Right: an example of a function g. Colored corners symbolize that a quasidegree was not nice, and we applied Lemma 4.32 and made blow-ups at these corners.

Lemma 4.23. (Used on pages [14]) Fix a presentation of $f_{\Omega_\varepsilon,P}$ is the small canonical form (Definition 4.13). We extend $f_{\Omega_\varepsilon,P}$ to Ω using this formula. In the hypothesis of the previous lemma, if $f_{\Omega,P}(\mathbf{p})\geq\varepsilon$ for each $\mathbf{p}\in P$, then we have $f_{\Omega,P}=f_{\Omega_\varepsilon,P}+\varepsilon$ on Ω_ε. Also $f_{\Omega_\varepsilon,P}+\varepsilon\geq f_{\Omega,P}$ on Ω.**verified**

Proof. On Ω_ε we have that $f_{\Omega,P}-\varepsilon\geq f_{\Omega_\varepsilon,P}$ by the definition of the latter. Then, two functions $f_{\Omega_\varepsilon,P}+\varepsilon, f_{\Omega,P}$ are equal on $\partial\Omega_\varepsilon$ and by the previous line the quasi-degree of $f_{\Omega_\varepsilon,P}$ is at most the quasi-degree of $(f_{\Omega,P}-\varepsilon)|_{\Omega_\varepsilon}$. Hence $f_{\Omega,P}$ is decreasing faster than $f_{\Omega_\varepsilon,P}$ when we move from $\partial\Omega_\varepsilon$ towards $\partial\Omega$. Therefore $f_{\Omega_\varepsilon,P}+\varepsilon\geq f_{\Omega,P}$ on $\Omega\setminus\Omega_\varepsilon$. Since $f_{\Omega_\varepsilon,P}+\varepsilon\geq 0$ on Ω we obtain the estimate $f_{\Omega_\varepsilon,P}+\varepsilon\geq f_{\Omega,P}$ on Ω which concludes the proof. □

Lemma 4.24. Let K be a compact subset of Ω, $M\in\mathbb{R}$. Let $(i,j)\in\mathbb{Z}^2\setminus\{(0,0)\}$. Let D be the set of all d such that $\{ix+jy+d=0\}\cap\Omega\neq\varnothing$ and $\exists(x_0,y_0)\in K$ such that $ix_0+jy_0+d\leq M$. Then there exists $\mathrm{C}>0$ such that $B_\mathrm{C}(K)\cap\Omega$ intersects the set $\{ix+jy+d=0\}$ for all $d\in D$.

Proof. Clearly, D is a finite interval in $\mathbb{R}$. The intersection $\Omega\cap\{-\max D\leq ix+jy\leq\min D\}$ is a convex set, therefore we can pick a point $z_0\in\Omega\cap\{ix+jy+\min D=0\}$ and a point $z_1\in\Omega\cap\{ix+jy+\max D=0\}$, and the interval z_0z_1 belongs to Ω. Then we can choose C such that $B_\mathrm{C}(K)$ contains z_0,z_1, and such C satisfies the assertion of this lemma. □

Proof of Lemma 3.20. If Ω is a compact set, then Lemma 3.20 follows from Lemmata 4.22, 4.23, because $B_{3\varepsilon}(K)\subset\Omega_\varepsilon$ for small $\varepsilon>0$. Suppose that Ω is unbounded. Let M be $\max_{z\in B_{3\varepsilon}(K)} f_{\Omega,P}(z)$. If follows from Lemma C.7 that the small canonical form (Definition 4.13) of a tropical series $f_{\Omega,P}|_{B_{3\varepsilon}(K)}$ on $B_{3\varepsilon}(K)$ contains only a finite set I of monomials.

Choose a compact set $K'=B_\mathrm{C}(K)\cap\Omega$ with $\mathrm{C}>0$. We want to prove that if C is big enough, then $f_{\Omega,P}=f_{K',P}$ on $B_{3\varepsilon}(K)$. Suppose that $f_{\Omega,P}$ and $f_{K',P}$ are not identical on $B_{3\varepsilon}(K)$. Pick the set of all monomial (i,j) such that $ix+jy$ has different coefficients a_{ij},a'_{ij} in $f_{\Omega,P},f_{K',P}$ respectively. We may assume that $(ix+jy+a'_{ij})|_\Omega\geq 0$, because $(i,j)\in I$, finite set, and we may choose C big enough by Lemma 4.24 to prevent that a line $\{ix+jy+a'_{ij}=0\}$ with $(ix+jy+a'_{ij})_K\leq M$ intersects Ω°.

Then we may take all the monomials of $f_{K',P}$ in the small canonical form of $f_{K',P}$ on $\mathcal{B}_{3\varepsilon}(K)$ and take their min with $f_{\Omega,P}$. The result will be less than $f_{\Omega,P}$ but still an Ω-tropical series, which is a contradiction.

Therefore we reduced the non-compact case for Ω to the compact case for $\Omega=K'$ with whom we know how to deal. □

4.4. **How to blow-up corners of a polygon.** Let $p_1,p_2,q_1,q_2\in\mathbb{Z}$ such that $p_1q_2-p_2q_1\neq 0$ and let

$$\Lambda=\{(x,y)\in\mathbb{R}^2|xp_1+yq_1\geq 0\ xp_2+yq_2\geq 0\}.$$

Consider the canonical form of l_Λ (see Definition 4.3)

$$l_\Lambda(x,y)=\min_{(i,j)\in\mathcal{A}}(ix+jy).$$

Denote by Λ^* the cone $\mathbb{R}_{\geq 0}(p_1,q_1)\oplus\mathbb{R}_{\geq 0}(p_2,q_2)$.

Lemma 4.25. The support $\mathcal{A}_\Lambda$ (Definition 4.1) of $l_\Lambda(x,y)$ is equal to the set

$$\Lambda^* \cap \mathbb{Z}^2.\textbf{verified}$$

Proof. Any vector $(p,q)\in\mathbb{Z}^2$ can be written as $(p,q)=\alpha\cdot(p_1,q_1)+\beta\cdot(p_2,q_2)$ with $\alpha,\beta\in\mathbb{R}$. Then, if $\alpha<0$ or $\beta<0$, then $px+qy$ is negative on one side of Λ. □

Let us now write l_Λ in the small canonical form (see Definition 4.13)

$$l_\Lambda(x,y)=\min_{(i,j)\in\mathcal{B}}(ix+jy).$$

Definition 4.26. (Used on pages [16]) Suppose that Δ is a $\mathbb{Q}$-polygon and $O=(0,0)$ is its vertex. Let $\varepsilon>0$. Let $(p_1,q_1),(p_2,q_2)\in\mathbb{Z}^2, p_1q_2-p_2q_1\neq 0$ be the primitive vectors in the directions of the edges of Δ at O, oriented outwards. Let

$$\Lambda=\{(x,y)\in\mathbb{R}^2|xp_1+yq_1\geq 0;\ xp_2+yq_2\geq 0\}.$$

Clearly, $\Delta\subset\Lambda$ and they coincide in a neighborhood of O. We say that $\Delta'\to\Delta$ is the ε-blowup of Δ in a direction $n\in\mathcal{A}_\Lambda$ if

$$\Delta'=\{(x,y)\in\Delta|n\cdot(x,y)-\varepsilon\geq 0\}.$$

We say that this blow-up is made *with respect to* the lattice point $n\in\mathbb{Z}^2$. Note that $\Delta'\subset\Delta$. We say that $\partial\Delta'\setminus\partial\Delta$ (i.e. the new side of Δ' obtained as cutting the corner at O) is the side dual to n.

Remark 4.27. Note that if Λ is smooth (Definition C.5) then $p_1q_2-p_2q_1=\pm 1$ and there exists a preferred direction (p_1+p_2,q_1+q_2) to perform a blow-up which produces two smooth corners near the vertex of Λ.

Remark 4.28. Note that n is not necessary a primitive vector. This will be important in Lemma 4.32.

Let f be any Λ-tropical polynomial written in the small canonical form (Definition 4.13). So, $\text{supp}(f)\subset\mathcal{A}_\Lambda$ (Definition 4.1) and is finite. Let $O=(0,0)$ be the corner of Λ.

Lemma 4.29. (Used on pages [16,16]) Consider any $\varepsilon>0$. There exist $\delta>0, N>0$ such that if

$$(p,q)\in(\Lambda^*)^\circ\cap\mathbb{Z}^2, \sqrt{p^2+q^2}>N,$$

then $px+qy-\delta>f$ on $\Lambda\setminus B_\varepsilon(O)$. **verified**

Proof. Each vector $(p,q)\in(\Lambda^*)^\circ\cap\mathbb{Z}^2$ can be written as $(p,q)=\alpha\cdot(p_1,q_1)+\beta\cdot(p_2,q_2)$ with $\alpha,\beta\in\mathbb{R},\alpha,\beta>0$. We consider the case $(p_1,q_1)=(1,0),(p_2,q_2)=(0,1)$, general case can be handled in the same way. Let $U=B_\varepsilon(O)\cap\Lambda$ be a neighborhood of O, we have

$$f|_U=\min_{(p_i,q_i)\in\mathcal{A}}(p_ix+q_iy)$$

where $\mathcal{A}\subset\mathcal{A}_\Lambda$. It is enough to prove the statement for $(p,q)=(1,N)$, i.e. that if N is big enough and $\delta>0$ is small enough, then

$$x+Ny-\delta>\min_{(p_i,q_i)\in\mathcal{A}}(p_ix+q_iy)\text{ for }(x,y)\in\Lambda\setminus U.$$

The cone Λ is dissected on regions where each of p_ix+q_iy is the minimal monomial. All these sectors except one satisfy $y>cx$ for a constant c depending on p_i,q_i. Therefore if N is big enough then $x+Ny>(p_i+1)x+(q_i+1)y>p_ix+q_iy+\delta$ if x or y bigger than δ. The only region where we do not have the estimate $y>cx$ is the region where the minimal monomial p_ix+q_iy satisfies $p_i=0$. In this region, again, $x+Ny>q_iy+\delta$ if x or y is bigger than δ and N is big enough.

□

Definition 4.30. (Used on pages [6,16,18,18,21]) Let f be a Δ-tropical series. We say that f is *nice* if all the corners of Δ are smooth (Definition C.5) and the quasi-degree (Definition 4.18) m_f is nice (Definition 4.20).

Lemma 4.31. (Used on pages [19]) Suppose that f is a nice Δ-tropical series for a $\mathbb{Q}$-polygon Δ. Then, at each corner of Δ, $C(f)$ has one edge of weight one.**verified**

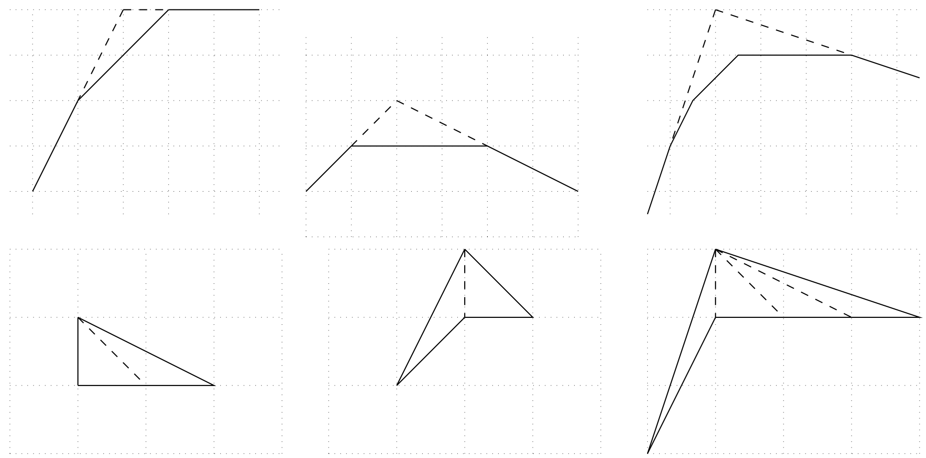


FIGURE 6. Above pictures show non-smooth corners Λ (dashed lines). Th corresponding below pictures present lattice points with respect to whom we should perform the blow-ups (the result is shown by continuous lines) in order to make all the corners smooth.

Proof. Suppose the contrary. Applying $SL(2,\mathbb{Z})$ transformation and translation we may assume that the corner in consideration is at the point $(0,0)$ and two neighboring vertices of Δ $(0,a)$ and $(b,0)$. Denote these neighboring sides by S_1, S_2. Suppose that $m_f(S_1) = 1, m_f(S_2) = k$. Then f is given by $f'(x,y) = \min(y, kx)$ in a neighborhood of $(0,0)$, and the tropical edge defined by f' has weight one. □

Lemma 4.32. (Used on pages [13,13,15,18]) Let Λ be a corner of a $\mathbb{Q}$-polygon and f be any Λ-tropical polynomial. Let $\varepsilon > 0$ be any small number. There exists a finite sequence of blowups (Definition 4.26)

$$\Lambda^n \to \cdots \to \Lambda^3 \to \Lambda^2 \to \Lambda^1 \to \Lambda,$$

and a tropical polynomial $\tilde{f}$ on Λ^n such that $f = \tilde{f}$ on $\Lambda \setminus B_\varepsilon(O)$, and $\tilde{f}$ is nice (Definition 4.30) on Λ^n, $|f - \tilde{f}| \leq \varepsilon$ on Λ^n. **verified**

Proof. Consider any ordering $\{(i_k, j_k)\}_{k=1}^{\infty}$ of primitive vectors in $\mathcal{A}_\Lambda \backslash \{(p_1,q_1),(p_2,q_2)\}$ such that $i_{k+1}^2 + j_{k+1}^2 \geq i_k^2 + j_k^2$ for any pair of consecutive (with respect to this order) primitive vectors. Choose $\delta > 0$ small enough and denote by Λ^k the δ-blow-up of Λ^{k-1} with respect to the vector $n_k(i_k, j_k)$ where $n_k \in \mathbb{N}$ is chosen in such a way that $||n_k(i_k,j_k)|| \geq N$ from Lemma 4.29.

Note that Λ^{k-1} contains k corners but only one of them can be blow-upped using the direction (i_k, j_k); so there is no ambiguity.

We construct the following sequence $\{f_k : \Lambda^k \to \mathbb{R}\}_{k=1}^{\infty}$ of functions. The function f_0 is taken to be f on $\Lambda^0 = \Lambda$. We take f_k to be

$$f_k(x,y) = \min(f_{k-1}(x,y), n_k(i_k x + j_k y) - \delta) \text{ on } \Lambda^k.$$

Because of the choice of n_k we know that f_k and f_{k-1} are equal outside of a small neighborhood of O. The number n_k represents the quasi-degree of $f_n, n > k$ on the side dual to the vector (i_k, j_k). By Lemma 4.29 for large k all this n_k can be chosen to be 1. From the construction it is clear that f_k is nice on Λ^k for some k big enough. □

4.5. **Dynamics on polyhedra.** In this section we study operators $G_{\mathbf{p}}$, the continuous analog of waves (c.f. Section B.2).

Definition 4.33. (Used on pages [6,21]) For an Ω-tropical series f in the canonical form (see (3.5), Definition 3.4), $(k,l)\in\mathcal{A}$, and $c\geq 0$ and we denote by $\mathrm{Add}_{kl}^c f$ the Ω-tropical series

$$(\mathrm{Add}_{kl}^c f)(x,y)=\min\left(a_{kl}+c+kx+ly,\ \min_{\substack{(i,j)\in\mathcal{A}\\(i,j)\neq(k,l)}}(a_{ij}+ix+jy)\right).$$

Remark 4.34. (Used on pages [17,17,18,18]) Suppose that $G_{\mathbf{p}}f=\mathrm{Add}_{ij}^c f$. We can include the operator Add_{ij}^c into a continuous family of operators

$$f\to\mathrm{Add}_{ij}^{ct}f,\ \text{where } t\in[0,1].$$

This allows us to observe the tropical curve *during* the application of Add_{ij}^c, in other words, we look at the family of curves defined by tropical series $\mathrm{Add}_{ij}^{ct}f$ for $t\in[0,1]$. See Figure 7.

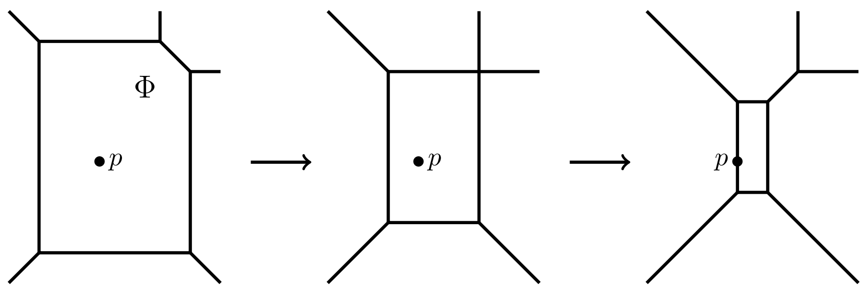


FIGURE 7. Illustration for Remark 4.34. The operator G_p shrinks the face Φ where p belongs to. Firstly, $t=0$, then $t=0.5$, and finally $t=1$ in $\mathrm{Add}_{ij}^{ct}f$. Note that combinatorics of the new curve can change when t goes from 0 to 1.

We denote by 0_Ω the function $f\equiv 0$ on Ω.

Lemma 4.35. (Used on pages) Let $f=\min_{(i,j)\in\mathcal{A}_\Omega}(ix+jy+a_{ij})\neq 0_\Omega$ be an Ω-tropical series in the canonical form, such that the curve $C(f)$ doesn't pass through a point $\mathbf{p}=(x_0,y_0)\in\Omega^\circ$. Then

- a) $G_{\mathbf{p}}f$ differs from f only in a single coefficient $d(\Phi)\in\mathcal{A}_\Omega$ dual to the face Φ of $C(f)$ (Proposition C.3) to which $\mathbf{p}$ belongs.
- b) Furthermore, $G_{\mathbf{p}}f=\mathrm{Add}_{kl}^c f$ with $c=f'(\mathbf{p})-kx_0-ly_0$, where f' is defined below. **verified**

Proof. Suppose that f is equal to $(kx+ly+a_{kl})$ near $\mathbf{p}$. Consider the function

$$f'(x,y)=\min_{(i,j)\in\mathcal{A}_\Omega,(i,j)\neq(k,l)}(ix+jy+a_{ij}).$$

Then $G_{\mathbf{p}}(f)$ is at most $\min\left(f',kx+ly+(f'(\mathbf{p})-kx_0-ly_0)\right)$ by definition and this proves a). Also, direct calculation shows that $\min(f',kx+ly+c)$ is smooth at $\mathbf{p}$ as long as $c<f'(\mathbf{p})-kx_0-ly_0$, which implies b). $\square$

Corollary 4.36. In the notation of Definition 4.3, for a point $\mathbf{p}\in\Omega^\circ$ we have $(G_{\mathbf{p}}0_\Omega)(\mathbf{z})=\min\{l_\Omega(\mathbf{z}),l_\Omega(\mathbf{p})\}$ for each $\mathbf{z}\in\Omega$.

Recall that $P=\{\mathbf{p}_i\}_{i=1}^n, P\subset\Omega^\circ$. Let $Q=\{\mathbf{q}_1,\mathbf{q}_2,\dots\}$ be an infinite sequence of points in P where each point $\mathbf{p}_i, i=1,\dots,n$ appears infinite number of times. Let g be any Ω-tropical series. Consider a sequence of Ω-tropical series $\{f_m\}_{m=1}^\infty$ defined recursively as

$$f_1=g, f_{m+1}=G_{\mathbf{q}_m}f_m.$$

Proposition 4.37. (Used on pages [11,22]) The sequence $\{f_m\}_{m=1}^\infty$ converges pointwise to $G_P g$.**verified**

Proof. First of all, $G_P g$ has an upper bound $g + nl_\Omega$ by arguments as in Proposition 4.10. Applying Lemma 4.12, induction on m and the obvious fact that $G_{\mathbf{p}_m} G_P g = G_P g$ we have that $f_m \leq G_P g$ for all m. Since the family $\{f_m\}_{m=1}^{\infty}$ is pointwise monotone and bounded, it converges to some function $f \leq G_P g$, which is an Ω-tropical series. Indeed, to find the canonical form of f we can take the limits (as $m \to \infty$) of the coefficients for f_m in their canonical forms (3.5).

It is clear that f is not smooth at all the points P. Therefore, by definition of G_P we have $f \geq G_P g$, which finishes the proof. □

Remark 4.38. Note that in the case when Ω is a lattice polygon and the points P are lattice points, all the increments c of the coefficients in $G_{\mathbf{p}} = \mathrm{Add}^c_{kl}$ are integers, and therefore the sequence $\{f_i\}$ always stabilizes after a **finite** number of steps.

Remark 4.39. Note that if $G_{\mathbf{q}_n} \dots G_{\mathbf{q}_1} g$ is close to the limit $G_P g$, then by Lemma 4.17 we see that the tropical curves are also close to each other.

Consider a sequence of operators $G_{\mathbf{q}_1}, G_{\mathbf{q}_2}, \dots, G_{\mathbf{q}_m}$ where $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$ are (not necessary distinct) points in Δ°, where Δ is a $\mathbb{Q}$-polygon. We will use the following notation

$$G = G_{\mathbf{q}_m} G_{\mathbf{q}_{m-1}} \dots G_{\mathbf{q}_1}. \tag{4.40}$$

Proposition 4.41. (Used on pages [22]) For each $\varepsilon > 0$ small enough there exists a smooth $\mathbb{Q}$-polygon $\Delta' \subset \Delta$ such that

- $G0_{\Delta'}$ is nice (Definition 4.30) on Δ',
- $0 \leq G0_\Delta - G0_{\Delta'} < \varepsilon$ on Δ'.
- $G0_\Delta \leq \varepsilon$ on $\Delta \setminus \Delta'$.

Proof. Consider $f = G_P 0_\Delta$. We apply Lemma 4.32 to f and each corner of Δ. This gives Δ' and a function $\tilde{f}$ on Δ' such that $\tilde{f}$ is nice on Δ' and $f = \tilde{f}$ near P. Therefore $G_P 0_{\Delta'} \leq \tilde{f}$ and hence $G_P 0_{\Delta'}$ is nice on Δ'. Clearly $G0_\Delta \geq G0_{\Delta'}$ and we might choose Δ' such that $G0_\Delta < \varepsilon$ on $\partial\Delta'$ which implies the second and third assessments. □

4.6. **Coarsening.** Let g be a nice Ω-tropical series, such that $C(g)$ is a smooth tropical curve. In the product $G_{\mathbf{q}_m} G_{\mathbf{q}_{m-1}} \dots G_{\mathbf{q}_1} g$ each $G_{\mathbf{q}_k}$ is the application of $\mathrm{Add}^{e_k}_{i_k,j_k}$ for some $e_k > 0$, i.e. we increase the coefficient in the monomial $i_k x + j_k y$ by e_k. So we have

$$\text{(Used on pages [18,19,19,19])}\quad G_{\mathbf{q}_m} G_{\mathbf{q}_{m-1}} \dots G_{\mathbf{q}_1} g = \mathrm{Add}^{e_m}_{i_m j_m} \mathrm{Add}^{e_{m-1}}_{i_{m-1} j_{m-1}} \dots \mathrm{Add}^{e_1}_{i_1,j_1} g. \tag{4.42}$$

Proposition 4.43 (The proof is on page 19)**.** (Used on pages [19,22,22]) Let $f = G_{\mathbf{q}_m} G_{\mathbf{q}_{m-1}} \dots G_{\mathbf{q}_1} g$. Suppose that the quasi-degrees m_f, m_g coincinde (in particular, f is also nice (Definition 4.30) on Δ). For a constant M we replace in (4.42)

$$G_{\mathbf{q}_k} = \mathrm{Add}^{e_k}_{i_k j_k} \text{ by } G^\circ_{\mathbf{q}_k} := \mathrm{Add}^{e_k - M\mathrm{h}}_{i_k j_k} \text{ for } k = 1, \dots, m.$$

Denote $f_0 = g, f_{k+1} = \mathrm{Add}^{e_k - M\mathrm{h}}_{i_k j_k} f_k = G^\circ_{\mathbf{q}_k}(f_k)$. Then, for each $\varepsilon > 0$, there exists a constant M such that for any $\mathrm{h} > 0$ small enough

- all the tropical curves defined by $f_k, k = 1, \dots, m$ are smooth or nodal (Definition C.6) on Δ as well as each tropical curve in the family during the application of $G^\circ_{\mathbf{q}_k}$ to f_k (Remark 4.34);
- the tropical curve defined by f_m is ε-close to the tropical curve defined by $G_{\mathbf{q}_m} G_{\mathbf{q}_{m-1}} \dots G_{\mathbf{q}_1} g$.**verified**

4.7. **Contracting a face.** Let a Δ-tropical polynomial f define a tropical curve $C(f) \subset \Delta$. Let $\mathbf{p}$ belong to the interior of a face Φ of the complement $\Delta \setminus C(f)$ of $C(f)$. Suppose that all corners of Φ are smooth (Definition C.5). We can find $c > 0$ and $(i, j) \in \mathbb{Z}^2$ such that

$$G_{\mathbf{p}} f = \mathrm{Add}^c_{ij} f.$$

Consider the family $\{\mathrm{Add}^{ct}_{ij} f\}_{t \in [0,1]}$ of tropical polynomials (see Remark 4.34). Denote by Φ^t the face of $\mathrm{Add}^{ct}_{ij} f$ to which $\mathbf{p}$ belongs.

Lemma 4.44 (The proof is on page 19)**.** (Used on pages [19,19,21]) In the above assumptions, all the vertices of $\Phi^t, t \in [0,1)$ are smooth or nodal (Definition C.4) vertices of the curve $C(\text{Add}_{ij}^{ct} f)$. If Φ is not contracted to a point or an interval by applying $G_{\mathbf{p}}$ to f, then the vertices of Φ^1 are smooth or nodal as well.**verified**

Consider a side S of the face Φ and two other sides S_1 and S_2 of Φ which are the neighbors of S. Applying $SL(2,\mathbb{Z})$-change of coordinates and homothety we may suppose that S is the interval with endpoints $(0,0),(1,0)$. We may assume, then, that the picture is locally given by

$$\tilde{f}(x,y) = \min(0, y, x + n_1 y + c_1, -x + n_2 y + c_2), n_1, n_2 \in \mathbb{Z}, c_1, c_2 \in \mathbb{R},$$

because both endpoints of S are smooth vertices of $C(f)$. Since the endpoints of S are $(0,0),(1,0)$ we see that $c_1 = 0, c_2 = 1$. We suppose that Φ is the face where the function $0 + 0x + 0y$ is the least monomial in $\tilde{f}$.

The curve $C(\text{Add}_{0,0}^{ct} f)$ in the neighborhood of S is given by the tropical polynomial

$$\tilde{f}_t(x,y) = \min(ct, y, x + n_1 y, -x + n_2 y + 1).$$

For small $t > 0$ denote by S^t the side of Φ^t (recall that Φ^t is a face of the curve $C(\text{Add}_{0,0}^{ct} f)$) which is close and parallel to the side S of the face Φ. It is easy to find the coordinates of the vertices of S^t by direct calculation: they are $(ct(1-n_1), ct)$ and$(ct(n_2-1)+1, ct)$. The length of S^t is therefore $ct(n_2-1)+1-ct(1-n_1) = 1 + ct(n_1+n_2-2)$. We just proved the following lemma.

Lemma 4.45. In the above notation, two facts are equivalent:

- S^t is shorter then S for small $t > 0$,
- $n_1 + n_2 < 2$.**verified**

Corollary 4.46. (Used on pages [19,19,19]) We have three different cases:

a) $n_1 + n_2 < 0$, corresponds to collapsing the face Φ to $\mathbf{p}$,
b) $n_1 + n_2 = 0$, corresponds to collapsing the face Φ to a (possibly degenerate) interval containing $\mathbf{p}$,
c) $n_1 + n_2 = 1$, note that in this case $(1, n_1) + (-1, n_2) = (0, 1)$.

Definition 4.47. We say that a continuous family of tropical curves has a nodal perestroika if all the curves, except one, are smooth, and non-smooth curve has only one nodal point, and the family near it is given by $\min(x, y, t, x + y)$ for $t \in [-\varepsilon, \varepsilon]$ up to $SL(2,\mathbb{Z})$ change of coordinates.

Proof of Lemma 4.44. The combinatorial type of Φ_t can only change when at least one of the sides of the Φ^t is getting shrinked to a point for some t. Choose the minimal such $t = t_0$, and denote one of the shrinking sides by S. Corollary 4.46 tells us that cases a), b) correspond to collapsing the face, hence in these cases the lemma is proven.

We assume that $t_0 < 1$ and the case c) in Corollary 4.46 takes place.

If neither S_1 nor S_2 gets contracted when we pass from $C(f)$ to $C(\text{Add}_{ij}^{ct_0} f)$, then we see a *nodal perestroika*(Definition 4.47). If S_2 is contracted by passing from $C(f)$ to $C(\text{Add}_{ij}^{ct_0} f)$, then the direct computation using Corollary 4.46 c) implies that the side S_3 of Φ, which is next after S_2, is parallel to S_2 and therefore the whole face Φ is contracted by $\text{Add}_{ij}^{ct_0}$ which is a contradiction. The case when S_1 is contracted is handled by the same argument. $\square$

Corollary 4.48. The edges of $C(\text{Add}_{ij}^{ct} f) \cap \Phi$ for $0 \le t < 1$ have weight 1.

Proof of Proposition 4.43. The only two possibilities how the tropical curve can become non-smooth during our procedure in (4.42) is appearance of a non-smooth vertex inside Δ° and appearance of an edge with weight bigger than one inside Δ° or at the corners of Δ.

It follows from Lemma 4.44 that a non-smooth vertex or an edge with weight bigger than one in Δ° can appear only by contracting a face. We can decrease the constants e_i in (4.42) by any small positive numbers, such that no $G_{\mathbf{q}_k}^\circ$ contracts a face, this eliminates a part of the problems with smoothness inside Δ°. For that it is enough to choose M such that mMh (total change of function) would be less than the minimal non-zero distance between one of the points $q_1, \dots, q_m$ and the tropical curve in the process (4.42). Finally, f_m is nice on Δ and, therefore, the tropical curve $C(f_m)$ is smooth at the corners of Δ by Lemma 4.31. $\square$

5. Proof of the lower estimate

In this section we introduce several important concepts and prove Proposition 3.22 modulo other statements which we prove later.

Lemma 5.1. (Used on pages) Let $G : \mathbb{Z}^2 \to \mathbb{Z}$ be a superharmonic function. Suppose that $G - G'$ is also superharmonic where $G' : \mathbb{Z}^2 \to \mathbb{Z}$ satisfies $0 \leq G' \leq 1$. Then for each connected component A of $\mathbb{Z}^2 \setminus D(G)$ we have $G'|_A \equiv 0$ or $G'|_A \equiv 1$.**verified**

Proof. Consider a point $v \in A$. Note that $4G(v) = \sum_{w\sim v} G(w)$. Suppose that $G'(v) = 1$. Since $G - G'$ is superharmonic and $G' \leq 1$, we must have $G'(w) = 1$ for all $w \sim v$. Repeating this for the neighbors of v, we prove the lemma. $\square$

Definition 5.2. We say that a function $F : \mathbb{Z}^2 \to \mathbb{Z}$ is *solid* if there exists $\mathrm{C} > 0$ with the following property. If a connected component T of $\mathbb{Z}^2 \setminus D(F)$ belongs to $B_n(D(F))$ for some n, then $T \subset B_{\mathrm{C}}(D(F))$. When we want to specify the constant C we write that F is C-solid.

Example 5.3. The functions $F = \Psi_{\text{edge}}, \Psi_{\text{vertex}}, \Psi_{\text{node}}$ (see (5.9),(5.9),(5.11)) are solid just because $\mathbb{Z}^2 \setminus D(F)$ has no components which belong to a finite neighborhood of $D(F)$.

Lemma 5.4. (Used on pages [20,25]) If F is C-solid, then for each $G \in \Theta_n(F)$ the set $\{F \neq G\}$ is contained in $B_{\max(n,\mathrm{C})}(D(F))$.**verified**

Proof. Let $A_n = \{v \in \mathbb{Z}^2 | G(v) = F(v) - n\}$. By the superhamonicity of F, G we see that if $v \in A_n \setminus D(F)$ then all neighbors of v belong to A_n. Therefore the connected component of v in $\mathbb{Z}^2 \setminus D(F)$ belongs to A_n, which, in turn, belongs to a finite neighborhood of $D(F)$ because F is solid. Therefore v is contained in $B_{\mathrm{C}}(D(F))$. By the same arguments, $A_{n-1} = \{G = F - n + 1\}$ is contained in the 1-neighborhood of $D(F) \cap A_n$ or in $B_{\mathrm{C}}(D(F))$, A_{n-2} is contained in the 1-neighborhood of A_{n-1}, etc. $\square$

Lemma 5.5. (Used on pages [20,25]) If F, G are two superharmonic functions on $\Gamma \subset \mathbb{Z}^2$, then $\min(F, G)$ is a superharmonic function on Γ.**verified**

Proof. Let $v \in \Gamma$. Without loss of generality, $F(v) \leq G(v)$. Then, $\Delta \min(F, G)(v) \leq \Delta F(v) \leq 0$. $\square$

Let $F : \mathbb{Z}^2 \to \mathbb{Z}$ be a superharmonic (i.e. $\Delta F \leq 0$ everywhere) function.

Definition 5.6. (Used on pages [6,22,24,24,24,25,31]) Suppose that F is solid. For each $n \in \mathbb{N}$ consider the set $\Theta_n(F)$ of all integer-valued superharmonic functions G such that $F - n \leq G \leq F$ and G coincides with F outside a finite neighborhood of the deviation set of D, i.e.

$$\Theta_n(F) = \{G : \mathbb{Z}^2 \to \mathbb{Z} | \Delta G \leq 0, F - n \leq G \leq F, \exists \mathrm{C} > 0, \{F \neq G\} \subset B_{\mathrm{C}}(D(F))\}.$$

Define $S_n(F) : \mathbb{Z}^2 \to \mathbb{Z}$ to be

$$S_n(F)(v) = \min\{G(v) | G \in \Theta_n(F)\}.$$

We call $S_n(F)$ *the n-smoothing of F.*

Proposition 5.7. (Used on pages [25,27]) If F is C-solid for some $\mathrm{C} > 0$, then the function $S_n(F)$ belongs to $\Theta_n(F)$.**verified**

Proof. By Lemma 5.5 the function $S_n(F)$ is superharmonic and by Lemma 5.4, $\{F \neq S_n(F)\} \subset B_{\mathrm{C}'}(D(F))\}$ for $\mathrm{C}' \geq \max(n, \mathrm{C})$. $\square$

Remark 5.8. For a fixed $\mathrm{h} > 0$ we can naturally extend the notion of smoothing to integer-valued superharmonic functions on $\mathrm{h}\mathbb{Z}^2$.

Let us fix $p_1, p_2, q_1, q_2, c_1, c_2 \in \mathbb{Z}$ such that $p_1q_2 - p_2q_1 = 1$. Consider the following functions on $\mathbb{Z}^2$:

(5.9) (Used on pages [20,20,25,25,25,25]) $\Psi_{\text{edge}}(x,y) = \min(0, p_1x + q_1y)$,

(5.10) (Used on pages [25,25,27]) $\Psi_{\text{vertex}}(x,y) = \min(0, p_1x + q_1y, p_2x + q_2y + c_1)$,

(5.11)
(Used on pages [20,25,25,27]) $\Psi_{\text{node}}(x,y) = \min\Big(0, p_1x+q_1y, p_2x+q_2y+c_1, (p_1+p_2)x+(q_1+q_2)y+c_2\Big)$.

Theorem 2 (See a proof on pages 25 (a) and 29 (b,c))**.** (Used on pages [4,4,21,21,21,25,27,27,29,29,30,30,31]) Let F be a) Ψ_{edge}, b)Ψ_{vertex}, or c)Ψ_{node}. The sequence of n-smoothings $S_n(F)$ of F stabilizes eventually as $n \to \infty$, i.e. there exists $N > 0$ such that $S_n(F) \equiv S_N(F)$ for all $n > N$. In other words, there exists a pointwise minimum θ_F (we call it *the canonical smoothing of* F) in $\bigcup \Theta_n(F)$. **verified**

5.1. **Waves and operators $G_{\mathbf{p}}$.**

Remark 5.12. Note that $\Delta\theta_F \geq -3$ because otherwise we could decrease θ_F at a point violating this condition, preserving superharmonicity of θ_F, and this would contradict to the minimality of θ_F in $\bigcup \Theta_n(F)$.

Consider a state $\phi = \langle 3 \rangle + \Delta\theta_F$. By the previous remark, $\phi \geq 0$. Let $v \in \mathbb{Z}^2$ be a point far from $\{\Delta\theta_F \neq 0\}$. Let F equal to $ix + jy + a_{ij}$ near v. Then, informally, sending a wave from v increases the coefficient a_{ij} by one.

Lemma 5.13. In the above conditions, $W_v\phi = \langle 3 \rangle + \theta_{F'}$ where W_v is the sending wave from v (Definition B.20) and $F' = \text{Add}^1_{ij} F$ (Definition 4.33).

Proof. In other words we want to prove that the toppling function H^v_ϕ (B.21) is equal to $\theta'_F - \theta_F$. Clearly, $\theta'_F - \theta_F \geq H^v_\phi$ thanks to Corollary B.32 and the fact that $\theta'_F - \theta_F = 1$ at v. On the other hand, if $\theta'_F - \theta_F > H^v_\phi$ then the function $\theta_F + H^v_\phi$ coincides with $\theta_{F'}$ far from $D(F')$, is superharmonic and less than F', which contradicts to the definition of $\theta_{F'}$. $\square$

Therefore, if $\phi = \langle 3 \rangle + \Delta F$ where F is a tropical polynomial such that $C(F)$ is smooth or nodal, then sending waves from $\mathbf{p}$ corresponds to the operators Add. Then, $(\phi + \delta_{\mathbf{p}})^\circ$ can be obtained by sending waves until v has less than three grains after the wave. This corresponds to applying Add until a point gets to the tropical curve, i.e. this corresponds to the operator $\Gamma_{\mathbf{p}}$. Note that we can not avoid perestroïki (Definition 4.47) because they happen during application of $G_{\mathbf{p}}$ (Lemma 4.44) therefore we study not only F defining a smooth tropical vertex or a node, but also nearby to the node configurations (Ψ_{node}).

Canonical smoothings in Theorem 2 provide us with the building blocks of the set $D(\phi^\circ_{\text{h}})$. The smoothing of Ψ_{edge} represents *a sandpile soliton*, becoming a tropical edge in the limit, the smoothing of Ψ_{vertex} represents *a sandpile triad*, becoming a smooth tropical vertex in the limit, and the smoothing of Ψ_{node} represents the degeneration of two sandpile triads into the union of two sandpile solitons. See Section 6 for details.

Definition 5.14 (Canonical smoothing of a tropical polynomial)**.** (Used on pages [6,22]) Let f be a nice (Definition 4.30) Δ-tropical polynomial, such that $C(f)$ is a smooth or nodal (Definition C.6) tropical curve. Then, for small enough $\text{h} > 0$, we define *the canonical smoothing* $\text{Smooth}_{\text{h}}(f) : \Gamma_{\text{h}} \to \mathbb{Z}$ as follows. We consider the lattice $\text{h}\mathbb{Z}^2 \subset \mathbb{R}^2$ and define $F(x,y) = [\text{h}^{-1}f(x,y)]$. This defines a piecewise linear function on Δ (cf. (B.18)) and we extend it to $\mathbb{R}^2$, using its formula. The curve $C(f)$ has a finite number of edges and vertices, which are smooth or nodal. Hence the same is true for $C(F)$ if h is small enough. Each local equation of vertices and edges of $C(F)$ belongs to the cases in Theorem 2. We apply Theorem 2 for all these local equations. Hence there exists $N > 0$ such that the smoothings of all the local equations of edges and vertices of $C(F)$ stabilize after N steps. Finally, we define $\text{Smooth}_{\text{h}}(f)$ as $S_N(F)$ restricted to Γ_{h}. We call this procedure the *canonical smoothing* of F (or the canonical smoothing of f with respect to h). Note that $\text{Smooth}_{\text{h}}(f)$ may be negative.

Remark 5.15. (Used on pages [22]) The canonical smoothing procedure changes F only in $B_{N\text{h}}(D(F))$ where N is an absolute constant, which only depends on the slopes of the edges of $C(f)$. Therefore if h is small enough, the smoothings of different vertices and edges never overlap. In this case we say that $\text{Smooth}_\text{h}(f)$ is well defined.

5.2. **Proof of Proposition 3.22.** The main idea is to construct a state on $\Gamma_\text{h} \cap \Delta$ whose toppling function is less than that of ϕ_h, and whose relaxation (via wave decomposition) is completely controlled by operators $G_{\mathbf{p}_i}$.

Constructing auxiliary state. Choose $\varepsilon > 0$. By Proposition 4.41 there exists a $\mathbb{Q}$-polygone $\Delta' \subset \Delta$ such that $f_{\Delta',P} = G_P 0_{\Delta'}$ is ε-close to $G_P 0_\Delta$ on Δ and $G_P 0_{\Delta'}$ is nice (Definition 4.20).

Lemma 4.21 asserts that there exist a Δ'-tropical polynomial g such that $g < \varepsilon$, a curve $C(g)$ is smooth and the quasi-degree m_g is equal to the quasidegree of $G_P 0_{\Delta'}$.Note that $G_P g = G_P 0_{\Delta'}$. Therefore while applying G_P to g we have no changes near the boundary of Δ'.

Now we will make reduction to a finite composition of $G_{\mathbf{p}_i}$ and smooth intermediate curves. Let $Q = \mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$ be a sequence of points in P such that each $\mathbf{p}_i$ appears in Q infinite number of times. By Proposition 4.37 we can choose m such that

$$f = G_{\mathbf{q}_m} G_{\mathbf{q}_{m-1}} \dots G_{\mathbf{q}_1} g$$

is ε-close to $G_P g = G_P 0_{\Delta'}$. Now we will use the notation of Proposition 4.43. We replace each $G_{\mathbf{q}_k} = \text{Add}_{i_k j_k}^{e_k}$ by $\text{Add}_{i_k j_k}^{e_k - M\text{h}}$ such that all the curves in the new process are always smooth or nodal. We have now a sequence of tropical polynomials $f_0 = g, f_1, \dots, f_m$.

The next step is to use the theory of smoothings. Using f_k we define $F_k = [\text{h}^{-1} f_k] : \Gamma_\text{h} \to \mathbb{Z}_{\geq 0}$ as in (B.18). We can choose $\text{h} > 0$ small enough such that all canonical smoothings $\text{Smooth}_\text{h}(f_k), k = 0, \dots, m$ are well defined (Remark 5.15). Recall Definition 5.14, we denote by $\text{Smooth}_\text{h}(f_k)$ the canonical smoothing of F_k. Define the states $\phi_k = 3 + \Delta\text{Smooth}_\text{h}(f_k)$.

The final step is to use the fact that waves commutes with smoothings (we proceed as in Lemma 5.13 but with the global notation). Namely, let $0 \leq k \leq m$. Fix the notation by $F_k(x, y) = \min_{(i,j)\in\mathcal{A}}(ix\text{h}^{-1} + jy\text{h}^{-1} + [a_{ij}\text{h}^{-1}])$ with finite $\mathcal{A}$. By Proposition 4.43, the points p_i do not belong to $C(f_k)$ and so do not belong to $C(F_k)$ as long as h is small enough. Let $v = \text{h}[\text{h}^{-1} q_k]$. Then $\phi_k(v) = 3$. Suppose that v belongs to the region where $i_0 x\text{h}^{-1} + j_0 y\text{h}^{-1} + [a_{i_0 j_0}\text{h}^{-1}]$ is the minimal monomial. Denote

$$F'(x, y) = \min_{(i,j)\in\mathcal{A}}(ix\text{h}^{-1} + jy\text{h}^{-1} + [a'_{ij}\text{h}^{-1}])$$

where $a'_{ij} = a_{ij}$ if $(i, j) \neq (i_0, j_0)$ and $a'_{i_0 j_0} = a_{i_0 j_0} + \text{h}$. Then $W_v \phi_k = \langle 3\rangle + \Delta\text{Smooth}(F')$. Indeed, this follows from the fact that in territory where v belongs we will have one toppling and $\Delta(F' - F)$ is the pointwise minimal integer-valued function with the property that $\langle 3\rangle + \Delta\text{Smooth}(F) + \Delta(F' - F)$ is a stable state.

Recall that $f_1 = \text{Add}_{i_1 j_1}^{\text{h}[e_1 \text{h}^{-1}] - M\text{h}} g$.

Therefore

$$\text{Smooth}_\text{h}(F_1) = W_{\text{h}[\text{h}^{-1}\mathbf{p}_i]}^{[e_1 \text{h}^{-1}] - M} \text{Smooth}(F_0).$$

Therefore, the toppling function of $\phi_0 + \sum_{p\in P} \delta_p$ is at least F_m. The proposition follows, since the toppling function of $\phi_0 + \sum_{p\in P} \delta_p$ is less then the toppling function of $\langle 3\rangle + \sum_{p\in P} \delta_p$. Note that thanks to the construction of g we had no topplings near the boundary of Δ' during this process. We finished the proof of Proposition 3.22.

6. Origin of patterns: smoothing of integer-valued superharmonic functions

Periodic patterns in sandpiles were discovered by S. Caracciolo, G. Paoletti, and A. Sportiello in a pioneer work [3], see also Section 4.3 of [4] and Figure 3.1 in [22]. Experimental evidence suggests that these patterns carry a number of remarkable properties: in particular, they are self-reproducing under the action of waves. That is why we call these patterns *solitons*. Solitons naturally appear during relaxations on convex domains, see examples for different directions in Figure 1. Solitons on pictures correspond to tropical edges of the limiting curve.

The fact that the solitons appear as "smoothings" of piece-wise linear functions was predicted in [26]. We provide a definition of the *smoothing* procedure (Definition 5.6). We will prove the existence (and uniqueness modulo translation) of solitons for all rational slopes and provide certain estimates on their shape.

We construct *triads* — three solitons meeting at a point — and triads satisfy similar properties. In the proof of Theorem 1 we show that the deviation locus of ϕ°_{h} is essentially glued out of solitons and triads along the graph $C(f_{\Omega,P})$. In this section we study only local pictures of sand relaxations, it happens that our main objects of study are integer-valued non-negative discrete harmonic functions on $\mathbb{Z}^2$.

6.1. **Descending to a cylinder.** Let G be an integer valued function on $\mathbb{Z}^2$ satisfying

$$G(i,j)=G(i+q,j-p) \text{ for all } i,j\in\mathbb{Z}$$

and fixed $p,q>0$. The function G naturally descends to the quotient of $\mathbb{Z}^2$ with respect to the additive action of the vector $(q,-p)$.

Definition 6.1. (Used on pages [25]) We consider the equivalence relation $(i,j)\sim(i+q,j-p)$ on $\mathbb{Z}^2$, it respects the graph structure on $\mathbb{Z}^2$, so we define a new graph

$$\Sigma=\mathbb{Z}^2/\sim,\ \text{where}\ \sim\ \text{is generated by}\ (i,j)\sim(i+q,j-p).$$

We identify Σ with the strip $[0,q-1]\times\mathbb{Z}$ where each vertex is connected with its neighbors and, additionally, $(0,i)$ is connected with $(q-1,i-p)$ for all $i\in\mathbb{Z}$. The notion of discrete harmonic function also easily descends to Σ.

Let G be an integer valued superharmonic function on Σ. Suppose that there exists a constant μ such that $0\le G(i,j)\le\mu j$ for $j>0, 0\le i\le q-1$. Suppose also that the number of points v with $\Delta G(v)<0$ is finite and denote $\mathcal{D}=\sum_{v\in\Sigma}\Delta G(v)<0$.

Lemma 6.2. (Used on pages [26]) Let $k\in\mathbb{Z}_{>0}$. There exists a constant $M=M(\mu,p,q,|\mathcal{D}|,k)$ such that if a function G is as above, then G is linear on $\Sigma'=[0,q-1]\times[m-k,m]$ for some $m\le M$.**verified**

Proof. Take N very big. Dissect $[0,q-1]\times[0,6N(|\mathcal{D}|+1)]$ on $|\mathcal{D}|+1$ parts

$$[0,q-1]\times[0,6N]$$
$$[0,q-1]\times[6N,12N], \text{etc.}$$

Then there exists $l\le|\mathcal{D}|+1$ and a part $A=[0,q-1]\times[6Nl,6N(l+1)]$ in this dissection where G is discrete harmonic. We have the estimate

$$0\le G|_A\le 6\mu(|\mathcal{D}|+1)(N+1).$$

Let v be the center of A. Applying Lemma A.2 for v we prove that derivatives $\partial_\bullet\partial_\bullet G$ are zeros in the middle part of A if N is big enough. ☐

6.2. **Properties of smoothing.** Consider a connected graph $\Gamma\subset\mathbb{Z}^2$

Let F,G be two superharmonic integer-valued functions on $\Gamma\setminus\partial\Gamma$. Suppose that $H=F-G$ is non-negative and bounded. Let m be the maximal value of H. Define the functions $H_k, k=0,1,\dots,m$ as follows:

$$H_k(v)=\chi(H\ge k)=\begin{cases}1, \text{ if } H(v)\ge k,\\ 0, \text{ otherwise.}\end{cases} \tag{6.3}$$

Lemma 6.4. In the above settings, the function $F-H_m$ is superharmonic.**verified**

Proof. Indeed, $F-H_m$ is superhamonic outside of the set $\{x|H(x)=m\}$. Look at any point v such that $H(v)=m$. Then we conclude by

$$4(F-H_m)(v)=4G(v)+4(m-1)\ge\sum_{w\sim v}G(w)+4(m-1)\ge\sum_{w\sim v}(F-H_m)(w).$$

☐

We repeat the procedure in Lemma 6.4 for $F - H_m$; namely, consider $F - H_m - H_{m-1}, F - H_m - H_{m-1} - H_{m-2}$, etc. We have

$$H = H_m + H_{m-1} + H_{m-2} + \cdots + H_1,$$

and it follows from subsequent applications of Lemma 6.4 that all the functions $F - \sum_{i=m}^{m-k+1} H_i$ are superharmonic, for $k = 1, 2, \ldots, m$. Also, it is clear that

$$0 \leq \left(F - \sum_{i=m}^{m-k+1} H_i\right) - \left(F - \sum_{i=m}^{m-k} H_i\right) = H_{m-k} \leq 1$$

at all $v \in \Gamma, k = 0, \ldots, m$.

Remark 6.5. It follows from definition of H_i that $\operatorname{supp}(H_m) \subset \operatorname{supp}(H_{m-1}) \subset \ldots$ and, hence, $|H_i - H_{i+1}| \leq 1$ for $i = 1, \ldots, m-1$.

Consider a superharmonic function F. We are going to prove that two consequitive smoothings (see Definition 5.6) of F differ at most by one at every point of $\mathbb{Z}$.

Proposition 6.6. (Used on pages [24,25]) For all $n \in \mathbb{N}$

$$|S_n(F) - S_{n+1}(F)| \leq 1.\textbf{verified}$$

Proof. By definition, $S_n(F) \geq S_{n+1}(F)$ at every point of $\mathbb{Z}^2$. If the inequality $|S_n(F) - S_{n+1}(F)| \leq 1$ doesn't hold, then the maximum M of the function $H = S_n(F) - S_{n+1}(F)$ is at least 2. We will prove now that in this case the following inequality holds:

$$S_n(F) - \chi(H \geq M) \geq F - n.$$

Namely, by Lemma 6.4 the function $S_n(F) - \chi(H \geq M)$ is superharmonic. Suppose that

$$S_n(F) - \chi(H \geq M) < F - n \text{ at a point } v \in \mathbb{Z}^2.$$

Since the support of $\chi(H \geq 1)$ contains the support of $\chi(H \geq M)$, we have that

$$S_n(F) - \chi(H \geq M) - \chi(H \geq 1) < F - (n+1) \text{ at } v.$$

But this contradicts to

$$S_n(F) - \chi(H \geq 1) - \chi(H \geq M) \geq S_{n+1}(F) \geq F - (n+1).$$

Therefore, $S_n(F) - \chi(H \geq M) \geq F - n$ (for χ see (6.3)). This implies that $S_n(F) - \chi(H \geq M) \in \Theta_n(F)$ which contradicts the minimality of $S_n(F)$.

□

Corollary 6.7. (Used on pages [25,25]) Proposition 6.6 implies that the function $S_{n+1}(F)$ can be characterized as the point-wise minimum of all superharmonic functions G such that $S_n(F) - 1 \leq G \leq S_n(F)$ and $S_n(F) - G$ vanishes outside some neighborhood of $D(S_n(F))$ (Definition 5.6). In other words, n-smoothing $S_n(F)$ of F is the same as 1-smoothing of $(n-1)$-smoothing $S_{n-1}(F)$ of F.

Corollary 6.8. (Used on pages [26,28]) If $S_n(F) \neq S_{n+1}(F)$ then there exists a point v_0 such that $S_{n+1}(F)(v_0) = F(v_0) - (n+1)$.

Indeed, if there is no such a point, then $S_{n+1}(F) \geq F - n$ and therefore $S_{n+1}(F) = S_n(F)$.

Remark 6.9. (Used on pages [25,30]) Let $F(x, y) = \min(x, y, 0)$ or $F(x, y) = \min(x, y, x+y, c)$ for $c \in \mathbb{Z}_{\geq 0}$. Then it is easy to check that $S_1(F)(x, y) = F(x, y)$ and therefore $\Theta_F = \{F\}$ (see Definition 5.6 for the notation).

Definition 6.10. (Used on pages [25,25,25,30,31]) Let $e \in \mathbb{Z}^2 \setminus \{(0,0)\}$. We say that a function $G : \mathbb{Z}^2 \to \mathbb{Z}$ is *e-increasing* if

a) G is a smoothing of a solid function F,
b) for each $v \in \mathbb{Z}^2$ we have $G(v) \leq G(v+e)$,
c) there exists a constant $\mathrm{C} > 0$ such that for each v for each $k \geq \mathrm{C} \cdot (\operatorname{dist}(v, D(G)) + 1)$ we have

$$G(v - k \cdot e) < G(v).$$

Recall that $D(G) = \{w \in \mathbb{Z}^2 | \Delta G(w) \neq 0\}$.

Example 6.11. (Used on pages [26]) Let $G(x, y) = \min(px + qy, 0)$ where $p < 0, q > 0, p, q \in \mathbb{Z}$. Note that G is $\mathbb{Z}$-increasing in the direction $(0, 1)$. Furthermore, if $e = (e_1, e_2) \in \mathbb{Z}^2$ is such that $0 \leq e_1 \leq q - 1$ and $e_2 \geq |p|$, then G is e-increasing.

Lemma 6.12. (Used on pages [31]) If G is e-increasing, then $S_1(G)$, the 1-smoothing of G, is also e-increasing.**verified**

Proof. Corollary 6.7 gives the property a) of Definition 6.10 for free, and, therefore by Proposition 5.7 the function $G - S_1(G)$ is supported on a finite neighborhood of $D(G)$, where, thus, and $D(S_1(G))$ dwells. Proposition 6.6 implies that $|S_1(G) - G| \leq 1$. This gives c), probably, with another constant. Therefore, we only need to prove that $S_1(G)$ satisfies b) in Definition 6.10.

We argue a contrario. Let $H = F - S_1(G)$. Suppose that the set

$$A = \{v \in \mathbb{Z}^2 | G(v - e) = G(v), H(v - e) = 0, H(v) = 1\}$$

is not empty. Since $H(v) = 1$ for $v \in A$, A belongs to some finite neighborhood of $D(G)$. Consider the set

$$B = \{v | H(v) = 0, \exists n \in \mathbb{Z}_{>0}, v + n \cdot e \in A, G(v) = G(v + n \cdot e)\}.$$

Take $v \in B$. Since $G(v) = G(v + n \cdot e)$ and $v + n \cdot e$ belongs to a fixed finite neighborhood of $D(G)$, then c) in Definition 6.10 implies that B belongs to a finite neighborhood of $D(G)$. Consider the following function

$$\tilde{G} = S_1(G) - \sum_{v \in B} \delta_v.$$

Note that for all $v \in \mathbb{Z}^2 \setminus B$ we have $\Delta\tilde{G}(v) \leq \Delta S_1(G)(v) \leq 0$. Take $v \in B$. We have $v + n \cdot e \in A$ for some $n \in \mathbb{Z}_{>0}$, therefore

$$4\tilde{G}(v) = 4S_1(G)(v + n \cdot e) \geq \sum_{w \sim v + n \cdot e} S_1(G)(w) \geq \sum_{w \sim v} \tilde{G}(w).$$

Therefore $\tilde{G}$ is superharmonic, and satisfies $G \geq \tilde{G} \geq G - 1$ by construction, which contradicts to the minimality of $S_1(G)$ in $\Theta_1(G)$. □

Corollary 6.13. (Used on pages [26,29,30]) Let F be one of $\Psi_{\text{edge}}, \Psi_{\text{vertex}}, \Psi_{\text{node}}$ (Eqs. (5.9),(5.10),(5.11)). Let $e \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. If F is e-increasing, then $S_n(F)$ is also e-increasing.**verified**

Lemma 6.14. (Used on pages [25]) Let $F = \Psi_{\text{edge}}$ (see (5.9)). Then for all $n \in \mathbb{Z}_{>0}$ smoothings $S_n(F)$ are periodic in the direction $e = (q_1, -p_1)$, i.e. $S_n(F)(v) = S_n(F)(v + e)$ for all $v \in \mathbb{Z}^2$.**verified**

Proof. Suppose, to the contrary, that $S_n(F)(v) > S_n(F)(v + e)$ for some $v \in \mathbb{Z}^2$. It follows from Lemma 5.5 that $\tilde{S}_n(F)(w) = \min(S_n(F)(w), S_n(F)(w + e)), w \in \Gamma$ belongs to $\Theta_n(F)$, but $\tilde{S}_n(F)(v) < S_n(F)(v)$ which contradicts to the minimality of $S_n(F)$ in $\Theta_n(F)$. □

Lemma 6.15 (The proof is on page 25)**.** (Used on pages [25,26,27]) Let F be one of $\Psi_{\text{edge}}, \Psi_{\text{vertex}}, \Psi_{\text{node}}$ (see (5.9),(5.10),(5.11)). Then for each n we have

$$\text{dist}\Big(D(F), \big\{F \neq S_n(F)\big\}\Big) \leq n.$$

verified

Proof of Lemma 6.15. It follows from Lemma 5.4 and Example 5.3. □

6.3. **Proof of Theorem 2 for Ψ_{edge}.**

Proof. For the sake of notation denote $F = \Psi_{\text{edge}}$ (see (5.9)), $p = p_1, q = q_1$. We will prove that the sequence $\{S_n(F)\}_{n=1}^{\infty}$ of n-smoothings (Definition 5.6) of F eventually stabilizes. It is easy to check that we cannot perform even a non-trivial 1-smoothings for F in the cases when $(p, q) = (\pm 1, 0)$ or $(0, \pm 1)$ (cf. Remark 6.9). Therefore, we conclude the proof of the theorem in this case by Corollary 6.7. From now on we suppose that $pq \neq 0$ and that the sequence $\{S_n(F)\}_{n=1}^{\infty}$ does not stabilize.

By Lemma 6.14, all $S_n(F)$ are periodic in the direction $(q,-p)$. Consider the quotient Σ of $\mathbb{Z}^2$ by translations by $(q,-p)$ (see Definition 6.1). Abusing notations, we think of $F, S_1(F), S_2(F),\dots$ as functions on Σ. Note that $\mathcal{D}=\sum_{v\in\Sigma}\Delta F(v)$ is finite. Indeed, $\min(0,px+qy)$ has only finite number of points in Σ where the laplacian is not zero.

Next we observe that $\sum_{v\in\Sigma}\Delta S_1(F)(v)=\mathcal{D}$. Indeed, $\Delta S_1(F)$ is zero far from $D(F)$, additionally F is equal to $S_1(F)$ far from $D(F)$, therefore we can sum up the laplacian on a finite neighborhood of $D(F)$, and by Lemma A.3, this sum can be found using only those values of F which lie near the boundary of such neighborhood. Similarly, we obtain $\sum_{v\in\Sigma}\Delta S_n(F)(v)=\mathcal{D}$ for all $n\in\mathbb{Z}_{>0}$ and because of superharmonicity of $S_n(F)$ we see that

$$|D(S_n(F))| = |\{v\in\Sigma|\Delta S_n(F)(v)\neq 0\}|\leq \mathcal{D}. \tag{6.16}$$

We supposed that the sequence $\{S_n(F)\}_{n=1}^{\infty}$ does not stabilize. Therefore, by Corollary 6.8 for each $n\in\mathbb{Z}_{>0}$ the set $A_n=\{v\in\mathbb{Z}^2|S_n(F)(v)=F(v)-n\}$ is not empty. Hence $A_1\supset A_2\supset A_3\dots$, and A_1 is finite because $A_1\subset D(F)$ by Lemma 6.15. Thus we can take $v\in\bigcap\limits_{n\geq 1}A_n$.

Without loss of generality we may suppose that $q>0, p<0$. Thanks to Example 6.11, F is $(0,1)$-increasing and by Corollary 6.13 so do all $S_n(F)$. By the same reason all $S_n(F)$ are (m,k)-increasing if $0\leq m\leq q-1$ and $k>|p|$. The property of (e_1,e_2)-increasing gives that

$$F(v)-n = S_n(F)(v)\geq S_n(v+(e_1,e_2))$$

which is less than $F(v+(e_1,e_2))$ for fixed (e_1,e_2) and n big enough. So we see that there exists a constant C_3 such that $\mathrm{supp}(F-S_n(F))$ contains $[0,q-1]\times[-\mathrm{C}_3 n,0]$ for all n big enough.

For such n let $j_0=\min\{j|\exists i,(i,j)\in\mathrm{supp}(F-S_n(F))\}$. Applying Lemma 6.2 to the function $S_n(F)-\min_i F(i,j_0)$ on the set $\{(i,j)\in\Sigma|j\geq j_0\}$ we conclude by saying that $S_n(F)$ is linear on a big subset A of Σ, and $A\subset\mathrm{supp}(F-S_n(F))$. So we reduced the proof to the following lemma. □

Lemma 6.17. Let $F=\min(pi+qj,0)$ and Σ as above. Suppose that

$$A\subset\{F\neq S_n(F)\}, A=[0,q-1]\times[x,x+p].$$

Suppose that $S_n(F)$ restricted to A is linear. Then $\gcd(p,q)>1$.**verified**

Proof. Since $S_n(F)$ is periodic in the direction $(q,-p)$, we conclude that $S_n(F)(i,j)|_A=k(pi+qj)+k'$ for some $k,k'\in\mathbb{Z}$. The property of $\mathbb{Z}$-increasing in the direction $(0,1)$ implies that $k\geq 0$.

Suppose that $k=0, S_n(F)=k'$ on A. Then $k'<0$ because $S_n(F)\leq F$. However, we arrive to the contradiction with superharmonicity by looking at the top part of A. Indeed, let (i_0,j_0) be the point with the maximal second coordinate in the region $\{(i,j)|S_n(F)(i,j)=k'\}$. Then we have $S_n(F)(i,j)\geq S_n(F)(i_0,j_0)$ for all neighbors (i,j) of (i_0,j_0) and $S_n(F)(i_0,j_0+1)>S_n(F)(i_0,j_0)$ by the construction, this contradicts to the superharmonicity of $S_n(F)$ at (i_0,j_0). Therefore $k>0$.

Consider the function $F'(i,j)=F(i,j)-pi-qj$. Then $S_n(F')(i,j)=S_n(F)(i,j)-pi-qj$ and we can repeat verbatim all the above consideration, which gives $k<1$.

Since $k(pi+qj)$ must have integer coefficients and $0<k<1$ we conclude that $\gcd(p,q)>1$. □

Remark 6.18. The following equality holds: $|\mathcal{D}|=p^2+q^2$. **verified**

Proof. Consider a function $G\colon\mathbb{Z}^2\to Z$ given by $G(x,y)=\min(0,px-qy)$ and the lattice rectangle $R=[0,q]\times[0,p]\cap\mathbb{Z}^2$. Then

$$\mathcal{D}=\sum_{R\backslash(q,p)}\Delta G.$$

On the other hand, the sum of Laplacians over the rectangle R is reduced to the sum along its boundary, i.e.

$$\sum_R\Delta G=\sum_{t=0}^{p}(G(0,t)-G(-1,t))+\sum_{t=0}^{p}(G(q,t)-G(q+1,t))+\sum_{t=0}^{q}(G(t,0)-G(t,-1))+\sum_{t=0}^{q}(G(t,p)-G(t,p+1)).$$

Since $\Delta G(q,p) = -p-q$ we have

$$-\mathcal{D} = -p-q-\sum_R \Delta G = -p-q+(p+1)p+(q+1)q.$$

$\square$

This equality was observed earlier in [3]. Note also that p^2+q^2 is the weight of an edge parallel to (p,q) in the definition of the symplectic area (see [9] for details).

Corollary 6.19. (Used on pages) Let $p,p',q,q',a,a' \in \mathbb{Z}$. Suppose that $\gcd(p-p',q-q')=1$. Then there exists the canonical smoothing $F_{p,q,a,p',q',a'}(x,y)$ of $F(x,y)=\min(px+qy+a, p'x+q'y+a')$. Furthermore,

$$F_{p,q,a,p',q',a'}(x,y) = F_{p-p',q-q',a-a',0,0,0}(x,y) = F_{p-p',q-q',0,0,0,0}(x+(a-a')p'', y+(a-a')q'')$$

where $(p'',q'')\in\mathbb{Z}^2$ satisfies $(p-p')q''+(q-q')p''=1$. **verified**

Proof. The operation $f(x,y)\to f(x,y)+p'x+q'y+a'$ of adding a linear function commutes with n-smoothings and

$$\min\big((p-p')x+(q-q')y+(a-a'),0\big) = \min\big((p-p')(x+(a-a')p'')+(q-q')(y+(a-a')q''),0\big).$$

$\square$

6.4. **Triads and their perestroiki**[1].

We use the notation of Theorem 2. Consider Ψ_{vertex} (see (5.10)). We denote by Ψ'_{vertex} the function

(6.20)
$$\Psi'_{\text{vertex}}(x,y) = \min\big(\theta_{\min(0,p_1x+q_1y)}(x,y), \theta_{\min(0,p_2x+q_2y+c_1)}(x,y), \theta_{\min(p_1x+q_1y,p_2x+q_2y+c_1)}(x,y)\big).$$

Consider Ψ_{node} (see 5.11). We denote by Ψ'_{node} the function

(6.21)
$$\Psi'_{\text{node}}(x,y) = \min\big(\theta_{\min(0,p_1x+q_1y)}(x,y), \theta_{\min(0,p_2x+q_2y+c_1)}(x,y),$$

(6.22)
$$\theta_{\min(p_1x+q_1y,(p_1+p_2)x+(q_1+q_2)y+c_2)}(x,y), \theta_{\min(p_2x+q_2y+c_1,(p_1+p_2)x+(q_1+q_2)y+c_2)}(x,y)\big).$$

Note that each of the functions $F=\Psi'_{\text{vertex}},\Psi'_{\text{node}}$ is C-solid for some C, because they are made of periodic patches. Therefore by applying Lemma 5.7 we obtain the following remark.

Remark 6.23. (Used on pages [28]) Lemma 6.15 holds for $F=\Psi'_{\text{vertex}},\Psi'_{\text{node}}$, if n is bigger than C.

Lemma 6.24. Let F be Ψ_{vertex} (resp. Ψ_{node}) and F' be Ψ'_{vertex} (resp. Ψ'_{node}). The following conditions are equivalent:

- The sequence of n-smoothings $S_n(F)$ of F stabilizes.
- The sequence of n-smoothings $S_n(F')$ of F' stabilizes.**verified**

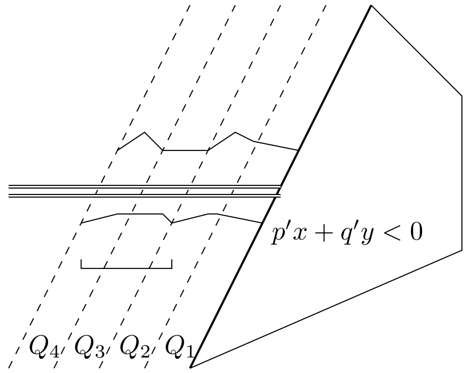
Proof. It is enough to note that F' coincides with F outside of a finite neighborhood of $D(F)$ because we have already proven Theorem 2 for the case of Ψ_{edge}. Hence there exists n such that $|F-F'|<n$. Therefore $S_n(F)\le F'\le F$ and smoothings of F' can be estimated by smoothings of F and vice versa. $\square$

We want to consider F' instead of F because of the following lemma.

Lemma 6.25. (Used on pages [28,29]) For all $k\in\mathbb{Z}_{\ge 0}$ the cardinality of the set $\{F'\neq S_k(F')\}$ is finite.**verified**

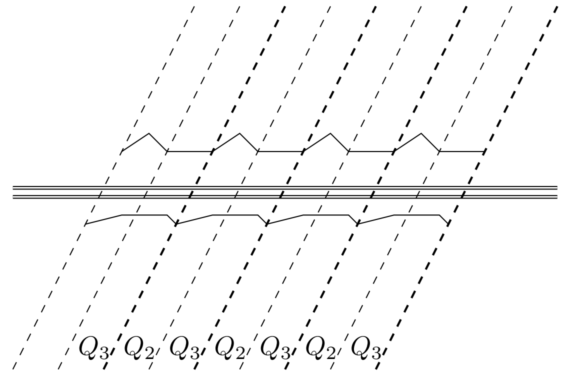
Proof. Each time we apply 1-smoothing, the set $\{H_k\neq 0\}$ (see Section 6.2, (6.3) for the definition of H_k) belongs to the finite neighborhood of $C(F)$. Therefore we only need to prove that $\{H_k\neq 0\}$ can not propagate far **along** a ray of $C(F)$, which is exactly the assertion in Lemma 6.26. $\square$

[1] Plural of *perestroika* (rus.), literally "rebuilding, restructuring". The word is coined by V. Arnold as a synonym to bifurcation.



Picture for Lemma 6.26. Horizontal line represents $\{\Delta\phi \neq 0\}$, broken lines along it represent the boundary of $\{G \neq S_1(G)\}$. Slices Q_2, Q_4 are identical.

FIGURE 8



Taking the region in between of Q_2, Q_4 we repeat it, thus obtaining a smoothing of $\theta_{px+qy,0}$ which is a contradiction.

FIGURE 9

Lemma 6.26. (Used on pages [27,27,28]) Let $(p, q), (p', q')$ be primitive vectors such that $pq' - p'q = 1$. Denote $A = \{(x, y)|p'x + q'y \leq 0\}$. Let $G : \mathbb{Z}^2 \to \mathbb{Z}$ be equal to $\theta_{\min(0,px+qy)}$ in the region $p'x + q'y \geq 0$. Then there exists a constant C such that

$$\{G \neq S_1(\phi)\} \setminus B_1(A) \text{ is contained in } B_{\mathrm{C}}(A).$$

Proof. We know that $\{\phi \neq S_1(\phi)\}$ is contained in the union of $B_1(A)$ and $B_1(\Delta\phi \neq 3)$. Therefore we need to prove that $\{\phi \neq S_1(\phi)\} \setminus B_1(A)$ (which is in 1-neighborhood of $\{\Delta\phi \neq 3\}$) can not be far from A. Suppose the contrary.

The pattern in $\mathrm{supp}(\Delta\phi)$ is periodic, so we can cut this pattern into periodic pieces, Figure 8. We look at $\phi - S_1(\phi)$ on the periodic pieces of the pattern and find two patterns such that the restriction of $\phi - S_1(\phi)$ on them is the same. Then it means that we could smooth more the initial function $\theta_{\min(0,px+qy)}$: indeed, take all the pieces in between of these two, repeat that all along as in Figure 9, and decrease $\theta_{\min(0,px+qy)}$ according to $\phi - S_1(\phi)$ periodically. $\square$

Lemma 6.27. (Used on pages [29]) Let $G : \mathbb{Z}^2 \to \mathbb{Z}, p, q, r \in \mathbb{Z}$. Let $G'(i, j) = G(i, j) - pi - qj - r$. Then $S_n(F)(i, j) - (pi + qj + r) = S_n(G)(i, j)$.

Definition 6.28. For a subset $A \subset \mathbb{Z}^2$ we define $r(A)$ as $\max_{(i,j)\in A}(\sqrt{i^2 + j^2})$, i.e. the maximal distance between A and $(0, 0)$.

Lemma 6.29. (Used on pages [29]) The sequence $R_n = r\Big(\{F' \neq S_n(F')\}\Big)$ grows at most linearly in n, i.e. there exists a constant C_4 such that $R_n \leq C_4 n$ for all $n \in \mathbb{Z}_{>0}$. **verified**

Proof. Let c_n be the minimal number such that the support of $F' - S_n(F')$ is contained in $B_{c_n}(O)$. It is enough to prove that there exists a constant C_4 such that $c_{n+1} \leq c_n + \mathrm{C}_4$ for all n. Now, look at how the support of $F' - S_{n+1}(F')$ differs from the support of $F' - S_n(F')$ outside of $B_{c_n}(O)$. It follows from Remark 6.23 that $\mathrm{supp}(F' - S_n(F'))$ belongs to the $n + 1$-neighborhood of $\mathrm{supp}(\Delta F')$ for n big enough. Then we use Lemma 6.26. $\square$

Lemma 6.30. (Used on pages [29]) There exist $\mathrm{C}_5 > 0$ such that for any point $v \in \mathbb{Z}^2$ the set $\{k|H_k(v) = 0\}$ (see Section 6.2 for the definition of H_i) contains less than $\mathrm{C}_5|v|$ elements. In particular, for any v there exist m such that $H_k(v) = 1$ for all $k > m$.**verified**

Proof. Let $v \in \mathbb{Z}^2$. We argue as in Section 6.3. We supposed that the sequence $\{S_n(F)\}_{n=1}^{\infty}$ does not stabilize. Therefore, by Corollary 6.8 for each $n \in \mathbb{Z}_{>0}$ the set $A_n = \{v \in \mathbb{Z}^2|S_n(F)(v) = F(v) - n\}$

is not empty. Hence $A_1 \supset A_2 \supset A_3 \ldots$, and Lemma 6.25 tells that A_1 is finite. Thus we can take $v_0 \in \bigcap_{n\geq 1} A_n$. Consider the vector $\tilde{v} = v - v_0$.

Let C be the tropical curve defined by F. Draw $\tilde{v}$ from the vertex of C. There exist a choice of two primitive vectors e_1 and e_2 spanning two edges of C such that $\tilde{v} = k_1e_1 + k_2e_2$ for some non-negative integers k_1 and k_2. By an integer change of coordinates we can send e_1 to $(-1,0)$, e_2 to $(0,-1)$ and $F'(x,y)$ to $\min(0,-x,-y)$. In this coordinates F' is $(0,1)$-increasing and $(1,0)$-increasing. By Corollary 6.13 F'_n is also $(0,1),(1,0)$-increasing. This implies

$$F'(v_0) - n = S_n(F')(v_0) \geq S_n(F')(k_1e_1 + k_2e_2 + v_0) \geq S_n(F')(v_0 + \tilde{v}) = S_n(F')(v) = F'(v) - \tilde{H}_n(v),$$

where $\tilde{H}_n = \sum_{k=0}^{n-1} H_k$. And therefore, $\tilde{H}_n(v) \geq n + F'(v_0) - F'(v)$. To conclude, we note that there exists a constant C_5 such that $|F'(v_0) - F'(v)| \leq \mathrm{C}_5|v|$. □

Lemma 6.31. (Used on pages [29]) The sequence $r_n = \max\Big\{r|B_r(O) \subset \{S_n(F') \neq S_{n+1}(F')\}\Big\}$ grows at least linearly in n, i.e. there exists a constant C_6 such that $r_n \geq \mathrm{C}_6 n$ as long as $n \geq N$.**verified**

Proof. In the notation of Lemma 6.30 we take v_0 which enjoys the property $S_n(F')(v_0) = F'(v_0) - n$. Take any point $v \in \mathbb{Z}^2$. Note that we may always suppose that F is $(v_0 - v)$-increasing in the direction by adding to F a suitable linear function, this affects F' by adding the same linear function. Hence we may suppose that F' is $(v_0 - v)$-increasing. Then $S_n(F')(v) \leq S_n(F')(v_0) \leq F'(v_0) - n$. Therefore there exists a constant C_6 (depending on the slopes of the linear parts of F) such that if $|v - v_0| < \mathrm{C}_6 n$ then $F'(v) \geq F'(v_0) - n$. For such v, clearly, $S_n(F')(v) < F'(v)$ which, with the fact that $|v_0|$ is a fixed finite number, concludes the lemma. □

Lemma 6.32. (Used on pages [30]) There exists a constant ρ such that the number of points v in $B_{n\mathrm{C}_4}(0,0)$ with $\Delta S_n(F')(v) < 0$ is at most ρn for n big enough.**verified**

Proof. The functions $F', S_n(F')$ coincide outside $B_{n\mathrm{C}_4}(O)$ and superharmonic, therefore it follows from Lemma A.3 that

$$\sum_{v\in B_{n\mathrm{C}_4}(0,0)} |\Delta S_n(F')(v)| = \sum_{v\in B_{n\mathrm{C}_4}(0,0)} \Delta S_n(F')(v) = \sum_{v\in B_{n\mathrm{C}_4}(0,0)} \Delta F'(v).$$

Then, outside of a finite neighborhood of $(0,0)$ the function $\Delta F'(v)$ coincide locally with $\Delta\theta_{\min(p'x+q'y,c)}$ in each direction, and $\sum_{v\in B_{n\mathrm{C}_4}(0,0)} \Delta\theta_{\min(p'x+q'y,c)}$ is linear in n for any coprime $p', q' \in \mathbb{Z}^2$. This works both for Ψ_{vertex} and Ψ_{node}. □

Lemma 6.33. (Used on pages [30]) Let F be Ψ_{vertex} or Ψ_{node}. Then there exists $k > 0$ that for each $n > 0$ for each square S of size $k \times k$ inside the set $\{F \neq S_n(F)\}$ the function $S_n(F)$ is not linear on S.

Proof. Suppose that $S_n(F)$ is linear on such a square S. Denote by V the set of primitive integer vectors v going in the directions of the edges of the tropical curve corresponding to F, in the direction "from the vertex" of this tropical curve. Take k bigger than twice the maximal length of the vectors in V. Using Lemma 6.27, for each directions $v \in V$ we can add a linear function to F, and so we can suppose that F is v-increasing. It follows from the proof of Lemma A.8 that if (after adding a linear function) $S_n(F)$ is v-increasing for each $v \in V$, then $S_n(F)$ is a linear function on some part of $\{F \neq S_n(F)\}$, and this linear function can not have integer coefficients which is contradiction. □

The main idea of the proof of the Theorem 2 is as follows. Using the fact that r_n grows linearly, we see that the set where $S_n(F') \neq F'$ encircles a figure with the area of order n^2. In this figure, the number of points where $\Delta S_n(F') \neq 0$ is linear, and we can find a part where $S_n(F')$ is harmonic. A positive harmonic function of at most linear grows is linear. This will contradict to the Lemma A.8.

Proof of Theorem 2. (For $F = \Psi_{\text{vertex}}$, and for $F = \Psi_{\text{node}}$.)

We suppose that the sequence $\{F_n\}$ of n-smoothings of F does not stabilize as $n \to \infty$. Therefore, by Lemma 6.24 the sequence of $\{S_n(F')\}$ of n-smoothings of F' does not stabilize. Lemma 6.25 asserts that the support of $F' - S_n(F')$ is finite, and Lemmata 6.29,6.31 tell us that the set $\{F' \neq S_n(F')\}$ grows at most and at least linearly in n. Refer to Figure 10: the grey region is $\{F' \neq S_n(F')\}$, internal (resp. external) circle has radius $r_n = \mathrm{C}_6 n$, (resp. $R_n = \mathrm{C}_4 n$) and represents a subset (resp. superset) of $\{F' \neq S_n(F')\}$.

Remark 6.9 eliminated several simple cases. So, after change of coordinates $x \to y, y \to x$, if necessary, we may assume that the tropical curve defined by F does not contain the vertical ray (the dashed line on Figure 10) and the top part of the dashed line belongs to the region where $F \equiv 0$. Note that F is $(0,1)$-increasing (Definition 6.10), and, thus, F' and all $S_n(F')$ are $(0,1)$-increasing by Corollary 6.13.

By Lemma 6.32 the number of points v in $B_{n\mathrm{C}_6}(O)$ with $\Delta S_n(F')(v) < 0$ is bounded from above by ρn for some fixed ρ. We draw a rectangular R like in Figure 10. Namely, the vertical sides of R lie on different sides of the dashed line $x = 0$, the horizontal sides has length cN with some fixed c, R does not intersect the set $\Delta F' < 0$ outside of $B_{\mathrm{C}_6 n}(O)$.

Take N big enough and consider $S_N(F')$. Then we choose l big enough and cut R into squares of size $\frac{cN}{l}$ (later we refer to them as small squares). We consider the intersection $R' = R \cap \{F' \neq S_N(F')\}$ and pick all the small square which belongs to this intersection.

Comparing the area of $R' \sim N^2$ with ρN we see that there exists a small square S which contains at most $\frac{cN}{k}$ points v with $\Delta S_N(F')(v) < 0$. Let M be the minimum of $S_N(F')$ on R'. Then Lemma A.7 implies that $S_n(F') - M$ should be linear on this small square $k \times k$ which is a subset of S. To conclude we use Lemma 6.33. $\square$

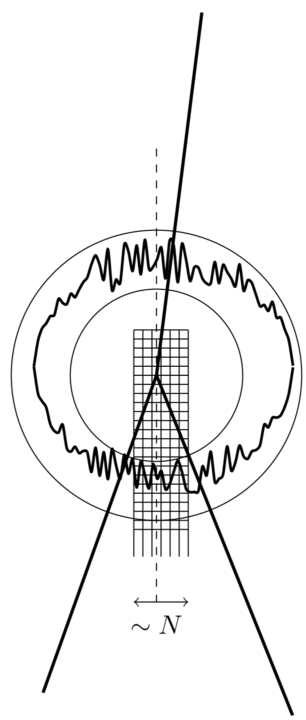


FIGURE 10. An illustration for the proof of Theorem 2.

7. The weights of the edges via a weak convergence.

In the notation of Theorem 1, define $\psi_{\mathrm{h}}(x,y) = \mathrm{h}^{-1}\left(3 - \phi^\circ_{\mathrm{h}}(\mathrm{h}[\mathrm{h}^{-1}x], \mathrm{h}[\mathrm{h}^{-1}y])\right)$. Note that ϕ_{h} is not zero only near $D(\phi^\circ_{\mathrm{h}})$.

Theorem 3 (Theorem 2 announced in [10])**.** There exists a $*$-weak limit ψ of the sequence ψ_{h} as $\mathrm{h} \to 0$. Moreover, there exists a unique assignment of weights m_e for the edges e of $C(f_{\Omega,P})$ such that for all smooth functions Φ supported on Ω we have

$$\psi(\Phi) = \lim_{\mathrm{h}\to 0} \int_{\mathbb{R}^2} \psi_{\mathrm{h}} \Phi = \sum_{e\in E} \left(||l_e|| \cdot m_e \cdot \int_e \Phi \right),$$

where E is the set of all edges of $C(\Omega, P)$ and l_e is a primitive vector of $e \in E$, i.e. the coordinates of l_e are coprime integers and l_e is parallel to e.

Proof. We know that outside of $C(f_{\Omega,P})$ the $*$-weak limit of ψ_{h} is zero, because for any $\varepsilon > 0$ for h small enough $\psi_{\mathrm{h}} \equiv 3$ outside of ε-neighborhood of $C(f_{\Omega,P})$ by Theorem 1. Consider an edge e of $C(f_{\Omega,P})$ and a strict subinterval e' of it, $e' \subset e$. Consider a small parallelogram Q containing e' whose one side is parallel to e and another one is infinitesimally small, $Q \cap C(f_{\Omega,P}) = e'$. Choosing h small enough and using Lemma A.3 (letting $\Gamma = Q$ and $\partial\Gamma = \mathbb{Z}^2_{\mathrm{h}} \setminus Q$) we see that $\int_Q \psi_{\mathrm{h}} \sim \mathrm{h}^{-1}|l_e| \cdot |e'|$, because only the contribution over the long sides of Q matters. We just need to show that the contribution of infinitesimally small sides of Q is small. Here we use Lemma 3.18: the total amount of topplings is bounded by a constant C, therefore the sum of defects along these sides is bounded by Ch^{-1} times the sum of their length. Therefore, choosing the lengths of these sides significantly smaller than the length of e' we conclude the proof.

□

8. Discussion

8.1. **About generalizations for higher dimensions.** It seems that all the results can be extended to higher dimensions: instead of a plane $\Omega \subset \mathbb{R}^2$ we consider a convex set in $\mathbb{R}^n$. Let us mention how one should change the cornerstones of our proofs. Modified Lemma A.8 will take as its main object a lattice polyhedron without lattice points except vertices. It seems that Theorem 2 can be extended to higher dimensions more or less by induction. There seem to be some problems in generalizing Section 4.4 to the higher dimension case — the pictures appear to be as in the resolution of singularities for toric varieties, but we actually need less then smoothness, so at the current state we see no conceptual problems.

8.2. **Relation to the other convergences in sandpiles.** We defined the procedure of smoothing (Definition 5.6), and found several properties of smoothing of piece-wise linear functions. It seems that the pictures in [14] are the smoothings of piece-wise quadratic functions, mixed with linear functions. In order to prove that one should generalize the definition and properties of $\mathbb{Z}$-increasing (which means more or less monotonicity) (Definition 6.10, Lemma 6.12) for higher discrete derivatives.

8.3. **Sand dynamic on tropical varieties, divisors.** Let G be a graph and $V = \{v_1, v_2, \dots, v_k\}$ be a collection of some of its vertices. Consider the following state $\phi_V = \sum_{v\in G} v \cdot (\deg(v) - 1) - \sum_{v\in V} \delta_v$. It corresponds to the divisor $V = \sum_{v\in V} \delta_v$. Let $P = \{p_1, p_2, \dots, p_n\}$ be another collection of vertices of G. There there exists a divisor linearly equivalent to V and containing P if and only if the relaxation of $\phi_V + \sum_{p\in P} \delta_p$ terminates.

In this article we studied sandpile on $\mathrm{h}\mathbb{Z}^2 \cap \Omega$. We can produce the same type of problems for a tropical variety, if we have a sort of grid on it. The convergence results are expected to be formulated and proven in the same way.

What is an interesting aspect of the possible applications is the tropical divisors. Indeed, using relaxation in sand dynamic we can understand if there exist a divisor linearly equivalent to a given tropical divisor L, passing through prescribed set of points. For that we represent L as a collection of sand-solitons glued with help of sand triads (because locally L looks like a tropical plane curve and we know which sand-solitons should we take), then we add sand to the points $p_1, p_2, \dots, p_n$ and relax the obtained state. If the relaxation terminates, it produces the divisor which is linearly equivalent to L. If not, that means that such a divisor does not exist.

8.4. **Continuous limit.** Tropical curves appear as limits of algebraic curves under the map $\log_t|\cdot|$ when $t\to\infty$. It is natural to ask how we can obtain a continuous family of "sandpile" models which converges to the pictures we studied in this paper. An attempt to present such model was made in [27] and it is yet to be understood how translate the results and methods of this paper in this new framework.

Appendix A. Discrete harmonic functions.

A.1. Discrete superharmonic integer-valued functions and auxiliary statements.

Lemma A.1. ([5], Theorem 5) (Used on pages [32,32,33]) Let $R>1, v\in\mathbb{Z}^2$, and $F:B_R(v)\cap\mathbb{Z}^2\to\mathbb{R}$ be a discrete **non-negative** harmonic function. Let $v'\sim v$, then

$$|F(v')-F(v)|\leq\frac{\mathrm{C}_0\cdot F(v)}{R},$$

where C_0 is an absolute constant.

Note that in [5] this result is formulated for a discrete harmonic function F on $\mathbb{Z}^3$ lattice, but the two-dimensional case follows if we substitute $F(x,y,z)=F(x,y)$ for all $z\in\mathbb{Z}$. Morally, this lemma provides an estimate on a derivative of a discrete harmonic function. We call $\partial_x F(x,y)=F(x+1,y)-F(x,y)$ the *discrete derivative* of F in the x-direction. The derivative ∂_y in the y-direction is defined in a similar way. We denote by $\partial_\bullet F$ the discrete derivative of a function F in any of directions x or y.

Lemma A.2 (Integer-valued discrete harmonic functions of sublinear growth)**.** (Used on pages [23,33]) Let $v\in\mathbb{Z}^2$ and $\mu>0$ be a constant. Let $R>4\mu\mathrm{C}_0^2$ (C_0 is from Lemma A.1). For a discrete **integer-valued** harmonic function $F:B_{3R}(v)\cap\mathbb{Z}^2\to\mathbb{Z}$, the condition $|F(v')|\leq\mu R$ for all $v'\in B_{3R}(v)$ implies that F is linear in $B_R(v)\cap\mathbb{Z}^2$. **verified**

Proof. Consider F which satisfies the hypothesis of the lemma. Note that $0\leq F(v')+\mu R\leq 2\mu R$ in $B_{3R}(v)$ and applying Lemma A.1 for each $v'\in B_{2R}(v)$ yields

$$|\partial_\bullet F(v')|\leq\frac{\mathrm{C}_0\cdot 2\mu R}{R}=2\mu\mathrm{C}_0,\text{ for all } v'\in B_{2R}(v).$$

The functions $\partial_\bullet F(v')+2\mu C_0$ are discrete harmonic too, and are bounded from both sides by 0 and $4\mu\mathrm{C}_0$ respectively, hence

$$|\partial_\bullet\partial_\bullet F(v')|\leq\frac{4\mu\mathrm{C}_0^2}{R}<1,\text{ for } v'\in B_R(v)\text{ if } R>4\mu\mathrm{C}_0^2.$$

Since F is integer-valued, all the derivatives $\partial_\bullet\partial_\bullet F$ are also integer-valued. Therefore all the second derivatives of u are identically zero in $B_R(v)$, which implies that F is linear in $B_R(v)$. □

Let Γ be a finite subset of $\mathbb{Z}^2$, $\partial\Gamma$ be the set of points in Γ which have neighbors outside Γ. Let F be any function $\Gamma\to\mathbb{Z}$.

Lemma A.3. (Used on pages [26,29,31]) In the above hypothesis the following equality holds:

$$\sum_{v\in\Gamma\setminus\partial\Gamma}\Delta F(v)=\sum_{\substack{v\in\partial\Gamma,\\ v'\in\Gamma\setminus\partial\Gamma, v\sim v'}}\big(F(v)-F(v')\big).\textbf{verified}$$

Proof. We develop left side by definition of ΔF. All the terms $F(v)$, except for the vertices v near $\partial\Gamma$, cancel each other. So we conclude by a direct computation. □

Definition A.4. (Used on pages [33]) For $v\in\mathbb{Z}^2$ we denote by $G_v:\mathbb{Z}^2\to\mathbb{R}$ the function with the following properties:

- $\Delta G_v(v)=1$,
- $\Delta G_v(w)=0$ if $w\neq v$,
- $G_v(v)=0$,
- $G_v(w)=\frac{1}{2\pi}\log|w-v|+c+O\left(\frac{1}{|w-v|^2}\right)$ when $|w-v|\to\infty$, where c is some constant.

It is a classical fact that G_v does exist and is unique ([28], (15.12), or [13], p.104, see [6], Remark 2, for more terms in the Taylor expansion).

Corollary A.5. Let $v=(0,0)$. By direct calculation we conclude that there exists a constant C_1 such that

$$|\partial_\bullet\partial_\bullet G_v(i,j)|\leq\frac{\mathrm{C}_1}{(i^2+j^2+1)}.$$

Lemma A.6. There exists a constant C_2 such that the following inequality holds for all $N\in\mathbb{Z}_{>0}, v\in\mathbb{Z}^2$:

$$\sum_{-N\leq i,j\leq N}|\partial_\bullet\partial_\bullet G_v(i,j)|\leq \mathrm{C}_2\ln N.\textbf{verified}$$

Proof. The maximum of this sum is attained when $v=(0,0)$. Then the sum is estimated from above by

$$\int\limits_{1\leq x^2+y^2\leq 2N^2}\frac{2\mathrm{C}_1\,dxdy}{x^2+y^2}+\mathrm{C}_1<\mathrm{C}_1\int_{r=1}^{2N}\frac{rdr}{r^2}+\mathrm{C}_1\leq \mathrm{C}_2\ln N.$$

□

Lemma A.7. (Used on pages [30]) Let $k,\mu\in\mathbb{N}$. Then there exists $N>0$ with the following property. Let F be any non-negative integer-valued function on the square $\Gamma=\big([0,N]\times[0,N]\big)\cap\mathbb{Z}^2$ such that for all $z\in\Gamma$

$$|F(z)|\leq\mu(|z|+1).$$

Let $v_1,v_2,\dots v_N$ be points in $\mathbb{Z}^2$ (not necessary distinct) and suppose that $G=F+\sum_{i=1}^N G_{v_i}$ (see Definition A.4) is a discrete harmonic function on Γ. Then there exists a square of size $k\times k$ in Γ such that F is linear on this square. **verified**

Proof.

Applying Lemma A.1 for $v\in\Gamma'=\left[\frac{N}{5},\frac{4N}{5}\right]\times\left[\frac{N}{5},\frac{4N}{5}\right]$ we obtain $|\partial_\bullet G|\leq\frac{\mu\mathrm{C}_0(N+1)}{N/5}$.

Proceeding as in Lemma A.2, we see that in the square

$$\Gamma''=\left[\frac{2N}{5},\frac{3N}{5}\right]\times\left[\frac{2N}{5},\frac{3N}{5}\right]$$

the second discrete derivatives $\partial_\bullet\partial_\bullet G$ are at most

$$\frac{2\mu\mathrm{C}_0^2(N+1)}{(N/5)^2}$$

by the absolute value, which is less than $\frac12$ if N is big enough.

Since $\sum_{v\in\Gamma}\partial_\bullet\partial_\bullet G_{v_i}(v)$ is at most $\mathrm{C}_2\ln N$ (Lemma A.6), we obtain by the direct calculation that

$$\sum_{i=1}^N\sum_{w\in\Gamma}\partial_\bullet\partial_\bullet G_{v_i}(w)\leq \mathrm{C}_2N\ln N.$$

We cut Γ'' on $(\frac{N}{5k})^2$ squares of size $k\times k$. Therefore for N big enough we can find a square $\Gamma'''\subset\Gamma''$ of size $k\times k$ such that

$$\sum_{i=1}^N|\partial_\bullet\partial_\bullet G_{v_i}(v)|\leq 1/3 \text{ at every point } v\in\Gamma'''.$$

The estimates for $|\partial_\bullet\partial_\bullet G|$ and $\sum_{i=1}^N|\partial_\bullet\partial_\bullet G_{v_i}|$ imply that for all second derivatives of F we have $\partial_\bullet\partial_\bullet F(v)=0$ for $v\in\Gamma'''$. Thus F is linear on Γ'''. □

Suppose that the area of a lattice triangle with vertices $(0,0),(p,q),(p'q')\in\mathbb{Z}^2$ is $1/2$. Let $c,c',d,d'\in\mathbb{Z}$ be constants. Consider the functions

$$H_1(i,j)=\min(pi+qj+c,p'i+q'j+c',d),$$
$$H_2(i,j)=\min(pi+qj+c,p'i+q'j+c',d,(p+p')i+(q+q')j+d').$$

Lemma A.8. (Used on pages [29,29,31]) There exists no linear function $H'(i,j) = p''i + q''j + a''$ with

$$(p'', q'') \in \mathbb{Z}^2 \setminus \{(0,0), (p,q), (p',q')\}$$

such that the set $\{(i,j)|H'(i,j) < H_1(i,j)\}$ is bounded. There exists no linear function $H'(i,j) = p''i + q''j + a''$ with

$$(p'', q'') \in \mathbb{Z}^2 \setminus \{(0,0), (p,q), (p',q'), (p+p', q+q')\}$$

such that the set $\{(i,j)|H'(i,j) < H_2(i,j)\}$ is bounded.**verified**

Proof. Applying $\mathrm{SL}(2,\mathbb{Z})$-change of the coordinates and a parallel translation, we may restrict ourselves to the model case

$$H_1(i,j) = \min(i,j,0) \text{ and } H'(i,j) = Pi + Qj + R$$

for some $P, Q \in \mathbb{Z}, R \in \mathbb{R}$. Consider the restriction of H' to the ray $(t,0)_{t\in\mathbb{R}_{\geq 0}}$. Since H' must be bigger than H far from zero, we see that $P \geq 0$. By considering rays $\{(0,t)\}_{t\in\mathbb{R}_{\geq 0}}$and $\{(-t,-t)\}_{t\in\mathbb{R}_{\geq 0}}$, we conclude that $Q \geq 0$ and $P + Q \leq 1$. Using the fact that $(p'', q'') \in \mathbb{Z}^2$ we arrive to a contradiction. The second part of the statement can be proven similarly by reduction to the model case $H_2 = \min(i, j, i+j, 0)$. □

Appendix B. Locally finite relaxations and waves

In this section we study the relaxations and stabilizability issues. The main goal here is to establish The Least Action Principle (Proposition B.16) and wave decomposition (Proposition B.28 and Corollary B.32) for **locally-finite** relaxations (Definition B.6) on infinite graphs. We also have to prove that given a finite upper bound on a toppling function of a state, there exists a relaxation sequence of this state which converges pointwise to a stable state (Lemma B.13).

The proofs are the same as in the finite case, but in the absence of references we give all the details here. Sandpiles on infinite graphs were previously considered, for example, in [**?**, **?**, **?**], but only from the distribution point of view: in their approach the relaxation (after adding a grain to a random configuration in a certain class) is locally finite almost sure with respect to a certain distribution. However, there are a lot of similarities between this section and [8].

B.1. **The Least Action Principle for locally finite relaxations, relaxability.** Let Γ be at most countable set, $\tau\colon \Gamma \to \mathbb{Z}_{>0}$ be a function and $\gamma\colon \Gamma \to 2^\Gamma$ be a set-valued function such that

- $v \notin \gamma(v)$,
- if $v \in \gamma(w)$, then $w \in \gamma(v)$,
- $|\gamma(v)| \leq \tau(v)$ for all $v \in \Gamma$, where $|\gamma(v)|$ denotes the number of elements in the set $\gamma(v)$.

We call τ the *threshold* function and interpret $\gamma(v)$ as the set of neighbors of a point $v \in \Gamma$. We write $u \sim v$ instead of $u \in \gamma(v)$, because it is a symmetric relation. The laplacian Δ is the operator on the space $\mathbb{Z}^\Gamma = \{\phi\colon \Gamma \to \mathbb{Z}\}$ of *states* on Γ given by

$$\Delta\phi(v) = -\tau(v)\phi(v) + \sum_{u\sim v} \phi(u).$$

A function ϕ is called *superharmonic* if $\Delta\phi \leq 0$ everywhere.

Remark B.1. (Used on pages [36]) Note that condition $|\gamma(v)| \leq \tau(v)$ for all $v \in \Gamma$ is equivalent to the superharmonicity of the function $\phi \equiv 1$.

Example B.2. In our main situation, Γ is a subset of $\mathrm{h}\mathbb{Z}^2$ and $|\gamma(v)| = \tau(v) = 4$ for all $v \in \Gamma \setminus \partial\Gamma$. In this case we obtain the standard definition of a laplacian on Γ°:

$$\Delta\phi(v) = -4\phi(v) + \sum_{u\sim v} \phi(u). \tag{B.3}$$

Definition B.4. For a point $v \in \Gamma$, we denote by T_v the *toppling* operator acting on the space of states $\mathbb{Z}^\Gamma$. It is given by

$$T_v\phi = \phi + \Delta\delta(v),$$

where $\delta(v)$ is the function on Γ taking 1 at v and vanishing elsewhere.

Definition B.5. (Used on pages [8,35]) A *relaxation sequence* $\phi_\bullet$ of a state $\phi \in \mathbb{Z}^\Gamma$ is a sequence of functions $\phi_0, \phi_1, \phi_2, \dots$ such that $\phi_0 = \phi$ and for each $k \geq 0$ there exists $v_k \in \Gamma$ such that $\phi_k(v_k) \geq \tau(v_k)$ and $\phi_{k+1} = T_{v_k}\phi_k$. The *toppling function* $H_{\phi_\bullet}\colon \Gamma \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ of the relaxation sequence $\phi_\bullet$ is given by

$$H_{\phi_\bullet} = \sum_{k=0}^{\infty} \delta(v_k),$$

it counts the number of topplings at every point during this relaxation. We also refer to $\{v_1, v_2, \dots\}$ as a relaxation sequence. We may consider finite relaxation sequences $\{v_1, v_2, \dots, v_n\}$, with a natural modification of the definition of the toppling function, $H_{\phi_\bullet} = \sum\limits_{k=0}^{n} \delta(v_k)$.

Definition B.6. (Used on pages [34,36]) A relaxation $\phi_\bullet$ is called *locally-finite* if $H_{\phi_\bullet}(v)$ is finite for every $v \in \Gamma$. The *result of a locally-finite relaxation* is the state ϕ' given by the point-wise limit

$$\phi' = \phi_0 + \Delta H_{\phi_\bullet} = \lim_{k\to\infty} \phi_k.$$

Lemma B.7. (Used on pages [35,36,36,38,38,38,39]) Consider a locally-finite relaxation $\phi_\bullet$ for a state ϕ and a function $F\colon \Gamma \to \mathbb{Z}_{\geq 0}$ such that $\phi + \Delta F < \tau$. Then $H_{\phi_\bullet}(v) \leq F(v)$ for all $v \in \Gamma$. **verified**

Proof. We use the notation from Definition B.5. Consider the relaxation sequence $\phi_\bullet$ and the corresponding sequence of functions H_n for $n = 1, \dots$ given by

$$H_n = \sum_{k=1}^{n} \delta(v_k). \tag{B.8}$$

Let $H_0 \equiv 0$. It suffices to show that $H_n \leq F$ for every n, and $H_0 \equiv 0 \leq F$. Suppose that $n \geq 0$ and $H_{n-1} \leq F$. Since $H_n = H_{n-1} + \delta(v_n)$, it is enough to show that $H_{n-1}(v_n) < F(v_n)$. We know that $\phi_n(v_n) \geq \tau(v_n)$ and $\phi_n(v_n) = \phi_0(v_n) + \Delta H_{n-1}(v_n)$. Therefore,

$$\begin{aligned}\tau(v_n) \leq &\phi_0(v_n) - \tau(v_n)H_{n-1}(v_n) + \sum_{u\sim v_n} H_{n-1}(u) \leq \\ \leq &\phi_0(v_n) - \tau(v_n)H_{n-1}(v_n) + \sum_{u\sim v_n} F(u) = \\ = &\phi_0(v_n) + \Delta F(v_n) + \tau(v_n)\big(F(v_n) - H_{n-1}(v_n)\big).\end{aligned}$$

Since $\phi_0(v_n) + \Delta F(v_n) < \tau(v_n)$ (by the hypothesis of the lemma) and $\tau(v_n) > 0$, we conclude that

$$1 \leq F(v_n) - H_{n-1}(v_n).$$

□

Corollary B.9. Consider a state ϕ. If there exist a function $F\colon \Gamma \to \mathbb{Z}_{\geq 0}$ such that $\phi + \Delta F < \tau$, then all relaxation sequences of ϕ are locally finite.

Proof. Applying Lemma B.7 twice, we have $H_{\phi^1_\bullet} \leq H_{\phi^2_\bullet}$ and $H_{\phi^1_\bullet} \geq H_{\phi^2_\bullet}$. □

Lemma B.10. (Used on pages [36,36]) Consider a state ϕ and the set Ψ of all its relaxations $\psi_\bullet$. Then there exists a relaxation $\phi_\bullet$ of ϕ such that

$$H_{\phi_\bullet}(v) = \sup_{\psi_\bullet\in\Psi} H_{\psi_\bullet}(v), \forall v \in \Gamma.\textbf{verified}$$

Proof. Consider the set $W = \{(v,k)\} \subset \Gamma \times \mathbb{Z}_{\geq 0}$ which contains all pairs (v,k) such that there exists a relaxation sequence $\phi^{v,k}_\bullet \in \Psi$ which has k topplings at the vertex $v \in \Gamma$. Clearly, if $(v,k) \in W, k > 0$ then $(v, k-1) \in W$. The set W is at most countable, so we order it as $\{(v_n, k_n)\}_{n=1,2,\dots}$ in such a way that $(v, k-1)$ appears earlier than (v,k) for all $(v,k) \in W, k > 0$.

Take any relaxation sequence $\phi_\bullet$. We construct relaxation sequences $\phi^0_\bullet, \phi^1_\bullet, \dots$ in such a way that $\phi_\bullet = \phi^0_\bullet$, all $\phi^{\geq n}_\bullet$ coincide at first n topplings, and the toppling function of $\phi^n_\bullet(v_n)$ is at least k_n for all $n \geq 0$.

Let $\phi_\bullet^{n-1}$ be already constructed, $n \geq 1$, we construct $\phi_\bullet^n$ as follows.

If the toppling function of $\phi_\bullet^{n-1}$ at v_n is at least k_n, we are done. If not, take $\phi_\bullet^{v_n,k_n}$ and consider its toppling functions $H^i_{\phi_\bullet^{v_n,k_n}}$ as in (B.8). Take the first i such that there exists $w \in \Gamma$ such that $H_{\phi_\bullet^{n-1}}(w) < H^i_{\phi_\bullet^{v_n,k_n}}(w)$. Note that there is a moment $j > n$ when

$$H^j_{\phi_\bullet^{n-1}}(w') \geq H^i_{\phi_\bullet^{v_n,k_n}}(w')$$

for all $w' \sim w$. So we add to $\phi_\bullet^{n-1}$ the toppling at w somewhere j-th toppling, and denote the obtained relaxation sequence as $\phi_\bullet^{n-1}$ again. Note that after repeating this cycle of arguments a finite number of times, we will be done. □

Definition B.11. (Used on pages [8]) A state ϕ is called *stable* if $\phi < \tau$ everywhere. A state ϕ is called *relaxable* if there exist a locally-finite relaxation $\phi_\bullet$ of ϕ such that ϕ'(Definition B.6) is stable. Such a relaxation $\phi_\bullet$ is called *stabilizing*.

Corollary B.12. If ϕ is relaxable, then $H_{\phi_\bullet^1} = H_{\phi_\bullet^2}$ for any pair of stabilizing relaxations $\phi_\bullet^1$ and $\phi_\bullet^2$ of ϕ. In particular, $(\phi_\bullet^1)^\circ = (\phi_\bullet^2)^\circ$.

Lemma B.13. (Used on pages [34,36]) If all relaxations of a state ϕ are locally-finite, then ϕ is relaxable.**verified**

Proof. Consider a point $v \in \Gamma$. We will prove that there exist $N > 0$ such that $H_{\phi_\bullet}(v) < N$ for all relaxations $\phi_\bullet$ of ϕ. Suppose the contrary. Then there exists a sequence of relaxations $\phi_\bullet^n$ such that $\lim_{n\to\infty} H_{\phi_\bullet^n}(v) = \infty$. Applying Lemma B.10 to the sequence $\phi_\bullet^n$ we see that there exists a relaxation of ϕ, that is not locally-finite.

Therefore, for any $v \in \Gamma$ there exist a relaxation $\phi_\bullet^v$ such that $H_{\phi_\bullet}(v) \leq H_{\phi_\bullet^v}(v)$ for all relaxations $\phi_\bullet$ of ϕ. Applying Lemma B.10 again to the family of relaxations $\{\phi_\bullet^v\}_{v\in\Gamma}$ we find a relaxation sequence $\tilde\phi_\bullet$ such that $H_{\phi_\bullet}(v) \leq H_{\tilde\phi_\bullet}(v)$ for all relaxations $\phi_\bullet$.

We claim that $\tilde\phi_\bullet$ is a stabilizing relaxation. Suppose that $\phi + H_{\tilde\phi_\bullet}$ is not stable, i.e. there exists $v \in \Gamma$ such that $\phi(v) + H_{\tilde\phi_\bullet}(v) \geq \tau(v)$. Therefore, we can make an additional toppling at v after the moment when all the topplings at v and its neighbors in $\tilde\phi_\bullet$ are already made. This contradicts to the maximality of $\tilde\phi_\bullet$. □

Proposition B.14. (Used on pages [36,38]) A state ϕ is relaxable if and only if there exists a function $F\colon \Gamma \to \mathbb{Z}_{\geq 0}$ such that $\phi + \Delta F < \tau$.**verified**

Proof. If ϕ is relaxable then we can take F to be H_ϕ. On the other hand, if such F exists, then by Lemma B.7 all the relaxations of ϕ are locally-finite. Therefore, ϕ is relaxable by Lemma B.13. □

Definition B.15. (Used on pages [8]) Consider a relaxable state ϕ. Denote by H_ϕ *the toppling function of* ϕ, where H_ϕ is a toppling function of some stabilizing relaxation of ϕ. Define *the relaxation of* ϕ to be the state $\phi^\circ = \phi + \Delta H_\phi$.

Proposition B.16 (The Least Action Principle)**.** (Used on pages [34,37]) Let ϕ be a relaxable state and $F\colon \Gamma \to \mathbb{Z}_{\geq 0}$ be a function such that $\phi + \Delta F$ is stable. Then $H_\phi \leq F$. In particular, H_ϕ is the **pointwise** minimum of all such functions F. **verified**

Proof. Straightforward by Lemma B.7. □

Lemma B.17. (Used on pages [37]) Consider a stable state ϕ and a point $v \in \Gamma$. Then the state $T_v\phi$ is relaxable. **verified**

Proof. Consider a function $F(z) = 1 - \delta(v)$ for every $z \in \Gamma$. Then $T_v\phi + \Delta F = \phi + \Delta\delta(v) + \Delta(1 - \Delta\delta(v)) = \phi + \Delta 1$. Applying Remark B.1 we see that $T_v\phi + \Delta F$ is stable. Thus, $T_v\phi$ is relaxable by Proposition B.14. □

The least action principle allows us to construct an upper bound for the toppling function of ϕ_{h}.

Proof of Proposition 3.12. Recall that the piece-wise linear function $f_{\Omega,P}:\Omega\to\mathbb{R}_{\geq 0}$ is not smooth at the points $\{\mathbf{p}_1,\mathbf{p}_2,\dots,\mathbf{p}_n\}=P\subset\Omega$. We consider the function (cf. Definition 3.11)

$$F(x,y)=[\mathrm{h}^{-1}f_{\Omega,P}(x,y)],(x,y)\in\Gamma_{\mathrm{h}}.$$

Note that on Γ_{h} we have

$$\text{(Used on pages [21,22])}\ \mathrm{h}F(x,y)=\min_{(i,j)\in\mathcal{A}}(ix+jy+\mathrm{h}[a_{ij}\mathrm{h}^{-1}]) \tag{B.18}$$

where we take a_{ij} from $f_{\Omega,P}(x,y)=\min_{(i,j)\in\mathcal{A}}(ix+jy+a_{ij})$.

The difference between corresponding coefficients a_{ij} and $\mathrm{h}[a_{ij}\mathrm{h}^{-1}]$ is at most h. It follows from Lemma 4.17 that for each $\mathbf{p}\in C(f_{\Omega,P})$ there exists an h-close to $\mathbf{p}$ point $\mathbf{p}^{\mathrm{h}}\in\Gamma_{\mathrm{h}}$ such that $\mathbf{p}^{\mathrm{h}}$ and one of its neighbors in Γ_{h} belong to different regions of linearity of F.

This implies that $\Delta F(\mathbf{p}_i^{\mathrm{h}})<0$ for $i=1,\dots n$ for

$$P^{\mathrm{h}}=\{\mathbf{p}_1^{\mathrm{h}},\mathbf{p}_2^{\mathrm{h}},\dots,\mathbf{p}_n^{\mathrm{h}}\}\subset\Gamma_{\mathrm{h}}$$

chosen as explained above, i.e. P^{h} is a set of proper roundings. □

Proof of Proposition 3.13. The choice in the above proof depends only on arbitrary small neighborhood of $\mathbf{p}\in P$ on $C(\Omega,P)$. Therefore we can fix some choice (for example: "take the nearest point in Γ_{h} from the south-east region of $\mathbf{p}$") for all possible neighbors of a points in a tropical curve. □

Proof of Proposition 3.17. If $P\subset\mathbb{Z}^2,\mathrm{h}^{-1}\in\mathbb{N}$ and Ω is a lattice polygon, then $a_{ij}\in\mathbb{Z}$ in (3.5). Indeed, near the boundary of Ω that holds because Ω is a lattice polygon, and then when a linear function with $a_{ij}\in\mathbb{Z}$ is equal to another linear function at $\mathbf{p}\in P$ this guarantees that its coefficient is also integer. Therefore $\mathrm{h}[\mathrm{h}^{-1}a_{ij}]=a_{ij}$ in the proof of the Proposition 3.12, so $\mathbf{p}^{\mathrm{h}}=\mathbf{p}$ for all $\mathbf{p}\in P$. □

Proof of Proposition 3.15. This proposition follows from the definition of proper roundings, Definition 3.14, their existence, Proposition 3.12, by applying the Least Action Principle, Proposition B.16, because $F\geq 0$ on Γ, $\Delta F\leq 0$ on Γ°, and $\Delta F(\mathbf{p}^{\mathrm{h}})<0$ for $\mathbf{p}\in P$. □

Proof of Lemma 3.18. Consider a point $z\in D(\psi^\circ)$. Suppose that z does not belong to $D(\psi)$ or $\partial\Gamma_{\mathrm{h}}$. Then $\Delta H_\psi(z)<0$, therefore there exists a neighbor z_1 of z such that $H_\psi(z_1)<H_\psi(z)$. If z_1 does not belong to $D(\psi)$ or $\partial\Gamma_{\mathrm{h}}$, then $\Delta H_\psi(z_1)\leq 0$, and $H_\psi(z_1)<H_\psi(z)$ implies that z_1 has a neighbor z_2 such that $H_\psi(z_2)<H_\psi(z_1)$. We repeat this argument and find z_3,z_4, etc. Since $H_\psi\leq \mathrm{C}$, we can not have such a chain of length bigger than $\mathrm{C}+1$. Therefore, starting with any point $z\in D(\psi^\circ)$ and passing each time to a neighbor we reach $D(\psi)$ or $\partial\Gamma_{\mathrm{h}}$ by at most C steps, which concludes the proof. □

Remark B.19. A piecewise linear analog of Lemma 3.18 is Lemma 4.17.

B.2. **Waves, their action.** Sandpile waves were introduced in [7], see also [12].

Definition B.20. (Used on pages [21]) Let v be a point in Γ. The *wave* operator W_v, acting on the space of the stable states on Γ, is given by

$$W_v\phi=(T_v\phi)^\circ.$$

The *wave-toppling* function H_ϕ^v of ϕ at v is given by

$$H_\phi^v=\delta(v)+H_{T_v\phi}. \tag{B.21}$$

Remark B.22. Note that if v has 3 grains and has a neighbor in $\Gamma\setminus\partial\Gamma$ with 3 grains, then the result $W_v\phi$ is also a stable state.

Indeed, $T_v\phi$ has -1 grain at v, but the neighbor of v has 4 grains and will topple. So, eventually, we will have non-negative amount of sand at v.

Remark B.23. It is clear that $W_v\phi=\phi+\Delta H_\phi^v$.

Corollary B.24 ([24])**.** (Used on pages [38,38]) For any $u\in\Gamma$ the value $H_\phi^v(u)$ is either 0 or 1. Furthermore, $H_\phi^v(v)=1$.

Proof. It follows from the proof of Lemma B.17 that $H_{T_v\phi} \leq 1 - \delta(v)$. $\square$

Lemma B.25. (Used on pages [38]) Suppose that ϕ is a stable state and v a point in Γ. If $\phi + \delta(v)$ is relaxable and not stable, then the toppling function for the wave from v is less or equal than the toppling function for a relaxation of $\phi + \delta(v)$, i.e.

$$H^v_\phi(w) \leq H_{\phi+\delta(v)}(w) \ , \forall w \in \Gamma^\circ.\textbf{verified}$$

Proof. It is clear that $(\phi + \delta(v))(w) = \phi(w) < \tau(w)$ for all $w \neq v$ and $(\phi + \delta(v))(v) = \tau(v)$. Therefore, T_v is the first toppling in any non-trivial relaxation sequence for $\phi + \delta(v)$ and $H_{\phi+\delta(v)}(v) \geq 1$. In particular, the function $H_{\phi+\delta(v)} - \delta(v)$ is non-negative and $H_{T_v\phi} \leq H_{\phi+\delta(v)} - \delta(v)$ by Lemma B.7 since

$$T_v\phi + \Delta\big(H_{\phi+\delta(v)} - \delta(v)\big) = \phi + \Delta\delta(v) + \Delta\big(H_{\phi+\delta(v)} - \delta(v)\big) = \phi + \Delta H_{\phi+\delta(v)} = \big(\phi + \delta(v)\big)^\circ - \delta(v) < \tau.$$

$\square$

Definition B.26. (Used on pages [9]) Let ϕ be a relaxable state, H_ϕ be its toppling function. Let $0 \leq F \leq H_\phi$. The state $\phi + \Delta F$ is called a *partial relaxation* of ϕ.

Lemma B.27. (Used on pages [9,38]) Consider a relaxable state ϕ and an integer-valued function F on Γ such that $0 \leq F \leq H_\phi$. Then the state $\phi + \Delta F$ is relaxable and

$$H_{\phi+\Delta F} = H_\phi - F.\textbf{verified}$$

Proof. By Proposition B.14 the state $\phi + \Delta F$ is relaxable because

$$\phi + \Delta F + \Delta(H_\phi - F) = \phi + \Delta H_\phi = \phi^\circ < \tau$$

and $H_\phi - F$ is non-negative. In particular, $H_\phi - F \geq H_{\phi+\Delta F}$ by Lemma B.7. On the other hand, since $H_{\phi+\Delta F} + F \geq 0$, we have

$$\phi + \Delta(H_{\phi+\Delta F} + F) = \phi + \Delta F + \Delta H_{\phi+\Delta F} = (\phi + \Delta F)^\circ < \tau.$$

Applying again Lemma B.7, we have $H_\phi \leq H_{\phi+\Delta F} + F$. $\square$

We applied this lemma in the proof of Theorem 1, page 9, while considering the relaxation of $\phi^\varepsilon_{\mathrm{h}}$ as a partial relaxation of ϕ_{h}.

Proposition B.28. (Used on pages [34]) Let ϕ be a stable state and v be a point in Γ. Suppose that $\phi + \delta(v)$ is relaxable. Then the relaxation of $\phi + \delta(v)$ can be decomposed into sending n waves from v, i.e.

$$(\phi + \delta(v))^\circ = \delta(v) + W^n_v\phi,$$

where $n = H_{\phi+\delta(v)}(v)$ and $W^n_v(\phi) = W_v(W_v(\dots(\phi))\dots)$, n-th power of W_v. On the level of toppling functions, this gives

$$H_{\phi+\delta(v)} = \sum_{k=0}^{n-1} H^v_{(W^k_v\phi)}.\textbf{verified}$$

Added parenthesis in the subscript are for better readability only.

Proof. Combining Lemmata B.25 and B.27 we have

$$H_{\phi+\delta(v)} = H^v_\phi + H_{(W_v\phi+\delta(v))}.$$

If the state $W_v\phi+\delta(v)$ is not stable, then we can apply the same lemmata again. We complete the proof by iteration of this procedure and using Corollary B.24 (each wave has one toppling at v, therefore we have n waves). $\square$

Lemma B.29. (Used on pages [38]) If ϕ is a stable state and $v_1, \dots, v_m$ are vertices of Γ such that v_i is adjacent to v_{i+1} and $\phi(v_i) = \tau(v_i) - 1$ for all $i = 1, 2, \dots, m$, then $H^{v_1}_\phi = H^{v_m}_\phi$.**verified**

Proof. It follows from the simplest case $m = 2$, for which it is just a computation. $\square$

Definition B.30. (Used on pages) In a given state ϕ, a *territory* is a maximal by inclusion connected component of the vertices v such that $\phi(v) = \tau(v) - 1$. Given a territory $\mathcal{T}$, we denote by $W_{\mathcal{T}}$ the wave which is sent from a point in $\mathcal{T}$ (by Lemma B.29 it does not matter from which one).

Basically, Corollary B.24 tells us that a wave from v increases the toppling function exactly by one in the territory to which v belongs to, and by at most one in all other vertices.

Proposition B.31. (Used on pages [39]) Let ϕ be a stable stable, v be a point in Γ, and $F\colon \Gamma \to \mathbb{Z}_{\geq 0}$ be a function such that $F(v) > 0$ and $\phi + \Delta F$ is stable. Then $F \geq H^v_\phi$. **verified**

Proof. Similar to Lemma B.7. □

Corollary B.32 (Least Action Principle for waves)**.** (Used on pages [21,34]) Suppose that a state ϕ is stable. We send n waves from a vertex v. Let $H = \sum_{k=0}^{n-1} H^v_{(W^k\phi)}$ be the toppling function of this process. Let F be a function such that $\Delta F \leq 0, F(w) \geq 0$ for all w, and $F(v) \geq n$. Then $F(w) \geq H(w)$ for all w.**verified**

Proof. We apply Proposition B.31 n times, each time decreasing F by $H^v_{W^k(\phi)}$ for $k = 0, 1, \ldots, n-1$. □

Appendix C. Tropical series, tropical analytic curves

In this section we briefly recall the basic notions of tropical geometry; see general introductions to tropical geometry [2], [19], or [1] for details and motivation. Also we define and study tropical analytic series and curves.

C.1. **Tropical Laurent polynomials and tropical curves.** A tropical Laurent polynomial (later just *tropical polynomial*) f in two variables is a function on $\mathbb{R}^2$ which can be written as

$$f(x,y) = \min_{(i,j)\in\mathcal{A}} (ix + jy + a_{ij}), \tag{C.1}$$

where $\mathcal{A}$ is a **finite** subset of $\mathbb{Z}^2$. Each point $(i,j) \in \mathcal{A}$ corresponds to a *monomial* $ix + jy + a_{ij}$, the number $a_{ij} \in \mathbb{R}$ is called *the coefficient* of f of the monomial corresponding to the point $(i,j) \in \mathcal{A}$. The locus of the points where a tropical polynomial f is not smooth is a *tropical curve* (see [19]). We denote this locus by $C(f) \subset \mathbb{R}^2$.

Definition C.2. The Newton polygon $\text{Newt}(f)$ of f is the collection of all integer points in the convex hull of $\mathcal{A}$, i.e. $\text{Newt}(f) = \mathbb{Z}^2 \cap \text{ConvHull}(\mathcal{A})$.

We construct the extended Newton polytope

$$\widetilde{\text{Newt}}(f) = \text{ConvexHull}\{(i,j,t)|(i,j) \in \mathcal{A}, t \geq a_{ij}\} \subset \mathbb{R}^2 \times \mathbb{R}.$$

A subdivision $\text{Sub}(f)$ of $\text{Newt}(f)$ is defined by the images of the faces of $\widetilde{\text{Newt}}(f)$ under the projection along the third coordinate.

Proposition C.3. (Used on pages [6,6,17,40,41]) (See [2] for a proof and Figures 11,12 for an illustration.) This subdivision $\text{Sub}(f)$ of $\text{Newt}(f)$ is dual to $C(f)$ in the following sense:

- each connected component Φ (we call them *faces*) of the complement of $C(f)$ in $\mathbb{R}^2$ corresponds to a vertex $d(\Phi)$ in $\text{Sub}(f)$,
- each edge E of $C(f)$ corresponds to an edge $d(E)$ in $\text{Sub}(f)$,
- each vertex V of $C(f)$ corresponds to a face $d(V)$ in $\text{Sub}(f)$.

All these correspondences are bijections; abusing notation we denote the maps in both directions by d. If I, J are two endpoints of an edge of $C(f)$, then the faces $d(I), d(J)$ of $\text{Sub}(f)$ share a common edge, this latter edge is dual to the edge IJ. In other words, $d(I) \cap d(J)$ is $d(IJ)$.

Not all integer points in the Newton polygon are necessarily the vertices of the subdivision defined by f. If a point $(i,j) \in \mathcal{A}$ is not a vertex of the subdivision, i.e. it belongs to the interior of a face or an edge of $\text{Sub}(f)$, then increasing of the coefficient c_{ij} in the expression (C.1) does not change the values of the function f, and does not change the tropical curve $C(f)$. In particular, this implies that a function represented by a tropical polynomial doesn't determine the coefficients of the polynomial in general. So, similar to Definition 3.4, we always assume that $\mathcal{A} = \text{Newt}(f)$ and the coefficients in (C.1) are the minimal possible. With this new requirement, decreasing any coefficient of the monomial in f changes the curve $C(f)$.

Definition C.4. (Used on pages [18,40]) A vertex V of a tropical curve is *smooth* if $d(V)$ is a triangle of area $1/2$. A vertex V of a tropical curve is called *a node* if $d(V)$ is a parallelogram of area 1. An edge E of a tropical curve has weight m if $d(E)$ contains $m+1$ lattice points (in other words $d(E)$ has the *lattice* length m). See Figure 11 for examples of smooth and non-smooth vertices.

The concept of smoothness and nodality of a vertex plays a central role in our construction of the lower bound for the toppling function (see Section 5).

It follows from Proposition C.3 that at every vertex of a tropical curve the *balancing condition* is satisfied, i.e. the weighted sum of the outgoings primitive vectors in the directions of edges is zero, see Figure 11.

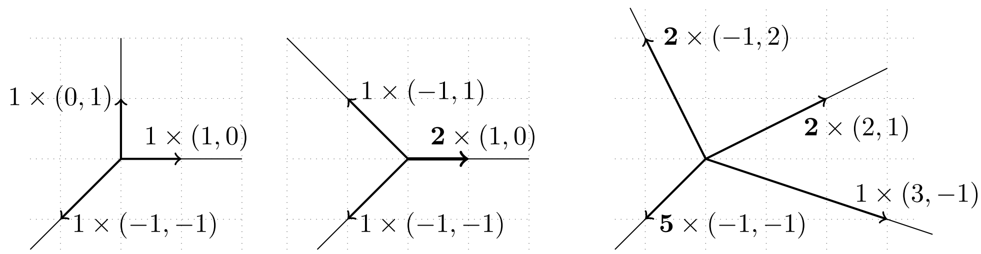


FIGURE 11. Examples of balancing condition in local pictures of tropical curves near vertices. The notation $\mathbf{m}\times(p,q)$ means that the corresponding edge has the weight m and the primitive vector (p,q). The vertex on the left picture is smooth, the vertices in the middle and right pictures are neither smooth nor nodal.

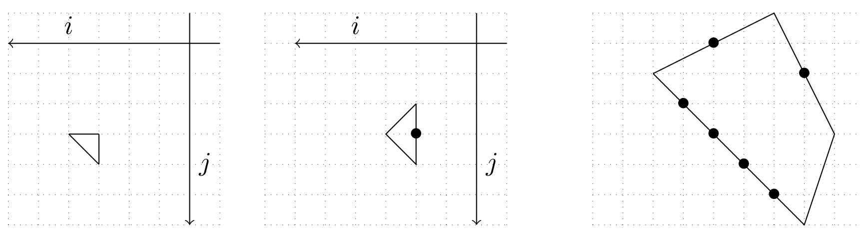


FIGURE 12. Polygons dual to the local models of tropical curves on Figure 11. A dual polygon is defined up to a translation, its lattice points (i,j) correspond to the monomials $ix+jy+a_{ij}$ (in some tropical polynomial defining this tropical curve) which contribute to the value of the tropical polynomial at the vertex. Note that, to make the duality visible, we need to reverse coordinate axes because of the "min" (instead of more conventional here "max") agreement in (C.1), in the definition of the tropical curve. Sides of polygons are orthogonal to the edges of curves. Moreover the lattice length of a side, computed as one plus the number of lattice points in its interior, is the weight of the corresponding dual edge.

Definition C.5. (Used on pages [13,15,15,18]) A corner of a $\mathbb{Q}$-polygon Δ is called *smooth* if the primitive vectors of the directions of the edges of Δ at this corner give a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. A $\mathbb{Q}$-polygon is smooth if all its corners are smooth.

Definition C.6. (Used on pages [13,18,21]) A Δ-tropical curve is called *smooth or nodal* if all its vertices in Δ° are smooth or nodal (see Definition C.4). In particular, this curve has no edges of weight bigger than one.

C.2. **Basic facts about convergence of tropical series.** Tropical series in one variable appeared in [29]. Tropical series on $\mathbb{R}^n$ received some attention in [11]. We do not know any earlier references.

Lemma C.7. (Used on pages [14,41,41,42,42]) Let U be an open subset of $\mathbb{R}^2$ and K be a compact subset of U. For any $\mathrm{C}>0$ the set

$$\mathcal{M}=\big\{(i,j)\in\mathbb{Z}^2|\exists d\in\mathbb{R},\exists(x_0,y_0)\in K,(ix+jy+d)|_U\geq 0,(ix_0+jy_0+d)\leq \mathrm{C}\big\}$$

is finite. **verified**

Proof. If $U=\mathbb{R}^2$, $\mathcal{M}=\{(0,0)\}$. So, let $R>0$ denote the distance between K and $\mathbb{R}^2\backslash U$. Then $(ix+jy+d)|_K\geq R\cdot\sqrt{i^2+j^2}$ for any i,j and d such that $(ix+jy+d)|_U\geq 0$. Therefore, $i^2+j^2\leq \mathrm{C}^2R^{-2}$ for all $(i,j)\in\mathcal{M}$. □

Proof of Lemma 3.3. Suppose that for a point $(x_0,y_0)\in\Omega^\circ$ and each $(i,j)\in\mathcal{A}$ the value of the monomial $a_{ij}+ix_0+jy_0$ is distinct from the value of the infimum

$$\inf_{(i,j)\in\mathcal{A}}(a_{ij}+ix_0+jy_0).$$

Thus, there exists $\mathrm{C}>0$ such that we have $a_{ij}+ix_0+jy_0<\mathrm{C}$ for infinite number of monomials $(i,j)\in\mathcal{A}$. Since $(a_{ij}+ix+jy)|_\Omega\geq 0$ for all $(i,j)\in\mathcal{A}$, applying Lemma C.7 yields a contradiction. □

Definition C.8. (Used on pages [13,41,42]) Let U be an open non-empty subset of $\mathbb{R}^2$. A function $f:U\to\mathbb{R}$ is called *a tropical series* if for each $(x_0,y_0)\in U$ there exists an open neighborhood $W\subset U$ of (x_0,y_0) such that $f|_W$ is a tropical polynomial. If we write that a function $f:U\to\mathbb{R}$ is a tropical series for a non-open U, it means that f is a continuous function on U and $f|_{U^\circ}$ is a tropical series.

Definition C.9 (Cf. Definition 3.2)**.** A *tropical analytic curve* in U is the locus of non-linearity of a tropical series f on U°. We denote this curve by $C(f)\subset U^\circ$.

Example C.10. The standard grid – the union of all horizontal and vertical lines passing through lattice points, i.e. the set

$$C=\bigcup_{k\in\mathbb{Z}}\{(k,y)|y\in\mathbb{R}\}\cup\{(x,k)|x\in\mathbb{R}\},$$

is a tropical analytic curve in $\mathbb{R}^2$. Similar examples are tropical Θ-divisors [20].

The following example shows that a tropical series on Ω° in general cannot be extended to $\partial\Omega$.

Example C.11. Consider a tropical analytic curve C in the square $[0,1]\times[0,1]$, presented as

$$C=\bigcup_{n\in\mathbb{N}}\Big\{\big(1/n,y\big)|y\in[0,1]\Big\}\cup\Big\{\big(x,1/2\big)|x\in[0,1]\Big\}.$$

For all tropical series f with $C(f)=C$, the sequence of values of $f(x,y)$ tends to $-\infty$ as $x\to 0$.

Example C.12. See examples of tropical analytic curves on Figure 3 in [4], first two rows. It is a mystery for us why the most of the slopes of the tropical curves on those pictures are $(1,0),(0,1),(1,1),(-1,1)$.

Lemma C.13. (Used on pages) An Ω-tropical series (Definition 3.2) is a tropical series on Ω in the sense of Definition C.8.**verified**

Proof. We take any point $(x_0,y_0)\in\Omega^\circ$ and a small compact neighborhood $K\subset\Omega^\circ$ of (x_0,y_0). There exists $\mathrm{C}>0$ such that our Ω-tropical series f restricted to K is less than C. We apply Lemma C.7 and see that $f|_K$ is a tropical polynomial. □

Lemma C.14. (Used on pages [42]) Suppose that a continuous function $f:\Omega\to\mathbb{R}$ on an admissible Ω satisfies two conditions: 1) $f|_{\Omega^\circ}$ is a tropical series, and 2) $f|_{\partial\Omega}=0$. Then f is an Ω-tropical series (Definition 3.2).**verified**

Proof. Let $f|_U=ix+jy+a_{ij}$ for an open $U\subset\Omega^\circ$. Note that f is convex (since it is locally convex and Ω is convex) and so $f(x,y)\leq ix+jy+a_{ij}$ on Ω. Therefore in Ω° we have

$$f(x,y)=\min\{ix+jy+a_{ij}|(i,j,a_{ij}),\exists\text{ open }U\subset\Omega^\circ,f(x,y)|_U=ix+ij+a_{ij}\}.$$

□

Corollary C.15. If f is a tropical series on Ω, then a duality similar to Proposition C.3 holds: we present f in a canonical form and construct the extended Newton polytope which defines a subdivision of $\mathcal{A}_\Omega$ (Definition 4.1). The vertices of this subdivision correspond to the connected components of $\Omega^\circ \setminus C(f)$, the edges correspond to the edges of $C(f)$, and the faces correspond to the vertices of $C(f)$.

Recall that Ω in the above lemma is admissible, in particular, it is convex. Tropical series on non-convex domains exhibit the behavior as in the following example.

Example C.16. Let $U_1 = \big([0,5]\times[0,1]\big)\cup\big([4,5]\times[1,2]\big), U_2 = \big([0,5]\times[2,3]\big)\cup\big([4,5]\times[1,2]\big), U = (U_1\cup U_2)^\circ$. The function $f(x,y)=\min(3,x+[y])$ is a tropical series on U, but $f|_{U_1^\circ}=\min(3,x), f|_{U_2^\circ}=\min(3,x+2)$, and the monomial x appears with different coefficients $0,2$ in the different parts of U.

Proof of Lemma 4.11. Let $g\in V(\Omega,P,f)$, $z_0\in\Omega^\circ$ and $K\subset\Omega^\circ$ be a compact set such that $z_0\in K^\circ$. Denote by $\mathrm{C}>0$ the maximum of g on K. Consider the set $\mathcal{M}$ of all $(i,j)\in\mathbb{Z}^2$ for which there exist $d\in\mathbb{R},(x_0,y_0)\in K$ such that $0\le(xi+yj+d)|_{\Omega^\circ}, ix_0+jy_0+d\le \mathrm{C}$. The set $\mathcal{M}$ is finite by Lemma C.7. Therefore, the restriction of any tropical series $g\in V(\Omega,P,f)$ to K can be expressed as a tropical polynomial $\min_{(i,j)\in\mathcal{M}}(ix+jy+a_{ij}(g))$. In particular, if we denote by b_{ij} the infimum of $a_{ij}(g)$ for all $g\in V(\Omega,P,f)$ then

$$G_Pf|_K=\min_{(i,j)\in\mathcal{M}}(ix+jy+b_{ij}).$$

It follows from Proposition 4.10, that $G_Pf\le f+n\cdot l_\Omega$. Then, $l_\Omega|_{\partial\Omega}=0$ by Lemma 4.5. Therefore $G_Pf|_{\partial\Omega}=0$ and, thus, Lemma C.14 concludes the proof that $f_{\Omega,P}$ is an Ω-tropical series. □

Lemma C.17. The function l_Ω is a tropical series in Ω° (Definition C.8). **verified**

Proof. Consider $z\in\Omega^\circ$. It follows from Lemma C.7 that for any constant $\mathrm{C}\in\mathbb{R}$ for each $z\in\Omega^\circ$ there exist only finite number of $(i,j)\in\mathbb{Z}^2$ such that $l_\Omega^{ij}(z)\le\mathrm{C}$ which concludes the proof. In particular, for $z\in\Omega^\circ$, $l_\Omega(z)$ is actually $\min_{(i,j)\in\mathcal{A}_\Omega\setminus\{(0,0)\}} l_\Omega^{ij}(z)$. □

Proof of Proposition 4.2. It is easy to verify that if Ω is not admissible, then $\Omega^\circ=\varnothing$ or $\mathcal{A}_\Omega=\{(0,0)\}$. Let us prove the statement in another direction. If $\Omega\ne\mathbb{R}^2$, then there exists a boundary point $\mathbf{p}$ of Ω and a support line l at $\mathbf{p}$. If the slope of l is rational, then $\mathcal{A}_\Omega$ contains the corresponding lattice point; if this slope is irrational but l does not belong to the boundary of Ω, then there exists another support line of Ω with a rational slope. So, we may suppose that l is contained in the boundary of Ω. If there is no other boundary points of Ω, then Ω is a half-plane and is not admissible. If there exists another boundary point, then we repeat the above arguments and find a support line of Ω with rational slope or prove that Ω is a strip between two lines of the same irrational slope. □

Proof of Lemma 4.5. It is enough to prove that l_Ω is zero on $\partial\Omega$. It is clear that $l_\Omega=0$ on zero sets of functions l_Ω^{ij} for all $(i,j)\in\mathcal{A}_\Omega$. Suppose that there exists a point $z\in\partial\Omega$ where the only support line L is of irrational slope α.

Since Ω is admissible, there exists a point $z'=(x_2,y_2)\in L$ which does not belong to $\partial\Omega$. Using continued fractions for α, we get two sequences of numbers, $p_{2n}/q_{2n}<\alpha<p_{2n+1}/q_{2n+1}$ such that

$$|\alpha-p_m/q_m|<1/q_m^2,\ \text{for all } m,$$

and q_m tends to infinity. Either for all even i, or for all odd i the line through z' with the slope p_i/q_i does not intersect Ω, so $(-p_i,q_i)\in\mathcal{A}_\Omega$. Thus, for such i the linear function

$$l_i(x,y)=q_i\left(y-\frac{p_i}{q_i}x-\big(y_2-\frac{p_i}{q_i}x_2\big)\right)$$

estimates $l^{-p_i,q_i}(x,y)$ from above. The absolute value of l_i at $z=(x_1,y_1)$ is

$$\begin{aligned}|-p_ix_1+q_iy_1+(p_ix_2-q_iy_2)|=&|p_i(x_2-x_1)+q_i(y_1-y_2)|=\\ &|(x_1-x_2)(-p_i+q_i\alpha)|=|(x_1-x_2)(\alpha-p_i/q_i)q_i|\le\left|\frac{x_1-x_2}{q_i}\right|,\end{aligned}$$

which tends to zero as $i\to\infty$. Therefore, we can construct a sequence of functions $l_i, i\to\infty$, whose values at z_1 tend to zero, and $l_\Omega\le l_i$. Therefore $l_\Omega(z_1)=0$ and $l_\Omega|_{\partial\Omega}=0$.

□

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