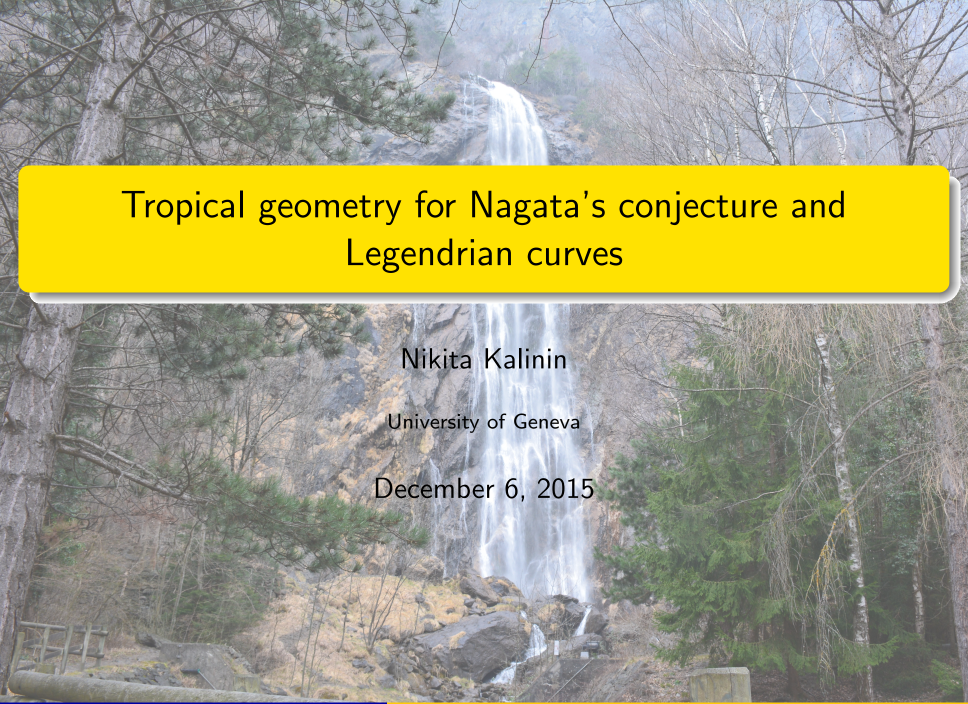

Tropical geometry for Nagata's conjecture and Legendrian curves

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Advocating

Tropical geometry has been recently heavily used in

- the proof of longstanding Rota log-concavity conjecture [Adiprasito, Huh, Katz, 2015]. In particular it implies that the coefficients a_i of the chromatic polynomial of a graph form a log-concave sequence, i.e. $a_i^2 \geq a_{i-1}a_{i+1}$.
- a uniform estimate for the number $N(d,g)$ of rational points on curves of small Mordell-Weil rank [Katz, Rabinoff, Zureick-Brown, 2015], as well as unconditional estimate for the number of torsion points of the images of curves in Jacobians.

Sandpile models: emergence of tropical objects

Topplings

Definition

A sandpile is a collection of indistinguishable sand grains distributed among a finite subset Γ of $\mathbb{Z}^2$, that is a function $f : \Gamma \to \mathbb{N}_0$. A vertex v is unstable if $f(v) \geq 4$. An unstable vertex can topple by sending one grain of sand to each of 4 neighbours.

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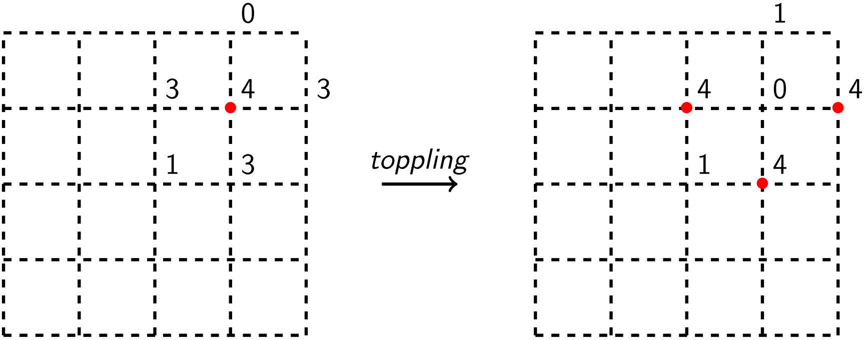

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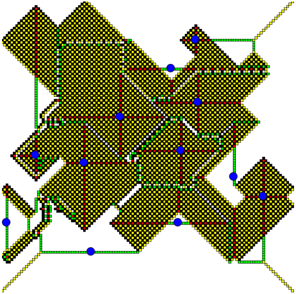
Definition

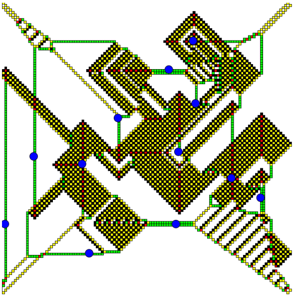
We chose a boundary $\partial\Gamma \subset \Gamma$, where we never do topplings. For example, $\Gamma = [0, \dots, N] \times [0, \dots, N]$, $\partial\Gamma$ is the border of this square.

We start from the maximal stable distribution of sand, i.e. $f \equiv 3$, then we add a grain somewhere. What is the result of relaxation? And if we add a number of grains to different places?

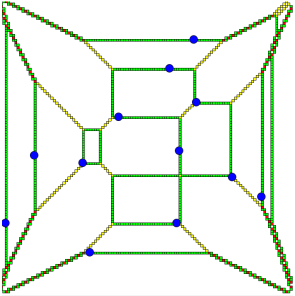
Simulation, results



Just after beginning...



... in the middle...



Et voilà ! Final result!

White: 3 grains, Green: 2 grains, Yellow: 1 grain, Red: no grains, Black: more than 3 grains.

Explanation: look at the number of topplings

Consider the number $h(x, y)$ of topplings at a point (x, y) during the relaxation.

Proposition

If the number of sand grains at (x, y) after relaxation is the same as before the relaxation, then
$h(x-1, y) + h(x+1, y) + h(x, y-1) + h(x, y+1) - 4h(x, y) = 0.$

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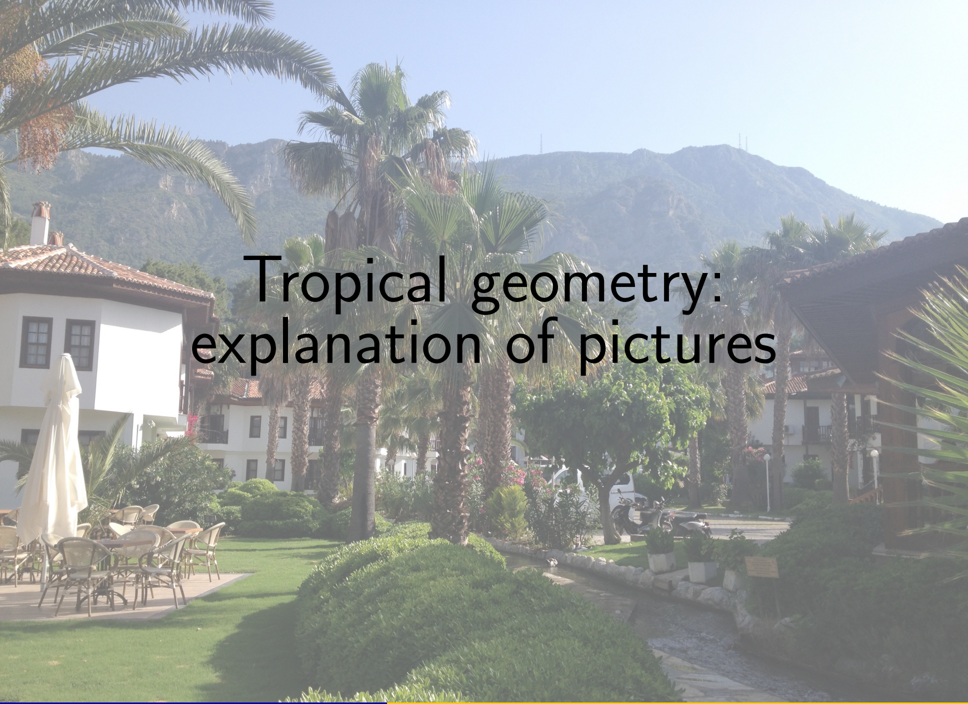
Proof.

Indeed, count the number of incoming and outgoing grains. □

That is, h is harmonic at (x, y). In fact, $h(x, y)$ is a "piece-wise" linear function.



Tropical geometry: explanation of pictures

Tropical curves

Definition

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^2$. A tropical polynomial is a function $h(x,y) = \min_{(i,j)\in\mathcal{A}} (a_{ij} + ix + jy), a_{ij} \in \mathbb{R}$. Note that $h : \mathbb{R}^2 \to \mathbb{R}$.

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Consider a family of polynomials $F_t(x,y) = \sum_{(i,j)\in\mathcal{A}} t^{a_{ij}} x^i y^j$.
Define the corresponding curves $C_t = \{(x,y) | F_t(x,y) = 0\} \subset \mathbb{C}^2$.

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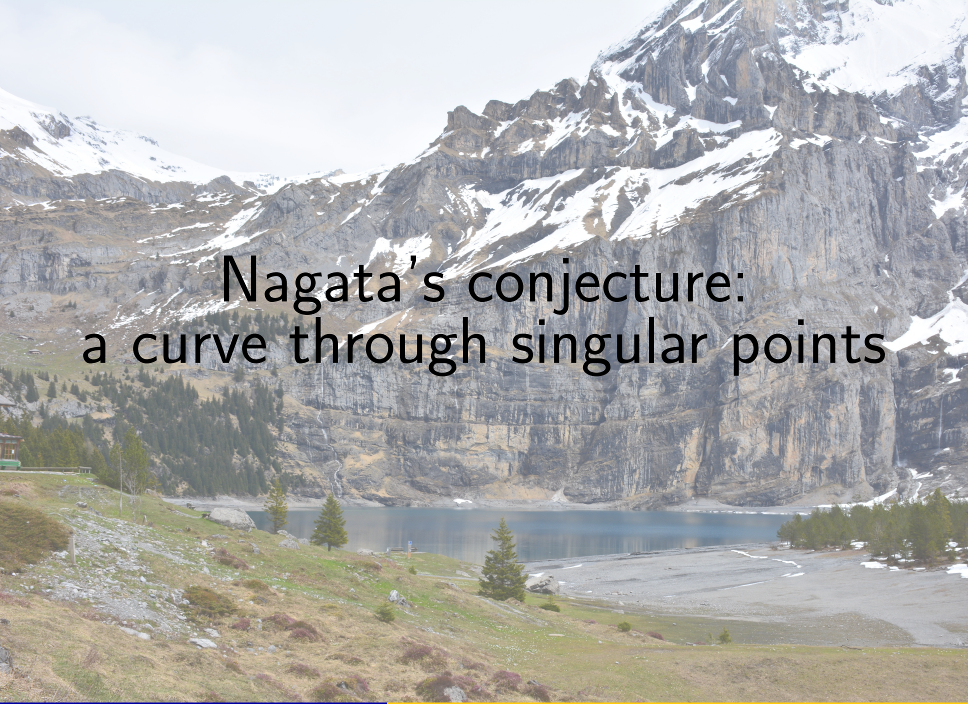
Proposition

Then $C = \lim_{t\to 0} log_t C_t$ *is the tropical curve defined by* $h(x,y) = \min_{(i,j)\in\mathcal{A}}(a_{ij} + ix + jx)$.

Nagata's conjecture: a curve through singular points



Singular points

We consider a polynomial $F(x, y) = \sum a_{ij}x^i y^j, a_{ij} \in \mathbb{C}, 0 \leq i, j, i + j \leq d$ of degree d, it defines a curve C as its zero-set, $C = \{(x, y)|F(x, y) = 0\}$.

Definition

A point $p \in \mathbb{C}^2$ is of multiplicity m for C if all the partial derivatives of F up to order $m - 1$ vanish at p.

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Example

The point $(0, 0)$ is of multiplicity

- 1 if C passes through $(0, 0)$
- 2 if $(0, 0)$ is a singular point of C
- m if $a_{ij} = 0$ for $i + j \le m - 1$.

Dimension counting

As we see, the fact that a point p is of multiplicity m for a curve C imposes $\frac{m(m+1)}{2}$ linear conditions on the coefficients a_{ij}.

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A curve of degree d has $\frac{(d+1)(d+2)}{2}$ coefficients. Therefore, if $p_1, p_2, \dots, p_n$ are in general position, a curve C passes through $p_1, p_2, \dots, p_n$ with some multiplicities $m_1, m_2, \dots m_n$, then we should expect

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$$\frac{(d+1)(d+2)}{2} - 1 \geq \sum_{1 \leq i \leq n} \frac{m_i(m_i+1)}{2}.$$

Unfortunately, these linear conditions are **not** independent.

- A line (d=1) passes though 2 points. $\frac{2\cdot 3}{2} - 1 \geq \frac{1\cdot 2}{2} + \frac{1\cdot 2}{2}$ (still true).
- A doubled conic ($d=4$) passes through 5 points with multiplicities 2. $\frac{5\cdot 6}{2} - 1 \geq 5 \cdot \frac{2\cdot 3}{2}$ (not true).

Nagata's conjecture

As a reasonable estimate, Nagata conjectured that if $n \geq 10$ and the above conditions are satisfied, then $d > \frac{\sum m_i}{\sqrt{n}}$.

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- Open even for $n = 10, m_i = m$. (That is $d > \sqrt{n} \cdot m$)
- Nagata proved it himself for $n = k^2$. The proof gives a counterexample for 14th Hilbert Problem:
  Is the ring of invariants of an algebraic group acting on a polynomial ring is always finitely generated?

Duality between tropical curves and subdivisions of Δ

A tropical curve comes as the locus where a function $h(x,y) = \min_{(i,j)\in\Delta}(a_{ij} + ix + jy)$ is not smooth.

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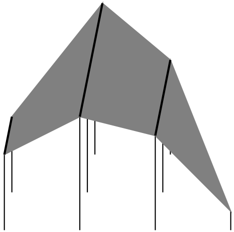
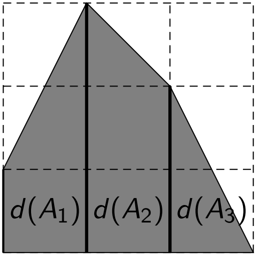
Proposition

This subdivision is dual to the tropical curve C defined by h.

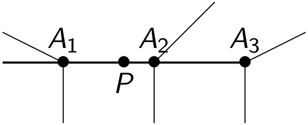
Example



(A)



(B)



(C)

The extended Newton polyhedron of the curve C is drawn in (A). The projection of its faces gives us the subdivision of the Newton polygon of the curve, see (B). The tropical curve C is drawn in (C).

Influenced regions

Definition

For a point P on a tropical curve C we define the region of influence of P in the Newton polygon as follows. Draw straight lines through all the edges of C which pass through P. Take all the vertices of C lying on the latter lines. Take the dual cells to these vertices. The union of these cells is the region of influence of P.

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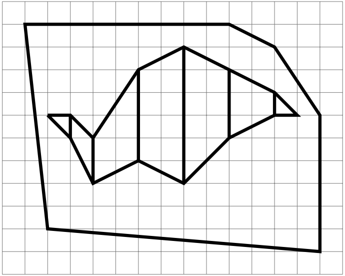
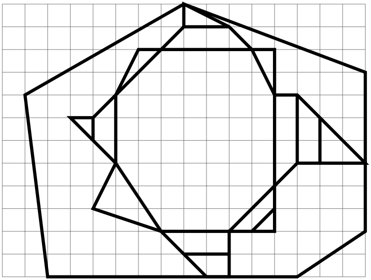
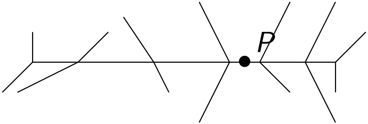
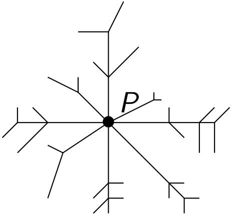
On the previous slide P is the tropicalization of a singular point of multiplicity 3. The influenced region is the whole Newton polygon.

Examples









Theorems

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If P is of multiplicity m on C, then the area of the influenced region of the Newton polygon is bigger or equal than

- $\frac{3}{8}m^2$ *if* $\mathrm{Trop}(P)$ *is a vertex of* $\mathrm{Trop}(C)$
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Theorem

If points $P_1, P_2, \dots P_n$ are chosen generically, then all triple intersections of their regions of influence are empty.

Implications

These tropical theorems imply some estimates in Nagata's conjecture.

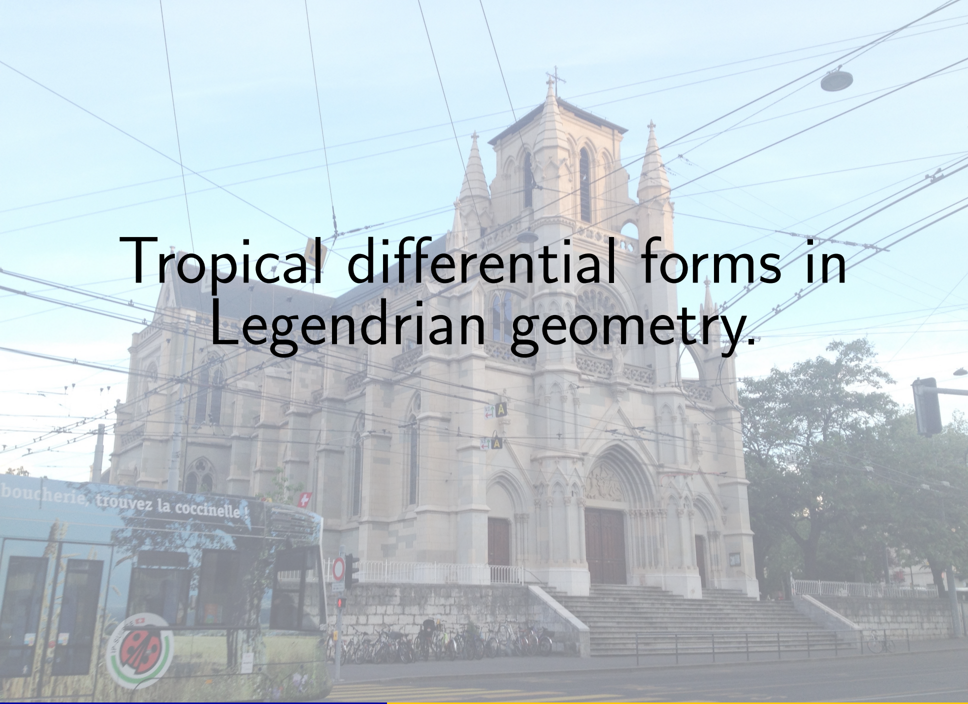
Proposition

The degree d of a curve passing through generic $p_1, p_2, \dots, p_n$ with multiplicities m is subject to the following inequality:

$$d^2 \geq (n - \sqrt{n} - 1)m^2$$

There are more tight estimates (Exercise: use Bezout's Theorem) .

Tropical differential forms in Legendrian geometry.



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Definition

A curve C is contact or Legendrian if $\omega(TC) = 0$.

How many algebraic Legendrian curves of given degree pass through given points in $\mathbb{C}P^3$?

Answer: **three** rational Legendrian cubics through 3 points and one line.

Exact formulae

All the curves tangent to $ydx - xdy + wdz - zdw$ and passing through three points $(0,0,0,1),(1,1,1,1),(-1,1-1,1) \in \mathbb{C}P^3$ are: $(3t - t^3, 2t^2 + 2\mu(t - t^3), 2t^3, 1 + t^2 - 2\mu(t - t^3)), t \in \mathbb{C}P^1, \mu \in \mathbb{C}$.

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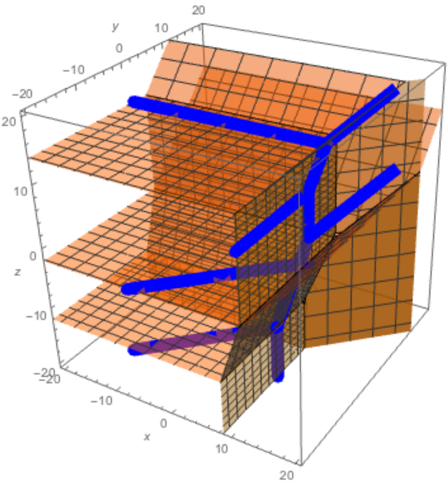
Question

Is it true that there are at least d rational legendrian curves of degree d through generic d points in $\mathbb{C}P^3$ and a line?

Tropical Legendrian curves



The tropicalization of the surface swept by legendrian cubics through three points. A particular tropical curve inside.

Abstract tropical curves

Proposition

Note that the amoeba $\log_t(C)$ *is given by integration of* $\frac{dx}{x}, \frac{dy}{y}$ *on* C *and rescaling by* $\ln t$.

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Definition

A tropical curve S is a degeneration of a family $S_t, t \to \infty$ of complex surfaces if the asymptotics of the lengths of the geodesics in the pair-of-pants decomposition are like $\frac{1}{a_i \ln t}$. The corresponding edges of S have lengths a_i.

The degeneration comes with the contraction maps $S_t \to S$, where we contract the hyperbolic collars of the geodesics.

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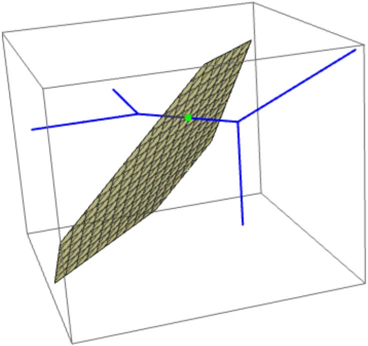
Proposition

If $\lim \int_{\gamma_t} \omega_t = L$ *then* $\lim \frac{1}{\ln t} \int_{\Gamma_t} \omega_t = aL$. *So,* ω *should be though as* $L\frac{dx}{x}$

Application of tropical differential forms



A part of the tropicalization of a legendrian curve. This part looks like a tropical line.

We can prove that the plane $x + y = z$ divides the interval between two vertices in the middle.

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- Sandpile models and their evolution.
- For the square in $\mathbb{Z}^2$ the vertices with less than 3 grains constitute a tropical curve
- From the matroidal point of view, tropical geometry is about subdivisions of the Newton polygon of a curve.
- Constraints, imposed by singular points on the coefficients, are visible on this subdivision.
- A definition of an abstract tropical curve and a tropical differential form.

Thank you for your attention!

Geneva, 2015