Mikhail Shkolnikov, Research Statement

I have quite broad interests in low-dimensional Geometry and Topology, such as Knot theory, Tropical, Hyperbolic, Convex and Non-commutative geometry, Combinatorics and Mathematical Physics, including sandpile model, string theory and condensed matter physics. In my early years I've been doing some homological algebra in relation with TQFT's. One of my major current interests is the interplay between Number theory, Enumerative Geometry and non-archemidean dynamics.

I'm finishing my PhD under the supervision of Grigory Mikhalkin in the University of Geneva, Switzerland. During my master program we started to work on a new topic in the theory of amoebas. The usual logarithmic amoebas were introduced in the theory of discriminants (see [1]) as images of algebraic subvarieties unsder the map $Log\colon (\mathbb{C}^*)^n \mapsto \mathbb{R}^n$. Group theoretically this map can be described as factorization by the maximal compact subgroup $(S^1)^n$ in the algebraic torus. Replacing an algebraic group $(\mathbb{C}^*)^n$ by a group $PSL_2\mathbb{C}$ in this description, we arrive to the definition of hyperbolic amoebas (to appear in [21]). This name comes from the fact that $PSL_2\mathbb{C}$ has $SO(3)$ as its maximal subgroup and the quotient is $\mathbb{H}^3$. In my thesis, I show that non-commutativity of $PSL_2\mathbb{C}$ (oposed to commutativity of algebraic torus) induces certain critical phenomena manifested at the level of hyperbolic amoebas that can be used to approach topological problems in the theory of real algebraic surfaces and projective knots.

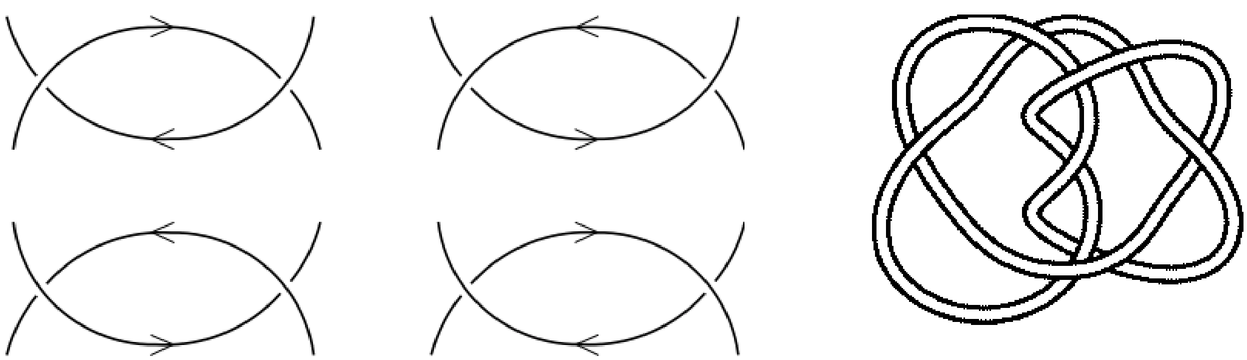

Figure 1: Matched pairs of crossings (on the left) and the minimal diagram 9_{46} of the preztzel knot P(3,3,-3) (on the right). The second Alexander ideal of this knot is spaned by 3 and t+1 in the ring of integral Laurent polynomials and, thus, the knot is not bipartite, i.e. doesn't admit the diagram consisting exclusively of matched pairs.

Before arriving to Geneva, I had been working with Sergey Duzhin in Saint-Petersburg. One of my favourite achievements was the proof [5] of a long-standing Przytycki conjecture stated in [3] which is included to the renowned Kirby list of problems in low-dimensional topology [2]. A knot is called bipartite if it admits a matched planar diagram. An oriented link diagram is called matched if the crossings come in pairs shown in the figure 1 on the left. All knots up to 7 crossings are bipartite. Our attention to this class of knots is due to a general observation that certain quantum invariants have a particular nice form for such knots. This observation has been exploited in [4, 6] for Homflypt and Kauffman polynomials on rational links and, in a similar way, in [7] by Krasner for Khovanov-Rozansky homology on matched tangles. In [5] we show that if a knot is bipartite then its higher Alexander ideal cannot contain $t+1$ provided the ideal is proper in $\mathbb{Z}[t, t^{-1}]$. Therefore, we have shown that there exist non-bipartite knots which was exactly the content of the conjecture mentioned above. Our technique recently has been used by Lewark and Lobb in [8] to show that there exist infinite families of bipatite and non-bipartite pretzel knots.

During my stay in Geneva I had a succefull collaboration with Nikita Kalinin, another PhD student of my working group. The work was initiated when our advisor, Grigory Mikhalkin, gave me a copy of the paper [14], a short text on experimental condensed matter physics recomended him by one of the authors. The model was a cellular automaton called abelian sandpile model and the main discovery was a numerical evidence for presense of certain conservation law similar to one obeyed by strings or tropical curves (the balancing condition [19]). Armed with newly studied tools from tropical and toric geometry, discrete harmonic analysis and number theory we attempted to study this model. As the main result of our work, we have shown that there exist a new type of scaling limits in sandpiles [16] similar to [12]. Moreover, the limits are effectively described in terms of tropical idempotent dynamics which we develop in [17]. This dynamics, in a sense, can be seen as a continous analog of the original self-organized criticality [9], since as we found emprically by means of supercomputer (to appear in [22]) the system demonstrates power-laws for the sizes of avalanches and doesn't have any continous tuning parameters such as temperature or magnetic field in similar models.

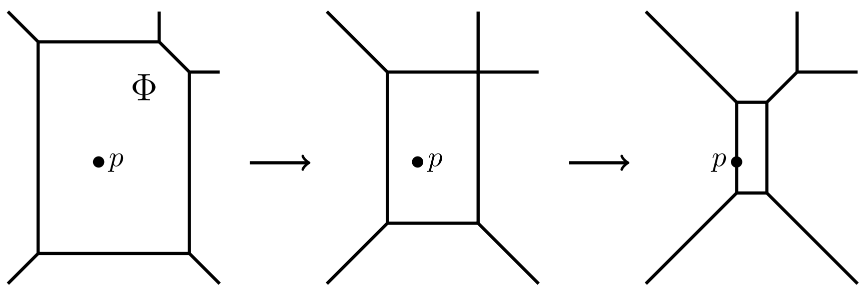

Figure 2: The operator G_p, the combinatorial shadow of quasi-algebraic S_p, shrinks the face Φ of the tropical curve where p belongs to. The resulting curve solves the tropical Steiner problem for one inner points p and five outer points (the ends of the curve).

The tropical idempotent dymamics can be lifted to quasi-algebraic non-archemedian dynamics in characteristic 2. The lift is easy to describe. Let k be a filed of characteristic *two*. For every point $p \in (k^*)^2$ there is a well defined idempotent operator S_p acting on the space of polynomials over k with a fixed support of coefficients $A \subset \mathbb{Z}^2$. This operator given by the formula

$$S_p f(z) = f(z) + \sqrt{f(z^2p^{-1})f(p)}.$$

The square root is the inverse to the Frobenius (we assume it is an isomorphism). The operator S_p is an algebra-geometric analog of the idempotent perturbation

$$\phi \mapsto (\phi + \delta_z)^\circ - \delta_z,$$

of [CPS11], where $z = val(p)$ and ψ° denotes the relaxation of ψ. Note that $(S_p f)(p) = 0$ for any f and if $f(p) = 0$ then $S_p f = f$. And so, S_p is a canonical projector on the space of polynomials vanishing at p. In fact, if k is a non-archemedian field with the valuation val, then

$$Val(S_p f) = G_{val(p)} Val(f)$$

and $G_{val(p)}$ is an idempotent operator (see Figure 2) of the tropical dynamics arising from the scaling limit of the idempotent perturbation acting on large sandpiles (as we discovered in [17]). Now I'm working on certain stabilization and convergence conjectures concerning mutual dynamics of finite collection of such idempotent operators. It is a problem of great importance, to find a characteristic zero (or any other) analog of S_p, and I'm looking for the ideas that might provide a better understanding of these questions.

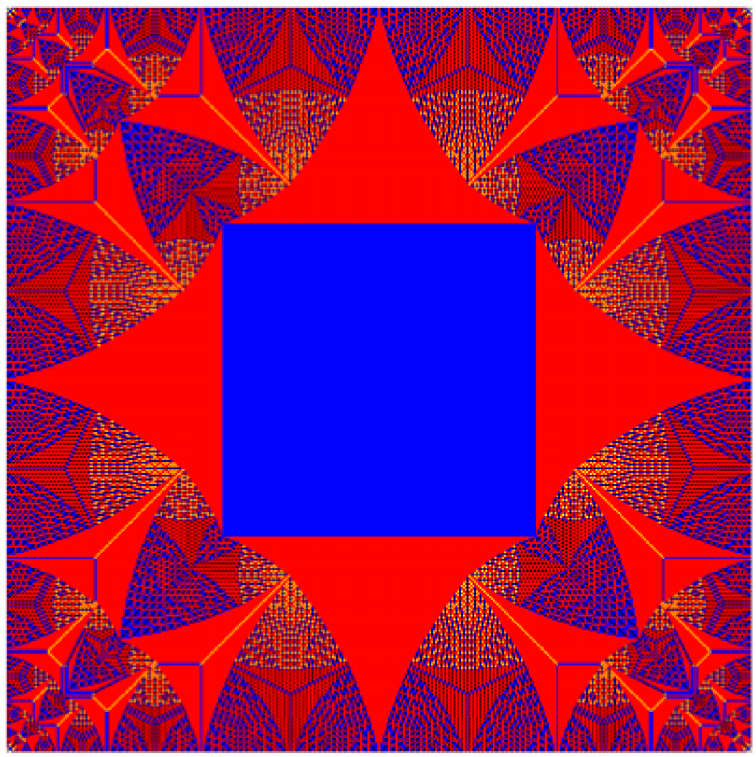

Figure 3: The neutral element of the sandpile group, combinatorial incornation of the Jacobian variety, on a big square. The cites are coloured with represent to the values {0,1,2,3} of this sandpile at every cite. The fractality here might be of the non-archemedian origin. Mind the tropical curves popping up in the triangles.

The original sandpile model inroduced in the celebrated paper [9] is a principal example of a discrete stochastic dynamical system demonstrating a variety of sapid properties such as self-organized criticality and power-laws manifested regardless of any tuning parameters. This makes it similar to some natural phenomena such as earthquakes, solar flares or noises in electrophysiological signals and could be usefull in their study. The evolution operator

$$\phi \mapsto (\phi + \delta_p)^\circ$$

of this dynamics acts on the space of all stable sandpiles $\phi\colon \Gamma \to \{0, 1, 2, 3\}$ on a fixed large portion Γ of the square grid $\mathbb{Z}^2$ and $p \in \Gamma$ in δ_p (the function which is equal to one at p and vanishes elsewhere) is chosen randomly at each step. A general sandpile ψ is a non-negative itegral function

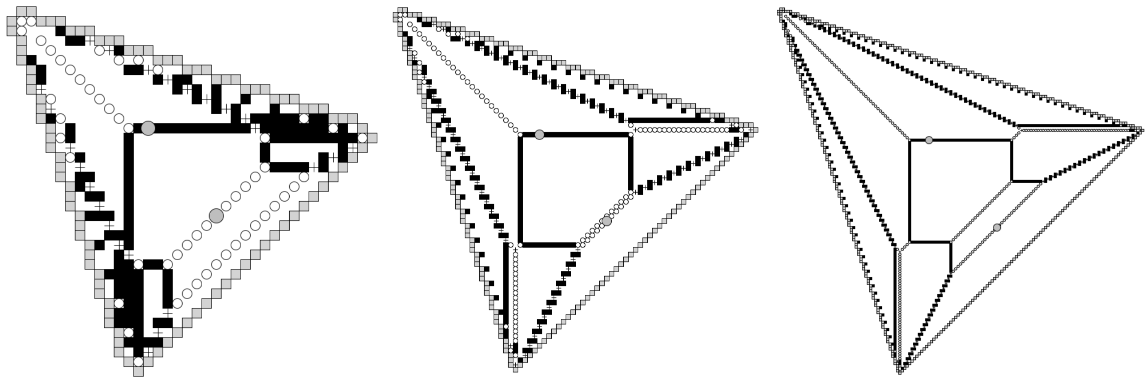

Figure 4: The deviation locus (black – less than 3 grains, white – 3) for the maximal stable state on the triangular domain perturebed at two points converging to a tropical curve passing throug the perturbation points where. We add a single grain at each of the perturbation points. This curve minimizes the symplectic area of the curve and therofore solves a tropical analog of the Steiner tree problem. The scale of the last picture is four times bigger than the scale of the first and two times bigger than the scale of the second. This is a prototipe of the scalinig limit introduced in [16].

on Γ representing a distribution of sand grains and by $(\psi)^\circ$ we denote the relaxation of ψ, i.e. the stable sandpile derived by performing as much topplings as possible. A toppling at certain cite of Γ is just the redistribution of 4 grains of sand among its neighbours. Note that some number of grains can leave the system during the relaxation. Clearly, we have $\psi = \psi^\circ$ if and only if $\psi \leq 3$, and such a sandpile is called stable. Therefore, one can vaguely interpret the original model as a randomly growing sandpile on a given finite domain.

The usage of the term "self-organized criticality" is justified by the fact that the system has an attractor consisting of the so-called recurrent sandpiles first studied mathematically in [10]. The sandpile group of a finite graph is one of the cutting-edge objects of the modern combinatorics intimately related to Jacobians of algebraic curves and domino-tillings (see [11]). It is an abelian group formed by all recurrent sandpiles on Γ and the operation is given by

$$(\phi, \psi) \mapsto (\phi + \psi)^\circ.$$

One might think that the neutral element in this group is equal to the empty sandpile. The reality is much more complicated, and for a general Γ the sandpile which is identically zero is not reccurent: in fact it cannot appear in the dynamics after the first step. An example of the unit element in the sandpile group for a square $\Gamma = \mathbb{Z}^2 \cap [0, N]^2$ is shown on the Figure 3. Although, the picture itself suggests a variety of nice geometric conjectures such as the existence of the fractal-like scaling limit almost nothing is known rigorously about the shape of this particular sandpiles for an arbitrary N.

In contrast, a big progress is made in [12, 13] where the scaling limit for the process of adding a grain at a single cite on the whole grid is shown to exist (the approximation to this limit is shown on Figure 5). Moreover, some of its fractal nature is explained there.

If we look carefully at Figures 3 and 5 pictures we may notice some piecewice linear graphs inside of peripherical triangles. In fact, these are not simply graphs but tropical curves, combinatorial counterparts of holomorphic and non-archemedian curves, they have rational slopes and satisfy the balancing condition at every vertex (see [19] for the introduction to the subject). The emergence

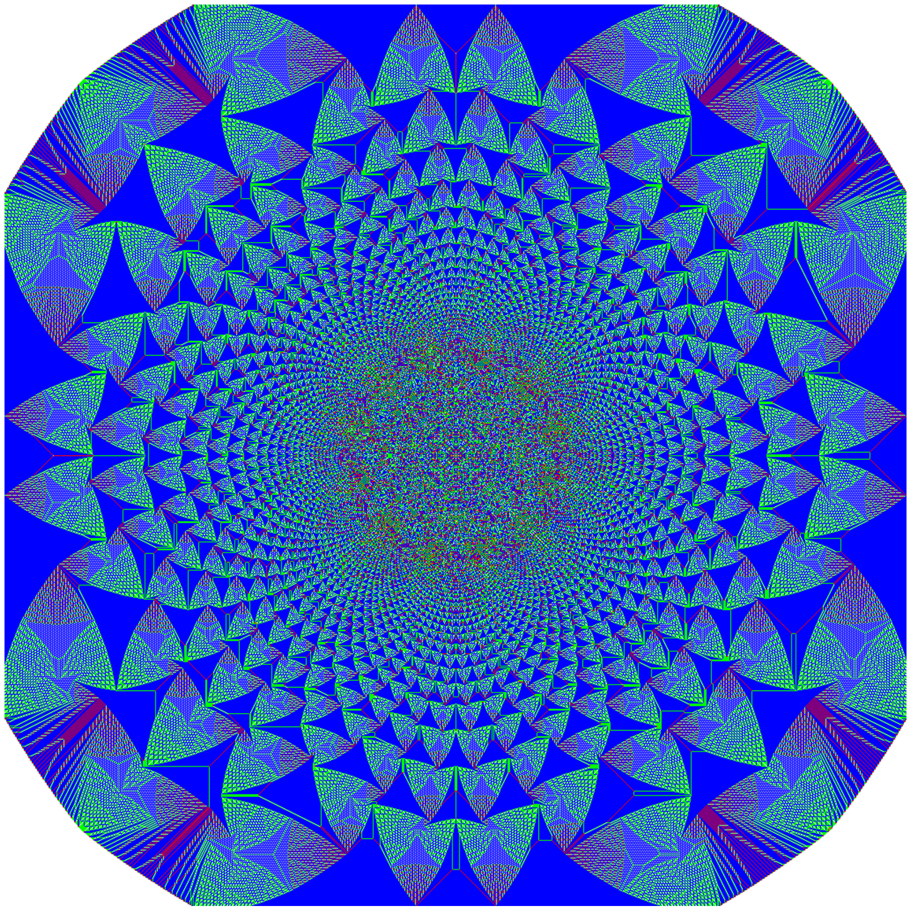

Figure 5: The result of the relaxation for the sandpile $28 \cdot 10^6 \delta_{(0,0)}$ on $\mathbb{Z}^2$. The fractality here is described in terms of circle paking (see [13]). The emergense of tropical curves here is still not clear.

of tropical geometry in these sandpiles is not accidental and is partially explained in our work [17]. The new scaling limit for sandpiles is introduced in this paper. The limits are completely described in terms of tropical analytic curves. In the case of lattice polygons these curves appear to be finite graphs providing solutions of a certain minimization problem of Steiner type (see [16]). Also, we may think that our work provides a description of the heavy layer of the sandpile group, identifying it with a space of soliton curves on the domain.

Inspired by the study of inner structure of curves arising in sandpiles, we have been able define a canonical pro-object in the category of toric schemes. This construction seems to be very similar to [20] (or to be even a generalization), where they constructed non-commutative toric manifolds having irrational moment polytopes. This direction looks very promising for the applications to geometry of numbers and convex geometry. For example, we have found natural coordinates on the space of all convex domains on the plane modulo $Sl_2\mathbb{Z}$. Namely, we found a canical embedding of this quotient to the space of all metrized trees. The later space can be viewed as a boundary of $M_{0,\infty}$, the moduli of an infinetly punctured sphere. This observation can lead to some fruitfull applications to classical convex geometry.

In [18] we prove an analogue of the scaling limit theorem in a particular hyperbolic case, where we consider specific convex piece of the triangular heptagonal lattice. The proof appears to be much simpler than in the flat case since the deviation locus for the maximal stable state on a finite hyperbolic disc looks like a complement of the graph to a single ball. In a very similar way as in the case of tropicals curves and sandpiles on a grid, the hyperbolic analogues of amoebas in $\mathbb{H}^3$ of hypersurfaces in $\mathrm{PSL}_2\,\mathbb{C}$ have a similar shape (see [21]) as the deviation locus of perturbation of the maximal stable state. This again motivates a deeper study of the relation between sandpiles on lattices and limits of algebraic varieties.

References

[1] I. Gelfand, M. Kapranov, A. Zelevinsky *Discriminants, Resultants, and Multidimensional Determinants,* Springer Science & Business Media, 2008

[2] R. Kirby, *Problems in Low-Dimensional Topology,* 1995.

[3] T. Przytycka and J. H. Przytycki, *Signed dichromatic graphs of oriented link diagrams and matched diagrams* (Univ. of British Columbia), notes 1987.

[4] S. Duzhin, M. Shkolnikov, *A formula for the HOMFLY polynomial of rational links*, Arnold Mathematical Journal 1 (4), 345-359, 2015

[5] S. Duzhin, M. Shkolnikov, *Bipartite knots*, Fundamenta Math. 225, 2014

[6] M. Shkolnikov *Kauffman polynomial of rational knots*, PDMI notes, 2011

[7] D. Krasner, *Integral HOMFLY-PT and sl(n)-link homology*, Int. J. Math. Math. Sci., 2010

[8] L. Lewark, A. Lobb, *New quantum obstructions to sliceness*, Proc. London Math. Soc. (3) 112, 2016

[9] P. Bak, C. Tang, and K. Wiesenfeld. *Self-organized criticality: An explanation of the 1/f noise.* Physical review letters, 59(4):381, 1987.

[10] D. Dhar. *Self-organized critical state of sandpile automaton models.* Phys. Rev. Lett., 64(14):1613–1616, 1990.

[11] L. Levine, J. Propp What is... a Sandpile ?, Notices of the AMS, Volume 57, number 8, 2010

[12] W. Pegden, C. K. Smart *Convergence of the Abelian sandpile* arXiv:1105.0111

[13] L. Levine, W. Pegden, C. K. Smart *Apollonian structure in the Abelian sandpile* arXiv:1208.4839

[14] S. Caracciolo, G. Paoletti, and A. Sportiello *Conservation laws for strings in the Abelian Sandpile Model.* arXiv:1002.3974

[15] S. Caracciolo, G. Paoletti, and A. Sportiello *Multiple and inverse topplings in the abelian sandpile model.* arXiv:1112.3491

[16] N. Kalinin, M. Shkolnikov, *Tropical curves in sandpiles*, Comptes Rendus Mathematique 354, 2016

[17] N. Kalinin, M. Shkolnikov *Tropical curves in sandpile models (in preparation)*. arXiv:1502.06284

[18] N.Kalinin, M. Shkolnikov, *Sandpiles on the heptagonal tiling*, Journal of knot theory and its ramifications 25, 12, 2016

[19] E. Brugallé, I. Itenberg, G. Mikhalkin, and K. Shaw. *Brief introduction to tropical geometry.* Proceedings of 21st Gökova Geometry-Topology Conference., 2015.

[20] L. Katzarkov, E. Lupercio, L. Meersseman, A. Verjovsky *Non-commutative Toric Varieties.* arXiv:1308.2774

[21] G. Mikhalkin, M. Shkolnikov *Hyperbolic amoebas (in preparation).*

[22] N. Kalinin, E. Lupercio, Y. Prieto, A. G. Saenz, M. Shkolnikov, *Self-organized criticality and tropical geometry*,(in preparation)