TROPICAL SERIES AND SANDPILES IN ARBITRARY CONVEX DOMAINS

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Abstract. This is a draft of the second part of the article entitled “Tropical curves in 2-dimensional sandpile model”. Here we study the case of an arbitrary boundary.

1. Sandpile models

For an introduction, see our article [6] and references therein.

Here we prove that the limiting procedure for sandpiles defined on an convex body gives almost the same answer as for polygonal boundary. Namely, the critical locus tends to a tropical curve, but it has infinite number of vertices now. For that we develop some polytope theory and study different convergency questions.

2. Tropical series

Definition 1. We say that a polygon $\Delta \subset \mathbb{R}^2$ is a $\mathbb{Q}$-polygon if all its sides have rational slopes.

Definition 2. [10] A function $F : \Omega \to \mathbb{R}$ is called *a tropical series* if for each $x \in \Omega^\circ$ there exists a neighbourhood $U \subset \Omega$ of x such that $F|_U$ is a tropical polynomial.

Let Ω be a convex domain and $\bar{p} = \{p_1, \dots, p_n\}$ be a collection of points in Ω°. We always assume that the collection of points $\bar{p}$ is non-empty. Denote by $V(\Omega, \bar{p})$ the set of tropical series F on Ω such that $F|_\Omega \geq 0$ and F is not smooth at each of the points p_i.

Definition 3. For $x \in \Omega^\circ$ define $G^\Omega_{\bar{p}}(x) = \inf\{F(x) | F \in V(\Omega, \bar{p})\}$.

Lemma 1. The function $G^\Omega_{\bar{p}}$ is a tropical series.

Proof. Let us take a point $q \in \Omega^\circ$. If in a neighbourhood of q the function $G^\Omega_{\bar{p}}$ is the minimum of a finite number of tropical polynomials then it is itself a tropical polynomial. It remains to show that near q the function $G^\Omega_{\bar{p}}$ can not be a minimum of infinite number of series. If it is indeed the case, then, using the fact that all our linear functions are of the type $ix + jy + a_{ij}, i, i \in \mathbb{Z}$, we obtain that $G^\Omega_{\bar{p}}(q') = -\infty$ for some q' arbitrary close to q what is impossible since $G^\Omega_{\bar{p}} \geq 0$. □

In the notation of [6] the function $G^\Omega_{\bar{p}}$ is $G_{p_1,\dots,p_n}(0)$.

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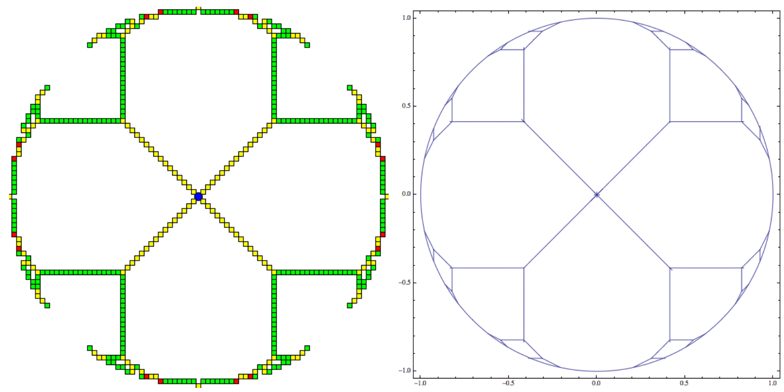

FIGURE 1. Additional grain of sand thrown to the center. Sand picture and limiting example.

Remark 1. Each tropical series f is a concave, piece-wise linear with integer slopes real-valued function on Ω°, in general f cannot be extended to $\overline{\Omega}$.

Corollary 1. The functional $F \mapsto \int_\Omega F dxdy$ defined on the space $V(\Omega, \bar{p})$ has a unique minimum $G_{\bar{p}}^{\Omega}$.

Definition 4. Let $C_{\bar{p}}^{\Omega}$ be a tropical analytic curve defined as the set of points where $G_{\bar{p}}^{\Omega}$ is not smooth.

Note that $G_{\bar{p}}^{\Omega}$ is a tropical polynomial if Ω is a lattice polygon, as we proved in [6].

Definition 5. A tropical analytical curve in Ω is the locus of non-linearity of a tropical series on Ω.

Definition 6. For any $w \in \mathbb{Z}^2$ denote by c_w the infimum of $w \cdot x$ over $x \in \Omega$. Let A_Ω be the set of all such w that $c_w \neq -\infty$. If $w \in A_\Omega$ define function l_Ω^w to be

$$l_\Omega^w(x) = w \cdot x - c_w.$$

Note that l_Ω^w is positive on Ω° and it is actually a support function for Ω.

Definition 7. The weighted distance function l_Ω on Ω is defined by

$$l_\Omega(x) = \inf_{w \in A_\Omega} l_\Omega^w(x) = \text{“} \sum_{w \in A_\Omega} c_w x^w \text{”}$$

Since l_Ω is an infimum of non-negative functions, it s a non-negative concave continous function on Ω. For each point $x \in \Omega^\circ$

Definition 8. Denote by C^Ω the tropical analytic curve defined by l_Ω.

Definition 9. Let R_Ω be a set of points on $x \in \partial\Delta$ such that $l_\Omega^w(x) = 0$ for some $w \in A_\Omega$.

Lemma 2. The set $\partial\Omega \backslash \bar{R}_\Omega$ is a disjoint union of intervals with irrational slope.

Proof. Consider a connected component I of $\partial\Omega\backslash\bar{R}$. Note that if I is not a straight interval, then we can find a line $\{l^w_\Omega = 0\}$ such that its intersection with I is not empty. Thus, I is a straight interval that cannot have a rational slope because otherwise $I \subset R$. □

Lemma 3. For a smooth convex function f with $f(0) = 0$ we have $f((f')^{-1}(x)) < x$ for x close to 0.

Proof. Indeed, denote $y = (f')^{-1}(x)$, then we have $f(y) < f'(y)$ which easy follows from convexity for $y < 1$. □

Lemma 4. For each convex function f such that $f > 0$ on $(0, \varepsilon]$ there exists a smooth convex function $\bar{f} > 0$ on $(0, \varepsilon'], \varepsilon' > 0$ such that $f < \bar{f}$ on $[0, \varepsilon']$.

Proof. We can consider the function $f_1(x) = f(x) \cdot x^2$, then choose a sequence $p_i \to 0$ and then draw piece-wise linear f_2 through points $f_1(p_i)$ and smoothen it. Namely, the fact that the interval between $(p_i, f_1(p_i))$ and $(p_{i-1}, f_1(p_{i-1}))$ is below the graph of f follows from $p_{i-1}\frac{f(p_i)}{p_i} \geq f(p_{i-1})p^2_{i-1}$, so we choose $p_1 = \varepsilon$ and p_i from the condition $\frac{f(p_i)}{p_i} = 2f(p_{i-1})p_{i-1}$. Clearly $p_i < p_{i-1}$ and the only limiting point p^* of the sequence of p_i is $p^* = 0$ since $\frac{f(p^*)}{p^*} = 2f(p^*)p^*$ implies $f(p^*) = 0$ or $p^* = 1/\sqrt{2}$ but we chose ε' small enough. Then we smooth this piecewise linear function.

□

Lemma 5. The function l_Ω can be continuously extended to $\partial\Omega$ by zero.

Proof. It is clear that l_Ω is zero on the closure of zero sets of functions $l^w, w \in \mathbb{Z}^2$. It means that it is enough to consider the situation near irrational smooth edges. Consider an edge of a boundary with irratioanl slope α. By the previous lemma we can suppose that the boundary near the edge is given by a garph of a smooth function $\alpha x + f(x)$. By continued fractions for an irrational number α we get two sequences of numbers, $p_{2n}/q_{2n} < \alpha < p_{2n+1}/q_{2n+1}$ such that $\alpha - p_{2n}/q_{2n} < 1/q^2_{2n}, p_{2n+1}/q_{2n+1} - \alpha < 1/q^2_{2n+1}$. Take an approximation $p/q = p_i/q_i$ for α.

Then for a tangency of slope p/q we have $\alpha + f'(x_0) = p/q$, then the equation of the line is $y - \alpha x_0 - f(x_0) = p/q(x - x_0)$ and we want its value at $(-t, -\alpha t)$. Id est, $x_0 = (f')^{-1}(p/q - \alpha)$,

$$q(y - \alpha(f')^{-1}(p/q - \alpha) - f((f')^{-1}(p/q - \alpha))) = p(x - (f')^{-1}(p/q - \alpha)).$$

$$-t(\alpha q - p) + (p - \alpha q)((f')^{-1}(p/q - \alpha)) - qf((f')^{-1}(p/q - \alpha))$$

Since $|\alpha q - p| < 1/q$ we have $| - t(\alpha q - p)| < t/q$,

$$(p - \alpha q)((f')^{-1}(p/q - \alpha)) < 1/q(f')^{-1}(1/q^2),$$

which tends to zero as $q \to \infty$, and since $f((f')^{-1}(x)) < x$ for x small enough, we have

$$qf((f')^{-1}(p/q - \alpha)) < q(p/q - \alpha) < 1/q.$$

□

Remark 2. If $f(x, y) = ix + jy + a_{ij}, i, j \in \mathbb{Z}, a_{ij} \in \mathbb{R}$, $f|_\Omega \geq 0$ and $(i, j) \neq (0, 0)$, then $f \geq l_\Omega$ on Ω.

Remark 3. For a point $q \in \Omega^\circ$ we have $G^\Omega_{\{q\}}(x) = \min\{l_\Omega(x), l_\Omega(q)\}$.

Proposition 1. It follows from the previous remark that there exists $\varepsilon > 0$ such that $G_{\bar{p}}^{\Omega}(x) \geq l_{\Omega}(x)$, for any x in the ε-neighbourhood of $\partial\Omega$.

Proposition 2. For any $x \in \Omega$, and $\bar{p} = \{p_1, \ldots, p_n\}$

$$G_{\bar{p}}^{\Omega}(x) \leq \sum G_{\{p_i\}}^{\Omega}(x) \leq n \cdot l_{\Omega}(x)$$

3. Limits of toppling functions for an arbitrary boundary

Lemma 6. Consider a function F on Ω given by $F(x) = \inf G_{\bar{p}}^{\Delta}(x)$, where the infimum is taken over all the $\mathbb{Q}$-polygons Δ containing Ω. Then F coincides with $G_{\bar{p}}^{\Omega}$.

Proof. Consider an arbitrary point $q \in \Omega^{\circ}$. Since $G_{\bar{p}}^{\Delta}|_{\Omega} \in V(\Omega, \bar{p})$ for any $\Delta \supset \Omega$, we have $F(q) \geq G_{\bar{p}}^{\Omega}(q)$. On the other hand, there exist only a finite number of monomials in $G_{\bar{p}}^{\Omega}$ which contribute to the values of $G_{\bar{p}}^{\Omega}$ at all the points $p_i \in \bar{p}$ and q. Thus, eliminating all the other monomials from $G_{\bar{p}}^{\Omega}$, we get a new tropical polynomial F_0. Note that F_0 is not smooth at the points p_i and $F_0(q) = G_{\bar{p}}^{\Omega}(q)$. Moreover, the set of points where $F_0 \geq 0$ contains a $\mathbb{Q}$-polygon $\Delta_0 \supset \Omega$, because $F_0|_{\Omega} \geq 0$. This implies that $F_0 \in V(\Delta_0, \bar{p})$ from which it immidiatly follows that

$$G_{\bar{p}}^{\Omega}(x) = F_0(x) \geq G_{\bar{p}}^{\Delta_0}(x) \geq F(x).$$

□

Lemma 7. For any $\varepsilon > 0$ the set $\Omega_{\varepsilon} = \{x \in \Omega | G_{\bar{p}}^{\Omega} \geq \varepsilon\}$ is a $\mathbb{Q}$-polygon and $G_{\bar{p}}^{\Omega}|_{\Omega_{\varepsilon}}$ is a tropical polynomial.

Proof. Since $G_{\bar{p}}^{\Omega}$ is continous and vanishes at the boundary of Ω, the set $G_{\bar{p}}^{\Omega} = \varepsilon$ is a curve disjoint from $\partial\Omega$. We claim that the intersection of Ω_{ε} with the tropical analytic curve defined by the series $G_{\bar{p}}^{\Omega}$ is a finite part of a tropical curve. Indeed, if the intersection is supported on an infinite graph, then we can find a sequence of vertices convergent to a point $y \in \Omega^{\circ}$. Thus, there is no neighbourhood of y where the series $G_{\bar{p}}^{\Omega}$ can be represented by a tropical polynomial, which it is a contradiction.

The finiteness of the number of vertices implies that there is only a finite number of monomials participating in the restriction of $G_{\bar{p}}^{\Omega}$ to the domain Ω_{ε}. □

Remark 4. For a convex domain Ω we constructed a canonical series of polygon $\Omega_{\varepsilon} \to \Omega$. This could lead to new toric insights.

Lemma 8. In the hypothesis of the previous lemma, for $\varepsilon > 0$ such that $G_{\bar{p}}^{\Omega}(p_i) \geq \varepsilon$ for each $p_i \in \bar{p}$ we have $G_{\bar{p}}^{\Omega_{\varepsilon}} = G_{\bar{p}}^{\Omega} - \varepsilon$.

Proof. We can canonically extend $G_{\bar{p}}^{\Omega_{\varepsilon}} + \varepsilon$ to Ω, using the fact that it is a tropical polynomial. Since $G_{\bar{p}}^{\Omega_{\varepsilon}} + \varepsilon$ is linear near each side of $\partial\Omega_{\varepsilon}$ and $G_{\bar{p}}^{\Omega_{\varepsilon}} + \varepsilon \leq G_{\bar{p}}^{\Omega}$ inside Ω_{ε}, we have $G_{\bar{p}}^{\Omega_{\varepsilon}} + \varepsilon \geq G_{\bar{p}}^{\Omega}$ on $\Omega \setminus \Omega_{\varepsilon}$. Then, $G_{\bar{p}}^{\Omega_{\varepsilon}} + \varepsilon \in V(\Omega, \bar{p})$, that finishes the proof. □

Lemma 9. Consider a function F on Ω given by $F(x) = \sup G_{\bar{p}}^{\Delta}(x)$, where the supremum is taken over the set of all lattice polygons $\Delta \subset \Omega$ containing x. Then F coincides with $G_{\bar{p}}^{\Omega}$.

Proof. It is clear that $G_{\bar{p}}^{\Omega} \geq F$ on Ω. Consider a point $q \in \Omega^{\circ}$ and $\varepsilon > 0$ such that $G_{\bar{p}}^{\Omega}(q) > \varepsilon$ and $G_{\bar{p}}^{\Omega}(p_i) > \varepsilon$ for each $p_i \in \bar{p}$. We are going to prove that $F(q) + \varepsilon \geq G_{\bar{p}}^{\Omega}(q)$. Lemma 7 implies that the set $\Omega_{\varepsilon} = \{x \in \Omega | G_{\bar{p}}^{\Omega}(x) \geq \varepsilon\}$ is a $\mathbb{Q}$-polygon. Evidently, $q \in \Omega_{\varepsilon}$ and $F \in V(\Omega_{\varepsilon}, \bar{p})$, therefore $F(q) \geq G_{\bar{p}}^{\Omega_{\varepsilon}}(q) = G_{\bar{p}}^{\Omega}(q) - \varepsilon$. □

Theorem 1. A sequence of functions $F_N^{\Omega} \colon \Omega \to \mathbb{R}$ given by

$$F_N^{\Omega}(x, y) = \frac{1}{N} Toppl_{\phi_N}^{\Omega}([Nx], [Ny])$$

uniformly converges to $G_{\bar{p}}^{\Omega}$.

Proof. It is clear that $F_N(\Delta)(x, y) \leq F_N(\Omega)(x, y) \leq F_N(\Delta')(x, y)$ for polygons Δ, Δ' such that $\Delta \subset \Omega \subset \Delta'$. Then, using Theorems from the previous article we get $G_{\bar{p}}^{\Delta} = \lim F_N(\Delta)(x, y) \leq \lim F_N(\Delta')(x, y) = G_{\bar{p}}^{\Delta'}$. Then, applying Lemmata 6 and 9 by standard squeeze convergency arguments we get

$$\lim_{\Delta} G_{\bar{p}}^{\Delta} = \lim_{N} F_N(\Omega)(x, y) = \lim_{\Delta'} G_{\bar{p}}^{\Delta'} = G_{\bar{p}}^{\Omega}.$$

□

Theorem 2. The sequence of sets $\frac{1}{N} E_N$ has a limit $\tilde{C}$ in Ω in the Hausdorff sense.

Proof. Indeed, $\frac{1}{N} E_N$ tends to $G_{\bar{p}}^{\Omega_{\varepsilon}}$ on Ω_{ε} for each $\varepsilon > 0$. □

Corollary 2. We can do all the same for all convex domains, possibly unbounded.

Corollary 3. Furthermore, under the above hypothesis, $C = \tilde{C} \backslash \partial\Omega$ is the zero set of an infinite tropical series $G_{\bar{p}}^{\Omega}$. Moreover, C passes through the points $p_1, \ldots, p_n$ and $\partial\Delta \setminus \overline{C}$ is the collection of intervals with rational slopes.

Remark 5. Consider a halfplane with irrational slope of its boundary. That is unique example where we have no limit.

4. Tropical symplectic length of curves defined by tropical series

Definition 10 (See [3])**.** The tropical symplectic area of a finite segment l with a rational slope is given by $Area(l) = Length(l) \cdot Length(v)$, where $Length(-)$ denotes a Euclidean length and v is a primitive integer vector parallel to l. If C' is a finite part of a tropical curve, then its symplectic area is the weighted sum of areas for its edges, i.e.

$$Area(C') = \sum_{e \in E(C')} Area(e) \cdot Weight(e).$$

Definition 11. We say that G is *antirational* if ∂G contains a connected part r which contains no subsets of rational slope.

Remark 6. For Δ which are neither $\mathbb{Q}$-polygons nor antirational, the $Area(C^{\Delta})$ can be infinite or finite.

Lemma 10. If ∂G is antirational then $Area(C^{\Omega})$ is infinite.

Proof. This lemma is somewhat similar to Theorem 1 in [3] but has no direct relation with it. Since Ω is antirational, there is a part r of $\partial\Omega$ which contains no sides of rational slope. We consider the polygons Ω_ε. Let us choose $r_\varepsilon \subset \partial\Omega_\varepsilon$ such that $r_\varepsilon \to r$. Recall that $Area(C^\Omega \cap \Omega_\varepsilon)$ equals to the sum of euclidean lengths of the sides of Ω_ε with the weights, given by their directions. Since r contains no sides of rational slopes, then for each $N > 0$ the sum s_i of euclidean lengths of edges in r_ε with slopes $(p, q), \sqrt{p^2 + q^2} < N$ tends to zero. This implies that $Area(r_\varepsilon) \to \infty$ as $\varepsilon \to \infty$, because $Area(r_\varepsilon) \geq N \times (r_\varepsilon - s_\varepsilon)$. □

Question: what is the euclidean length of this tropical curves?

Since the tropical symplectic area of C^Ω in infinite, it is natural to consider some kind of regularisation.

Definition 12. *Regularized* area of $C^\Omega_{\bar{p}}$ is the limit by compacts $K \subset \Omega$ of $Area(C^\Omega_{\bar{p}} \cap K) - Area(C^\Omega \cap K)$.

Lemma 11. If $\partial\Omega$ contains no sides with rational slopes then the regularised area of each $C^\Omega_{\bar{p}}$ is zero.

Proof. Indeed, if we consider the weights on $\partial\Omega_\varepsilon$ for $C^\Omega_{\bar{p}}$ and C^Ω, they differ only on finite number of intervals. Since $\partial\Omega$ contains no sides with rational slopes, all these intervals tends to zero. □

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