TROPICAL CURVES IN SANDPILES

NIKITA KALININ, MIKHAIL SHKOLNIKOV

Abstract. We study a sandpile model on the set of the lattice points in a large lattice polygon. A small perturbation ψ of the maximal stable state $\mu \equiv 3$ is obtained by adding extra grains at several points. It appears, that the result ψ° of the relaxation of ψ coincides with μ almost everywhere; the set where $\psi^\circ \neq \mu$ is called the deviation locus. The scaling limit of the deviation locus turns out to be a distinguished tropical curve passing through the perturbation points.

Nous considérons le modèle du tas de sable sur l'ensemble des points entiers d'un polygone entier. En ajoutant des grains de sable en certains points, on obtient une perturbation mineure de la configuration stable maximale $\mu \equiv 3$. Le résultat ψ° de la relaxation est presque partout égal à μ. On appelle lieu de déviation l'ensemble des points où $\psi^\circ \neq \mu$. La limite au sens de la distance de Hausdorff du lieu de déviation est une courbe tropicale spéciale, qui passe par les points de perturbation.

1. Introduction

1.1. **Sandpile model on a finite set.** Consider the standard lattice $\mathbb{Z}^2$ on the plane. For $v \in \mathbb{Z}^2$ we denote by $n(v)$ the set of all four closest points to v in $\mathbb{Z}^2$. Let Γ be a finite subset of $\mathbb{Z}^2$. A *state (or a configuration)* $\phi \in \mathbb{Z}^\Gamma$ of a sandpile on Γ is a non-negative integer-valued function on Γ. For a state ϕ and $v \in \Gamma$ we interpret $\phi(v)$ as a number of sand grains at the site v. For each $v \in \Gamma$ we define the *toppling* operator $T_v \colon \mathbb{Z}^\Gamma \to \mathbb{Z}^\Gamma$ at v given by

$$T_v\phi = \phi - 4\delta_v + \sum_{w \in \Gamma \cap n(v)} \delta_w,$$

where δ_v is a function on the lattice defined to be 1 at v and 0 otherwise. A toppling T_v is called *legal* for a state ϕ if $\phi(v) \geq 4$, i.e. $T_v\phi$ is again a state. In this case we think of T_v as a redistribution of sand from the overfilled site v to its neighbors. If some neighbors are missing in Γ, i.e. $n(v) \not\subset \Gamma$, then at least one grain leaves the system after the toppling. If $\phi(v) < 4$ for all $v \in \Gamma$, then ϕ is called a *stable* state. The state μ which is defined to be equal 3 at every point of Γ is called the *maximal stable state.*

A *relaxation* for a state ϕ is a sequence of states $\phi = \phi_0, \phi_1, \dots \phi_m$ such that ϕ_{i+1} is the result of applying a legal toppling to ϕ_i and ϕ_m is a stable state. It is well known that for any state ϕ there exists a relaxation and the last state ϕ_m depends only on ϕ (see [1, 2]). We denote ϕ_m by ϕ° and call it *the result of the relaxation* of ϕ. Informally, in order to find the result of the relaxation it doesn't matter which legal topplings to apply at each step of a relaxation sequence.

1.2. **Motivation.** Let $\Omega \subset \mathbb{R}^2$ be a non-degenerate (i.e. with non-zero area) lattice polygon. Let Γ be the intersection of $\mathbb{Z}^2$ with Ω. Consider a set $P \subset \Gamma$. We add extra grains to the state μ at the points $P = \{p_1, p_2, \dots, p_n\}$. After the relaxation, this gives the state $\phi = (3 + \sum_{p \in P} \delta_p)^\circ$.

In the examples shown in Figures 1,2 and 3 we see that the set of points where ϕ is not maximal constitutes some sort of a graph passing through P. As it is seen in Figures 2 and 3, the picture becomes more regular when the cardinality of P is small with respect to the size of Ω. In the next section we state certain theorems formalizing this concept. Some of this results were already proven in [3]. Far going generalizations of the theorems and their proofs will appear in [4]. This short note can be seen as an introduction to the subject and an announcement of [4].

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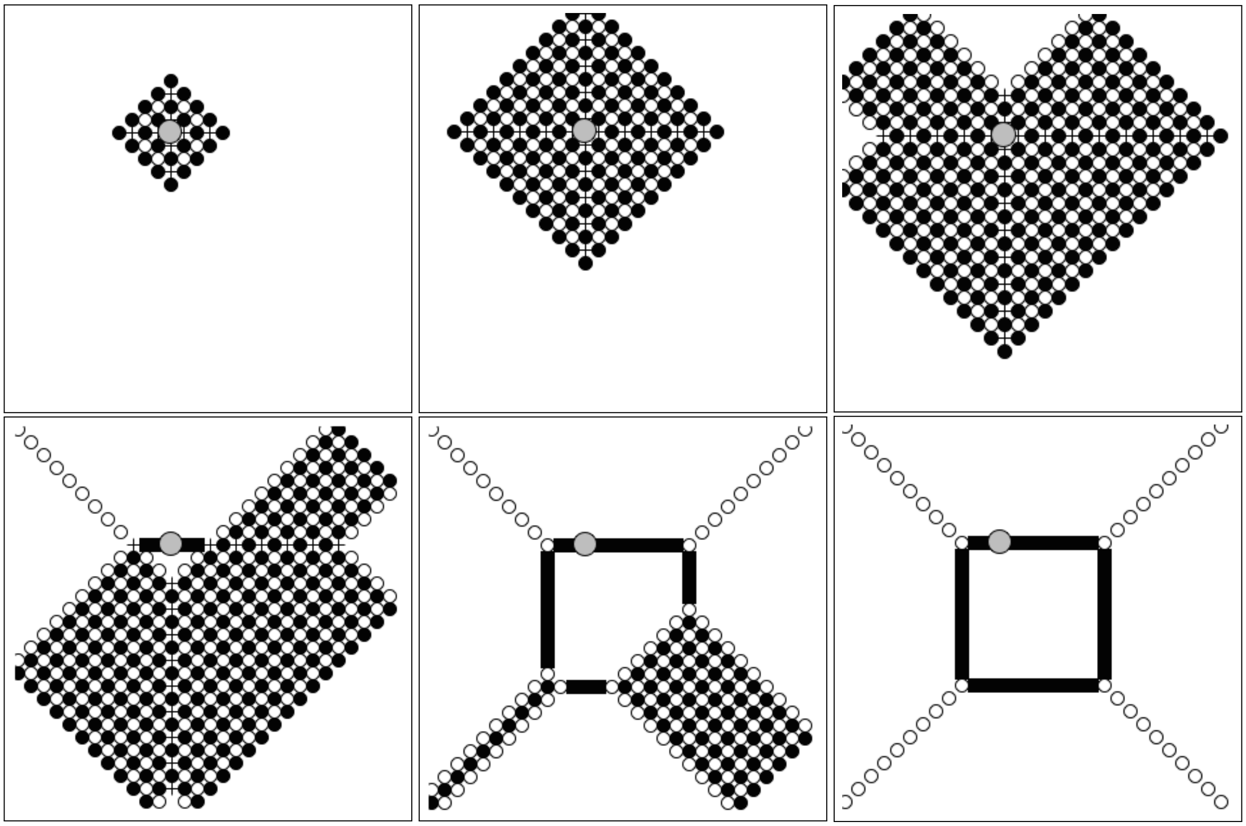

Figure 1. Snapshots during the relaxation for the state $\phi \equiv 3$ on a square after adding an extra grain at one point p (the big grey point). Black rounds represent x with $\phi(x) \geq 4$, black squares (which are arranged along the vertical and horizontal edges on the final picture) represent the value of sand equal to 2, white rounds (arranged along diagonals on the final picture) are 1, and whites cells are 3. Rare cells with zero grains are marked as crosses, one can see them during the relaxation on the vertical and horizontal lines through p. The value of the final state at p is 3.

Instantanés pendant la relaxation de la configuration $\phi \equiv 3$ sur un carré, après ajout d'un grain additionnel au point p (le gros point gris). Les ronds noirs représentent x avec $\phi(x) \geq 4$, les carrés noirs (qui se trouvent sur les arêtes verticales et horizontales dans la configuration finale) représentent les points oú le nombre de grains de sable égale 2. Les cercles (sur les diagonales dans la configuration finale) possèdent un grain de sable et les cellules blanches en ont 3. Les cellules rares avec zéro grains sont marquées par des croix; on peut les voir pendant la relaxation sur les lignes verticales et horizontales passant par p. La valeur au point p dans la configuration finale est 3.

1.3. **Related activities.** What we described above is quite close, at least at the basic level, to the original Back-Tang-Wiesenfeld model [1]. The fundamental difference is that their framework had a probabilistic flavor. Their subject was a stochastic growth of a sandpile simulated by an iterative process of adding an extra grain at a random site followed by a relaxation.

In the paper [2] it has been proven that for any initial configuration most of the states stop occurring after some step of the BTW-process and the other states (the *recurrent* states) occur with equal probabilities. Dhar has also shown that the set of all recurrent states forms an abelian group (called the sandpile group) under the action $\phi \oplus \psi = (\phi + \psi)^{\circ}$. Study of this group is an important and fruitful part of the theory of abelian sandpiles [5]. Unfortunately, it is little known about the structure of particular elements of the sandpile group [6]. There is a simple characterization of a recurrent state. A stable state ϕ is

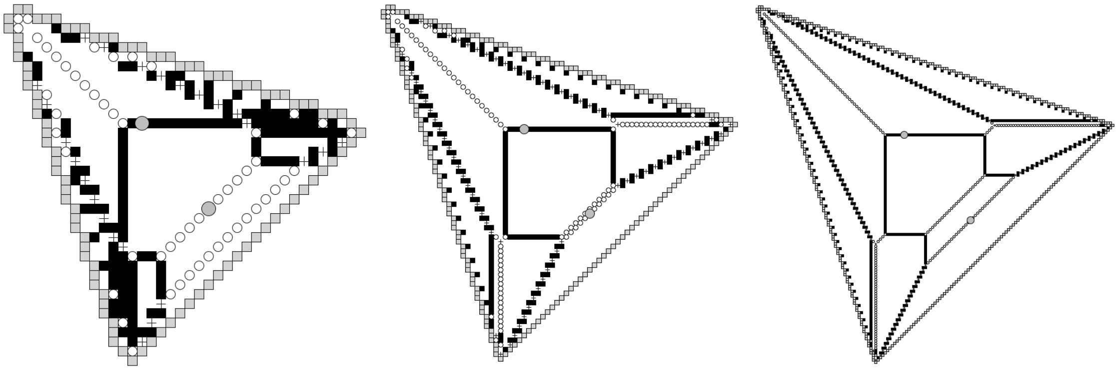

FIGURE 2. Triangular domain and two perturbation points. The scale of last picture is four times bigger than the scale of the first and two times bigger than the scale of the second.
Un domaine triangulaire, deux point de perturbation. L'échelle de la figure de droite est quatre fois celle de la figure de gauche et deux fois celle de la figure centrale.

recurrent if and only if it can be represented as $\phi = (3+\psi)^\circ$ for some state ψ. Therefore, the states that we study in this paper are recurrent.

Our initial inspiration was the paper [7]. They considered a modification of the BTW-process. It was clear for us that some pictures that they obtained have depicted tropical curves. At the beginning our principal goal was to understand reasons why these curves do appear.

We use scaling limits since in our case this is the most concise way to give the precise statements. It makes our results particularly close to [8, 9, 10] where the scaling limit of the states $(N\delta_0)^\circ$ was shown to exist.

2. THE RESULTS

Let Ω be a non-degenerate lattice polygon and P be a finite non-empty subset of Ω°. For any $N \in \mathbb{N}$ consider a set $\Gamma_N = N\Omega \cap \mathbb{Z}^2$. Denote by $[p] \in \mathbb{Z}^2$ the coordinate-wise rounding down of a point $p \in \mathbb{R}^2$. Define the state $\phi_N = (3 + \sum_{p\in P} \delta_{[Np]})^\circ$ on Γ_N and the *deviation* set

$$C_N = \frac{1}{N}\{v \in \Gamma_N | \phi_N(v) < 3\}.$$

Experimental evidence suggests that when N grows the shape of sets C_N stabilizes, see Figure 2.

Theorem 1. The sequence of sets $C_N \subset \Omega$ Hausdorff converges to a set $\tilde{C}_{\Omega,P}$. The set $\tilde{C}_{\Omega,P}$ is a planar graph passing through the points P. Each edge of $\tilde{C}_{\Omega,P}$ is a straight segment with a rational slope.

Denote by $C_{\Omega,P}$ the closure of $\tilde{C}_{\Omega,P} \cap \Omega^\circ$. We have a canonical mapping associating the graph $C_{\Omega,P}$ to the configuration of points P. Theorem 4 later represents $C_{\Omega,P}$ as a solution of some variation problem.

First of all we note that $C_{\Omega,P}$ is a weighted graph, i.e. there is a canonical choice of weights for its edges. This choice comes from averaging an amount of sand in ϕ_N along edges of $C_{\Omega,P.}$ Namely, we define a sequence of functions $\phi'_N \colon \mathbb{R}^2 \to \mathbb{R}$ given by $\phi'_N(x) = N(3 - \phi_N([Nx]))$ if $[Nx] \in \Gamma_N$ and $\phi'_N(x) = 0$ otherwise.

Theorem 2. There exists a $*$-weak limit ψ of the sequence ϕ'_N. Moreover, there exists a unique assignment of weights $m_e^{\Omega,P} \in \mathbb{N}$ for edges of $C_{\Omega,P}$ such that for all smooth functions ψ supported on Ω°

$$\lim_{N\to\infty} \int_{\mathbb{R}^2} \phi'_N \psi = \sum_{e\in E} ||l_e|| m_e^{\Omega,P} \int_e \psi,$$

where E is the set of all edges of $C_{\Omega,P}$ and l_e is a primitive vector of $e \in E$, i.e. the coordinates of l_e are coprime integers and l_e is parallel to e.

In other words, edges with high weights are "wider" than edges with low weights (see Figures 2 and 3). Theorem 2 motivates the following definition.

Definition 1. Let C be a weighted planar graph whose edges have rational slopes. We define *the tropical symplectic area* of C by

$$Area(C)=\sum_{e\in E}||l_e||m_e||e||.$$

It appears that $C_{\Omega,P}$ minimizes its area in the class of weighted graphs that we call Ω-*tropical* curves.

Definition 2. A finite planar weighted graph $C\subset\Omega$ is called an Ω-tropical curve if

- each edge e of C has a rational slope and an integer weight m_e,
- the intersection of C with $\partial\Omega$ is the set of all vertices of Ω,
- if $v\in\Omega^\circ$ is a vertex of C, then

$$\sum_{e\in E_v} m_e l_e=0, \tag{1}$$

  where E_v is the set of all edges of $C_{\Omega,P}$ incident to v and l_e is the primitive vector of $e\in E_v$ oriented out of v,
- there exist a unique labeling $d_s\in\mathbb{Z}_{>0}$ for each side s of Ω such that for each vertex v of Ω

$$\sum_{e\in E_v} m_e l_e=d_{s_1}l_{s_1}+d_{s_2}l_{s_2}, \tag{2}$$

  where s_1 and s_2 are the two sides of Ω incident to v, l_{s_i} is the primitive vector of s_i oriented out of v.

Condition (1) is well known in tropical geometry under the name *balancing condition,* see [11]. We call (2) the *outer balancing* condition.

The condition (1) implies that an Ω-tropical curve can be always represented as the intersection of Ω with a *tropical curve* (see [12, 11]). A *plane tropical curve* is a planar graph, whose edges are intervals with rational slopes and prescribed positive integer weights. The edges are allowed to be unbounded, the number of edges and vertices is finite and at each vertex the balancing condition is satisfied. If C is a subgraph of a tropical curve, then its area can be seen as the limit of the usual symplectic areas for a family of holomorphic curves that degenerates to C (see [3, 13]). On the other hand, the area can be seen as a particular normalization for the Euclidean length of C. In this interpretation $C_{P,\Omega}$ gives a solution to the analog of the Steiner tree problem.

Theorem 3. *The graph $C_{\Omega,P}$ is an Ω-tropical curve. Furtherefore, $C_{\Omega,P}$ has the minimal area among all Ω-tropical curves passing through the configuration of points P.*

In certain cases, the curve $C_{\Omega,P}$ can be found by means of this minimization property. This happens essentially when Ω is "simple" enough and the number of points $|P|$ is small. In general, for any lattice polygon Ω there are *generic* configurations P for which $C_{\Omega,P}$ is not a unique minimizer of the symplectic area. Fortunately, the curve $C_{\Omega,P}$ can be characterized by the property of minimizing its tropical polynomial.

A *tropical polynomial* is a function $F\colon\mathbb{R}^2\to\mathbb{R}$ given by

$$F(x_1,x_2)=\min_{(k,l)\in A}(kx_1+lx_2+a_{k,l}),$$

where A is a finite subset of $\mathbb{Z}^2$ and $a_{k,l}\in\mathbb{R}$. For a tropical polynomial F we consider its corner locus C, i.e. the set of points where F is not locally linear. The set C has a natural structure of a tropical curve [11] and we say that the curve C is defined by the polynomial F.

Consider an Ω-tropical curve C. Consider an extension of C to a tropical curve C', i.e. $C=C'\cap\Omega$. The outer balancing condition (2) implies that there exist a unique tropical polynomial F' vanishing at $\partial\Omega$ which defines C'. It is easy to see that $F=F'|_\Omega$ doesn't depend on C'. Therefore, we call F *the Ω-tropical polynomial of the curve C.* Denote by $F_{\Omega,P}$ the Ω-tropical polynomial of $C_{\Omega,P}$.

Theorem 4. *The polynomial $F_{\Omega,P}$ is the point-wise minimum of all polynomials F of all Ω-tropical curves that pass through P.*

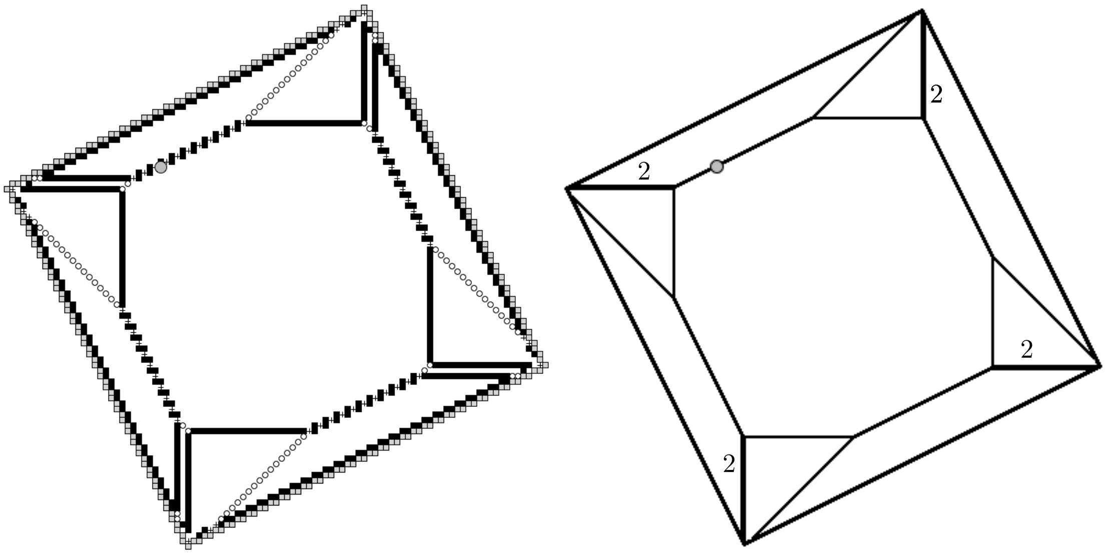

FIGURE 3. The result of adding a grain in two different scales. The polygon Ω is a square with edges having slopes 2 and 1/2.
Résultat de l'ajout d'un grain de sable à deux échelles différentes. Le domaine Ω est un carré avec côtés de pente 2 et 1/2 respectivement.

Theorem 4 can be seen as a manifestation of the least action principle for sandpiles [8]. Consider a state ϕ on a finite set $\Gamma \subset \mathbb{Z}^2$. Consider a set $\mathcal{H}$ of functions $H\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ such that H vanishes outside Γ and $\phi + \Delta H \leq 3$ on Γ, where ΔH is the discrete Laplacian of H given by

$$\Delta H(v) = -4H(v) + \sum_{w \in n(v)} H(w).$$

Denote by F the point-wise minimum of all functions $H \in \mathcal{H}$. Then $\phi^\circ = \phi + \Delta F$. Furthermore, $F(v)$ counts the number of topplings at $v \in \mathbb{Z}^2$ in any relaxation sequence for ϕ. Therefore, we call the function F the *toppling function* (or odometer) of the state ϕ.

Let F_N be the toppling function of the state $\phi_N = (3 + \sum_{p \in P} \delta_{[Np]})^\circ$ on $\Gamma_N = N\Omega \cap \mathbb{Z}^2$. Consider a function $\tilde{F}_N\colon \Omega \to \mathbb{R}$ given by

$$\tilde{F}_N(x) = \frac{1}{N} F_N([Nx]).$$

Theorem 5. The tropical polynomial $F_{\Omega,P}$ is the scaling limit of toppling functions, i.e. the sequence $\tilde{F}_n$ converges pointwise to $F_{\Omega,P}$.

This theorem essentially implies the previous ones (see [3]). Note that Theorem 2 is not stated explicitly in [3]. The main idea of the proof of Theorem 5 is that the functions $\tilde{F}_N$ are bounded by the piecewise linear function $F_{\Omega,P}$ and are harmonic on the regions where $\phi_N = \mu$. Then, F_N is harmonic almost everywhere for large N, and its growth is tied by linear functions. This finally gives that $\tilde{F}_N$ is a piecewise linear function almost everywhere. Detailed proofs and generalizations of our results will appear in the paper [4]. In this forthcoming paper, the lattice polygon Ω will be replaced by an arbitrary convex domain. In this case, all the convergence results still hold without significant changes. The set $C_{\Omega,P}$ is still a union of straight edges, but the number of edges is usually infinite. It appears that $C_{\Omega,P}$ is a *tropical analytic curve* and $F_{\Omega,P}$ is a *tropical series*.

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Université de Genève, Section de Mathématiques, Route de Drize 7, Villa Battelle, 1227 Carouge, Switzerland

E-mail address: `Nikita.Kalinin{at}unige.ch`
`mikhail.shkolnikov{at}gmail.com`