Chapter 1

A guide to tropical modifications

Substance is by nature prior to its modifications.
... nothing is granted in addition to the understanding,
except substance and its modifications.
Ethics. Benedictus de Spinoza.

This is a draft of an expository chapter in my thesis. I would be really glad to receive any comments and suggestions (Nikita.Kalinin at unige.ch). Probably, this text is intended to be published somewhere, but definitely not in this state and not soon. If you know some appearance of tropical modification, not mentioned here, or some practical advices about the exposition, or any questions, do not hesitate to contact me. The most recent version is available at https://`https://www.unige.ch/math/tggroup/lib/exe/fetch.php?media=guide.pdf`

This chapter is dedicated to *tropical modifications*, which have already become a folklore in tropical geometry. Tropical modifications are used in tropical intersection theory and in study of singularities. They admit interpretations in various contexts such as hyperbolic geometry, Berkovich spaces, and non-standard analysis.

We cite [10]: "*Tropical modifications ... can be seen as a refinement of the tropicalization process, and allows one to recover some information about X sensitive to higher order terms.*"

One must say that the name "modification" is used in two different senses: the modification as a well-defined *operation*; and a modification along X as a *method* that reveals a behaviour of other varieties in an infinitesimal neighborhood of X. Namely, performing the modification of Y along $X \subset Y$, we know how Y changes, but the objects of codimension 1 in Y may behave differently, depending on their behavior near X. We will clarify this distinction with examples.

Our main goal is to mention different points of view, give references, and demonstrate the abilities of tropical modifications. We assume that the reader have already met "tropical modifications" somewhere and wants to understand them better. There are novelties here: a tropical version of Weil's reciprocity law and the study of non-transversal intersections. For a preliminary introduction to tropical geometry, see [8], [9] and [31], where tropical modifications are also discussed. We are glad to mention other texts, promoting modifications from different perspectives: [10] (examples, construction of curves with inflection points), [3] (repairing the j-invariant of elliptic curves), and [40] (intersection theory on tropical surfaces).

We define of tropical modifications via multivalued operations. Then we discuss several examples indicating principal features of the following observations. Then we obtain several structure theorems and list the set of applications. In Section 1.3 we summarize the interpretations of the tropical modifications, so a curious reader may start there and only after it return to Section 1.0.1.

1.0.1 Definition: tropical modification via the graphs of functions

Recall that the tropical semi-ring $\mathbb{T}$ is defined as $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, with the operation addition ("+") and order as for the real numbers, we extend addition by the rule $-\infty + A = -\infty$ for all $A \in \mathbb{T}$, and the order by the rule $-\infty < A$ for all $A \in \mathbb{R}$. The fastest way to define the tropical modifications is via multivalued tropical addition.

Definition 1.0.1. [45]. Define tropical addition $+_{\text{trop}}$ and multiplication $\cdot_{\text{trop}}$ on the set $\mathbb{T}$ as follows:

- $A \cdot_{\text{trop}} B = A + B$,
- $A +_{\text{trop}} B = \max(A, B)$ if $A \neq B$, and
- $A +_{\text{trop}} A = \{x | x \leq A\}$.

We can say, equivalently, that the operation max is redefined to be multivalued in the case of equal arguments, i.e. $\max(A, A) = \{X | X \leq A\}$.

Definition 1.0.2. A tropical monomial is a function $f : \mathbb{T}^n \to \mathbb{T}$, $f(X_1, X_2, \dots, X_n) = A + i_1 X_1 + i_2 X_2 + \dots + i_n X_n$, where $A \in \mathbb{T}, (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n$. A tropical polynomial is a tropical sum (i.e. we use the operation $+_{\text{trop}}$) of a finite number of tropical monomials. A point $X' = (X_1, X_2, \dots, X_n)$ belongs to the zero set of a tropical polynomial f if $0 \in f(X')$. A tropical hypersurface (as a set) is the zero set of a tropical polynomial on $\mathbb{T}^n$.

Remark 1.0.3. In order to have the balancing condition satisfied, one has to provide a tropical hypersurface with weights on its faces of the maximal dimension. We assume that the reader understands how to do it.

We suppose that the reader knows the definition of an abstract tropical variety, if it is not the case, refer to [30].

Definition 1.0.4. Let V be a tropical hypersurface in $\mathbb{T}^n$, let f be a tropical polynomial on $\mathbb{T}^n$ and W be the zero set of f. The modification of $\mathbb{T}^n$ along W is the set $m_W(\mathbb{T}^n) = \{(X, Y) \in \mathbb{T}^n \times \mathbb{T} | Y \in f(X)\}$, i.e. the graph of the multivalued function f. We call a tropical subvariety $V' \subset m_W(\mathbb{T}^n)$ **a** modification of V if the natural projection $p : \mathbb{T}^n \times \mathbb{T} \to \mathbb{T}^n$ restricted to V' is a tropical morphism $p : V' \to V$ of degree one.

Proposition 1.0.5 ([30], 1.5 B,C)**.** The set $m_W(\mathbb{T}^n)$ coincides with the zero set of the polynomial $f(X) +_{\text{trop}} Y : \mathbb{T}^n \times \mathbb{T} \to \mathbb{T}$.

Definition 1.0.6. For an abstract tropical variety Z and its subvariety $W \subset Z$ defined by a tropical (multivalued) function $f : Z \to \mathbb{T}$, we define the tropical modification $m_W(Z)$ of Z along W as the graph of f in $Z \times \mathbb{T}$. A subvariety $V' \subset m_W(Z)$ is called a modification of V along W if the natural projection $V' \to V$ is of degree one.

Now we explain how the given definitions appear through limiting procedures. Consider two algebraic curves $C_1, C_2 \subset (\mathbb{C}^*)^2$ defined by equations $F_1(x,y) = 0, F_2(x,y) = 0$, respectively. We build the map $m_{C_2} : (x,y) \to (x,y,F_2(x,y)) \in (\mathbb{C}^*)^3$. The set $m_{C_2}((\mathbb{C}^*)^2)$ is the graph of F_2, $z = F_2(x,y)$. Now the intersection $C_1 \cap C_2$ can be easily recovered as $m_{C_2}(C_1) \cap \{(x,y,0)\} = \{z = 0\}$. For the complex curves this seems to be not very interesting, but during the tropicalization process the plane $(x,y,0)$ goes to the plane $(X,Y,-\infty) = \{Z = -\infty\}$, and the intersection of tropical curves will be represented by certain rays going to minus infinity by Z coordinate.

We look now on the limiting procedure. Given two tropical curves $C_1, C_2 \subset \mathbb{T}^2$, we start with $C_{1,t}, C_{2,t}$ – two families of plane algebraic curves, which tropicalize to C_1, C_2, i.e., in the Gromov-Hausdorff sense we have

$$C_1 = \lim_{t\to\infty} \mathrm{Log}_t(C_{1,t}), C_2 = \lim_{t\to\infty} \mathrm{Log}_t(C_{2,t}),$$

where we apply $\mathrm{Log}_t : \mathbb{C} \to \mathbb{T}$ coordinate-wise, i.e. $\mathrm{Log}_t(C_1) = \{(\log_t(|x|), \log_t(|y|))|(x,y) \in C_1\}$. Let $F_{2,t}$ be the equation of $C_{2,t}$.

Proposition 1.0.7. The tropical modification $m_{C_2}\mathbb{T}^2$ of $\mathbb{T}^2$ along C_2 is the limit of surfaces $S_t = \mathrm{Log}_t\{(x,y,F_{2,t}(x,y)) \in \mathbb{C}^3 | x, y, \in C^*\}$, i.e. $m_{C_2}\mathbb{T}^2 = \lim_{t\to\infty} S_t$.

Proposition 1.0.8. The tropical limit of the curves $m_{C_{2,t}}(C_{1,t}) \subset m_{C_{2,t}}((\mathbb{C}^*)^2)$, i.e. $m_{C_2}C_1 = \lim_{t\to\infty} \mathrm{Log}_t m_{C_{2,t}}(C_{1,t})$, is a tropical modification $m_{C_2}C_1$ of C_1.

Eventhough the families $C_{1,t}, C_{2,t}$ are included in the data, the graph $m_{C_2}\mathbb{T}^2$ does not depend on this choice. However, for given tropical curves C_1, C_2 we can construct different families $C_{1,t}, C_{2,t}$ and the result $m_{C_2}C_1$ can be different.

Note that we always suppose that an algebraic hypersurface comes with a defining equation. Also, instead of taking the limit we can consider non-Archimedean amoebas of the varieties defined over valuation fields.

Definition 1.0.9. Let $V \subset (\mathbb{K}*)^n$ be a variety over a valuation field $\mathbb{K}$. Let $W \subset (\mathbb{K}*)^n$ be an algebraic hypersurfaces defined by an equation $f(x) = 0, x \in (\mathbb{K}*)^n$. The modification $m_W(V)$ is the non-Archimedean amoeba of the set $\{(x, f(x)|x \in (\mathbb{K}*)^n)\} \subset (\mathbb{K}*)^{n+1}$. The modification of $\mathrm{Val}(V)$ along $\mathrm{Val}(W)$ is the non-Archimedean amoeba of $\{(x, f(x)|x \in V\} \subset (\mathbb{K}*)^{n+1}$.

Note that given **only** tropical curves C_1, C_2 it is often not possible to uniquely "determine" the image of C_1 after the modification along C_2. Still, as it is proved below (and equivalent to the tropical Weil reciprocity law), we know the sum of coordinates of all the legs of $m_{F_2}(C_1)$ going to minus infinity by Z -coordinate, i.e. for each connected component C of $C_1 \cap C_2$ we know the valuation of the product of intersections of C_1, C_2 which are tropicalized to C, just looking on behavior of C_1 and C_2 near C.

That is why a modification of a curve along another curve is rather *a method.* The strategy is the following: given two tropical curves, we lift them in a non-Archimedean field (or present them as limits of complex curves, that is the same), then we construct the graph as above and tropicalize the result. Depending on the conditions we imposed on lifted curves (be smooth or singular, be tangent to each other, etc), we have a set of possible results for modification of one curve along the second, see examples below.

1.0.2 Examples.

In this section we calculate examples of the modification, treated as a method.You should not be scared with these horrific equations, they are reverse-engineered, starting from the pictures. All the calculations are quite straightforward.

First of all, we consider how modifications resolve indeterminacy that happens when the intersection of tropical objects is non-transversal. Also this example promotes the point of view that a tropical modification is the same as adding a new coordinate.

In the second example, a modification helps to recover the position of the inflection point. Also, the usefulness of the tropical momentum is demonstrated. The tropical Weil theorem which shortens the combinatorial descriptions of possible results of a modification is proved in Section 1.1.1.

In the third example we study the influence of a singular point on the Newton polygon of a curve. The same method suits for higher dimension and different types of singularities, but nothing is yet done there, due to complicated combinatorics. In the same example we describe how to find all possible valuations of the intersections of a line with a curve, knowing only their stable tropical intersection – the answer is Vieta theorem. The same arguments may be applied for non-transversal intersections of tropical varieties of any dimension.

Example 1.0.10. Modification, root of big multiplicity, Figure 1.1a.

In this example we see two tropical curves with non-transverse intersection which hides tangency and genus. Consider the plane curve C, given by the following equation: $F(x,y)=0$,

$$F(x,y) = (x-t^{1/3})^3(x-t^{-2}) + t^{-4}xy^2 + (t^{-4}+2t^{-5})xy + (t^{-5}+t^{-6})x.$$

Its tropicalization[1] is the curve, given by the set of non-smooth points of

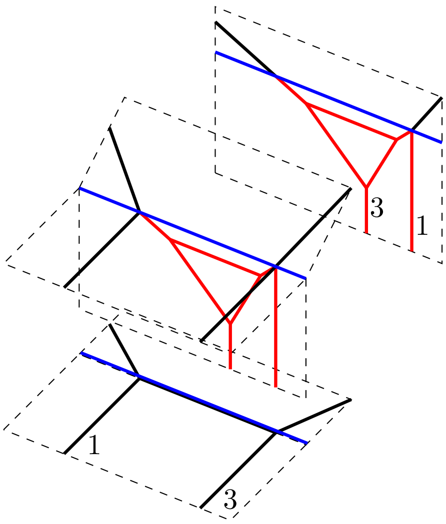
$$\mathrm{Trop}(F) = \max(1, 6+x, 5+x+y, 4+x+2y, 5/3+2x, 2+3x, 4x).$$

We want to know what is the intersection of C with the line L given by the equation $y+t^{-1}=0$. Tropicalizations of C and L are drawn on Figure 1.1a, below, as well as the Newton polygon of C. The intersection is not transverse, hence we do not know the tropicalization of $C\cap L$.

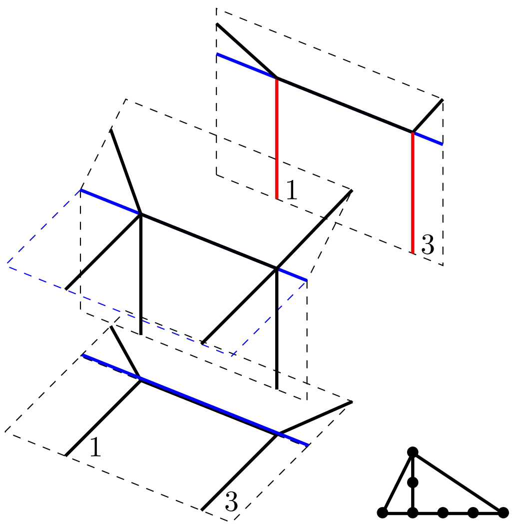
Then, let us consider the map $m_L : (x,y) \to (x,y,y+t^{-1})$. On Figure 1.1a, in the middle, we see the tropicalization of the set $\{(x,y,y+t^{-1})\}$ and the tropicalization of the image of C under the map m_L. Let $G(x,z)$ be the equation of the projection of $m_L(C)$ on the xz-plane. So, $F(x,y)=0$ implies that for the new coordinate $z=y+t^{-1}$ we have $G(x,z)=0, G(x,z)=(x-t^{1/3})^3(x-t^{-2})+t^{-4}xz+t^{-4}xz^2$. Therefore the curve $C'=pr_{xz}m_L(C)$ is given by the set of non-smooth points of $\max(1, 4+X+Y, 4+X+2Y, 2+3X, 4X)$, we see C' on the projection onto the plane XZ on the left part of Figure 1.1a. One can notice, that in order to have transversal intersection of non-Archimedean amoebas we did nothing else as a change of coordinates.

Definition 1.0.11. As we see in this example, a tropical curve in $\mathbb{T}^n$ typically contains infinite edges. We call them *legs* of a tropical curve. For each leg we have a canonical parametrization $(a_0+p_0s, a_1+p_1s, a_2+p_2s)$ where $a_i\in\mathbb{R}, p_i\in\mathbb{Z}, s\in\mathbb{R}, s\geq 0$, where the vector (p_0,p_1,p_2), the *direction* of the leg, is primitive.

[1] One can think that we have a family of curves C_t with parameter t and its tropicalization is the limit of amoebas $\lim_{t\to 0}\mathrm{Log}_t(\{(x,y)|F(x,y)=0\})$, or that we have a curve C over Puiseux series $\mathbb{C}\{\{t\}\}=\mathbb{K}$ given by $\sum a_{ij}x^iy^j=0, a_{ij}\in\mathbb{K}$. Its non-Archimedean amoeba is given by the set of non-smooth points of the function $\max_{ij}(\mathrm{val}(a_{ij})+ix+jy)$. Both ways lead to the same result.



(a) Initial picture is below.In the center we see the limit of the graphs of the functions $F_{2,t}$. On the picture behind we see the projection of the graph to the plane XZ. Numbers on the edges are the corresponding weights.



(b) Notation is the same as for the picture on the left. We see the result of the modification in the case when the stable intersection is the actual intersection. The Newton polygon of the curve C is depicted below.

Figure 1.1: Example of a modification along a line

Now, on the tropicalization of C' we see a vertical leg of of weight 3, i.e. z coordinate is zero at this point. That happens because we have the tangency of order 3 between C and L, and z as a function of x has a root of order 3.

Note that this leg can not mean the point is a singular point of C, because the curve C (according to criteria of [26] or, more generally [18]) has no singular points, eventhough the tropicalization of C has an edge of multiplicity 3.

Thus, this new tropicalization restores the valuations of intersection. We see that the modification of the plane (i.e. amoeba of the set $\{(x, y, y + t^{-1})\}$) is defined, but in codimension one this procedure shows order of roots and more unapparent structures like hidden genus. One can think that this cycle was close to intersection, but after change of coordinates it becomes visible on the picture of the amoeba of C'.

Remark 1.0.12. Nevertheless, for a general choice of representative in Puiseux series for these two tropical curves $\mathrm{Trop}(C), \mathrm{Trop}(L)$, after modification we will have Fig. **??**, which represents stable intersection of the curves.

Example 1.0.13. Modification, inflection point, momentum map.

We consider a curve and its tangent line at an inflection point. Suppose, that the intersection of their tropicalizations is not transverse. How can we recover the presence of the inflection point?

We consider a curve C with the equation $F(x, y) = 0$ where

$$F(x,y) = y+t^{-3}xy+(t^{-1}+4+6t+4t^2+t^3)x^2+(-t^{-3}-3-t-t^2)xy^2+(t^{-2}-t^{-1}-2+t^2+t^3)x^2y+x^2y^2,$$

and a line L with the equation $y = 1 + tx$. The equation of the curve is chosen just in such a the way that the restriction of F on the line L is $t^2(x - 1)^3(x - t^{-1})$, i.e. the point $(1, 1 + t)$ is the inflection point of the curve and L is tangent to C at this point.

Tropicalization of the curve is given by the following equation:

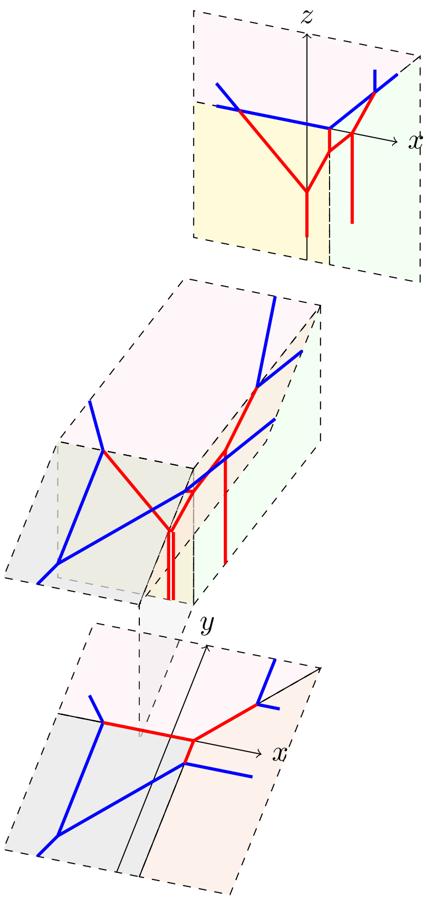
$$\mathrm{Trop}(F) = \max(y, x + y + 3, 2x + 1, 2x + y + 2, x + 2y + 3, 2x + 2y). \tag{1.1}$$

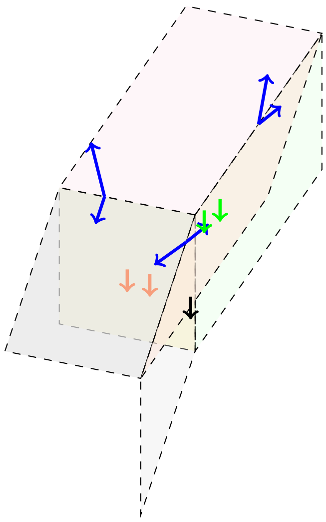
On the Fig. 1.2.1 we see the non-Archimedean amoeba of the image of the curve under the map $(x, y) \to (x, y, y - 1 - tx)$.

In order to find X-coordinates of the possible legs we can apply the tropical momentum: see Fig.1.1.2.

Definition 1.0.14. The moment of a leg $(A_0 + P_0s, A_1 + P_1s, A_2 + P_2s)$ with respect to a point (B_0, B_1, B_2) is the vector product $(A_0 - B_0, A_1 - B_1, A_2 - B_2) \times (P_0, P_1, P_2)$.

We will prove a (simple) theorem that the sum of the moments of the legs, counted with their weights, is zero. Note, that in our case, all the legs we do not know are of the form $(X, Y, Z_0 - s)$, because they are vertical. So, we take the vertex O of the tropical plane, and sum up the vector products $OX_i \times X_iY_i$ where X_iY_i are blue edges (which we know),beige, black, and green edges (which are projected to points on the initial curve). Computation gives us $(-4, 0, 0) \times (-1, 1, 1) + (-4, 0, 0) \times (0, -1, 0) + (0, -1, 0) \times (-1, -1, 0) + (0, -1, 0) \times (1, 0, 1) + (2, 2, 2) \times (1, 0, 1) + (2, 2, 2) \times (0, 1, 1) + (X, 0, 0) \times (0, 0, -1) + (0, Y, 0) \times (0, 0, -1) + (Z + 1, Z, 0) \times (0, 0, -1) = 0$, i.e. $(1, -2, 0) + (Y + Z + 1, X + Z, 0) = 0$, where X stands for the sum of the X-coordinates of the legs under the





(a) Application of the tropical moment: we know the direction of the infinite blue rays emanating from the tropical curve, therefore an application of tropical momentum gives the sum of x- and y- coordinates of legs, going vertically to the bottom (these legs present exactly the intersection of two considered curves.)

Figure 1.2: Example of modification in the case of inflection point. The point $(0, 0)$ on the bottom picture is the tropicalization of the inflection point.

line $(1-s,0,0)$, Y stands for the sum of the Y-coordinates of the legs under the line $(1,-s,0)$, Z stands for the sum of the Y-coordinates of the legs under the line $(1,s,s)$.

On the left picture we see that in fact we have one beige leg (which represent the inflection point), one green leg, and no black legs. But, since modification of a tropical curve C along a tropical curve C' is not canonically defined[2].

For example, a modification of these tropical curves could differ from the initial curve by adding vertical legs at four vertices of the blue curve: this would correspond to stable intersection (which is always realizable in the sense that there exist curve in Puiseux series, such that, etc.)

Example 1.0.15. Singular point, its unique position, and possible liftings of intersection

Consider a curve C' defined by the equation $G(x,y)=0$, where

$$G(x,y) = t^{-3}xy^3 - (3t^{-3}+t^{-2})xy^2 + (3t^{-3}+2t^{-2}-2t^{-1})xy - (t^{-3}+t^{-2}-2t^{-1}-3t^2)x+ \\ +t^{-2}x^2y^2 - (2t^{-2}-t^{-1})x^2y + (t^{-2}-t^{-1}-3t^2)x^2 + t^{-1}y - (t^{-1}+t^2) + t^2x^3.$$

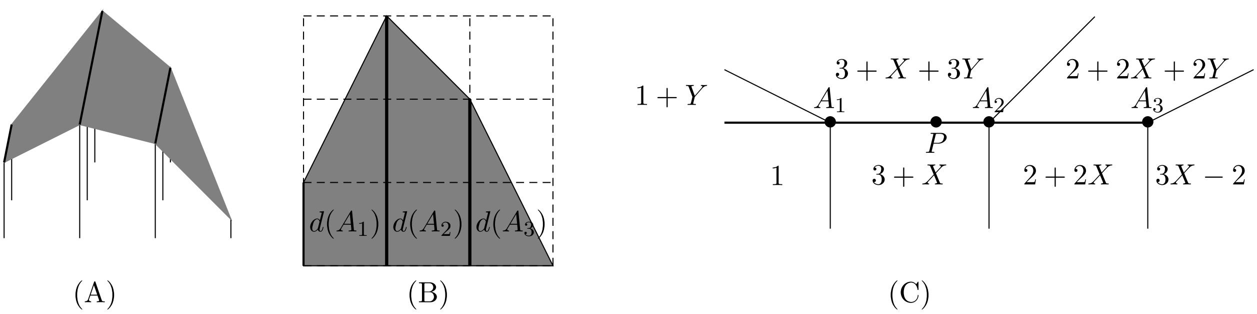
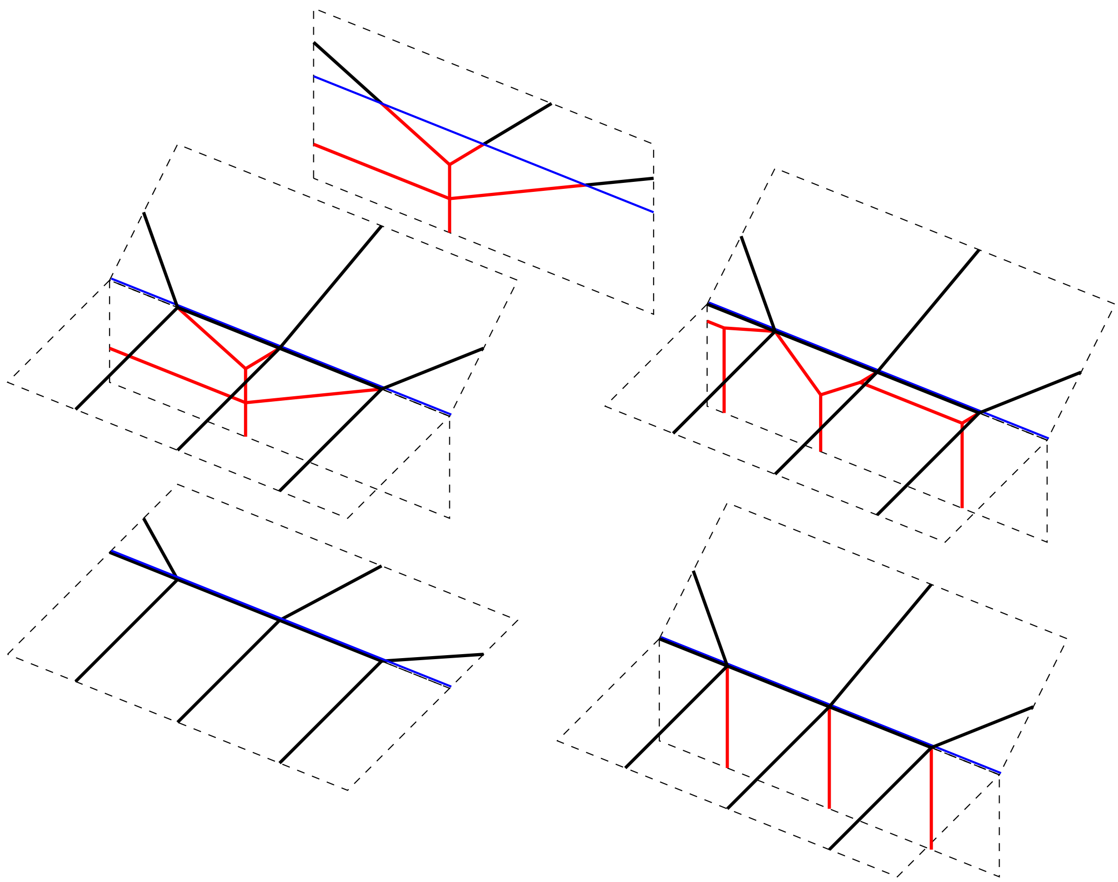


Figure 1.3: The extended Newton polyhedron $\widetilde{\mathcal{A}}$ of the curve C' is drawn in (A). The projection of its faces gives us the subdivision of the Newton polygon of C'; see (B). The tropical curve $\mathrm{Trop}(C')$ is drawn in (C). The vertices A_1,A_2,A_3 have coordinates $(-2,0),(1,0),(4,0)$. The edge A_1A_2 has weight 3, while the edge A_2A_3 has weight 2. The point P is $(0,0)=\mathrm{Val}((1,1))$.

Let us make the modification along the line $y=1$. For that we draw the graph of the function $z(x,y)=y-1$.

Note that we can easily find the number (with multiplicities) of the vertical legs. Indeed, each edge from A_1,A_2,A_3 going up in direction (i,j) becomes after the modification a ray going in the direction (i,j,j), therefore, the total moment of the vertical legs is the sum of y-moments of the edges going up from A_1,A_2,A_3, that is, 3. Then, if we know that after the modification our curve has a leg of multiplicity 3, then its unique position can be found from the tropical momentum theorem.

[2] If the intersection $C\cap C'$ is transverse, then the modification is uniquely defined.



1.1 Some structural theorems about tropical modification

Proposition 1.1.1. Suppose that a horizontal edge E contains a point P. Suppose that on the dual subdivision of the Newton polygon the vertical edge $d(E)$ is dual to E. Let the endpoints of E be A_1, A_2 and two faces $d(A_1), d(A_2)$ adjacent to $d(E)$ have no other vertical edges. Let the sum of widths of the faces $d(A_1), d(A_2)$ equals to m. Then the stable intersection of E with a horizontal line through E is m.

Proof. Refer to Example 1.0.15 and Figure 1.1b. Let L be a tropical line containing E and with vertex not coinciding with the endpoints of E. Making the modification along the line l we see that the sum S of vertical components of edges going upward from A_1, A_2 equals the sum m of the y-components of them.

Then, the sum of vertical components of edges going downward equals S by the balancing condition for tropical curves. Sum of y-components of edges in the vertex v is exactly the width in the $(1,0)$ direction of the dual to v face $d(v)$ in the Newton polygon. □

Here we repeat some definitions from Section **??** (see also [18]).

The *multiplicity* $m(P)$ of the point P of the intersection of two lines in directions $u, v \in P(\mathbb{Z}^2)$ is $|u_1v_2 - u_2v_1|$ where $u \sim (u_1, u_2), v \sim (v_1, v_2)$ (Def. **??**).

Given two tropical curves $A, B \subset \mathbb{T}^2$ we define their *stable intersection* as follows. Let us choose a generic vector v. Then we consider the curves $T_{tv}A$ where $t \in \mathbb{R}, t \to 0$ and T_{tv} is translation by the vector tv. For a generic small positive t, the intersection $T_{tv}A \cap B$ is transversal and consists of points $P_i^t, i = 1, \dots, k$ with multiplicities $m(P_i^t)$.

Definition 1.1.2 (cf. [38])**.** For each connected component X of $A \cap B$, we define *the local stable intersection of A and B along X* as $A \cdot_X B = \sum_i m(P_i^t)$ for t close to zero, where the sum runs over $\{i | \lim_{t\to 0} P_i^t \in X\}$. For a point $Q \in A$, we define $A \cdot_Q B$ as $A \cdot_X B$, where X is the connected component of Q in the intersection $A \cap B$.

Theorem 1.1.3. For two curves $C_1, C_2 \in \mathbb{K}^2$ we consider a connected component X of the intersection $\text{Trop}(C_1) \cap \text{Trop}(C_2)$. Then, $\sum_{x \in C_1 \cap C_2, \text{Val}(x) \in X} m(x) = \text{Trop}(C_1) \cdot_X \text{Trop}(C_2)$ where $m(x)$ is the multiplicity of the point x in the intersection $C_1 \cap C_2$.

Proof. Consider the equation $F(x, y) = 0$ of C_2. We construct the non-Archimedean amoeba $m_{C_2}C_1$ of $\{(x, y, F(X, y)) | (x, y) \in C_1)\}$. Then $\text{Trop}(C_1) \cdot_X \text{Trop}(C_2)$ is the sum of the weights of the vertical legs of $m_{C_2}C_1$ under X. The latter is equal to $\sum_{x \in C_1 \cap C_2, \text{Val}(x) \in X} m(x)$. □

1.1.1 Tropical Weil reciprocity law and the tropical momentum map

The aim of this section is to establish another fact in tropical geometry, obtained by word-by-word repetition of a fact in classical algebraic geometry. Weil reciprocity law can be formulated as

Theorem 1.1.4. Let C be a complex curve and f, g are two meromorphic functions on C with disjoint divisors. Then $\prod_{x \in C} f(x)^{\text{ord}_g x} = \prod_{x \in C} g(x)^{\text{ord}_f x}$, where $\text{ord}_f x$ is the minimal degree in the Taylor expansion (in local coordinates) of the function f at a point x: $f(z) = a_0(z-x)^{\text{ord}_f x} + a_1(z-x)^{\text{ord}_f x + 1} + \dots, a_0 \neq 0$.

Remark 1.1.5. The products in this theorem is finite because $\text{ord}_g x, \text{ord}_f x$ equal to zero everywhere except finite number of points.

Definition 1.1.6. If f and g share some points in their zeros and poles sets, then we state this theorem as $\prod\limits_{x\in C}[f,g]_x = 1$ and define the term $[f,g]_x =$ "$\frac{f(x)^{ord_g x}}{g(x)^{ord_f x}}$" $= \frac{a_n^m}{b_m^n}\cdot(-1)^{nm}$ at a point x, where $f(z) = a_n(z-x)^n + \dots, g(z) = b_m(z-x)^m + \dots$ are the Taylor expansions at the point x.

Khovanskii studied various generalizations of the Weil reciprocity law and reformulated them in terms of logarithmic differentials [21], [22],[23]. The final formulation is for toric surfaces and seems like a tropical balancing condition, what is, indeed, the case. The symbol $[f,g]_x$ is related with Hilbert character and link coefficient, and is generalized by Parshin residues. Mazin treated them [27] in geometric context of resolutions of singularities[3].

In order to study what happens after a modification we consider a tropical version of Weil theorem. We need to define tropical meromorphic function and $\text{ord}_f x$, see also see [31].

Definition 1.1.7 ([30])**.** A tropical meromorphic function f on a tropical curve C is a piece-wise linear function with integer slope. The points, where the balancing condition is not satisfied, are poles and zeroes, and $\text{ord}_f x$ is the defect in the balancing condition by definition.

Example 1.1.8. The function $f = \max(0, 2x)$ on $\mathbb{T}P^1$ has a zero of multiplicity 2 at 0, i.e. $\text{ord}_f(0) = 2$ and $\text{ord}_f(+\infty) = -2$.

Theorem 1.1.9. Let C be a tropical curve and f, g are two meromorphic tropical functions on C. Then $\sum\limits_{x\in C} f(x)\cdot\text{ord}_g x = \sum\limits_{x\in C} g(x)\cdot \text{ord}_f x$.

Example 1.1.10. Let C be $\mathbb{C}P^1$ and f, g are polynomials $f(x) = A\prod_{i=1}^n(x-a_i), g(x) = B\prod_{j=1}^m(x-b_j)$ with $a_i \neq b_j$. Then, $\prod\limits_{x\in C} g(x)^{\text{ord}_f x} = B^{nm}\prod_{i=1,j=1}^{n,m}(a_i - b_j)$, for the second product we have $A^{nm}\prod_{i=1,j=1}^{n,m}(b_j - a_i)$ and the difference is corrected by the term $[f,g]_\infty$, because f, g have a common pole there, see Def. 1.1.6.

Word-by-word repetition proves this case in tropical context because a tropical polynomial $f : \mathbb{T}\to\mathbb{T}$ can be presented as $f(x) = \sum \max(a_i, x)$ where a_i are the roots of f.

For the general statement there are many proofs (and one can proceed by studying piece-wise linear functions on a graph), we give here the shortest[4] one, via so-called *tropical momentum.*

Suppose C is a planar tropical curve, its infinite edges are $E_1, \dots, E_k$ with directions given by primitive[5] integer vectors $v_1, \dots, v_n$. Suppose that each edge E_i has weight w_i and the direction of each v_i is chosen to be "to infinity" (there are two choices and for us the orientation of v_i will be important). Let A be a point on the plane. Let EB_i be the perpendicular from A to the line l_i containing E_i and $B_i \in l_i$.

[3]Unfortunately, tropical analog of this problem has no big interest: Parshin residues are destined for non-transversal intersection, in order to define local residues. That suggests that Mazin's resolution of singularities is a classical version of tropical modifications. Probably, tropical approach can repeat classical results, and better visualize the different types of non-transversality for higher-dimensional varieties.

[4]And which is using tropical modification

[5]i.e. non-multiple of another integer vector

Definition 1.1.11. (Due to G. Mikhalkin) Tropical momentum for the point A with respect to C is given by $\rho_C(A) = \sum det(v_i, AB_i) \cdot w_i$.

Lemma 1.1.12. If a tropical curve C has only one vertex, then $\rho_C(A) = 0$ for any point A on the plane.

Proof. First of all, $\rho_C(A)$ does not depend on point A, because if we translate A by some vector u, each summand in $\rho_C(A)$ will change by $\det(v_i, u) \cdot w_i$ and the sum of changes is zero because of the balancing condition. Therefore, $\rho(A) = 0$, because we can place A in the vertex of this curve. □

Lemma 1.1.13. For an arbitrary plane tropical curve C and each point A in the plane $\rho_C(A) = 0$

Proof. Note that the moment for a curve is the sum of moments for all vertices (a summand corresponding to an edge between two vertices will appear two times with different signs). So, this lemma follows from the previous one. □

Definition 1.1.14. We consider a tropical curve $C \subset \mathbb{T}^3$. Let $e_1, e_2, \dots, e_n$ be its infinite edges. We define the moment of C with respect to A as $\rho(A) = \sum_{i=1}^{n} (v_i \times AB_i) \cdot w_i$ where $\times$ stands for the vector product, v_i is the primitive vectors of an edge e_i, w_i is the weight of e_i and B_i is a point on e_i.

Theorem 1.1.15. For a tropical curve $C \subset \mathbb{T}^3$ and any point A, the tropical moment $\rho(A)$ of C with respect to A is zero.

Proof. The same as in the planar case, we show that $\rho(A)$ does not depend on A because of the balancing condition, if C has only one vertex, then the claim is trivial, in general case we sum up by all the edges, and the terms for internal edges appear two times with different signs. □

This remark was demonstrated in Example 1.0.13.

1.1.2 Application of the tropical momentum to modifications.

Example 1.1.16. Consider the graph of a tropical polynomial $f(x) = \max(a_0, a_1+X, \dots, a_n+nX)$. Suppose that we know only a_0 and a_n. Definitely, the positions of the roots of f may vary, being dependent on the coefficients of f. Nevertheless, we can apply the tropical momentum theorem for the graph of f. We will calculate the moment with respect to $(0, 0)$. This graph has one infinite horizontal edge with moment a_0 and one edge of direction $(1, n)$ with the moment $-a_n$. Also, for each root $p_i \in \mathbb{T}$ of f we have an infinite vertical edge with the moment $-p_i$. Application of the tropical moment theorem gives us $\sum p_i = a_0 - a_n$, which is simply a tropical version of the Vieta theorem — the product of the roots of a polynomial $\sum_{i=1}^{n} a_i x^i$ is a_0/a_n.

On the Fig. **??**,1.2.1, *a priori* we know only the sum of directions of edges with endpoints on the modified curve. We know that there is no horizontal infinite edges (in these examples). Generally it is possible, if the intersection of two initial curves is non-compact. Therefore by Weil theorem (or tropical moment map, they are the same) we know the sum of x-coordinates of vertical infinite edges. The sum of weights for vertical edges equals the sum of vertical components of the green edges on the Figure.

Lemma 1.1.17. While doing a modification along horizontal line, the total vertical slope is the total horizontal slope on the dual picture.

Proof. The same as for Proposition 1.1.1. □

1.1.3 Proof of the tropical Weil theorem

We carry on with a proof of the tropical Weil theorem. Given two tropical meromorphic functions f, g on a tropical curve C we want to define the map $C \to \mathbb{T}P^2, x \to (f(x), g(x))$ and then use tropical momentum theorem. Here we have to use tropical modification, because *a priori*, the image of tropical curve under the map $(f, g) : C \to \mathbb{T}^2$ with f, g tropical meromorphic functions, is **not** a plane tropical curve: balancing condition is not satisfied near zeroes and poles of f and g

Definition 1.1.18. We call a triple (C, f, g) of a tropical curve C and two meromorphic function $f, g : C \to \mathbb{T}P^1$ on it *admissible* if all the zeroes and poles of f, g are attained at different one-valent vertices of C.

Lemma 1.1.19. Given a triple (C, f, g) of a tropical curve C and two meromorphic function $f, g : C \to \mathbb{T}P^1$ on it, we can always find a modification C' of C and extend the function f, g there, such that the obtained triple (C', f', g') is admissible and $\sum\limits_{x\in C} f(x) \cdot \mathrm{ord}_g x - \sum\limits_{x\in C} g(x) \cdot \mathrm{ord}_f x = \sum\limits_{x\in C'} f'(x) \cdot \mathrm{ord}_{g'} x - \sum\limits_{x\in C'} g'(x) \cdot \mathrm{ord}_{f'} x.$

Proof. We perform tropical modifications of C in order to have all zeros and poles of f, g at the vertices of valency one. Namely, for a point p such that p is in the corner locus of f we add to C an infinite edge l emanating from p. We define f on l as a linear function with integer slope such that the sum of slopes of f over the edges from p is zero, i.e. $f(p') = f(p) - ord_f p \cdot p'$ where p' is a coordinate on l such that $p' = 0$ at p and then grows. We define g on this edge as the constant $g(p)$. We perform this operation for all roots and poles of f. Then we do the symmetric procedure for g. □

Proof of the tropical Weil theorem. By the lemma above we may suppose that the triple (C, f, g) is admissible. Now f, g define a map $C \to \mathbb{T}^2$ and the image is a tropical curve C': indeed, at every vertex of the image the balancing condition is satisfied. Now it is easy to verify that $g(x) \cdot \mathrm{ord}_f(x)$ is a term in the definition of the moment of C' with respect to $(0, 0)$: if $\mathrm{ord}_f(x) \neq 0$, then C' has a horizontal infinite edge, and its y-coordinate is $g(x)$. Finally,

$$\sum_{x\in C'} f(x) \cdot \mathrm{ord}_g x - \sum_{x\in C'} g(x) \cdot \mathrm{ord}_f x = \rho((0, 0)) = 0. \tag{1.2}$$

□

Remark 1.1.20. If f, g come as limits of complex functions f_i, g_i, having $\mathrm{ord}_{f_i}(p_i) = k, \mathrm{ord}_{g_i}(p_i) = m, \lim p_i = p$, then the tropicalization of $\{(f_i(x), g_i(x)) | x \in C_i\}$ might not have vertical (with multiplicity k) and horizontal (with multiplicity m) leg from a common divisor point p of f and g, but can have a tree of legs, growing from p, with sum of slops still equal to (k, m). Nevertheless, because of the tropical momentum theorem or the balancing condition, it has no influence on the (1.2).

1.1.4 Difference between stable intersection and any other realizable intersection

After examples considered, one may ask if the only obstruction for a modification is the tropical momentum theorem? As we will see in this section, not at all.

Let us start with a variety $V' \subset \mathbb{K}^n$ and a hypersurface $W' \subset \mathbb{K}^n$ and their non-Archimedean amoebas $V, W \subset \mathbb{T}^n$. We suppose that the intersection of V with a tropical hypersurface W is not transverse. We ask: how does the non-Archimedean amoeba of of intersections of $V' \cap W'$ looks like?

First of all, as a divisor on V (or W) it should be rationally equivalently to the stable intersection of V and W, as it shown for the case of curves in [32]. In the general case in follows from the results of this section.

It it easy to find some additional necessary conditions. Let us restrict on V' the equation f of W', and take the valuations of all these objects V', W', f. We get some function $\mathrm{Trop}(f)$ whose behavior on a neighborhood of $V \cap W$ is fixed but its behavior on W is under the question.

Definition 1.1.21. Let V be an abstract tropical variety and $\iota : V \to \mathbb{T}^n$ be its realization as an tropical subvariety of $\mathbb{T}^n$. Let f is a tropical function on $\mathbb{T}^n$. We define the pull-back of $\iota^*(f)$ to V as $f \circ \iota$. We call $\iota^*(f)$ *frozen* at a point $p \in V$ if f is smooth at $\iota(p)$.

Note that in general, the slopes of f along $\iota(V)$ does not coincide with slopes of $\iota^*(f)$ on V. From now on we consider tropical functions which have frozen points, the motivation is explained in the following definition.

Definition 1.1.22. A principal divisor P on an abstract tropical variety V is called *positively* equivalent to a principal divisor Q, which is defined by a tropical meromorphic function f, if P can defined by a tropical meromorphic function h, which satisfies $h \le f$ and $h = f$ at the points where f is frozen.

Remark 1.1.23. As it is easy to see, the fact of positively equivalence depends only on P, Q, and does not depend on particular choice of f, h as long as the sets of frozen points for f and h coincide.

Example 1.1.24. Refer to Example 1.0.15.

Now, let us start from the tropical curve W given by $\max(3+X+3Y, 3+X+2Y, 3+X+Y, 3+X, 2+2X+2Y, 2+2X+Y, 2+2X, 1+Y, 1, 3X-2)$ and a horizontal line V given by $\max(Y, 0)$. We want to understand the valuations of possible intersections of $V' \cap W'$ where $\mathrm{Trop}(V') = V, \mathrm{Trop}(W') = W$.

We can rewrite the equation of W as $F(x, y) = (t^{-1} + \alpha_0 + t^{-1}y) + x(t^{-3} + \alpha_1 + t^{-3}y^3) + x^2(t^{-2} + \alpha_2 + t^{-2}y) + x^3(t^2 + \alpha_3)$, where $\mathrm{val}(\alpha_0) < 1, \mathrm{val}(\alpha_1) < 3, \mathrm{val}(\alpha_2) < 2, \mathrm{val}(\alpha_3) < -2$ It is clear, that by choosing y of the $1 + \alpha, \mathrm{val}(\alpha) < 0$ and then careful choice for $\alpha_1, \alpha_2, \alpha_3$ we can obtain any tropical polynomial $\mathrm{Trop}(F)(x, 1 + \alpha) = \max(A, B + X, C + 2X, -2 + 3X$ with $A \le 1, B \le 3, C \le 2$.

Note that in this case we have also choice for the constant term. If the intersection is a compact set, then the constant term is also fixed. Note that for the stable intersection our tropical function is $\max(1, 3 + X, 2 + 2X, -2 + 3X)$.

In this example the set $X \ge 4$ on V is frozen for $\mathrm{Trop}(F)$.

Now we prove the following theorem whose proof consists only in a reformulation of the statement on the language of tropical modifications.

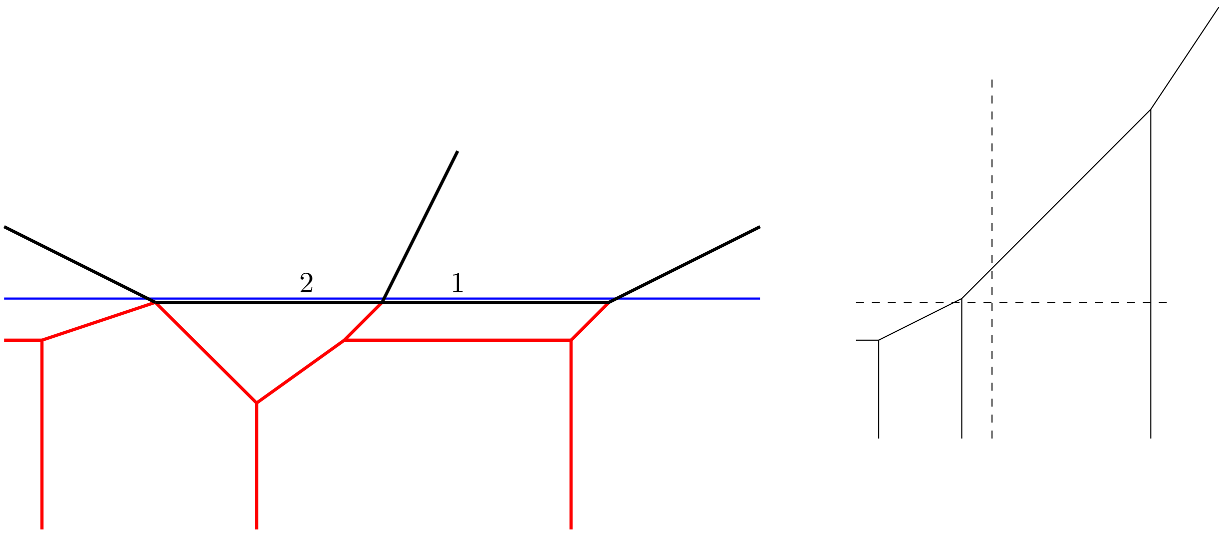


Figure 1.4: On the left figure we see the vertical part of the modification of the curve given by $F(x,y) = (-t^{-1}+t^{5/3}+t^{-1}y)+x(t^{-3}y-(t^{-3}+t^{-5/6})+x^2(t^{-2}y-t^{-2}+t^{-3/2})+x^3t^2$ along the line $y=1$. On the right figure we see the restriction of F on $y=1$, i.e. the function $\max(3X-2, 2X+1.5, X+5/6, -5/3)$.

Theorem 1.1.25. For an abstract tropical variety X, $\iota : X \to \mathbb{T}^n$ and a tropical hypersurface $F \subset \mathbb{T}^n$, given by a tropical function f, the pullback of the divisor of the stable intersection of $\iota(X)$ and F is given by $\iota^*(f)$. Furthermore, if F' and X are such that $\text{Trop}(F') = F$, F' is given by an equation $g = 0$, and $\text{Trop}(X') = X$, then the pullback of $\text{Trop}(F' \cap X')$ is positively equivalent to the divisor of $\iota^*(f)$.

Proof. Let us make the modification of $\mathbb{T}^n$ along F. Look at the image $m_f(X)$ of X under this map. The natural projection $m_f(X) \to X$ can be interpreted as a function on X. Note that this function is exactly $\iota^*(f)$. Now we look at the tropicalization of the restriction $g|_X$ of the equation of F' on X'. Clearly, $\text{Trop}(g|_X)$ coincides with f at the points where f is smooth. At other points, the graph of $\text{Trop}(g|_X)$ belongs to $m_F(\mathbb{T}^2)$. Therefore the pullback of $\iota^*(\text{Trop}(g|_x))$ is at most $\iota^*(f)$ everywhere, and $\iota^*(\text{Trop}(g|_x)) = \iota^*(f)$ at the points where $\iota^*(f)$ is frozen. So, the divisor of $\iota^*(\text{Trop}(g|_x))$ on X is positively equivalent to the pullback of the stable intersection. The graph of $\text{Trop}(g|_X)$ can be less than the graph of F_X because when we substitute the points on X' to g, some cancellation can occur, but not in the general case. □

1.1.5 Interpretation with chips

In the case of curves we can represent the divisors on curves as collections of chips. So, we proved the theorem that any realizable intersection is positively equivalent to the stable intersection. This gives a method to produce all realizable intersections (but not of the produced divisore will be necessary realizable).

We start with the stable intersection, it is a divisor (collection of chips) on our curve. Then we allow the following movement: pushing continuously together two neighbor chips on an edge, with equal speed. We donÕt allow the opposite operation Ñ when we slide continuously two points apart from each other.

This corresponds to the following: we look at the modification of the first curve V along the second, given by $f(x, y) = 0$, and by decreasing the coefficients of $\iota^*(f)$ on V we can obtain any function less than $\iota^*(f)$.

Remark 1.1.26. Note that if the stable intersection is not compact that we need to add a chip at infinity (or to treat infinity as a point with one chip). Now let A, B be two chips, A is at infinity and B is on the leg of V going to A. Then, pushing together A, B moves only B towards infinity. This corresponds to decreasing the constant term in Example 1.1.24.

Example 1.1.27. Big order tangency without perturbations. ([10], Lemma 3.15). We consider a line $y - \alpha x - \beta = 0, \mathrm{val}(\alpha) = 0, \mathrm{val}(\beta) = 0$ and a curve $a_0 + a_1 y + a_2 x y^l = 0$ with $\mathrm{val}(a_0) = 0, \mathrm{val}(a_1) = 0, \mathrm{val}(a_2) = 0$.

Clearly, we have non-stable intersection, we can perform substitution $y = \alpha x + \beta$, that gives $a_0 + a_1(\alpha x + \beta) + a_2 x(\alpha x + \beta)^l = (a_0 + a_1\beta) + x(a_1\alpha + a_2\beta^l) + \sum_{i=2}^{l+1} x^i(a_2\beta^{l+1-i}\alpha^{i-1}$. The only contraction may appear at two coefficients before x and the constant. So we have only two degrees of freedom. So, when $\mathrm{val}(a_0 + a_1\beta) < \mathrm{val}(a_0)$, this correspond to the movement in Remark 1.1.26. Also we can push two chips together by decreasing the valuation of $a_1\alpha + a_2\beta^l$. This reasoning can be applied to the intersection of any two tropical varieties, if one of them is a complete intersection. We restrict the equations of the second variety on the first, that give us a stable intersection, then we have a situation similar to Definition 1.1.22.

Here we have only two degrees of freedom because we have only two degrees of freedom in the equation $a_0 + a_1 y + a_2 x y^l = 0$.

Question 1. The following suggestions seems to be reasonable for the realizability of intersections. While defining $\iota^*(f)$ we keep track of all the monomials m_i of f and then in Definition 1.1.22 we allow g to contain only monomials of the type $\iota^*(m_i)$. I.e. if $f = \sum a_{ij}x^i y^j$, then we only allow g of the type $\max(c_{ij} + \iota^*(x^i y^j))$ with $c_{ij} \leq \mathrm{val}(a_{ij})$ which coincides with f on the frozen set of f.

Example 1.1.28. Difference between multiplicity leg and root. Take $1 + (t^{-1} + t)x + (2t^{-1} + t^2 + t^4)x^2 + (t^3 + 2t^4)x^3 + t^{-1}xy + 2t^{-1}x^2 y$ in intersection with the line $t^5 x + y + 1 = 0$. The same example can be constructed for the similar Newton polygon $(0, 0) - (1, 1) - (n, 1) - (n + 1, 0)$.

1.2 Applications of a tropical modification as a method

1.2.1 Inflection points.

An inflection point of a curve is either its singular point, or a point where the tangent line has order of tangency at least 3. It was known before that the number of real inflection points on a curve of degree d is at most $d(d - 2)$ and the maximum is attainable. The question, attacked in [10] is *which topological types of planar real algebraic curves admits the maximal number of real inflection points?* Using classical way to construct algebraic curves – Viro's patchworking method – the authors construct examples, for what they study possible local pictures of tropicalizations of inflection points. The property to be verified is tangency, but intersection of tropical curve with a tangent line at some point in most cases is not transversal and it is not visible what is the actual order of tangency. To see that, the authors do tropical modifications.

1.2.2 The category of tropical curves

See also [1, 2]. Namely, Mikhalkin defines the morphisms in the category of tropical curves as all the maps, satisfying the balancing and Riemann-Hurwitz conditions (see, for example [6]) and subject to the *modifiability* condition:

Definition 1.2.1. A morphism $f : A \to B$ of tropical curves A, B is said to be *modifiable* if for any modification B' of B there exists a modification A' of A and a lift f' of f which makes the obtained diagram commutative.

Theorem 1.2.2. The modifiability condition ensures that a morphism came as a degeneration of maps between complex curves.

Sketch of a proof. After a number of modifications we may have the map f' contracting no cycles. Then we construct a family of complex curves B_i such that $\lim B_i = B'$ in the hyperbolic sense (see section 1.4.1). Finally, since f' should come as a tropicalization of a covering, the complex curves A_i with $\lim A_i = A'$ are constructed as coverings over $f_i : A_i \to B_i$ where the combinatorics (ramification profiles, local degrees at points of tori contracting to tropical edges) of f_i is prescribed by f'. Balancing and Riemann-Hurwitz conditions follow. □

1.2.3 Realization of collection of lines, $(4, d)$-nets

Which configuration of lines and points in $\mathbb{P}^2$ with given incidence relation are possible? That is a classical question and even for seemingly easy data the answer is often not clear.

Definition 1.2.3. A $(4, d)$-net in $\mathbb{P}^2$ is four collections by d lines each of them, such that exactly four lines pass through any point of intersection of two lines from different collections, all these four lines are from different collections.

It is not clear whether a $(4, d)$-net exists for $d \geq 5$. In [15] the authors proved, using tropical geometry, that there is no $(4, 4)$-net.

The one of the key ingredients is the following: if some net exists in the classical world, then it exists in the tropical world. The problem is the following: if we have more than three tropical lines through a point on a plane, then the intersection will be non-transversal. But thanks to modifications we always can have transversal intersection, but probably in the space of bigger dimension. For that we just do modification along lines which has non-transversal intersection, after this modification, all intersections with it become transversal and the modified lines goes to infinity. Then, let us think about the following theorem, announced by the authors of [15], from the point of view of modifications:

Theorem 1.2.4. If for some combinatorial data of intersection of linear spaces can be realized in $\mathbb{P}^k$, then there is a tropical configuration of tropical linear spaces which realize the same data in $T\mathbb{P}^{k'}$ with $k' \geq k$.

Indeed, consider the realization in $\mathbb{P}^k$. By passing to the tropical limit we obtain a tropical configuration, but the intersection dimensions may increase. Then, by doing the modifications, we want repair the right dimensions.

1.2.4 A point of big multiplicity on a planar curve.

In its most general form, this question could be formulated as follows: given a type of subvariety S in a bigger variety, how many singularities S may have? For example, is it possible for a surface of degree 4 in $\mathbb{C}P^4$ to have four double points and three two fold lines?

There are several reasons for tropical geometry could provide tools for such questions. We will demonstrate these tools in the case of curves, where this deed has been already done. Combinatorics of a tropical curve is encoded in the subdivision of its Newton polygon. In fact, a singular point of multiplicity m influences a part of the subdivision of area of order m^2, what is in accordance with the order of the number of linear conditions ($\frac{m(m+1)}{2}$) that a point of multiplicity m imposes on the coefficients of the curve's equation. For a general treatment of the tropical sibgularities, see [17],[18].

In this section we will only demonstrate how to apply modification technic in this problem, though we will obtain weaker estimation – but still of order m^2.

The idea is the following: if a curve C has a point p of multiplicity m, then for each curve D, passing through p, the local intersection of C and D at p is at least m. The multiplicity of a local intersection of C and D can be estimated from above by studying the connected component, containing $\mathrm{Val}(p)$, of the stable intersection $\mathrm{Trop}(C)\cap\mathrm{Trop}(D)$ for the non-Archimedean amoebas of C and D.

So, the method: we take the polynomial F defining D, and use the fact that the image of C under the map $m_F : (x, y) \to (x, y, F(x, y))$ intersects the plane $z = 0$ with multiplicity at least m. That implies **existence** of a modification of $\mathrm{Trop}(C)$ along $\mathrm{Trop}(D)$, which has a leg of multiplicity m going in the direction $(0, 0, -1)$, exactly under the point $\mathrm{Val}(p)$. The latter modification is obtained just by taking the non-Archimedean amoeba of $m_D(C) \subset m_D(\mathbb{P}^2)$.

Now we reduce the problem for its combinatorial counterpart: is it possible for two given tropical curves, that after the modification along the second, the first curve will have a leg of multiplicity m, which projects exactly on the given point $\mathrm{Val}(p)$? After some work with intrinsically tropical objects, we will get an estimate of this point's influence on the Newton polygon of the curve.

We are not going to consider this problem in the full generality, so we will have a close look at the simplest interesting example. Suppose that $\mathrm{Val}(p)$ is inside some edge E of the tropical curve $\mathrm{Trop}(C)$ and this edge is horizontal.

Suppose that p is of multiplicity m for C. Let us take a line D through p, such that $\mathrm{Trop}(D)$ contains $\mathrm{Val}(p)$ inside its vertical edge. Clearly the intersection $\mathrm{Trop}(C)\cap\mathrm{Trop}(D)$ is one point, and of multiplicity at least m. That immediately implies that the weight of E is at least m. Hence the lattice length of $d(E)$ is at least m.

If we consider the modification along the horizontal line, then the contribution to the edge of direction $(0, 0, -1)$ consists of the horizontal components of the edges which intersect $\mathfrak{E}$ at exactly one point, see Example 1.0.15.

What to do if there is a rational component through $\mathrm{Val}(p)$? We do the modification along the horizontal line L. If a part of the curve goes to the minus infinity, that means that we can divide the equation of F by some degree of L. That means that the Newton polygon of C has two parallel vertical sides. The components which do not go to the minus infinity do not contribute to the singularity.

Let $\mathfrak{E}$ be the stable intersection of $\mathrm{Trop}(C)$ and the horizontal line; clearly $E \subset \mathfrak{E}$. Now, let's compute sum of areas of faces corresponding to the singular point p. By that, we mean the sum of areas of $d(V)$ where V runs over all vertices on $\mathfrak{E}$. It can be possible that more than two faces correspond to one singular point, if the edge with the singular point has an extension, see again Example 1.0.15.

First of all, we consider the simplest case.

Proposition 1.2.5. Suppose an horizontal edge E contains a point $\mathrm{Val}(p)$ of multiplicity m. Suppose that on the dual subdivision of the Newton polygon the vertical edge $d(E)$ is dual to E. Let the endpoints of E be A_1, A_2 and two faces $d(A_1), d(A_2)$ adjacent to $d(E)$ have no other vertical edges. Therefore the sum of widths of the faces $d(A_1), d(A_2)$ is at least m, so their total area is at least $m^2/2$.

Proof. Let us look at the dual picture in the Newton polygon. Two faces $d(A_1), d(A_2)$ adjacent to the vertical edge have the sum of width in the $(1,0)$ direction at least m (by Proposition 1.1.1), $d(E_1)$ has length m, so the sum of the areas of $d(A_1), d(A_2)$ is at least $m^2/2$. □

Remark 1.2.6. Suppose that a tropical curve has edges $A_1A_2, A_2A_3, \dots, A_{k-1}A_k$ and $A_1, A_2, \dots, A_k$ are situated on a horizontal interval $A_1A_k = \mathfrak{E}$. Suppose that p, point of multiplicity m, is on the edge A_sA_{s+1}. The method above doesn't work.

Explanation. Making a modification along **a line** containing A_1A_k in the horizontal ray we estimate only common width of faces corresponding to $A_1, A_2, \dots A_k$, which gives no good estimate for the sum of areas of $d(A_i)$.

The new idea is to make a modification along a quadric.

Lemma 1.2.7. In the hypothesis above the sum of areas of all faces $d(A_1), d(A_2), \dots, d(A_k)$ is at least $m/2 + m^2/4$.

Proof. Let a_i be the width of i-th face on the right of the egde dual to A_sA_{s+1}, b_i be the width of i-th face on the left, c_i be the length of i-th vertical edge on the right, d_i be the length of the i-th vertical edge on the left. Therefore, $\sum\limits_{i=1}^{k} a_i = A_k, \sum\limits_{i=1}^{k} b_i = B_k$. With the same calculations as above, making the modification along a piece of a quadric with vertices on $A_{s-j}A_{s+1-j}$ and $A_{s+i}A_{s+1+i}$ we get $A_i + c_i + B_j + d_j \geq m$. Denote $\min\limits_i(c_i + A_i) = A, \min\limits_i(d_j + B_j) = B$, so $A + B \geq m$.

Then, $c_i \geq A - A_i, d_j \geq B - B_j$. Sum S of areas can be estimated as

$$2S \geq (m + c_1)A_1 + \sum(A_{i+1} - A_i)(c_i + c_{i+1}) + (m + d_1)B_1 + \sum(B_{i+1} - B_i)(d_i + d_{i+1})$$

$$2S \geq (m+A-A_1)A_1+\sum(A_{i+1}-A_i)(A-A_i+A-A_{i+1})+(m+B-B_1)B_1+\sum(B_{i+1}-B_i)(B-B_i+B-B_{i+1}) \geq$$

$$\geq A_1(m - A) + A^2 + B_1(m - B) + B^2 \geq m + m^2/2.$$

So, $S \geq m/2 + m^2/4$.

□

1.3 Motivation

Tropical modifications were introduced in the seminal paper [28] as the main ingredient in the tropical equivalence relation. Namely, two tropical varieties are *equivalent* if they are related by a chain of tropical modifications and reverse operations[6].

The underlying idea is the following. Recall, that a tropical variety V can be decomposed into a disjoint union of a compact part V_c and a non-compact part V_∞, and $V = V_c \cup V_\infty$. Moreover, V retracts on V_c. Then, the set V_∞ consists of "tree-like" unions of hyperplanes' parts. We call these parts *legs*, by analogy with the one-dimensional case. For tropical curves, V_∞ is a union of half-lines. For example, for a tropical elliptic curve (see Fig. 1.5, left side) the set V_c is the ellipse, and V_∞ is the set of trees growing on the ellipse.

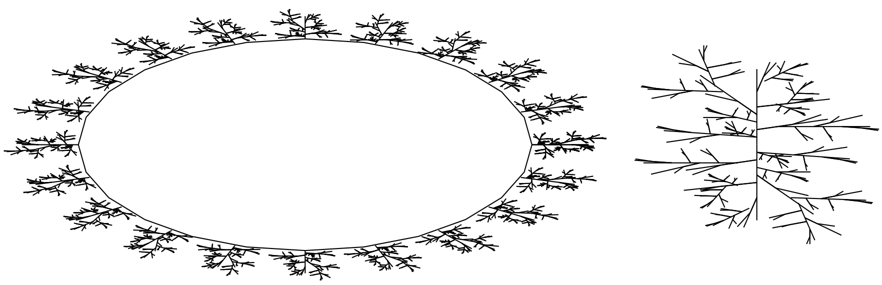


Figure 1.5: On the left side we see a tropical elliptic curve V which is a part of the analytification of an elliptic curve. The ellipse is V_c and the union of tree-like pieces is V_∞. On the right side we see a tropical rational curve V, which is equal to V_∞. We can chose each point of V as V_c, because V contracts onto any of its point $x \in V$.

Remark 1.3.1. On a tropical rational[7] variety V, each point can serve as V_c, see Fig. 1.5 right side. But there is a canonical way to define V_c for tropical varieties: we define V_∞ as the set of points v of V such that the shortest path from v to "infinity" does not pass through the cells of the codimension one of the natural cell subdivision of V. Then, $V_c = V \setminus V_\infty$.

Consider the tropical limit V of algebraic varieties $W_{t_i} \subset (\mathbb{C}^*)^n$, i.e. $V = \lim_{t_i \to \infty} \mathrm{Log}_{t_i}(W_{t_i})$, where we apply the map $\mathrm{Log}_{t_i} : \mathbb{C}^* \to \mathbb{R}, x \to \log_{t_i} |x|$ coordinate-wise. In this case the set V_∞ encodes the topological way of how W_i approach some compactification of $(\mathbb{C}^*)^n$. For the moment, the particular choice of the compactification does not matter[8].

Besides, for i big enough the Bergman fan $B(W_i) := \lim_{t\to\infty} \mathrm{Log}_t(W_i)$ of W_i is equal to $\lim_{t\to\infty} \frac{1}{t}V$. The latter limit is obtained by contracting the compact part V_c of V, so the Bergman fan can be restored by V_∞. Note, that V came here with a particular immersion to $\mathbb{R}^n$.

[6] For the full definition of an abstract tropical variety, see [31] and [29].

[7] Basically, rational tropical varieties are the contractible ones. They are not well studied even in small dimensions. For example, there exist 3 dimensional cubic hypersurfaces which are not rational. It is not known whether we can see this tropically.

[8] For a fixed compactification, see the notion of sedentarity in [42] and [8], p. 44.

Example 1.3.2. If curves $W_i, i = 1, 2, \dots$ in $(\mathbb{C}^*)^2$ all have branches with asymptotic (s^k, s^l) with a local parameter $s \to 0$, then the tropical limit V of this family lies in $\mathbb{R}^2$, and V has the infinite leg (half-line) in the lattice direction $(-k, -l)$.

Let us suppose that we have an algebraic map $f : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$, and f is in general position with respect to the family $\{W_i\}$, i.e. for each i big enough, the image $f(W_i)$ is birationally equivalent to W_i. Let V' be the tropical limit of the family $\{f(W_i)\}$. One can prove that V'_∞ differs from V_∞ by adding new half-planes and contracting other half-planes, see Section **??**. These half-planes grow along the tropicalization of zeros and poles of f on W_i. This consideration suggests the ideas of *modification* and *tropical birational equivalence*. The name "modification" was borrowed from complex analysis, and tropical modification is sometimes called "tropical blow-up".

In Section 1.2.2 we see how the notion of modifications allows us to define the *category of tropical curves*. This category keeps track of birational isomorphism in the category of complex algebraic curves. See also §1.1.1, where making modifications for curves simplifies a proof to some extent.

Tropical geometry can be thought as studying of skeletas of analytifications of algebraic varieties. We can obtain a tropical variety V as the non-Archimedean amoeba of an algebraic variety W over a non-Archimedean field. This approach (see section §1.4.2) finally suggests the same idea of equivalence up to modification, because the analytification W^{an} should be thought as the injective limit of all "affine" tropical modifications (i.e. along only principal divisors) of V([35]). Berkovich proved that W^{an} retracts on a finite polyhedral complex, so V_0 is a deformation retract of W^{an}. Even better, the metric on W^{an} agrees with the metric on V for the case of curves[9] ([5]).

Connection between tropical geometry and analytic geometry leads to the questions of *lifting* or *realisability*, i.e. what could be the intersection of two varieties if we know the intersection of their tropicalizations? In fact, if their tropicalizations intersect transversally, it is relatively simple, see [33]. If the intersection is non-transverse, then we can lift *the stable intersection* of these tropical varieties, see [34],[36].

This raised the following question: to what extent the only condition for a divisor on a curve to be realizable as an intersection is to be rationally equivalent to the stable intersection (cf. [32], Conjecture 3.4)?

Tropical modification (as a *method*) helps dealing with such questions. It is known that being rationally equivalent to the stable intersection is not enough. We consider other existing obstructions (in fact, equivalent to Vieta theorem) for what can happen in non-transverse tropical intersections, and prove, for that occasion, the tropical Weil reciprocity law by using the tropical momentum.

Consequently, modifications are used in tropical intersection theory ([40, 41]), to define the intersection product. Nevertheless, one must use modifications along non-Cartier divisors (Examples 1.1.37, 3.4.18 in [41], for moduli space of five points on rational curve) and even along non-realizable subvarieties – for a proof that they are non-realizable as tropical limits.

As we stated before, one should think that a tropical modification along X reveals asymptotical behavior of objects near X. We can find an analogy in non-standard analysis: the tropical line is the hyperreal line, the modification at a point is an approaching this point with *an infinitesimal telescope*, see Fig. 1.7 and Section 1.4.2. In order to define tropical Hopf manifolds one should also use the modifications to study their germs [39].

[9] That should be true for varieties of any dimension, modulo integer affine transformations, but no proof has appeared yet. For the skeletas in higher dimensions see [13, 14]

Given a surface with hyperbolic structure, we can make a puncture at x. This changes the hyperbolic structure and x goes, in a sense, to "infinity". A tropical curve can be obtained as a degeneration of hyperbolic structures, and making a puncture at x results as the modification at the limit of x, see Section §1.4.1.

A modification can be described as a graph of a function, if we use the convention about multi-valued addition, brought in tropical geometry by Oleg Viro ([45]), see the next section.

The other applications of tropical modification as a *method* are following. Passing to tropical limit squashes a variety, and some local features become invisible. In order to reveal them back we can do a modification[10]. For example, modifications allow us to restore transversality between lines if we have lost it during tropicalization (§1.2.3), then it allows us to see (-1)-curves on del Pezzo surfaces ([37]). Methods of lifting non-transverse intersections leads us to use modifications in questions about singularities: inflection points – [10], singular points – [26]. As an example, we use modification in the study of singular points of order m (but obtain weaker results than in [18]).

1.4 Interpretations

1.4.1 Hyperbolic approach and moduli spaces

Consider a tropical curve C given as the limit of complex curves C_i. From the point of view of hyperbolic geometry, a modification at a point x of tropical curve C means just making a puncture x_i in C_i, with condition that $x_i \to x$. To explain this we need to know how to directly construct tropical curves via limits of abstract surfaces with hyperbolic structure on them, without any immersions[11].

So, for details how tropical geometry can be built on on the ground of hyperbolic geometry, see [25]. Here we briefly sketch the construction.

The approach, proposed by L. Lang, uses the collar lemma ([11]). This lemma simply says that any closed geodesic of length l has a collar of width $\log coth(l/4)$ and what is more important, for different closed geodesics their collars do not intersect, see Fig. 1.6. That is also important that smaller geodesics have bigger collars (and, intuitively, a puncture has the collar if infinite width).

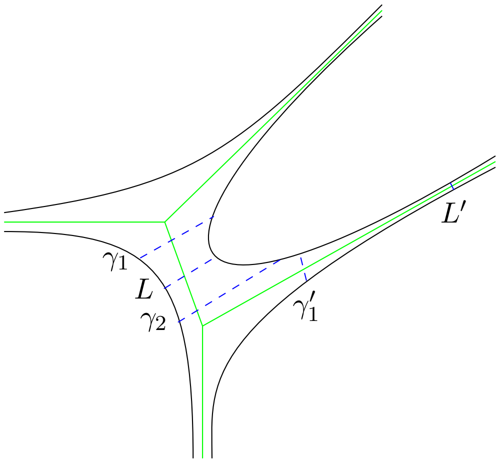
Thus, given a family of curves C_i (of the same genus), we consider a fixed pair-of-pants decomposition by geodesics L_i. The tropical curve is constructed as follows: its vertices are in one-to-one correspondence with the pair-of-pants, each shared boundary component between two pairs-of-pants correspond to an edge of the tropical curve, and the collar lemma furnishes us with the length of the edges of the tropical curve as the logarithms (with base t, and $t \to \infty$ as the hyperbolic structure degenerates) of widths of the collars of L_i's. Compare this approach with [7].

What will happen if we make a puncture ? A puncture is the limit of small geodesic circles. Cutting out a disk with radius t^l add a leaf of finite length, as it is seen from the above description. Therefore, cutting out a point results in adding an infinite edge, i.e. a modification.

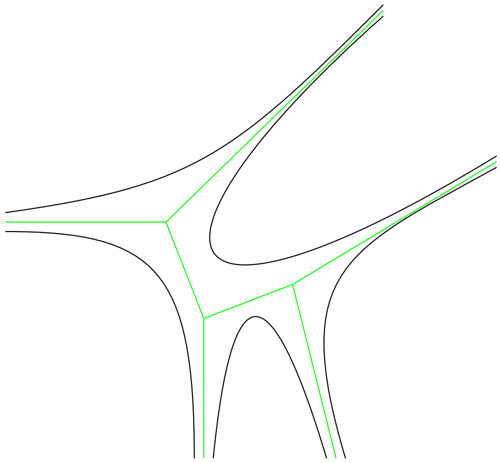
That explains why a permanent using of graphs for moduli space problems actually work ([24], then compare with tropical interpretation [20]). Tropical curves describe the part of boundary of a moduli space, and modification corresponds to marking a point (read [12] to see the hyperbolic view on moduli space problems), which are punctures from the hyperbolic point of view (see applications

[10] so the metaphor "look in an infinitesimal microscope" grasps the essence.

[11] Usually people consider curves C_i in toric variety X and then they consider degeneration of complex structures on X.



(a) Blue dashed lines γ_1, γ_2 depict the collar of geodesic L, γ_1' is a part of L''s collar.



(b) Modification subdivides old edge and adds a new edge of infinite lenght.

Figure 1.6: We draw the limits of hyperbolic surfaces, i.e. tropical curves. Modification adds a puncture to each curve in the family and a leg to the tropical curve.

to moduli space of points [29]). Tropical differential forms are also defined in this manner while taking a limit of hyperbolic structure [31].

1.4.2 Berkovich spaces, non-standard analysis

Non-standard analysis appeared as an attempt to formalize the notion of "infinitesimally small" variables (see §4 [43] for nice and short exposition).

There is an approach to tropical geometry via nonstandard analysis (cf. §1.4 [16]) and the following Fig. 1.7 shows that tropical modifications is similar to "infinitesimal microscope" for the hyperreal line in the terminology of [19], and this interpretation in computational sense is the same as for Berkovich spaces: doing modification at the point $x = 1$ on a curve is adding a leg to the tropical curve, which ranges points according their asymptotical distance to $x = 1$, i.e. $\text{val}(x-1)$, these pictures are also similar to the hyperbolic ones.

It is worth noting that there are still no applications of this point of view, neither in tropical geometry, nor in non-standard analysis. Still, Berkovich spaces can be treated as a more modern version of non-standard analysis, and tropical modification has applications there.

We should say that an important feature of tropical geometry is that it erects a bridge from a very geometric things (hyperbolic geometry) to very discrete things as p-adic valuations and non-Archimedean analysis.

See Figure 1.5, the analytification of an elliptic curve on the left, the analytification of $\mathbb{P}^1$ on the right. Ends of leaves represent the norms with "zero" radius; . Berkovich spaces appeared as a wish to have an analytic geometry on discrete spaces. The analytification X^{an} of a variety X is the set of all seminorms on functions on X. Each point $x \in X$ defines such a seminorm by measuring the order of vanishing of a function at x, on Fig.1.5 these points are represented by the ends of leafs.

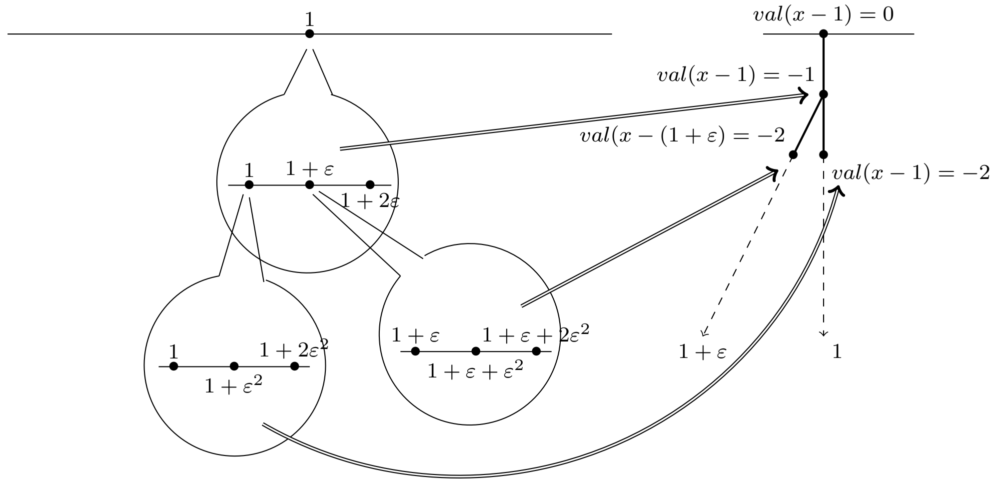


Figure 1.7: Similarity in the pictures of infinitesimal microscope (left) and tropical modification at points 1 and $1+\varepsilon$ (right).

For the sake of shortness, we refer the reader to a nice introduction in Berkovich spaces, with a bit of pictures [4],[44] and to [5] to see how it has been applied to tropical geometry (also, see on the page 7 in [5], using of log reminds hyperbolic approach). Also there exists Berkovich skeletas of analytifications, they correspond to the compact part V_c of a tropical variety, for example, for elliptic curves that will be a circle in both tropical and analytical cases, and its length is prescribed by j-invariant of a curve ([3]). The analytification of an elliptic curve is the injective limit of all modifications of its tropicalization, see Fig. 1.5.

Bibliography

[1] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. *Res. Math. Sci.*, 2:2:7, 2015. (page 1.2.2).

[2] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff. Lifting harmonic morphisms II: Tropical curves and metrized complexes. *Algebra Number Theory*, 9(2):267–315, 2015. (page 1.2.2).

[3] M. Angelica Cueto and H. Markwig. How to repair tropicalizations of plane curves using modifications. *ArXiv e-prints*, Sept. 2014. (pages 1 and 1.4.2).

[4] M. Baker. An introduction to berkovich analytic spaces and non-archimedean potential theory on curves. *p-adic Geometry (Lectures from the 2007 Arizona Winter School)*, 2008. (page 1.4.2).

[5] M. Baker, S. Payne, and J. Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *ArXiv e-prints*, Apr. 2011. (pages 1.3 and 1.4.2).

[6] B. Bertrand, E. Brugallé, and G. Mikhalkin. Tropical open Hurwitz numbers. *Rend. Semin. Mat. Univ. Padova*, 125:157–171, 2011. (page 1.2.2).

[7] B. H. Bowditch and D. B. A. Epstein. Natural triangulations associated to a surface. *Topology*, 27(1):91–117, 1988. (page 1.4.1).

[8] E. Brugallé, I. Itenberg, G. Mikhalkin, and K. Shaw. Brief introduction to tropical geometry. *Proceedings of 21st Gökova Geometry-Topology Conference*, arXiv:1502.05950, 2015. (pages 1 and 8).

[9] E. Brugallé and K. Shaw. A bit of tropical geometry. *ArXiv e-prints*, Nov. 2013. (page 1).

[10] E. A. Brugallé and L. M. López de Medrano. Inflection points of real and tropical plane curves. *J. Singul.*, 4:74–103, 2012. (pages 1, 1.1.27, 1.2.1, and 1.3).

[11] P. Buser. The collar theorem and examples. *Manuscripta Math.*, 25(4):349–357, 1978. (page 1.4.1).

[12] N. Do. *Intersection theory on moduli spaces of curves via hyperbolic geometry.* PhD thesis, Ph. D. Thesis, University of Melbourne, 2008. (page 1.4.1).

[13] W. Gubler, J. Rabinoff, and A. Werner. Skeletons and tropicalizations. 04 2014. (page 9).

[14] W. Gubler, J. Rabinoff, and A. Werner. Tropical skeletons. 08 2015. (page 9).

[15] M. Hakan Gunturkun and A. Ulas Ozgur Kisisel. Using Tropical Degenerations For Proving The Nonexistence Of Certain Nets. *ArXiv e-prints*, July 2011. (page 1.2.3).

[16] I. Itenberg, G. Mikhalkin, and E. Shustin. *Tropical algebraic geometry*, volume 35 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, second edition, 2009. (page 1.4.2).

[17] N. Kalinin. Tropical approach to Nagata's conjecture in positive characteristic. *arXiv:1310.6684*, Oct. 2013. (page 1.2.4).

[18] N. Kalinin. The Newton polygon of a planar singular curve and its subdivision. *Journal of Combinatorial Theory, Series A*, 137:226 – 256, 2016. (pages 1.0.10, 2, 1.2.4, and 1.3).

[19] H. J. Keisler. *Foundations of infinitesimal calculus*, volume 20. Prindle, Weber & Schmidt Boston, 1976. (page 1.4.2).

[20] M. Kerber and H. Markwig. Intersecting Psi-classes on tropical $\mathfrak{M}_{0,n}$. *Int. Math. Res. Not. IMRN*, (2):221–240, 2009. (page 1.4.1).

[21] A. Khovanskii. Logarithmic functional and the weil reciprocity laws. In *Proceedings of the Waterloo Workshop on Computer Algebra*, pages 85–108. World Scientific, 2006. (page 3).

[22] A. Khovanskii. *Logarithmic functional and reciprocity laws.* Amer Math Soc, 2008. (page 3).

[23] A. G. Khovanskii. Newton polygons, curves on torus surfaces, and the converse weil theorem. *Russian Mathematical Surveys*, 52(6):1251, 1997. (page 3).

[24] M. Kontsevich. Intersection theory on the moduli space of curves and the matrix airy function. *Communications in Mathematical Physics*, 147(1):1–23, 1992. (page 1.4.1).

[25] L. Lang. Geometry of tropical curves and application to real algebraic geometry. *ID: unige:43180*, http://archive-ouverte.unige.ch/unige:43180, 2014. (page 1.4.1).

[26] H. Markwig, T. Markwig, and E. Shustin. Tropical curves with a singularity in a fixed point. *Manuscripta Math.*, 137(3-4):383–418, 2012. (pages 1.0.10 and 1.3).

[27] M. Mazin. Geometric theory of parshin residues. (page 3).

[28] G. Mikhalkin. Enumerative tropical algebraic geometry in R2. *ArXiv Mathematics e-prints*, Dec. 2003. (page 1.3).

[29] G. Mikhalkin. Moduli spaces of rational tropical curves. In *Proceedings of Gökova Geometry-Topology Conference 2006*, pages 39–51. Gökova Geometry/Topology Conference (GGT), Gökova, 2007. (pages 6 and 1.4.1).

[30] G. Mikhalkin and G. Mikhalkin. Tropical geometry and its applications, international congress of mathematicians. *Journal of the European Mathematical Society*, Vol. II,:827–852, 2006. (pages 1.0.1, 1.0.5, and 1.1.7).

[31] G. Mikhalkin and I. Zharkov. Tropical curves, their jacobians and theta functions. *arXiv preprint math/0612267*, 2006. (pages 1, 3, 6, and 1.4.1).

[32] R. Morrison. Tropical images of intersection points. *Collectanea Mathematica*, pages 1–11, 2014. (pages 1.1.4 and 1.3).

[33] B. Osserman and S. Payne. Lifting tropical intersections. *ArXiv e-prints*, July 2010. (page 1.3).

[34] B. Osserman and J. Rabinoff. Lifting non-proper tropical intersections. *ArXiv e-prints*, Sept. 2011. (page 1.3).

[35] S. Payne. Analytification is the limit of all tropicalizations. *Math. Res. Lett.*, 16(2-3):543–556, 2009. (page 1.3).

[36] J. Rabinoff. Tropical analytic geometry, Newton polygons, and tropical intersections. *Adv. Math.*, 229(6):3192–3255, 2012. (page 1.3).

[37] Q. Ren, K. Shaw, and B. Sturmfels. Tropicalization of Del Pezzo Surfaces. *ArXiv e-prints*, Feb. 2014. (page 1.3).

[38] J. Richter-Gebert, B. Sturmfels, and T. Theobald. First steps in tropical geometry. In *Idempotent mathematics and mathematical physics*, volume 377 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 2005. (page 1.1.2).

[39] M. Ruggiero and K. Shaw. Tropical hopf manifolds and contracting germs. *arXiv preprint arXiv:1509.03787*, 2015. (page 1.3).

[40] K. Shaw. Tropical surfaces. 06 2015. (pages 1 and 1.3).

[41] K. M. Shaw. Tropical intersection theory and surfaces. *http://archive-ouverte.unige.ch/unige:22758*, 2011. (page 1.3).

[42] K. M. Shaw. A tropical intersection product in matroidal fans. *SIAM Journal on Discrete Mathematics*, 27(1):459–491, 2013. (page 8).

[43] T. Tao. *Compactness and contradiction.* American Mathematical Society, Providence, RI, 2013. (page 1.4.2).

[44] M. Temkin. Introduction to Berkovich analytic spaces. *ArXiv e-prints*, Oct. 2010. (page 1.4.2).

[45] O. Viro. On basic concepts of tropical geometry. *Proceedings of the Steklov Institute of Mathematics*, 273(1):252–282, 2011. (pages 1.0.1 and 1.3).

If you are in difficulties with a book,
try the element of surprise: attack it at an hour when it isn't expecting it.
Herbert George Wells