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Master Thesis

Phase Diagram of a Bosonic Ladder with a Single Correlated Impurity

by

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This is a frontier of exponentiating possibilities, forming a glorious continuum spanning from the simplest collective properties of the elements, out towards the most dramatic emergent phenomenon of all- that of life itself.

Piers Coleman

Many-Body Physics: Unfinished Revolution

In loving memory of my dear friend Alan.
To the joys we shared, and the sweet memories you gave us all.

Contents

Abstract

We considered the problem of a two-legged bosonic ladder with a single impurity correlated across the rungs of the ladder, i.e. an impurity that is coupled to the density of both legs at the same point. We studied the problem at absolute zero temperature and in the limit in which the impurity potential is the smallest energy scale of the problem, using a bosonization scheme. A renormalization group procedure determined the existence of a regime in which the antisymmetric modes become massive, and of another one in which they remain massless. For both regimes, a transition between conducting and insulating behaviors has been found, determined, in this limit, only by the interactions and not by the strength of the impurity. In the massless regime, the transition takes place at $K_s + K_a = 2$, while in the massive regime it occurs at $K_s = 2$ for attractive interchain coupling and $K_s = 1/2$ for repulsive coupling, K_s and K_a being the Luttinger liquid parameters of the symmetric and the antisymmetric modes of the ladder. Several comparisons between the bosonic ladder and a single chain have also been discussed.

1 Introduction

Since the emergence of quantum mechanics, understanding the properties of many-body interacting quantum systems has been one of the biggest challenges of physics. The combination of interference effects of quantum mechanics and the difficulties related to statistical mechanics make these situations difficult to analyze. In the efforts towards developing theories capable of describing these problems, one-dimensional systems have played a particular role: they are simple enough to allow for explicit and in some cases exact calculations, all while maintaining a non-trivial behavior giving rise to rich and complex physical phenomena. Specific techniques were thus designed to tackle one dimensional problems, such as the Bethe ansatz [1, 2], bosonization [3], and numerous others, both analytical and numerical [4, 5].

At first, one-dimensional many-body quantum problems were considered to be toy models, useful for theoreticians to understand some basic principles but not to represent reality. With time, experimental realizations of these models became available [4, 5], first in some materials like polymers and organic compounds, and afterwards in condensed matter systems where one dimensional structures were found inside bulk materials. Nanoscopic structures were also artificially constructed, such as quantum wires, Josephson junction arrays and edge states resulting from the quantum Hall effect. In the past decade, one of the biggest experimental developments of one-dimensional systems came from atomic physics, with the realization of chains of ultracold atomic gases confined in optical traps [6, 7, 8, 9]. This kind of systems allows unprecedented control and tunability of many parameters, in particular of interactions and disorder [10, 11], rendering accessible regimes that cannot be realized in condensed matter systems. Another advantage of cold gases is the capacity of isolating the system from external perturbations, and in particular to avoid phononic modes, which are always present in condensed matter systems. Being able to attain a high degree of isolation is essential to probe, for instance, the effects of disorder and out of equilibrium physics [11, 12]. All the recent experimental progresses in the field have fueled even more the theoretical research on one-dimensional systems.

A particular area of interest is the study of disordered systems. Adding disorder to the interplay of quantum interference, particle statistics, and interactions, gives rise to new phenomena. A disordered potential leads to the localization of particles [13, 14, 15], and this localization can be either reinforced or destroyed by the interactions [11], depending on the situation. In condensed matter systems, some degree of disorder is unavoidable, while in trapped ultracold gases a disordered potential can be created and controlled to a high degree.

A single impurity is a particular case of disorder, and provides a good description for systems like a chain with a constriction or with dilute impurities.

Inspired by the previous research in the field, we have turned our attention towards a system of two coupled one-dimensional chains (known as a ladder) populated by bosons, with a single impurity at the origin, correlated across both chains. The structure of this work is as follows: in section 2 we briefly introduce the theoretical tools necessary to solve the problem. In section 3 we propose a model and solve it for different initial conditions. The results are then discussed in section 4, and finally we present our conclusions and some proposals for future research in section 5.

2 Brief Review on 1-D Many-Body Physics

2.1 Qualitative Differences Between One Dimension and Higher Dimensions Many-Body Physics

One-dimensional many-body systems have some peculiarities that distinguish them from their higher-dimensional counterparts. Most high-dimensional many-fermion problems can be treated in the framework of Fermi liquid theory. In this theory, the interactions are often neither negligible nor dominant compared to the Fermi energy, and simply renormalize the mass of the particles. The excitations of these systems, known as quasiparticles, have a finite lifetime, and consist of electrons dressed by surrounding density fluctuations [16]. The quasiparticles can be treated as a system of essentially free particles, unlike the electrons themselves, with small residual interactions that can now be treated perturbatively. Quasiparticles are one of the elementary excitations of the system, the others being collective excitations of charge, like plasmons, and collective excitations of spin, like magnons. Under certain circumstances, these collective excitations are frozen at low temperature, giving rise to ordered ground states known respectively as charge density waves (CDW) and spin density waves (SDW). These ordered states are related to the divergence of the corresponding susceptibility. In 1-d, the situation is very different. A simple, naive picture is useful to illustrate this point: when the particles are restrained to move in only one direction, if a single one of them is moving, it will necessarily collide with it's neighbours due to interactions, and any excitation will quickly become collective, hence no single particle excitation can take place in 1-d [4, 5]. The only possible excitations are collective ones. In the case of spinful fermions, the collective modes of charge and spin do not necessarily propagate with the same velocity, so the excitations of this kind of systems split into one mode carrying charge and another one carrying spin [17, 18].

A further important peculiarity is related to the Mermin-Wagner-Hohenberg theorem, which states that in classical systems of dimension $d \leq 2$ thermal fluctuations prevent any spontaneous breaking of a continuous symmetry [19, 20]. In quantum systems, the imaginary time is formally equivalent to a classical spatial dimension of finite size β [21], and quantum fluctuations play the role of classical thermal fluctuations. In this way, quantum problems in d dimensions can be related to classical statistical problems in $d + 1$ dimensions. Thus, the Mermin-Wagner-Hohenberg theorem prevents the spontaneous breaking of a continuous symmetry in quantum 1-d systems as well. As a consequence, no ordered state like a CDW or SDW can fully settle in. In fact, in 1-d both of the susceptibilities related to these collective modes diverge due to the ever present nesting of the Fermi surface, and both instabilities drive the system towards different ordered states, each one preventing the actual realization of the other. The system is at the brink of a phase transition, and behaves like a system at a critical point [4]. In higher dimensions, even in the presence of a competition between different instabilities, the system is driven to the realization of one of the possible ordered states, for which the other susceptibilities no longer diverge.

In the case of bosons, a non-interacting system has a tendency to condense in the lowest energy state with momentum $k = 0$ below a critical temperature, forming a Bose-Einstein condensate (BEC). In this phase, the wave functions of a large number of particles are coherent through all the sample, giving rise to a macroscopic quantum state with no rigidity [22]. There is no analogue to a Fermi energy defining a high energy scale that would allow a perturbative treatment of the interactions with free bosons as a starting point, and interactions must be taken into account even in high dimensions. In the interacting problem, particles condense below a critical temperature into a superfluid state, which also presents, in high-dimensions, long range order of the phase of the wave-function, and is characterized by having zero viscosity. Nevertheless, the Mermin-Wagner-Hohenberg theorem implies that the

order given by a homogeneous phase, which breaks a continuous symmetry, cannot take place in 1-d, preventing the condensate to form, even if there is an instability towards a superfluid state [23, 24].

Another difference between physics in 1-d and in higher dimensions is the role played by the exclusion principle. In one-dimension, the permutation of two particles requires that they interact with each other, whereas in higher dimensions one can always find a way to exchange the positions of two particles such as they don't collide with each other. Since (repulsive) interactions already prevent two particles from occupying the same position regardless of whether the particles are bosons or fermions, the fact that interactions are unavoidable in 1-d means that certain operators, those which don't depend on the sign of the wavefunction, like the particle density are less sensitive to the Pauli principle. Nonetheless, the wavefunctions themselves obey of course the corresponding statistics [4]. A similar theory can therefore be developed for both kinds of particles, with slight necessary modifications when computing expectation values of physical observables.

Specific techniques to solve 1-d problems have been developed, ranging from exact analytical solutions and numerical methods, to low-energy effective theories. Only the latter is implemented in this work, thus a description of this method is given in the following sections.

2.2 Bosonization

Bosonization, or harmonic fluid approach, is a useful technique to analyze the low-energy properties of one-dimensional systems. It allows us to relate most problems, regardless of the particle statistics, to a universal (or at least quite general) description in terms of bosonic fields [3]. For a generic 1-d system, with x_n the position of the n-th particle, one can write the density operator $\rho(x)$ as:

$$\rho(x) = \sum_n \delta(x - x_n) \tag{1}$$

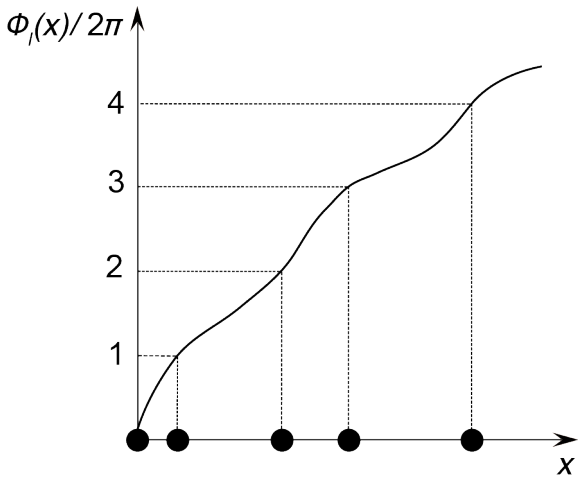

Figure 1: A possible realization of the labeling field $\phi_l(x)$, which takes values multiple of 2π at the positions of the particles, represented as black circles. It can always be chosen to be monotonically increasing.

A more convenient representation of the density operator can be written by introducing a labeling field $\phi_l(x)$, a continuous function of x that takes as value a multiple of 2π at the position of each particle, $\phi_l(x_n) = 2\pi n$, x_n being the position of the n-th particle. It can

always be chosen to be monotonically increasing, as in fig. 1, since particles can be numbered in a unique, unambiguous way in one dimension. In terms of such a field, the density operator is:

$$\begin{aligned}\rho(x) &= \sum_n \nabla\phi_l(x)\delta(\phi_l(x) - 2\pi n) \\ &= \frac{\nabla\phi_l(x)}{2\pi}\sum_p e^{ip\phi_l(x)}\end{aligned} \tag{2}$$

where we used Poisson's summation formula for the second equality, and the sum on p runs on all integers. The labeling field can also be expressed as a function of the deviation ϕ with respect to the perfect crystalline solution, $\phi_{crystal}(x) = 2\pi\rho_0 x$, ρ_0 being the average particle density, as:

$$\phi_l(x) = 2\pi\rho_0 x - 2\phi(x) \tag{3}$$

and the density operator is now:

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]\sum_p e^{i2p(\pi\rho_0 x - \phi(x))} \tag{4}$$

For a complete description, we need to be able to write the particle creation operator[1] as a function of the labeling field and an operator $\theta(x)$ representing the phase:

$$\psi^\dagger(x) = [\rho(x)]^{1/2} e^{-i\theta(x)} \tag{5}$$

The square root of a δ function is, up to a normalization constant, another δ function. This constant depends on the ultraviolet cutoff structure of the theory, and is thus non-universal. Neglecting the normalization constant, we get

$$\psi^\dagger(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]^{1/2}\sum_p e^{i2p(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)} \tag{6}$$

There is a relation between θ and the density operator, imposed by the commutation relations of the particle creation operators. For bosons,

$$\left[\psi(x), \psi^\dagger(x')\right] = \delta(x - x') \tag{7}$$

The field ϕ commutes with itself since the density at different points does so as well. Assuming that θ also commutes with itself, a sufficient condition to satisfy (7) is

$$\left[\rho(x), e^{-i\theta(x')}\right] = \delta(x - x')e^{-i\theta(x)} \tag{8}$$

In the long wavelength approximation, in which we neglect terms oscillating over distances smaller than ρ_0^{-1} (the average inter-particle distance), one can replace the density operator by its $p = 0$ component, the smeared density $\rho_{p\sim 0} = \rho_0 - \frac{1}{\pi}\nabla\phi(x)$. In that case, we have

$$\left[\frac{1}{\pi}\nabla\phi(x), \theta(x')\right] = -i\delta(x - x') \tag{9}$$

[1] This is the boson creation operator. A slight modification is needed for fermions to account for the correct statistics, namely a Jordan-Wigner transformation, see [4, 3].

The higher harmonics of the density give corrections that can be neglected in the continuum limit. This equation means that $\frac{1}{\pi}\nabla\phi(x)$ and $\theta(x)$ are canonically conjugated variables. Integrating (9) by parts, we can see that the same is true for $\frac{1}{\pi}\nabla\theta(x)$ and $\phi(x)$, hence

$$\Pi(x) = \nabla\theta(x)/\pi \tag{10}$$

is the canonical momentum of $\phi(x)$. This relation allows us to represent ϕ and Π in terms of bosonic creation and annihilation operators $b_k^\dagger$ and b_k, representing small oscillations of the density. Since ϕ and Π are real, each must contain both $b_k^\dagger$ and b_k. This representation is equivalent to that of a harmonic oscillator.

2.3 Tomonaga-Luttinger Liquid

Using (4) and (6), the Hamiltonian of a given problem can be rewritten in terms of the fields ϕ and θ. Interactions of the form $\int dx\rho(x)^2$ lead to terms proportional to $(\nabla\phi(x))^2$. The kinetic energy

$$H_{kin} = \int dx\ \frac{1}{2m}(\nabla\psi^\dagger(x))(\nabla\psi(x))) \tag{11}$$

gives terms proportional to $(\nabla\theta(x))^2$. These are the dominant, most relevant terms. Higher order, anharmonic terms can also be present, but they vanish at low energies, thus they are irrelevant in a renormalization group (RG) flow [3, 5]. They can nonetheless modify the prefactor in front of the quadratic terms. It is easy to see that the crossproduct $\nabla\phi(x)\nabla\theta(x)$ cannot be generated in the RG flow, at least not for systems with inversion symmetry. Under an inversion transformation $x \to -x$, physical operators must remain invariant, $\rho(x) \to \rho(-x)$ and $\psi(x) \to \psi(-x)$, implying that $\phi(x) \to -\phi(-x)$ and $\theta(x) \to \theta(-x)$. As a consequence, $\nabla\phi(x) \to \nabla\phi(-x)$ and $\nabla\theta(x) \to -\nabla\theta(-x)$. Crossed terms would change sign under inversion, and therefore cannot participate in the Hamiltonian, which must remain invariant. Taking into account all of these considerations, a universal low-energy effective Hamiltonian can be built:

$$\begin{aligned} H_{LL} &= \frac{1}{2\pi}\int dx\ \left[uK(\nabla\theta(x))^2 + \frac{u}{K}(\nabla\phi(x))^2\right] \\ &= \frac{1}{2\pi}\int dx\ \left[uK(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\phi(x))^2\right] \end{aligned} \tag{12}$$

where u and K are two model-dependent parameters that characterize the low energy spectrum and determine the asymptotic physical properties of a system. This is the Hamiltonian of the Tomonaga-Luttinger liquid, or Luttinger liquid (LL). The excitations of this model are non-interacting bosonic fluctuations similar to sound waves with a linear dispersion relation $\omega = u|k|$ [25] . One can thus see that u is the velocity of the excitations. The dimensionless Luttinger liquid parameter K is determined by the interactions. For bosonic systems, $K \geq 1$ for short-range repulsive interactions, being unity in the case of hard-core bosons, and tending to infinity when approaching the non-interacting case. $K < 1$ corresponds to long-range repulsive interactions. In the case of fermions, $0 < K < 1$ for repulsive interactions, $K = 1$ for non-interacting systems, and $K > 1$ for attractive interactions.

This remarkable result means that, at low energy, most 1-d problems can be reduced to a system of non-interacting particles, just as it happens in Fermi liquid theory in higher dimensions. Some interesting properties of the Luttinger liquid Hamiltonian are its invariance under Lorentz transformations and under the simultaneous substitution of $\phi(x) \to \theta(x)$ and $K \to 1/K$. Indeed, defining the canonical momentum associated to the field θ as

$$\Pi_\theta(x) = \nabla\phi(x)/\pi \tag{13}$$

the LL Hamiltonian can be rewritten in the so-called dual representation as:

$$H_{LL} = \frac{1}{2\pi}\int dx\ \left[\frac{u}{K}(\pi\Pi_\theta(x))^2 + uK(\nabla\theta(x))^2\right] \tag{14}$$

All that is left now to completely solve a problem is to compute the correlation functions. For this, we need the Euclidean action associated with the Hamiltonian (12)

$$S = \int_0^\beta d\tau \int dx \left[\frac{1}{2\pi}\left(uK(\nabla\theta(x,\tau))^2 + \frac{u}{K}(\nabla\phi(x,\tau))^2\right) - i\frac{1}{\pi}\nabla\theta(x,\tau)\partial_\tau\phi(x,\tau)\right] \tag{15}$$

where we now explicitly show the fields' dependence on the imaginary time τ, and β is the inverse temperature. It can be conveniently expressed in Fourier space as

$$S = \frac{1}{2\pi}\frac{1}{\beta\Omega}\sum_{\mathbf{q}} \begin{pmatrix}\theta^*(\mathbf{q}) & \phi^*(\mathbf{q})\end{pmatrix} \begin{pmatrix}k^2uK & ik\omega_n \\ ik\omega_n & k^2u/K\end{pmatrix} \begin{pmatrix}\theta(\mathbf{q}) \\ \phi(\mathbf{q})\end{pmatrix} \tag{16}$$

with Ω the length of the system, ω_n the Matsubara frequency, k the momentum and $\mathbf{q} = (k, \omega_n)$.

We can now compute the expectation value of an operator $O(\phi, \theta)$ with standard functional integration using a weight e^{-S},

$$\langle O(\phi,\theta)\rangle = \frac{1}{Z}\int \mathcal{D}\theta\mathcal{D}\phi\ O(\phi,\theta)e^{-S} \tag{17}$$

$$Z = \int \mathcal{D}\theta\mathcal{D}\phi\ e^{-S} \tag{18}$$

with Z the partition function. For operators independent of θ, the integration on this field can readily be done. After integrating out these degrees of freedom, the following effective action remains for ϕ

$$\begin{aligned} S_\phi &= \frac{1}{2\pi uK}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}[\omega_n^2 + u^2k^2]|\phi(k,\omega_n)|^2 \\ &= \frac{1}{2\pi uK}\int_0^\beta d\tau \int dx\left[(\partial_\tau\phi(x,\tau))^2 + u^2(\nabla\phi(x,\tau))^2\right] \end{aligned} \tag{19}$$

A correlation function that can be easily calculated using (19) is the correlation among products of exponentials of the field $\langle\prod_j e^{iA_j\phi(r_j)}\rangle$. To do so, we can complete the squares of the field inside the exponentials. The result is exactly zero unless $\sum_j A_j = 0$, in which case

$$\left\langle \prod_j e^{iA_j\phi(r_j)} \right\rangle = e^{\frac{1}{2}\sum_{i<j} A_iA_jKF_1(r_i - r_j)} \tag{20}$$

with

$$F_1(r) = \frac{1}{\beta\Omega}\sum_{\mathbf{q}}\left[1 - \cos(kx + \omega_n\tau)\right]\frac{2\pi u}{\omega_n^2 + u^2k^2} \tag{21}$$

The sum over k diverges. It is necessary to impose an energy cutoff Λ to regularize the ultra-violet behavior of the theory, which defines as well a spacial cutoff $\alpha = u/\Lambda$. In the zero temperature limit

$$F_1(r) = \frac{1}{2}\log\left[\frac{x^2 + (u|\tau| + \alpha)^2}{\alpha^2}\right] \tag{22}$$

Using this result we can see that some correlation functions decay with distance following a power-law, with a coefficient determined by K. For example,

$$\left\langle e^{ia\sqrt{2}\phi(r_1)} e^{-ia\sqrt{2}\phi(r_2)} \right\rangle \simeq \left(\frac{\alpha}{|r_1 - r_2|}\right)^{a^2 K} \tag{23}$$

up to a multiplicative constant, which depends on the microscopic model. The same result can be obtained for θ by changing $K \to 1/K$.

2.4 Sine-Gordon Hamiltonian

Luttinger liquids are fixed points in RG procedures for a wide variety of microscopic models. However, they are not the only ones, and other effective Hamiltonians are possible depending on the interactions. An important example, relevant for this work, is the sine-Gordon Hamiltonian,

$$H_{SG} = H_{LL} + 2\lambda \int dx \ \cos(2\sqrt{2}\phi(x)) \tag{24}$$

with H_{LL} defined by (12) and λ a parameter determined by the coupling constants of the interactions. For instance, for spin-1/2 fermions, λ is given by the coupling among antiparallel spins. If this parameter is small, it can be used to do an expansion in a power series, and the problem can be solved as a perturbation on a Luttinger liquid.

The relevance of the cosine term in (24) can be studied by an RG procedure. Up to second order in the coupling constant, and taking an isotropic cutoff α, the correlation function is

$$\mathcal{R}(r_1, r_2) \coloneqq \left\langle e^{ia\sqrt{2}\phi(r_1)} e^{-ia\sqrt{2}\phi(r_2)} \right\rangle \approx e^{-a^2 K_{eff} F_1(r_1 - r_2)} \tag{25}$$

where

$$K_{eff} = K - \frac{1}{2}K^2 g^2 \int_\alpha^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} \tag{26}$$

and

$$g = \frac{4\pi\alpha^2\lambda}{u} \tag{27}$$

Since the parameter K_{eff} fixes the critical exponent of the correlation functions, it must remain invariant under the RG transformation. Parametrizing the cutoff as $\alpha = \alpha_0 e^l$, the equations controlling the flow of K and g are:

$$\begin{aligned} \frac{dK(l)}{dl} &= -\frac{K^2(l)g^2(l)}{2} \\ \frac{dg(l)}{dl} &= (2 - 2K(l))g(l) \end{aligned} \tag{28}$$

From (28), a phase transition can be predicted at $K = 1$. Defining $K = 1 + \widetilde{K}/2$, in the lowest order for a small g the RG equations become

$$\frac{d\widetilde{K}(l)}{dl} = -g^2(l)$$
$$\frac{dg(l)}{dl} = -\widetilde{K}(l)g(l) \tag{29}$$

Equations (29) are symmetric under $g \to -g$, and have the quantity $A^2 \coloneqq \widetilde{K}^2 - g^2$ as a constant of motion. Hence, the flow is given by hyperbolae, as sketched in fig. 2. The phase diagram is separated in three regions, defined by the initial conditions. In region a, where $A^2 > 0$ and $K > 1$, g is irrelevant, and the theory becomes a Luttinger liquid with an effective parameter K^* given by the fixed point of (29). Region c is defined by $A^2 > 0$ and $K < 1$, and b by $A^2 < 0$. In these regions, the flow goes toward strong coupling, and the equations are valid only up to a given l^* for which the coupling is of order one, $|g(l^*)| \sim 1$. A separatrix with $|g| = \widetilde{K}$, where the coupling is marginally irrelevant and the fixed point is $K^* = 1$, separates regions a and b, while the (dotted) line where $|g| = -\widetilde{K}$ separates b from c and goes to strong coupling.

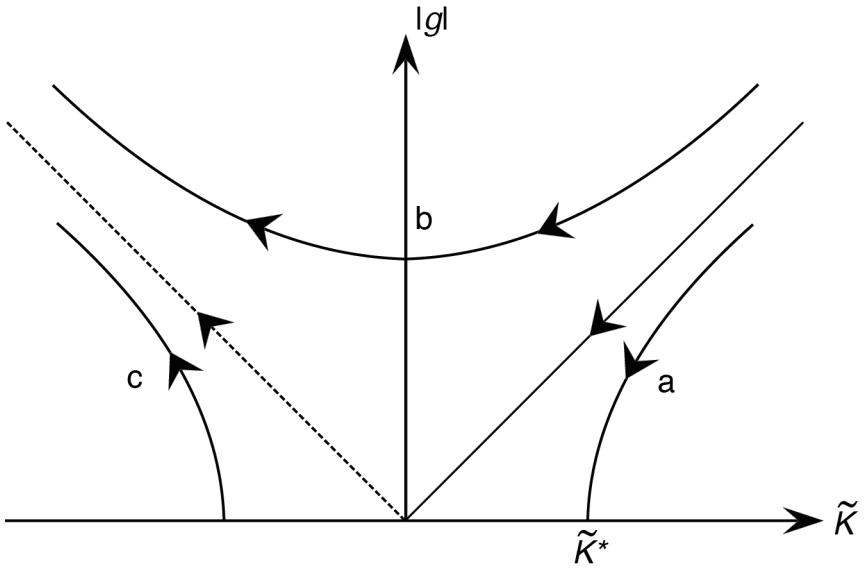

Figure 2: Flow defined by (29). In region a, g is irrelevant and the theory renormalizes to a Luttinger liquid, whereas in regions b and c, the RG flow gives a massive Hamiltonian.

The physics of the strong coupling phases can be determined by a semi-classical approximation, by expanding the cosine term in (24) around a solution that minimizes the energy. For negative g, this minimum is $\phi(x) = 0$. To the lowest order, the expansion gives the Hamiltonian

$$H_{SG} = H_{LL} + \frac{2|g|u}{\pi\alpha^2} \int dx\ \phi^2(x) \tag{30}$$

and after integrating over θ the effective action is

$$S^{\Delta} = \frac{1}{2\pi u K} \frac{1}{\beta\Omega} \sum_{\mathbf{q}} [\omega_n^2 + u^2k^2 + \Delta^2] |\phi(k, \omega_n)|^2 \tag{31}$$

The excitations of ϕ are massive in these regimes, with a mass parametrized by Δ. Indeed, the energy spectrum has now a gap proportional to the bandwidth, $\Delta_0 \coloneqq \Delta(l^*) \propto u/\alpha$. The

original gap before the renormalization, $\Delta(l=0) \approx e^{-l^*}\Delta_0$, can be obtained as a function of the initial parameters $g(l=0)$ and $K(l=0)$. Its behavior is obviously different in each of the different massive regions. Denoted as Δ_b for region b, Δ_c for region c, and Δ_{sep} for the line that separates b and c, the gap is given in these three cases by

$$\Delta_b(0) \approx \Delta_0 e^{-\pi/\sqrt{2g(0)(g(0)-\widetilde{K}(0))}} \tag{32}$$

$$\Delta_c(0) \approx \Delta_0 (g(0))^{\frac{1}{2-2K}} \tag{33}$$

$$\Delta_{sep}(0) \approx \Delta_0 e^{-\frac{1}{g(0)}} \tag{34}$$

The main feature of this model is the ordering of the field ϕ. Notice that the Mermin–Wagner–Hohenberg theorem doesn't forbid spontaneous breaking of discrete symmetries in 1-d, such as the $\phi \to \phi + \pi n/\sqrt{2}$ symmetry of the sine-Gordon Hamiltonian (24). The passage from the massless to the massive regime is thus a true quantum phase transition.

3 Impurity in a Bosonic Ladder

3.1 Defining the Problem

The interplay between disorder and interactions is today an important field of study, even more so in 1-d, where interactions play a fundamental role. An example of this interplay are bosonic systems, for which the superfluid phase is quite stable in the presence of disorder. However, disorder also induces the localization of particles via the celebrated Anderson localization [13], and leads to an insulating regime known as Bose-glass. The competition between both of these instabilities results in a rich phase diagram. For weak interactions, the system is in the insulating Bose-glass phase, and has a phase transition towards a superfluid state as the interaction strength increases. Further increase of the interactions leads to a second phase transition, bringing the system back to a Bose-glass state [26, 27, 28, 29]. The behavior of the correlation functions at the second transition, at moderate interactions and weak disorder, is reported to be given by a power law decay with universal critical exponents, as can be shown using low-energy calculations in a bosonization description of the problem. On the other hand, in the transition at weak interactions and strong disorder, the critical exponents were found to be non-universal [30, 31]. These exponents being non-universal means that the system might present a multicritical point in the boundary between the superfluid and the Bose-glass phases, separating the universal regime from the non-universal one. As a consequence of this possibility, it is not clear whether the two insulating phases are in fact the same or not.

The bosonization technique that predicts universal exponents is only valid for small density fluctuations, and therefore fails to describe the transition in the weak interaction and strong disorder regime, where we can expect larger density fluctuations. In an attempt to reconcile both results, we study a model that could emulate in some sort large density fluctuations but that can still be described by bosonization. Consider a ladder system, a collection of 1-d chains (the "legs" of the ladder) that are arranged on a plane in such a way that each leg interacts with its neighbours. Each leg can have small density fluctuations, and therefore allow for a bosonized description, while the sum of the densities of all the legs would present large fluctuations. In our case, we will consider a two-legged ladder. For this analogy between a ladder and a chain with large density fluctuations to work we consider as interchain interactions the coupling between the densities of both legs, mediated through a potential V. This potential (visually represented as the "rungs" of the ladder) adds a degree of freedom that allows to control the total density, i.e. the sum of the densities of both legs at a given point. Nevertheless, the mapping between this model and the single chain is not perfect, as particles in the ladder are distinguished by the leg index, whereas bosons in a single chain are indistinguishable. Other kind of interactions could also be taken into account, namely the usual particle hopping from one leg to another, which is necessary to restore the indistinguishability of the particles. We will further discuss some of the implications of such an interaction in section 3.2. For simplicity, we will only study the model without interchain hopping, as a first approach to the problem.

It is also necessary to consider how to include disorder in this model. To be able to use the ladder as a simulator of a chain with large density fluctuations, the disorder potential must be exactly the same for both legs of the ladder, coupling in the same way to the density of each leg and therefore affecting the total density consistently. This means that we must consider disorder correlated across the rungs. A first step in this direction has been taken in ref. [32] for extended disorder, and a similar bosonization scheme has been considered for clean chains in [33]. In the present work, we will focus on a single impurity affecting in the same way both chains at the origin through a potential V_0 coupled locally to the total density. A sketch of the problem is depicted in fig. 3.

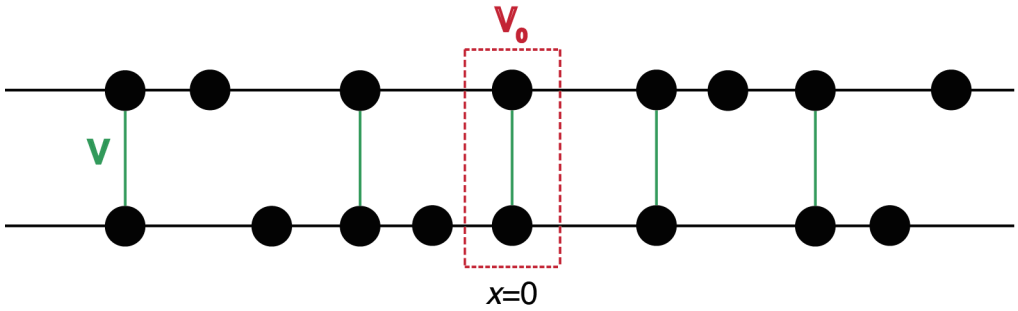

Figure 3: A bosonic ladder consisting of two bosonic chains coupled by an interaction potential V (represented by green vertical lines connecting the particles), with a single, point-like impurity potential V_0 at the origin (red box). The impurity potential is the same for both chains.

Let's consider now the Hamiltonian describing this situation. The density operator of the leg i is denoted by $\rho_i(x)$, with $i = 1, 2$. In accordance with the discussion above the Hamiltonian of the ladder is taken to be

$$H = H_1 + H_2 + V \int dx\ \rho_1(x)\rho_2(x) + V_0[\rho_1(x=0) + \rho_2(x=0)] \tag{35}$$

where H_i is the Luttinger liquid Hamiltonian for the chain i given by (12). The third term in the sum is the interchain interaction, which couples together the densities of the chains, and the last term is the localized impurity potential correlated between the two legs. Before solving the problem, let's make some comments about this Hamiltonian. The index $i = 1, 2$, which labels the legs, can be identified to the spin index $\sigma = \uparrow, \downarrow$ of a spin-1/2 fermionic chain, meaning that it is possible to identify as well the density of a leg with the density of carriers with a given spin in the spinful fermion chain, say $\rho_1(x) \to n_\uparrow(x)$ and $\rho_2(x) \to n_\downarrow(x)$. With this substitution, the interchain coupling becomes

$$V \int dx\ \rho_1(x)\rho_2(x) \to V \int dx\ n_\uparrow(x) n_\downarrow(x) \tag{36}$$

and defining the total charge density as $n(x) = n_\uparrow(x) + n_\downarrow(x)$, for the impurity we have

$$V_0[\rho_1(0) + \rho_2(0)] \to V_0[n_\uparrow(0) + n_\downarrow(0)] = V_0 n(0) \tag{37}$$

It is clear now that, in terms of the spin-1/2 fermionic chain, our model corresponds to a continuum version of the Hubbard model, with a nonmagnetic impurity that couples to the total charge density. There are, however, a few differences. The ranges of the parameter K are different whether we talk about fermions or bosons, as discussed in section 2.3. On top of that, and maybe more importantly, the fermionic chain in this model is likely to have an $SU(2)$ spin rotation symmetry. Since V controls the interactions between different spin species, and K those among equal spins, this symmetry imposes a strict relation between these two variables, and therefore V is not an independent parameter for the fermionic chain. Furthermore, the symmetry must be conserved under a renormalization group transformation, so the parameters must follow a determined RG flow. The boson ladder lacks this symmetry: there is a priori no reason for the interchain interaction to be necessarily the same as the interactions between the bosons of a particular chain, so there is no strict relation between K and V, the latter is a free parameter that can be chosen without restriction[2], and K and V are free to follow any possible flow under RG. The problem of transport across an impurity

[2] At least, no restriction other than that it must be small enough for the system to remain trapped and effectively 1-d.

for both spin-less and spinful electrons has already been studied in [34, 35, 36], and the effect of magnetic impurities in spin-1/2 antiferromagnetic chains in [37].

To finish the comparison with the fermionic chain, let us consider what the eventual interchain hopping with hopping amplitude t would mean for the fermions. Noting $\psi_i^\dagger(x)$, $i = 1, 2$, the boson creation operators of the ladder, and $\psi_\sigma^\dagger(x)$, $\sigma = \uparrow, \downarrow$, the spin σ fermion creation operators of the chain, this term is given by

$$-t \int dx \left[\psi_1^\dagger(x)\psi_2(x) + \psi_2^\dagger(x)\psi_1(x)\right] \to -t \int dx \left[\psi_\uparrow^\dagger(x)\psi_\downarrow(x) + \psi_\downarrow^\dagger(x)\psi_\uparrow(x)\right] \tag{38}$$

which, in terms of the fermions, corresponds to a magnetic field perpendicular to the spin quantization axis.

3.2 Solving the Problem Without Impurity

Let's go back to the original problem. From the particle creation and the density operators, the bosonization fields $\phi_i(x)$ and $\theta_i(x)$ can be defined using (4) and (5). Since we are interested in the low energy properties of the problem, we can consider a long-wavelength approximation, neglecting oscillations with wavenumber $2\pi\rho_0$ and higher, keeping only those with wavenumber close to zero. This approximation corresponds to considering only the terms with $p = 0$ and $p = \pm 1$ in (4). Keeping only the most relevant operators, i.e. those with the least products of fields, we get

$$\rho_i(x) \approx \rho_0 - \frac{1}{\pi}\nabla\phi_i(x) + 2\rho' \cos(2\phi_i(x) - 2\pi\rho_0 x) \tag{39}$$

where we have used the same average density ρ_0 for both legs, and ρ' parametrizes the coefficient in front of the cosine term, which depends on the UV structure of a given microscopical model, and that can be renormalized by the higher harmonics. It is convenient as well to define the symmetric and antisymmetric combinations of the fields,

$$\phi_s(x) \coloneqq \frac{1}{\sqrt{2}}[\phi_1(x) + \phi_2(x)] \tag{40}$$

$$\phi_a(x) \coloneqq \frac{1}{\sqrt{2}}[\phi_1(x) - \phi_2(x)] \tag{41}$$

and similarly for θ. Notice that in the correspondence between the two-legged bosonic ladder and a chain of spin-1/2 fermions, ϕ_s plays the role of the charge component of the fermion chain, while ϕ_a corresponds to the spin component. The spin-charge separation in the fermionic leg lets us expect the independent propagation of ϕ_s and ϕ_a.

It is easy to prove that $H_1 + H_2 = \widetilde{H}_s + \widetilde{H}_a$, with $\widetilde{H}_s$ and $\widetilde{H}_a$ the Luttinger liquid Hamiltonians for ϕ_s and ϕ_a, respectively. For the other terms in the Hamiltonian, neglecting the high-energy terms oscillating with wavevector $2\pi\rho_0$ as well as additive constants that globally shift the energy, we obtain

$$\begin{aligned}\rho_1(x)\rho_2(x) = &[\rho_0 - \frac{1}{\pi}\nabla\phi_1(x) + 2\rho' \cos(2\phi_1(x) - 2\pi\rho_0 x)] \\ &\times[\rho_0 - \frac{1}{\pi}\nabla\phi_2(x) + 2\rho' \cos(2\phi_2(x) - 2\pi\rho_0 x)] \\ \approx &-\frac{\sqrt{2}\rho_0}{\pi}\nabla\phi_s(x) + \frac{1}{2\pi^2}[(\nabla\phi_s(x))^2 - (\nabla\phi_a(x))^2] + 2\rho'^2 \cos(2\sqrt{2}\phi_a(x))\end{aligned} \tag{42}$$

and

$$\begin{aligned}\rho_1(0)+\rho_2(0) &= \rho_0 - \frac{1}{\pi}\nabla\phi_1(0) + 2\rho' \cos(2\phi_1(0)) \\ &\quad + \rho_0 - \frac{1}{\pi}\nabla\phi_2(0) + 2\rho' \cos(2\phi_2(0)) \\ &\approx -\frac{\sqrt{2}}{\pi}\nabla\phi_s(0) + 4\rho' \cos(\sqrt{2}\phi_s(0))\cos(\sqrt{2}\phi_a(0))\end{aligned} \tag{43}$$

The terms quadratic in the gradients of the fields can be regrouped with those coming from the Luttinger liquid Hamiltonian,

$$\begin{aligned}\frac{1}{2\pi}\int dx\ \ \frac{u}{K}(\nabla\phi_a(x))^2 - \frac{V}{2\pi^2}\int dx\ (\nabla\phi_a(x))^2 &= \frac{1}{2\pi}\int dx\ \left(\frac{u}{K} - \frac{V}{\pi}\right)(\nabla\phi_a(x))^2 \\ &= \frac{1}{2\pi}\int dx\ \frac{u_a}{K_a}(\nabla\phi_a(x))^2\end{aligned} \tag{44}$$

where

$$\begin{aligned}K_a &\coloneqq \frac{K}{\sqrt{1-\frac{VK}{\pi u}}} \\ u_a &\coloneqq u\sqrt{1-\frac{VK}{\pi u}}\end{aligned} \tag{45}$$

which satisfy $u_aK_a = uK$. The quadratic terms of the antisymmetric fields can thus be regrouped together to give a Luttinger liquid Hamiltonian with renormalized parameters

$$H_a = \frac{1}{2\pi}\int dx\ \left[u_aK_a(\nabla\theta_a(x))^2 + \frac{u_a}{K_a}(\nabla\phi_a(x))^2\right] \tag{46}$$

A similar treatment can be done for the symmetric fields, with

$$\begin{aligned}K_s &\coloneqq \frac{K}{\sqrt{1+\frac{VK}{\pi u}}} \\ u_s &\coloneqq u\sqrt{1+\frac{VK}{\pi u}}\end{aligned} \tag{47}$$

However, besides the quadratic term there's still a term linear in $\nabla\phi_s$ coming from (42) and (43). Since $\nabla\phi_s$ controls density fluctuations, these linear terms are equivalent to a modification of the chemical potential, which we have not explicitly written in the Hamiltonian. Since the chemical potential already needs to be adjusted in order to fix the average density, it can be modified to compensate these linear terms, as long as the correct ρ_0 is used. The linear terms can thus be safely regrouped in the Luttinger liquid Hamiltonian by completing the squares and shifting the field,

$$\begin{aligned}&\frac{1}{2\pi}\int dx\ \frac{u_s}{K_s}(\nabla\phi_s(x))^2 - \frac{\sqrt{2}V\rho_0}{\pi}\int dx\ \nabla\phi_s(x) - \frac{\sqrt{2}V_0}{\pi}\nabla\phi_s(0) \\ &= \frac{1}{2\pi}\int dx\ \frac{u_s}{K_s}\left\{\left[\nabla\phi_s(x) - \frac{\sqrt{2}K_s}{u_s}\left(V\rho_0 + V_0f_\lambda(x)\right)\right]^2 - \frac{2K_s^2}{u_s^2}\left(V\rho_0 + V_0f_\lambda(x)\right)^2\right\}\end{aligned} \tag{48}$$

where we can choose $f_\lambda(x) = \mathrm{sech}^2(x/\lambda)/2\lambda$, or more generally any function approaching a delta function, $f_\lambda(x) \to \delta(x)$ as $\lambda \to 0$. Therefore, the linear term can be absorbed with the following redefinition:

$$\widetilde{\phi}_s(x) \coloneqq \phi_s(x) - \frac{\sqrt{2}K_s}{u_s}(V\rho_0 x + V_0 F_\lambda(x)) \tag{49}$$

where $F_\lambda(x) = \frac{1}{2}\tanh(\frac{x}{\lambda})$ is a primitive of $f_\lambda(x)$. The local adjustment of the chemical potential by the term $V_0 f_\lambda(x)$ reflects the tendency of either augmenting or reducing the density locally at the impurity site. This term can also be interpreted as the forward scattering due to the impurity, i.e. the $k \approx 0$ momentum transfers, which is known to not lead to the pinning of the fields [4]. Neglecting a shift in the energy, the contribution to the Hamiltonian is purely quadratic, and we obtain again a Luttinger liquid Hamiltonian with the parameters u_s and K_s,

$$H_s = \frac{1}{2\pi}\int dx \ \left[u_s K_s(\nabla\theta_s(x))^2 + \frac{u_s}{K_s}(\nabla\widetilde{\phi}_s(x))^2\right] \tag{50}$$

Note that the transformation (49) verifies $\widetilde{\phi}_s(0) = \phi_s(0)$, and therefore leaves invariant the interaction with the impurity. From now on, we will drop the tilde to lighten the notation, keeping in mind the redefinition of ϕ_s.

Putting together all the contributions back in (35), we obtain in terms of the symmetric and antisymmetric fields

$$H = H_s + H_a + 2\rho'^2 V\int dx \ \cos(2\sqrt{2}\phi_a(x)) + 4\rho' V_0 \cos(\sqrt{2}\phi_s(0))\cos(\sqrt{2}\phi_a(0)) \tag{51}$$

which, after integrating out the fields θ_a and θ_s as we did to find (19), gives the following effective action

$$\begin{aligned} S = &\frac{1}{2\pi u_s K_s}\int dx\ d\tau \left[(\partial_\tau\phi_s(x,\tau))^2 + u_s^2(\nabla\phi_s(x,\tau))^2\right] \\ &+ \frac{1}{2\pi u_a K_a}\int dx\ d\tau \left[(\partial_\tau\phi_a(x,\tau))^2 + u_a^2(\nabla\phi_a(x,\tau))^2\right] \\ &+ 2\rho'^2 V\int dx\ d\tau \ \cos(2\sqrt{2}\phi_a(x,\tau)) \\ &+ 4\rho' V_0\int d\tau \ \cos(\sqrt{2}\phi_s(0,\tau))\cos(\sqrt{2}\phi_a(0,\tau)) \end{aligned} \tag{52}$$

From (51) we can see that in the problem without impurity the fields are completely decoupled and their dynamics are given by a Luttinger liquid for the symmetric field and a sine-Gordon Hamiltonian for the antisymmetric field, with $\lambda = \rho'^2 V$ in (24), giving $g = 4\pi\alpha^2\rho'^2 V/u$ from (27). Equation (28) defines the flow of the parameters, and a phase transition takes place as discussed in section 2.4 when $|g| = \widetilde{K}_a$, with $K_a = 1 + \widetilde{K}_a/2$. The antisymmetric field becomes massive after the transition, with a mass given by eqs. (32) to (34). At the transition the Luttinger parameter flows towards the fixed point $K_a^* = 1$. This situation resembles the case of spinful fermions, for which the spin component becomes massive in a similar transition. A Luttinger liquid for which the spin component (or in our case, the antisymmetric mode) becomes massive is known as a Luther-Emery liquid [38].

If we consider the impurity potential to be the smallest energy scale of the problem, and therefore $|V_0| \ll |V|$, we can treat it as a perturbation over the pure ladder. There are three

important cases to analyze. The first one is the case where ϕ_a is massless. In this case, the problem is very similar to the one studied in [39]. The other two cases correspond to the massive phase, and differ from each other by the sign of V. Indeed, according to this sign, the minimum energy of the cosine term in the sine-Gordon Hamiltonian is given by either $\phi_a(x) = 0$ if $V < 0$ or $\phi_a(x) = \pi/(2\sqrt{2})$ if $V > 0$. In both cases, the effective action for ϕ_a is the same, given by (31), but the interaction with the impurity will be different, as we will se in section 3.4.

In the following sections, we will study the relevance of the impurity in these three cases of interest, but before doing that, let's analyze what are the implications for the bosonized fields of an interchain hopping term in the Hamiltonian. In the long-wavelength approximation, and taking only the most relevant terms, the particle creation operator can be approximated by

$$\psi_i^\dagger(x) \approx \rho_0^{1/2} e^{-i\theta_i(x)} \tag{53}$$

The bosonized version of the interchain hopping is therefore

$$-t \int dx\ [\psi_1^\dagger(x)\psi_2(x) + \psi_2^\dagger(x)\psi_1(x)] \approx -2t\rho_0 \int dx\ \cos(\sqrt{2}\theta_a(x)) \tag{54}$$

In the regions where this term is relevant, the field θ_a is massive. Were this the only interchain coupling, the Hamiltonian for θ_a would be a sine-Gordon Hamiltonian, with θ_a locked in a minimum of the cosine. However, in our Hamiltonian there is also a term $\cos(2\sqrt{2}\phi_a(x))$ that tries to fix the value of ϕ_a, which is essentially the conjugate of θ_a. There will be a competition between this two terms, that try each to impose an order incompatible with the other, and each will be more relevant (that is, will go faster towards strong coupling) in different regions of the phase diagram. We will not study this competition nor its consequences regarding the relevance of the impurity potential in the scope of this work. A detailed discussion on how to deal with this competition can be found in [40, 4]. From now on, we will disregard the interchain hopping, and use the Hamiltonian (51) to describe the problem.

3.3 Massless Case

The massless case takes place when the initial conditions $K_a > 1$ and $A^2 = \widetilde{K}_a^2 - g^2 > 0$ are met, with $K_a = 1 + \widetilde{K}_a/2$. In this regime, the pure problem consists of two independent Luttinger liquids. When the local impurity is introduced, we can expect it to alter the properties of the system only locally, and hence it is convenient to trace over the fields evaluated away from the impurity, leaving only $\phi_s(0,\tau)$ and $\phi_a(0,\tau)$ as variables. To do so, we define the local fields

$$\phi_s^0(\tau) \coloneqq \phi_s(0,\tau) \tag{55}$$

$$\phi_a^0(\tau) \coloneqq \phi_a(0,\tau) \tag{56}$$

Using the local fields, physical quantities can be computed by integrating on both $\phi_{s(a)}$ and $\phi^0_{s(a)}$ by imposing the relations (55) and (56) with the use of auxiliary (real) fields $\lambda_s(\tau)$, $\lambda_a(\tau)$. As an example, the partition function can be written as

$$\begin{aligned} Z = \int \mathcal{D}\phi_a \mathcal{D}\phi_s \mathcal{D}\lambda_a \mathcal{D}\lambda_s \mathcal{D}\phi_a^0 \mathcal{D}\phi_s^0\ \exp\Big[& -S_a - i\int d\tau\ \lambda_a(\tau)\left(\phi_a^0(\tau) - \phi_a(0,\tau)\right) \\ & -S_s - i\int d\tau\ \lambda_s(\tau)\left(\phi_s^0(\tau) - \phi_s(0,\tau)\right)\Big] \end{aligned} \tag{57}$$

where S_s and S_a are the Luttinger action (19) for each field, using the appropriate Luttinger parameters, that is K_s and the one corresponding to the fixed point of the RG flow of the sine-Gordon Hamiltonian, K_a^*. To simplify the notation, we will drop the asterisk in the following discussions. Completing the squares in the exponential, we get for ϕ_s

$$\begin{aligned}
&S_s + i\int d\tau\ \lambda_s(\tau)\left(\phi_s^0(\tau) - \phi_s(0,\tau)\right) \\
&= \frac{1}{2\pi K_s u_s}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}(\omega_n^2 + u_s^2k^2)\phi_s^*(\mathbf{q})\phi_s(\mathbf{q}) + i\left[\frac{1}{\beta}\sum_{\omega_n}\lambda_s^*(\omega_n)\phi_s^0(\omega_n) - \frac{1}{\beta\Omega}\sum_{\mathbf{q}}\lambda_s^*(\omega_n)\phi_s(\mathbf{q})\right] \\
&= i\frac{1}{\beta}\sum_{\omega_n}\lambda_s^*(\omega_n)\phi_s^0(\omega_n) + \frac{1}{2}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}\frac{\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s^*(\omega_n)\lambda_s(\omega_n) \\
&+ \frac{1}{2\pi K_s u_s}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}(\omega_n^2 + u_s^2k^2)\left[\phi_s^*(\mathbf{q}) - \frac{i\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s^*(\omega_n)\right]\left[\phi_s(\mathbf{q}) - \frac{i\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s(\omega_n)\right]
\end{aligned} \tag{58}$$

To calculate the effective action, we make the trace over ϕ_a, which amounts to making a Gaussian integral

$$\begin{aligned}
&\int \mathcal{D}\phi_s\mathcal{D}\lambda_s\mathcal{D}\phi_s^0\ \exp\left\{-\left[i\frac{1}{\beta}\sum_{\omega_n}\lambda_s^*(\omega_n)\phi_s^0(\omega_n) + \frac{1}{2}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}\frac{\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s^*(\omega_n)\lambda_s(\omega_n)\right.\right. \\
&\left.\left. + \frac{1}{2\pi K_s u_s}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}(\omega_n^2 + u_s^2k^2)\left[\phi_s^*(\mathbf{q}) - \frac{i\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s^*(\omega_n)\right]\left[\phi_s(\mathbf{q}) - \frac{i\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s(\omega_n)\right]\right]\right\} \\
&= \mathcal{A}\int \mathcal{D}\lambda_s\mathcal{D}\phi_s^0\ \exp\left\{-\left[i\frac{1}{\beta}\sum_{\omega_n}\lambda_s^*(\omega_n)\phi_s^0(\omega_n) + \frac{1}{2}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}\frac{\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s^*(\omega_n)\lambda_s(\omega_n)\right]\right\}
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
\mathcal{A} = \int \mathcal{D}\phi_s\ \exp\Bigg\{&-\frac{1}{2\pi K_s u_s}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}(\omega_n^2 + u_s^2k^2)\left[\phi_s^*(\mathbf{q}) - \frac{i\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s^*(\omega_n)\right] \\
&\times\left[\phi_s(\mathbf{q}) - \frac{i\pi K_s u_s}{(\omega_n^2 + u_s^2k^2)}\lambda_s(\omega_n)\right]\Bigg\}
\end{aligned} \tag{60}$$

is a prefactor resulting from the Gaussian integral that simplifies with the same term in the partition function in the denominator when computing expectation values of operators that depend only on the local fields.

Furthermore, since λ_s only depends on the frequency ω_n and not on the wavenumber k, the sum over k in the remaining terms can be effectuated in the thermodynamic limit using (A.6)

$$\frac{1}{\Omega}\sum_k \frac{1}{\omega_n^2 + u_s^2k^2} \to \frac{1}{2\pi}\int dk\ \frac{1}{\omega_n^2 + u_s^2k^2} = \frac{1}{2\pi u_s\omega_n}\arctan\left(\frac{u_s k}{\omega_n}\right)\Bigg|_{-\infty}^{\infty} = \frac{1}{2u_s|\omega_n|} \tag{61}$$

which leaves us with the following expression as argument of the exponential

$$
\begin{aligned}
& i\frac{1}{\beta}\sum_{\omega_n}\lambda_s^*(\omega_n)\phi_s^0(\omega_n)+\frac{1}{2}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}\frac{\pi K_s u_s}{(\omega_n^2+u_s^2k^2)}\lambda_s^*(\omega_n)\lambda_s(\omega_n) \\
& \to i\frac{1}{\beta}\sum_{\omega_n}\lambda_s^*(\omega_n)\phi_s^0(\omega_n)+\frac{1}{2\beta}\sum_{\omega_n}\frac{\pi K_s}{2|\omega_n|}\lambda_s^*(\omega_n)\lambda_s(\omega_n) \\
& = \frac{\pi K_s}{4}\frac{1}{\beta}\sum_{\omega_n}\frac{1}{|\omega_n|}\left[\lambda_s^*(\omega_n)+i\frac{2|\omega_n|}{\pi K_s}\phi_s^{0*}(\omega_n)\right]\left[\lambda_s(\omega_n)+i\frac{2|\omega_n|}{\pi K_s}\phi_s^0(\omega_n)\right] \\
& \quad +\frac{1}{\beta}\sum_{\omega_n}\frac{|\omega_n|}{\pi K_s}\phi_s^{0*}(\omega_n)\phi_s^0(\omega_n)
\end{aligned}
\tag{62}
$$

Finally, we can trace over λ_s in the same way we did with ϕ_s to obtain an effective action for ϕ_s^0

$$
S_s^0=\frac{1}{\pi K_s}\frac{1}{\beta}\sum_{\omega_n}|\omega_n|\phi_s^{0*}(\omega_n)\phi_s^0(\omega_n) \tag{63}
$$

The same calculations are obviously valid for ϕ_a^0, giving the corresponding effective action S_a^0. In terms of the local fields, in the massless regime the action (52) becomes

$$
S=S_s^0+S_a^0+4\rho' V_0\int d\tau\ \cos(\sqrt{2}\phi_s^0(\tau))\cos(\sqrt{2}\phi_a^0(\tau)) \tag{64}
$$

In (64), there is no longer any spatial dependence: the problem has become (0+1)-dimensional. We are now in the position of effectuating an RG transformation to determine the relevance of the impurity potential, considering it to be a small perturbation. We will denote the impurity's contribution to the action as

$$
S_I\coloneqq 4\rho' V_0\int d\tau\ \cos(\sqrt{2}\phi_s^0(\tau))\cos(\sqrt{2}\phi_a^0(\tau)) \tag{65}
$$

In a perturbative expansion, the first order contribution of S_I vanishes due to infrared divergencies in the theory,

$$
\begin{aligned}
\langle e^{\pm i\sqrt{2}\phi_s^0(\tau)}\rangle_{0,s} &= \frac{1}{Z_s^0}\int\mathcal{D}\phi_s^0\ e^{\left[-\frac{1}{\pi K_s}\frac{1}{\beta}\sum_{\omega_n}|\omega_n|\phi_s^{0*}(\omega_n)\phi_s^0(\omega_n)\pm i\frac{\sqrt{2}}{\beta}\sum_{\omega_n}\phi_s^0(\omega_n)e^{-i\omega_n\tau}\right]} \\
&= \exp\left\{-\frac{\pi K_s}{2}\frac{1}{\beta}\sum_{\omega_n}\frac{1}{|\omega_n|}\right\} \\
&\to \exp\left\{-\frac{K_s}{4}\int_{-\Lambda}^{\Lambda}d\omega_n\ \frac{1}{|\omega_n|}\right\} \\
&= \exp\left\{-\frac{K_s}{2}\int_0^{\Lambda}d\omega_n\ \frac{1}{\omega_n}\right\} \\
&= 0
\end{aligned}
\tag{66}
$$

where $\langle\ldots\rangle_{0,s}$ denotes the average taken with the action S_s^0, and Z_s^0 the corresponding partition function. We will use the corresponding notation $\langle\ldots\rangle_{0,a}$ for the antisymmetric field.

The first finite contribution is of second order in V_0. To compute it, the correlation functions of exponentials of the fields are needed. Letting $\epsilon_1, \epsilon_2 = \pm 1$, we have

$$\begin{aligned}
\langle e^{i\epsilon_1\sqrt{2}\phi_s^0(\tau)} e^{i\epsilon_2\sqrt{2}\phi_s^0(0)}\rangle_{0,s} &= \frac{1}{Z_s^0}\int \mathcal{D}\phi_s^0 \; e^{\left[-\frac{1}{\pi K_s}\frac{1}{\beta}\sum_{\omega_n}|\omega_n|\phi_s^{0*}(\omega_n)\phi_s^0(\omega_n)+i\frac{\sqrt{2}}{\beta}\sum_{\omega_n}(\epsilon_1 e^{-i\omega_n\tau}+\epsilon_2)\phi_s^0(\omega_n)\right]} \\
&= \exp\left\{-\frac{\pi K_s}{2}\frac{1}{\beta}\sum_{\omega_n}\frac{1}{|\omega_n|}(\epsilon_1 e^{-i\omega_n\tau}+\epsilon_2)(\epsilon_1 e^{i\omega_n\tau}+\epsilon_2)\right\} \\
&\to \exp\left\{-\frac{K_s}{2}\int_{-\Lambda}^{\Lambda} d\omega_n \; \frac{1}{|\omega_n|}(1+\epsilon_1\epsilon_2\cos(\omega_n\tau))\right\} \\
&= \exp\left\{-K_s\int_0^{\Lambda} d\omega_n \; \frac{1}{\omega_n}(1+\epsilon_1\epsilon_2\cos(\omega_n\tau))\right\}
\end{aligned} \tag{67}$$

If we take $\epsilon_1 = \epsilon_2$, the last integral diverges to $+\infty$, and therefore the correlator goes to zero. If, on the contrary, $\epsilon_1 = -\epsilon_2$, the integral is the special function Cosine integral, denoted Cin,

$$\int_0^{\Lambda} d\omega_n \; \frac{1-\cos(\omega_n\tau)}{\omega_n} = \int_0^{\Lambda|\tau|} dt \; \frac{1-\cos(t)}{t} = \mathrm{Cin}(\Lambda|\tau|) \tag{68}$$

with $t \coloneqq \omega_n|\tau|$. Since we are interested in the asymptotic properties of the theory, we must consider the limiting behavior of our results for large times $|\tau| \gg 1/\Lambda$, for which $\mathrm{Cin}(\Lambda|\tau|) \simeq \gamma + \ln(\Lambda|\tau|)$, γ being the Euler-Mascheroni constant. Using (68), and the time translation invariance of the action, the correlators have the asymptotic behavior

$$\langle\cos(\sqrt{2}\phi_s^0(\tau))\cos(\sqrt{2}\phi_s^0(\tau'))\rangle_{0,s} \propto \left(\frac{1}{\Lambda|\tau-\tau'|}\right)^{K_s} \tag{69}$$

where the multiplicative constants have been disregarded, since they depend on the specific UV structure. The correlation functions decay as a power law with exponent K_s as a function of (imaginary) time. We can now calculate the expectation value of the second order contribution of S_I

$$\begin{aligned}
\langle S_I^2\rangle_0 &= (4\rho' V_0)^2 \int d\tau \; d\tau' \; \langle\cos(\sqrt{2}\phi_s^0(\tau))\cos(\sqrt{2}\phi_s^0(\tau'))\rangle_{0,s}\langle\cos(\sqrt{2}\phi_a^0(\tau))\cos(\sqrt{2}\phi_a^0(\tau'))\rangle_{0,a} \\
&\propto (4\rho' V_0)^2 \int d\tau \; d\tau' \; \left(\frac{1}{\Lambda|\tau-\tau'|}\right)^{K_s+K_a} \\
&\propto g_0^2 \int \frac{dy}{\alpha}\;\frac{dy'}{\alpha}\;\left(\frac{\alpha}{|y-y'|}\right)^{K_s+K_a}
\end{aligned} \tag{70}$$

Here, $\langle\ldots\rangle_0$ represents the average with the unperturbed action, y is defined by (A.1), and

$$g_0 \coloneqq \frac{4\alpha\rho' V_0}{u} \tag{71}$$

Changing infinitesimally the cutoff from $\alpha \coloneqq \alpha_0 e^l$ to $\alpha' \coloneqq \alpha_0 e^{l+dl}$ and imposing that $\langle S_I^2\rangle_0$ remains invariant under this transformation gives the following condition

$$g_0(\alpha') = g_0(\alpha)\left(\frac{\alpha}{\alpha'}\right)^{\frac{K_s+K_a}{2}-1} \tag{72}$$

which gives the renormalization equation of g_0

$$\frac{dg_0(l)}{dl} = \left(1 - \frac{K_s + K_a}{2}\right) g_0(l) \tag{73}$$

This equation can be directly derived by looking at the scaling dimension of the operator that couples to the impurity in (70), which behaves like $L^{-(K_s+K_a)}$. The integrals on y and y' contribute with L^2, and therefore $\langle S_I^2 \rangle_0$ behaves as $g_0^2 L^{2-(K_s+K_a)}$, giving back (73). The solution of this differential equation can be easily found

$$g_0(l) = g_0(l=0) e^{\left(1 - \frac{K_s+K_a}{2}\right) l} \tag{74}$$

and it shows a phase transition at $K_s + K_a = 2$, with g_0 being relevant for $K_s + K_a < 2$ and irrelevant for $K_s + K_a > 2$, as depicted in fig. 4.

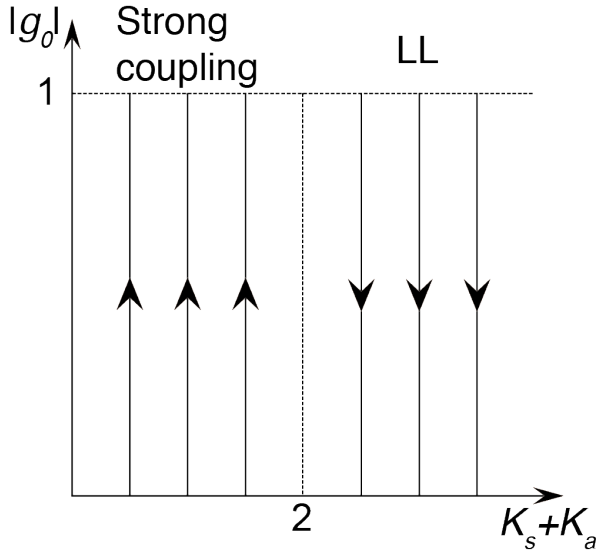

Figure 4: Renormalization of the impurity coupling constant g_0. The coupling is irrelevant for $K_s + K_a > 2$ and relevant for $K_s + K_a < 2$, with a phase transition at $K_s + K_a = 2$. In the relevant case, we stop the flow at the order $|g_0| \sim 1$.

The latter case is the simplest: both the interchain coupling and the impurity potential have been renormalized to zero, leaving just two decoupled chains described each by a Luttinger liquid. The former case is more complex, since the renormalization flow pushes the system towards strong coupling. Once again, this equation is only valid up to the order $|g_0(l^*)| \sim 1$. In this regime, the strong impurity is an obstacle in the propagation of the fields, and it has the effect of pinning them to the minimum of the operator $g_0 \cos(\sqrt{2}\phi_s^0(\tau)) \cos(\sqrt{2}\phi_a^0(\tau))$. In consequence, this phase is expected to be insulating. At the phase transition, the value of $K_s + K_a$ is universal, but each of the coefficients K_s and K_a can take any value accessible in the massless regime, as long as the sum of both gives 2. The correlation functions of each field will therefore decay as a power law with a non-universal exponent determined by one of these parameters, plus logarithmic corrections coming from the marginally irrelevant impurity coupling. This result agrees with the one found in [32] for extended disorder.

3.4 Massive Case with Attractive Interchain Coupling

We will now discuss the massive regime. This is the case of both regions b and c in fig. 2, so the initial conditions that lead to this regime are either $A^2 \coloneqq \widetilde{K}_a^2 - g^2 < 0$ (region c) or $A^2 > 0$ and $\widetilde{K}_a < 0$ (region b), with $g = 4\pi\alpha^2\rho'^2 V/u$. Let's consider first attractive interchain interactions, $V < 0$, which implies that $g < 0$.

Following the reasoning in section 2.4, the antisymmetric field, for which the free Hamiltonian is of the sine-Gordon type, orders at low energy taking a value of $\phi_a(x,\tau) \approx 0$, and presenting small harmonic oscillations around this value due to a quadratic term in the low-energy Hamiltonian. Given that this field represents the difference between the fields ϕ_1 and ϕ_2 of each chain, its ordering around zero means that the density fluctuations of both chains are in phase with each other, so particles of different chains tend to pair across the rungs as depicted in fig. 5.

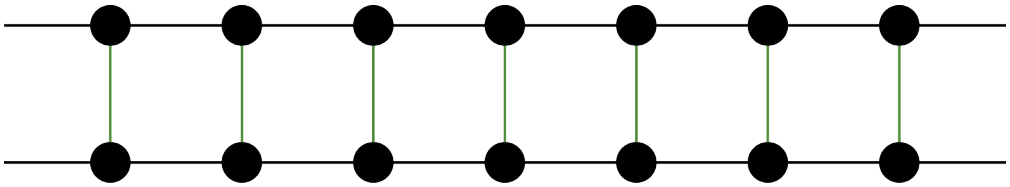

Figure 5: Configuration of the particles in the ladder for the ordering of the antisymmetric field at $\phi_a(x) = 0$. The bosons of both legs pair up across the rungs.

As in the preceding section, we want to analyze the effects of the local impurity, so we need to trace over the configurations of the fields away from the origin. The procedure to do so is essentially identical to the one in section 3.3, but now we take the massive action (31) for ϕ_a,

$$S^{\Delta} = \frac{1}{2\pi uK}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}[\omega_n^2 + u^2k^2 + \Delta^2]|\phi_a(k,\omega_n)|^2 \tag{75}$$

Adding the constraint $\phi_a^0(\tau) = \phi_a(0,\tau)$, we obtain

$$\begin{aligned}
&S^{\Delta} + i\int d\tau\ \lambda_a(\tau)\left(\phi_a^0(\tau) - \phi_a(0,\tau)\right)\\
&= \frac{1}{2\pi K_a u_a}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}(\omega_n^2 + u_a^2k^2 + \Delta^2)\phi_a^*(\mathbf{q})\phi_a(\mathbf{q})\\
&+ i\left[\frac{1}{\beta}\sum_{\omega_n}\lambda_a^*(\omega_n)\phi_a^0(\omega_n) - \frac{1}{\beta\Omega}\sum_{\mathbf{q}}\lambda_a^*(\omega_n)\phi_a(\mathbf{q})\right]\\
&= i\frac{1}{\beta}\sum_{\omega_n}\lambda_a^*(\omega_n)\phi_a^0(\omega_n) + \frac{1}{2}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}\frac{\pi K_a u_a}{(\omega_n^2 + u_a^2k^2 + \Delta^2)}\lambda_a^*(\omega_n)\lambda_a(\omega_n)\\
&+ \frac{1}{2\pi K_a u_a}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}(\omega_n^2 + u_a^2k^2 + \Delta^2)\left[\phi_a^*(\mathbf{q}) - \frac{i\pi K_a u_a}{(\omega_n^2 + u_a^2k^2 + \Delta^2)}\lambda_a^*(\omega_n)\right]\\
&\times\left[\phi_a(\mathbf{q}) - \frac{i\pi K_a u_a}{(\omega_n^2 + u_a^2k^2 + \Delta^2)}\lambda_a(\omega_n)\right]
\end{aligned} \tag{76}$$

Taking the trace over ϕ_a allows us to get rid of the last term of the sum, as usual. Now we can easily make the sum over the momenta k

$$\begin{aligned}\frac{1}{\Omega}\sum_k \frac{1}{\omega_n^2+u_a^2k^2+\Delta^2} &\to \frac{1}{2\pi}\int dk\ \frac{1}{\omega_n^2+u_a^2k^2+\Delta^2}\\ &= \frac{1}{2\pi u_a\sqrt{\omega_n^2+\Delta^2}}\arctan\left(\frac{u_a k}{\sqrt{\omega_n^2+\Delta^2}}\right)\Bigg|_{-\infty}^{\infty}\\ &= \frac{1}{2u_a\sqrt{\omega_n^2+\Delta^2}}\end{aligned} \tag{77}$$

The terms that are left are thus

$$\begin{aligned}&i\frac{1}{\beta}\sum_{\omega_n}\lambda_a^*(\omega_n)\phi_a^0(\omega_n)+\frac{1}{2}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}\frac{\pi K_a u_a}{(\omega_n^2+u_a^2k^2+\Delta^2)}\lambda_a^*(\omega_n)\lambda_a(\omega_n)\\ &\to i\frac{1}{\beta}\sum_{\omega_n}\lambda_a^*(\omega_n)\phi_a^0(\omega_n)+\frac{1}{2\beta}\sum_{\omega_n}\frac{\pi K_a}{2\sqrt{\omega_n^2+\Delta^2}}\lambda_a^*(\omega_n)\lambda_a(\omega_n)\\ &= \frac{\pi K_a}{4}\frac{1}{\beta}\sum_{\omega_n}\frac{1}{\sqrt{\omega_n^2+\Delta^2}}\left[\lambda_a^*(\omega_n)+i\frac{2\sqrt{\omega_n^2+\Delta^2}}{\pi K_a}\phi_a^{0*}(\omega_n)\right]\left[\lambda_a(\omega_n)+i\frac{2\sqrt{\omega_n^2+\Delta^2}}{\pi K_a}\phi_a^{0}(\omega_n)\right]\\ &\quad+\frac{1}{\pi K_a}\frac{1}{\beta}\sum_{\omega_n}\sqrt{\omega_n^2+\Delta^2}\phi_a^{0*}(\omega_n)\phi_a^0(\omega_n)\end{aligned} \tag{78}$$

All we have left to do to find the effective action is to trace over the auxiliary field λ_a, which gives

$$S_a^\Delta = \frac{1}{\pi K_a}\frac{1}{\beta}\sum_{\omega_n}\sqrt{\omega_n^2+\Delta^2}\phi_a^{0*}(\omega_n)\phi_a^0(\omega_n) \tag{79}$$

Comparing the integral (77) with (61) we can notice that the mass term Δ prevents the infrared divergencies that are otherwise present in the massless theory. As a consequence, some expectation values that vanished in the massless regime take now finite values, thanks to the ordering of the field. An example of such an expectation value is the one corresponding to the exponentials of the field, which we can explicitly calculate using the action (79). Noting by $\langle\ldots\rangle_\Delta$ the expectation values and Z_a^Δ the partition function computed with S_a^Δ, we have

$$\begin{aligned}
\left\langle e^{\pm i\sqrt{2}\phi_a^0(\tau)}\right\rangle_\Delta &= \frac{1}{Z_a^\Delta}\int \mathcal{D}\phi_a^0\; e^{\left[-\frac{1}{\pi K_a}\frac{1}{\beta}\sum_{\omega_n}\sqrt{\omega_n^2+\Delta^2}\phi_a^{0*}(\omega_n)\phi_a^0(\omega_n)\pm i\frac{\sqrt{2}}{\beta}\sum_{\omega_n}\phi_a^0(\omega_n)e^{-i\omega_n\tau}\right]} \\
&= \exp\left\{-\frac{\pi K_a}{2}\frac{1}{\beta}\sum_{\omega_n}\frac{1}{\sqrt{\omega_n^2+\Delta^2}}\right\} \\
&\to \exp\left\{-\frac{K_a}{4}\int_{-\Lambda}^{\Lambda}d\omega_n\;\frac{1}{\sqrt{\omega_n^2+\Delta^2}}\right\} \\
&= \exp\left\{-\frac{K_a}{2}\int_{0}^{\Lambda}d\omega_n\;\frac{1}{\sqrt{\omega_n^2+\Delta^2}}\right\} \\
&= \left[\frac{\Lambda}{\Delta}+\sqrt{\left(\frac{\Lambda}{\Delta}\right)^2+1}\right]^{-\frac{K_a}{2}}
\end{aligned} \tag{80}$$

Considering $\Lambda \gg \Delta$, we get

$$\left\langle e^{\pm i\sqrt{2}\phi_a^0(\tau)}\right\rangle_\Delta \propto \left(\frac{\Delta}{\Lambda}\right)^{\frac{K_a}{2}} \tag{81}$$

and therefore

$$\left\langle \cos\left(\sqrt{2}\phi_a^0(\tau)\right)\right\rangle_\Delta \propto \left(\frac{\Delta}{\Lambda}\right)^{\frac{K_a}{2}} \tag{82}$$

As for the correlation functions we have, with $\epsilon_1, \epsilon_2 = \pm 1$,

$$\begin{aligned}
\left\langle e^{i\epsilon_1\sqrt{2}\phi_a^0(\tau)}e^{i\epsilon_2\sqrt{2}\phi_a^0(0)}\right\rangle_\Delta &= \exp\left\{-\frac{\pi K_a}{2}\frac{1}{\beta}\sum_{\omega_n}\frac{2+2\epsilon_1\epsilon_2\cos(\omega_n\tau)}{\sqrt{\omega_n^2+\Delta^2}}\right\} \\
&\to \exp\left\{-\frac{K_a}{2}\int_{-\Lambda}^{\Lambda}d\omega_n\;\frac{1+\epsilon_1\epsilon_2\cos(\omega_n\tau)}{\sqrt{\omega_n^2+\Delta^2}}\right\} \\
&= \exp\left\{-K_a\int_{0}^{\Lambda}d\omega_n\;\frac{1}{\sqrt{\omega_n^2+\Delta^2}}\right\}\exp\left\{-K_a\int_{0}^{\Lambda}d\omega_n\;\frac{\epsilon_1\epsilon_2\cos(\omega_n\tau)}{\sqrt{\omega_n^2+\Delta^2}}\right\} \\
&= \left\langle e^{i\sqrt{2}\phi_a^0(\tau)}\right\rangle_\Delta^2\exp\left\{-\epsilon_1\epsilon_2 K_a\int_{0}^{\Lambda}d\omega_n\;\frac{\cos(\omega_n\tau)}{\sqrt{\omega_n^2+\Delta^2}}\right\}
\end{aligned} \tag{83}$$

We recognize the square of the expectation value (81), multiplied by an exponential factor. The argument of this exponential includes an integral for which finding an analytic expression for an arbitrary Λ is a difficult task. The problem comes from the brutally sharp cutoff procedure that we have used so far. The appropriate cutoff procedure is, for us, unknown, since it depends on the UV structure of each microscopic model. We took the freedom of assuming a sharp cutoff using the argument that we are only interested in the asymptotic, low-energy properties of the system, which are independent of its microscopic details, and in particular of the cutoff procedure. Evoking again the same argument, we will change this procedure for the calculation of the integral, choosing to integrate over all the frequencies, but multipliyng the argument by an exponential decay with characteristic energy Λ as follows

$$\int_0^\Lambda d\omega_n \, \frac{\cos(\omega_n \tau)}{\sqrt{\omega_n^2 + \Delta^2}} \to \int_0^\infty d\omega_n \, \frac{\cos(\omega_n \tau) e^{-\omega_n/\Lambda}}{\sqrt{\omega_n^2 + \Delta^2}} \tag{84}$$

A detailed calculation of this integral can be found in appendix A.2. From (83), and using (A.12), the correlator of the cosine can be computed

$$\begin{aligned} \left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \cos(\sqrt{2}\phi_a^0(0)) \right\rangle_\Delta &= \left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \right\rangle_\Delta^2 \times \frac{1}{2} \left(e^{K_a F(\tau,\Delta,\Lambda)} + e^{-K_a F(\tau,\Delta,\Lambda)} \right) \\ &= \left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \right\rangle_\Delta^2 \cosh\left(K_a F(\tau,\Delta,\Lambda)\right) \end{aligned} \tag{85}$$

This expression is only valid in the large τ regime due to the low-energy approximations. For $\tau \gg 1/\Delta \gg 1/\Lambda$, the function $F(\tau, \Delta, \Lambda)$ goes to zero. Therefore, the correlator $\left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \cos(\sqrt{2}\phi_a^0(0)) \right\rangle_\Delta$ saturates at a value of $\left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \right\rangle_\Delta^2$ after a time $\tau \sim 1/\Delta$. At large times the correlator is thus constant, instead of showing a power law decay like in the massless case. In fig. 6, we show the ratio

$$G(\tau) := \frac{\left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \cos(\sqrt{2}\phi_a^0(0)) \right\rangle_\Delta}{\left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \right\rangle_\Delta^2} = \cosh\left(K_a F(\tau,\Delta,\Lambda)\right) \tag{86}$$

as a function of τ. We can verify the saturation of this ratio at the value of 1 at large times.

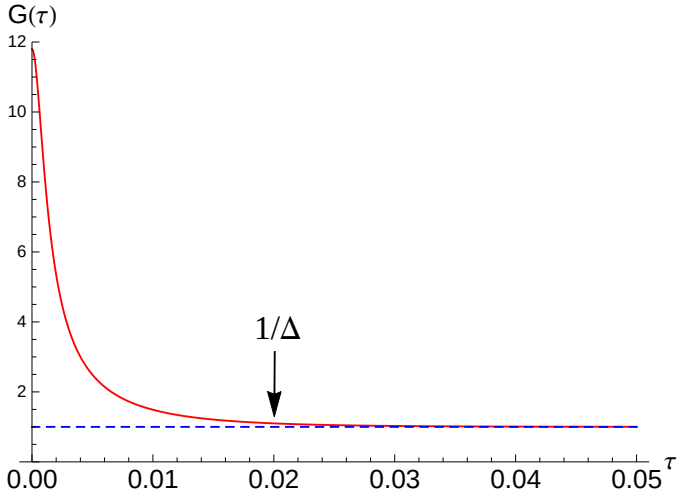

Figure 6: Ratio between $\left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \cos(\sqrt{2}\phi_a^0(0)) \right\rangle_\Delta$ and $\left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \right\rangle_\Delta^2$, denoted $G(\tau)$, as a function of imaginary time τ. In this particular example, we have taken the values $K_a = 1$, $\Delta = 50$, and $\Lambda = 1000$, with the energies expressed in inverse time units. At small τ, G decays until reaching a value of 1 (blue dotted line) at around $\tau = 1/\Delta = 0.02$, indicated in the figure by an arrow. For larger times, G saturates at this value.

We have so far analyzed the dynamics of the antisymmetric field in this regime. In conclusion, the field is essentially blocked in a minimum of the potential. Its quantum fluctuations will therefore play a much less important role than before. Some correlators now go to a constant instead of decaying with time. Analyzing the steps followed to derive the renormalization equation (73) in the massless regime, we see that the Luttinger liquid parameters define the flow of the impurity potential because they show up as the power in the correlators.

We can thus expect that in the massive regime, K_a will be absent from the corresponding renormalization equation, as we will show next.

The dynamics of the symmetric field remain the same as in the massless regime, described by the action S_s^0. The unperturbed action in this regime is therefore

$$S_0^\Delta = S_s^0 + S_a^\Delta \tag{87}$$

Since the symmetric field action presents IR divergencies, the expectation value of the impurity action remains zero,

$$\langle S_I \rangle_{S_0^\Delta} = 4\rho' V_0 \int d\tau \ \left\langle \cos(\sqrt{2}\phi_s^0(\tau)) \right\rangle_{0,s} \left\langle \cos(\sqrt{2}\phi_a^0(\tau)) \right\rangle_\Delta = 0 \tag{88}$$

The lowest non-zero contribution is at second order, for which

$$\begin{aligned}
\left\langle S_I^2 \right\rangle_{S_0^\Delta} &= (4\rho' V_0)^2 \int d\tau \ d\tau' \ \left\langle \cos(\sqrt{2}\phi_s^0(\tau)) \cos(\sqrt{2}\phi_s^0(\tau')) \right\rangle_{0,s} \left\langle \cos(\sqrt{2}\phi_a^0(\tau) \cos(\sqrt{2}\phi_a^0(\tau')) \right\rangle_\Delta \\
&\propto (4\rho' V_0)^2 \int d\tau \ d\tau' \left(\frac{1}{\Lambda|\tau - \tau'|} \right)^{K_s} \left(\frac{\Delta}{\Lambda} \right)^{K_a} \\
&\propto \left(g_0' \right)^2 \int \frac{dy}{\alpha} \frac{dy'}{\alpha} \left(\frac{\alpha}{|y - y'|} \right)^{K_s}
\end{aligned} \tag{89}$$

with

$$\begin{aligned}
g_0' \coloneqq & \frac{4\alpha\rho' V_0}{u} \left(\frac{\Delta}{\Lambda} \right)^{\frac{K_a}{2}} \\
= & g_0 \left(\frac{\Delta}{\Lambda} \right)^{\frac{K_a}{2}}
\end{aligned} \tag{90}$$

From the dimension of the operator in (89), which behaves like L^{2-K_s}, we can deduce the renormalization equation

$$\frac{dg_0'(l)}{dl} = \left(1 - \frac{K_s}{2} \right) g_0'(l) \tag{91}$$

The flow defined by this equation can be seen in fig. 7. It is similar to the one found in the massless case, but instead of depending on $K_s + K_a$, now it only depends on K_s, as expected. K_a does not determine in any way the transition point. However, it does change the value of the coupling constant by a multiplicative factor (a power of the ratio between the gap and the cutoff) as can be seen in (90).

The phase transition takes place at $K_s = 2$. For $K_s > 2$, the impurity is irrelevant, and the problem becomes that of a clean ladder, consisting of two decoupled fields (ϕ_s and ϕ_a), one described by a Luttinger liquid and the other by a sine-Gordon Hamiltonian blocked around the minimum of the cosine term. This is a Luther-Emery liquid. For $K_s < 2$, the impurity goes to strong coupling, so we can expect an insulating behavior, with ϕ_s^0 pinned at the minimum of the term $g_0' \cos(\sqrt{2}\phi_s^0)$ (notice that ϕ_a^0 has already a fixed value due to the sine-Gordon term). At the transition, the situation is now very different from the one in the massless case, since the Luttinger liquid parameter K_s is now fixed, and therefore the correlation functions of ϕ_s decay with a universal critical exponent $K_s = 2$. Qualitatively, a universal exponent is the result obtained using bosonization techniques for a single chain.

Since the antisymmetric mode of the ladder is essentially blocked, the ladder has effectively only one remaining free mode, so it can be expected that its behavior resembles that of a single chain in a first approximation.

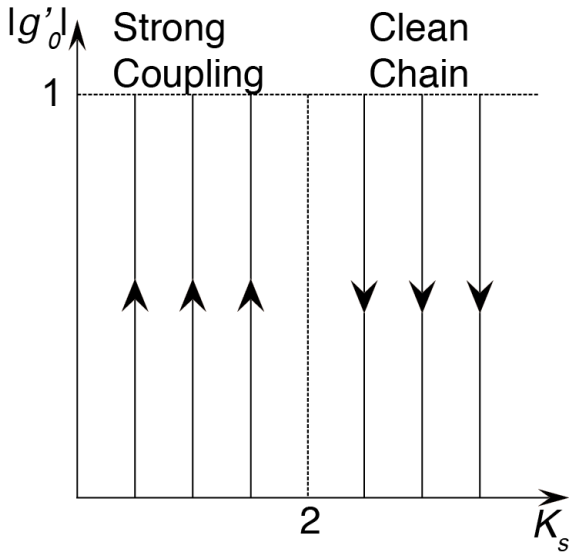

Figure 7: Renormalization flow of the impurity potential in the massive regime, for attractive interchain interactions. A phase transition takes place at $K_s = 2$, separating the strong-coupling regime from the clean chain regime.

3.5 Massive Case with Repulsive Interchain Coupling

The third and last case to analyze is the massive regime with repulsive interchain coupling. The initial conditions leading to this regime are either $A^2 \coloneqq \widetilde{K}_a^2 - g^2 < 0$ (region c) or $A^2 > 0$ and $\widetilde{K}_a < 0$ (region b), while $g > 0$ for repulsive coupling.

Since $g > 0$, the corresponding minimum of the cosine term in the sine-Gordon Hamiltonian is no longer at zero but at $\phi_a(x) = \pi/(2\sqrt{2})$, meaning that the particles in one leg of the ladder will be dephased with respect to the other leg, as shown in fig. 8.

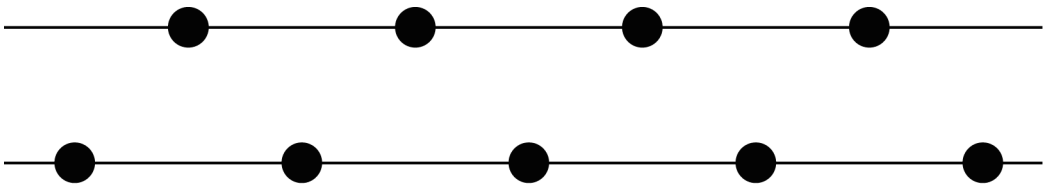

Figure 8: Configuration of the particles in the ladder when the antisymmetric field orders around $\phi_a(x) = \pi/(2\sqrt{2})$. The bosons in one leg are out of phase with respect to those in the other leg.

The expansion of the cosine around $\pi/(2\sqrt{2})$ in the Hamiltonian (24) gives

$$H_{SG} = H_{LL} + \frac{2gu}{\pi\alpha^2} \int dx \ \left(\phi_a(x) - \frac{\pi}{2\sqrt{2}} \right)^2 \tag{92}$$

With a redefinition of the field, $\widetilde{\phi}_a(x) \coloneqq \phi_a(x) - \frac{\pi}{2\sqrt{2}}$, which verifies $\nabla\widetilde{\phi}_a(x) = \nabla\phi_a(x)$, we find again for $\widetilde{\phi}_a(x)$, and without the impurity, the Hamiltonian (30), and therefore the action (31). We define once again the local fields $\phi_s^0(\tau) \coloneqq \phi_s(0,\tau)$ and $\phi_a^0(\tau) \coloneqq \widetilde{\phi}_a(0,\tau)$ (notice that we disregard the tilde in the local antisymmetric field to keep notation as simple as possible).

Tracing over the fields away from the impurity yields the exact same result as in section 3.4. So far, the problem is essentially the same as for the attractive coupling. The difference comes when we introduce the impurity. Indeed, in terms of $\widetilde{\phi}_a(x)$, this perturbation is

$$\begin{aligned} H_I &= 4\rho' V_0 \cos(\sqrt{2}\phi_a(0)) \cos(\sqrt{2}\phi_s(0)) \\ &= 4\rho' V_0 \cos\left(\sqrt{2}\widetilde{\phi}_a(0) + \frac{\pi}{2}\right) \cos(\sqrt{2}\phi_s(0)) \\ &= -4\rho' V_0 \sin(\sqrt{2}\widetilde{\phi}_a(0)) \cos(\sqrt{2}\phi_s(0)) \\ &= -4\rho' V_0 \sin(\sqrt{2}\phi_a^0) \cos(\sqrt{2}\phi_s^0) \end{aligned} \tag{93}$$

and the full action in this regime is therefore

$$S = S_0^\Delta + S_I \tag{94}$$

with

$$S_I = -4\rho' V_0 \int d\tau \ \sin(\sqrt{2}\phi_a^0(\tau)) \cos(\sqrt{2}\phi_s^0(\tau)) \tag{95}$$

The impurity in this case involves the sine of the antisymmetric field instead of the cosine. The expectation value of the sine is obviously zero from eq. (81). Its correlation functions are *asymptotically* zero as well, despite the absence of infra-red divergencies, as we can show using eq. (83)

$$\begin{aligned} \left\langle \sin(\sqrt{2}\phi_a^0(\tau)) \sin(\sqrt{2}\phi_a^0(0)) \right\rangle_\Delta &= \left\langle e^{i\sqrt{2}\phi_a^0(\tau)} \right\rangle_\Delta^2 \times \frac{1}{2}\left(e^{K_a F(\tau,\Delta,\Lambda)} - e^{-K_a F(\tau,\Delta,\Lambda)}\right) \\ &= \left\langle e^{i\sqrt{2}\phi_a^0(\tau)} \right\rangle_\Delta^2 \sinh(K_a F(\tau,\Delta,\Lambda)) \end{aligned} \tag{96}$$

Since $F(\tau,\Delta,\Lambda)$ goes asymptotically to zero, so does this correlation function. We could naively think that the impurity is thus necessarily irrelevant. Actually, the impurity is not always irrelevant, as we will show next, but indeed its effects are less important than in the attractive coupling case: the correlation functions will prove to decay faster with time in this regime. This can be understood by comparing the figs. 5 and 8: in fig. 5, a correlated localized impurity must interact simultaneously with particles on both legs, since they form bounded pairs across the rungs of the ladder, whereas in fig. 8 the impurity interacts with the particles of one leg at a time, due to the dephasing between the legs. The pinning due to the impurity is therefore more efficient in the first case (attractive interactions), giving rise to more slowly decaying correlation functions, which means that in average, the fields are pinned for longer times.

The reason the impurity is not necessarily irrelevant comes from the *short* time correlation functions. For times shorter than the cutoff, $\tau \ll \alpha/u$, the actual correlation functions cannot be computed with our low-energy approximation, in the same way as at small distances we cannot see the microscopic behavior of a system within such an approximation. In fact, we assume that the properties of the system don't change significantly at small distances: e.g. in the long-wavelength approximation, density variations are supposed to take place over distances much larger than the cutoff, and the density can be taken to be constant at small distances. In the same spirit, we consider the properties of the system to remain constant at short timescales. In some sense, it is as if within a low-energy approximation we don't have enough time "resolution" to distinguish events taking place in a timescale shorter than α/u.

With this in mind, we can compute an effective action for the symmetric field. Consider for instance the full partition function Z, computed by a perturbative expansion of the impurity action, with $Z_0^\Delta = Z_s^0 Z_a^\Delta$ the unperturbed partition function:

$$\begin{aligned} Z &= \int \mathcal{D}\phi_s^0 \mathcal{D}\phi_a^0 \; e^{-(S_0^\Delta + S_I)} \\ &\approx \int \mathcal{D}\phi_s^0 \mathcal{D}\phi_a^0 \; e^{-S_0^\Delta} \left(1 - S_I + \frac{1}{2} S_I^2\right) \\ &\approx Z_0^\Delta \left[1 - \frac{1}{Z_0^\Delta} \int \mathcal{D}\phi_s^0 \mathcal{D}\phi_a^0 \; e^{-S_0^\Delta} S_I + \frac{1}{2Z_0^\Delta} \int \mathcal{D}\phi_s^0 \mathcal{D}\phi_a^0 \; e^{-S_0^\Delta} S_I^2\right] \end{aligned} \tag{97}$$

After integrating over ϕ_a^0, we get:

$$\begin{aligned} Z &\approx Z_0^\Delta \left[1 - \frac{1}{Z_s^0} \int \mathcal{D}\phi_s^0 \; e^{-S_s^0} \langle S_I \rangle_\Delta + \frac{1}{2Z_s^0} \int \mathcal{D}\phi_s^0 \; e^{-S_s^0} \left\langle S_I^2 \right\rangle_\Delta\right] \\ &\approx Z_0^\Delta \left[1 + \frac{1}{2Z_s^0} \int \mathcal{D}\phi_s^0 \; e^{-S_s^0} \left\langle S_I^2 \right\rangle_\Delta\right] \end{aligned} \tag{98}$$

The linear term is zero since it includes the expectation value of the sine term. Let's focus on the quadratic term, considering the short time correlations as being constant for $|\tau - \tau'| \ll \alpha/u$ as discussed above

$$\begin{aligned} \left\langle S_I^2 \right\rangle_\Delta &= (4\rho' V_0)^2 \int d\tau d\tau' \; \left\langle \sin(\sqrt{2}\phi_a^0(\tau)) \sin(\sqrt{2}\phi_a^0(\tau')) \right\rangle_\Delta \cos(\sqrt{2}\phi_s^0(\tau)) \cos(\sqrt{2}\phi_s^0(\tau')) \\ &\approx (4\rho' V_0)^2 \left[\int d\tau \int_{|\tau'-\tau|<\alpha/u} d\tau' \; \left\langle \sin^2(\sqrt{2}\phi_a^0(\tau)) \right\rangle_\Delta \cos^2(\sqrt{2}\phi_s^0(\tau)) \right. \\ &\left. + \int d\tau \int_{|\tau'-\tau|>\alpha/u} d\tau' \; \left\langle \sin(\sqrt{2}\phi_a^0(\tau)) \sin(\sqrt{2}\phi_a^0(\tau')) \right\rangle_\Delta \cos(\sqrt{2}\phi_s^0(\tau)) \cos(\sqrt{2}\phi_s^0(\tau')) \right] \\ &\approx (4\rho' V_0)^2 \int d\tau \; \frac{2\alpha}{u} \left\langle \frac{1}{2} - \frac{1}{2}\cos(2\sqrt{2}\phi_a^0(\tau)) \right\rangle_\Delta \left[\frac{1}{2} + \frac{1}{2}\cos(2\sqrt{2}\phi_s^0(\tau))\right] \\ &\approx \frac{(4\rho' V_0)^2 \alpha}{2u} \left[1 - \left(\frac{\Delta}{\Lambda}\right)^{2K_a}\right] \int d\tau \; \left[1 + \cos(2\sqrt{2}\phi_s^0(\tau))\right] \end{aligned} \tag{99}$$

The integral with $|\tau' - \tau| > \alpha/u$ vanishes since it has as argument the long time correlator of the sine operator, which goes asymptotically to zero. Putting this result back in the partition function (neglecting the constant term), we get, at the lowest order in V_0,

$$
\begin{aligned}
Z &\approx Z_0^{\Delta}\left\{1+\frac{1}{2Z_s^0}\int \mathcal{D}\phi_s^0\ (e^{-S_s^0})\frac{(4\rho' V_0)^2\alpha}{2u}\left[1-\left(\frac{\Delta}{\Lambda}\right)^{2K_a}\right]\int d\tau\ \cos(2\sqrt{2}\phi_s^0(\tau))\right\}\\
&\approx Z_a^{\Delta}\left\{Z_s^0+\frac{1}{2}\int \mathcal{D}\phi_s^0\ (e^{-S_s^0})\frac{(4\rho' V_0)^2\alpha}{2u}\left[1-\left(\frac{\Delta}{\Lambda}\right)^{2K_a}\right]\int d\tau\ \cos(2\sqrt{2}\phi_s^0(\tau))\right\}\\
&\approx Z_a^{\Delta}\int \mathcal{D}\phi_s^0\ e^{-S_s^0}\left\{1+\frac{(4\rho' V_0)^2\alpha}{2u}\left[1-\left(\frac{\Delta}{\Lambda}\right)^{2K_a}\right]\int d\tau\ \cos(2\sqrt{2}\phi_s^0(\tau))\right\}\\
&\approx Z_a^{\Delta}\int \mathcal{D}\phi_s^0\ \exp\left\{-\left[S_s^0-\frac{(4\rho' V_0)^2\alpha}{2u}\left(1-\left(\frac{\Delta}{\Lambda}\right)^{2K_a}\right)\int d\tau\ \cos(2\sqrt{2}\phi_s^0(\tau))\right]\right\}
\end{aligned}
\tag{100}
$$

From the calculations above we see that the impurity enters the effective action at second order thanks to the expansion in powers of the impurity action, even if the contribution from the first order is strictly zero. This expansion is known as "operator product expansion" [41]. As anticipated, the impurity is weaker than for attractive interactions. We can write the effective impurity action in this regime as

$$
\widetilde{S}_I = -g_0''\frac{u}{\alpha}\int d\tau\ \cos(2\sqrt{2}\phi_s^0(\tau)) \tag{101}
$$

with

$$
\begin{aligned}
g_0'' &\coloneqq \frac{1}{2}\left[1-\left(\frac{\Delta}{\Lambda}\right)^{2K_a}\right]\left(\frac{4\alpha\rho' V_0}{u}\right)^2\\
&= g_0^2\left\langle \sin^2(\sqrt{2}\phi_a^0(\tau))\right\rangle_{\Delta}
\end{aligned}
\tag{102}
$$

In this regime, the role of the impurity will be to try to pin ϕ_s^0 to a minimum of the operator $\widetilde{S}_I$. Therefore, the correlation function that describes the properties of the system is

$$
\left\langle \cos(2\sqrt{2}\phi_s^0(\tau))\cos(2\sqrt{2}\phi_s^0(0))\right\rangle_{0,s} \propto \left(\frac{1}{\Lambda|\tau|}\right)^{4K_s} \tag{103}
$$

where we have used (20). This correlator decays with an exponent $4K_s$. Comparing with (69), which is the appropriate correlator for attractive interchain coupling and decays with an exponent K_s, the decay is now much faster in time, as expected. This confirms that the impurity pins the field more efficiently when the particles create pairs across the rungs.

Now we are in position to try an RG treatment to determine the relevance of the impurity. The expectation value $\langle \widetilde{S}_I\rangle_{0,s} = 0$ is still zero (since the symmetric field is massless), so we need the expectation value in the square of the action. Notice that this term, quadratic in g_0'', is quartic in V_0.

$$
\begin{aligned}
\left\langle \widetilde{S}_I^2\right\rangle_{0,s} &= \left(g_0''\frac{u}{\alpha}\right)^2\int d\tau d\tau'\ \left\langle \cos(2\sqrt{2}\phi_s^0(\tau))\cos(2\sqrt{2}\phi_s^0(\tau'))\right\rangle_{s,0}\\
&\propto \left(g_0''\frac{u}{\alpha}\right)^2\int d\tau d\tau'\ \left(\frac{1}{\Lambda|\tau-\tau'|}\right)^{4K_s}\\
&\propto (g_0'')^2\int \frac{dy}{\alpha}\frac{dy'}{\alpha}\ \left(\frac{\alpha}{|y-y'|}\right)^{4K_s}
\end{aligned}
\tag{104}
$$

This operator has the dimension L^{2-4K_s}, which gives us the renormalization equation for the coupling constant

$$\frac{dg_0''(l)}{dl} = (1 - 2K_s)\, g_0''(l) \tag{105}$$

The phase transition takes place at $K_s = 1/2$, with the impurity being irrelevant for $K_s > 1/2$ and relevant for $K_s < 1/2$. The flow is depicted in fig. 9. Once again, the correlation functions at the transition decay with a universal exponent, this time given by $K_s = 1/2$. The strong coupling regime is expected to be insulating, while the regime with irrelevant impurity is just a Luther-Emery liquid.

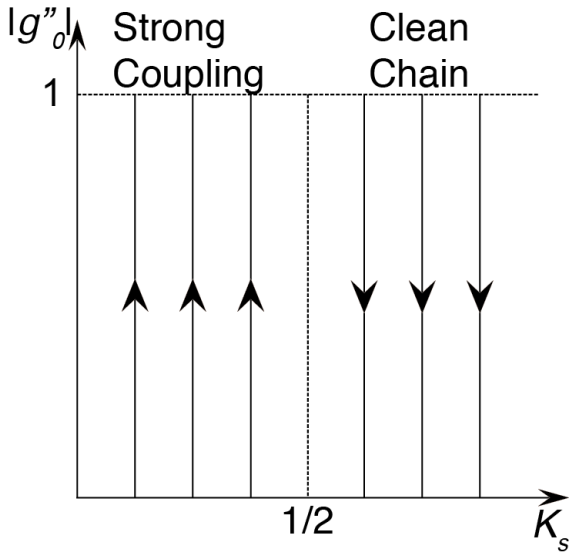

Figure 9: Flow of the impurity potential in the massive case with repulsive interchain coupling. The impurity is relevant for $K_s < 1/2$, and the system goes to strong coupling. For $K_s > 1/2$, the impurity is irrelevant, and the problem becomes that of a clean chain.

4 Discussions

Initial Conditions	**Interchain Coupling V**	**Impurity Potential V_0**	**Phase**
• $\widetilde{K}_a^2 > g^2$, $\widetilde{K}_a > 0$ • $K_s + K_a^* > 2$	Irrelevant	Irrelevant	Two decoupled Luttinger liquids
• $\widetilde{K}_a^2 > g^2$, $\widetilde{K}_a > 0$ • $K_s + K_a^* < 2$	Irrelevant	Relevant	Two Luttinger liquids coupled at one point through the impurity
• $\widetilde{K}_a^2 < g^2$ or both $\widetilde{K}_a^2 > g^2$ and $\widetilde{K}_a < 0$ • $V < 0$ • $K_s > 2$	Relevant	Irrelevant	Luther-Emery liquid
• $\widetilde{K}_a^2 < g^2$ or both $\widetilde{K}_a^2 > g^2$ and $\widetilde{K}_a < 0$ • $V < 0$ • $K_s < 2$	Relevant	Relevant	Luther-Emery liquid with impurity
• $\widetilde{K}_a^2 < g^2$ or both $\widetilde{K}_a^2 > g^2$ and $\widetilde{K}_a < 0$ • $V > 0$ • $K_s > 1/2$	Relevant	Irrelevant	Luther-Emery liquid
• $\widetilde{K}_a^2 < g^2$ or both $\widetilde{K}_a^2 > g^2$ and $\widetilde{K}_a < 0$ • $V > 0$ • $K_s < 1/2$	Relevant	Relevant through operator product expansion	Luther-Emery liquid with impurity at second order

Table 1: Phase diagram of the ladder. Here is indicated whether the interchain coupling and the impurity are relevant or not, as well as the conditions leading to each regime. $\widetilde{K}_a$ measures the deviation of the LL parameter K_a of the antisymmetric modes from the value $K_a = 1$, and g is the interchain coupling constant, proportional to the potential V. The values of $\widetilde{K}_a$ and g shown in the table are the ones before their renormalization, while K_a^* is the renormalized LL parameter. K_s is the LL parameter of the symmetric modes, which does not get renormalized by either interaction.

To summarise, in table 1 we show the obtained phase diagram. In the first column the initial conditions leading to each phase are listed. We have reintroduced the asterisk notation for the renormalized parameter K_a^*. The quantities $\widetilde{K}_a$ and g are to be calculated with the initial (not renormalized) parameters. In the second and third columns we indicate whether the interchain coupling and the impurity potential are relevant or not. Finally, some physical properties of each phase are listed in the fourth column.

Let's analyze now some of the physical implications of our results. First of all, let's focus on the physics of the ladder. The regimes where the impurity is irrelevant are described by the LL theory. When the impurity is relevant, the perturbative treatment becomes insufficient since the renormalization flow takes the coupling to orders of magnitude comparable to other energy scales of the system. In that case, the fields become strongly pinned at the impurity site. We can thus view this pinning as a boundary condition, and say that the role of the impurity was to cut the ladder into two semi-infinite parts. Given that, however strong, the impurity potential is nonetheless not infinite, tunneling events would take place between the two parts of the ladder, characterized by a small tunneling amplitude, which is now suitable for perturbative treatment. Such a development is out of the scope of this work, but a detailed account on how to put it in place can be found in [34, 35, 39, 4]. At zero temperature, the system is a perfect insulator, all of the excitations being perfectly reflected by the impurity. At finite temperatures, or in the presence of a driving force, tunneling events do take place, taking the fields from one minimum of energy to another. The trajectories that minimize the action during these events are kinks in the field as a function of time, with a characteristic small width. These kinks can be described as instantons [42]. In the massless regime, both ϕ_s and ϕ_a participate in transport. Intuitively, since the interchain coupling has been renormalized to zero in this regime, each leg is independent, and a particle can tunnel through the impurity in one of the legs regardless of what is happening in the other leg. The simultaneous tunneling of two particles, one per leg, is also permitted but with a smaller probability. The situation is different for the massive regimes, in which the instantons only involve the symmetric field, and therefore all of the tunneling events involve both legs. For attractive coupling, since the particles form pairs across the rungs of the ladder (due to the fact that the antisymmetric modes are pinned to zero), there can be only the simultaneous tunneling of two particles, one per leg, and no individual transport. On the other hand, for repulsive interactions, during a tunneling event there is a density fluctuation on both chains, but it is bigger in one of them since ϕ_a is not zero in this regime. The total density fluctuation corresponds to the tunneling of one particle, but not in a single leg. Given that the simultaneous tunneling of two particles occurs with a smaller probability than the tunneling of only one particle, this explains why the pinning by the impurity is more effective in the regime with attractive interchain coupling than in the one with repulsive coupling.

We can also compare our results to the ones previously obtained for a chain of spinful fermions. As stated before, the bosonic ladder can be mapped onto a single chain of spin 1/2 fermions. For the latter case, and in the massless regime for the spin part, Furusaki and Nagaosa found in [39] the transition at $K_\sigma + K_\rho = 2$, where the subindex σ denotes the spin modes, equivalent to our antisymmetric modes, and ρ the charge modes, corresponding to our symmetric modes. This is the same result we found in the massless case. In addition, in their procedure the expansion of the impurity potential was taken farther, and other transitions were predicted at $K_\sigma = 1/2$ and at $K_\rho = 1/2$, for which respectively either the spin modes or the charge modes became individually pinned by the potential, in contrast with the previous transition that pins both modes simultaneously. A similar expansion in our case would lead us to the same results. In fact, we employed a similar procedure in section 3.5, and we found the transition at $K_s = 1/2$ corresponding to the pinning of the symmetric modes. The two problems are hence formally identical, the main difference being the relevant range of

parameters: repulsive interactions for spin 1/2 fermions correspond to $K < 1$, whereas in the case of bosons the condition for repulsive short range interactions is $K > 1$, so in practice the initial conditions will often be naturally different for both cases.

There are other ways of relating the problem to a single chain. Since in the massive regimes the antisymmetric modes become massive and essentially frozen, the only degrees of freedom left correspond to ϕ_s, and we could think of this situation as if we had a single chain, of either bosons or spinless fermions, described by a Luttinger liquid with parameters K_s and u_s, perturbed by an impurity with a coupling constant of either g_0' or g_0'', depending on the minima at which ϕ_a is locked. For the spinless fermionic chain, Kane and Fisher [34, 35] showed that the phase transition takes place at $K = 1$, which corresponds to free fermions or hard-core bosons, whereas our result suggest a transition at $K_s = 2$ for the attractive interchain coupling. To find from where this difference comes, let's look at the problem of an impurity in the single chain. The impurity term, after reabsorbing the forward scattering and neglecting a constant shift in energy, gives

$$V_0\rho(0) \to 2\rho' V_0 \cos(2\phi(0)) \tag{106}$$

As we can see, the operator to which the impurity couples is $\cos(2\phi(0))$, whereas for the ladder it was $\cos(\sqrt{2}\phi_s(0))$. Since the correlators of these operators have a different exponent controlling their decay, they define different renormalization equations for the coupling constant, and hence different critical points. It is nevertheless still possible to make an exact mapping to the single chain. Consider the transformation $\phi_s(x) \to \sqrt{2}\phi'(x)$. With this transformation, the operator with which the impurity couples in the ladder transforms as $\cos(\sqrt{2}\phi_s(0)) \to \cos(2\phi'(0))$, the same as for a single chain. In order to leave invariant the canonical commutation relations the transformation needs to be completed by $\theta_s(x) \to \theta'(x)/\sqrt{2}$, and for the LL Hamiltonian to remain invariant as well we need $K_s \to 2K'$. In terms of the new parameters, the phase transition at $K_s = 2$ now takes place at $K' = 1$, just as in a single chain. With this mapping, both problems become indeed identical. For the case with repulsive interchain coupling, a similar mapping can be done, namely $\phi_s(x) \to \phi''(x)/\sqrt{2}$, $\theta_s(x) \to \sqrt{2}\theta''(x)$, and $K_s \to K''/2$. In this case the transition at $K_s = 1/2$ becomes $K'' = 1$ as well, with the difference that now the coupling constant is given by g_0'', which is quadratic in the strength of the impurity.

Let's try to extend the comparison with the single chain with large fluctuations. First of all, we have to consider that, if our results depend on the fact that the particles are distinguishable by the leg index, we have no guarantee of their validity once indistinguishability is restored. For instance, adding an interchain hopping term to the model (necessary to restore indistinguishability, as discussed before) will renormalize the parameter K_a in a different way, and will also give different asymptotic behaviors to the correlation functions. The question then arises as to whether or not restoring indistinguishability could significantly change the conditions of the transition. We could nevertheless reasonably expect at least one regime in which the hopping would be irrelevant, and in that regime our considerations would remain valid but with different (renormalized) values for the parameters. To simplify the comparison, let's consider hardcore bosons. In the massive case with repulsive interchain coupling, as we can see in fig. 8, at any position there can be either no particles, or just one particle in one of the legs and none in the other. Therefore, the total density can oscillate only between 0 and 1 particles. Hence, even if interesting by itself, this case will not help to understand the effects of large density fluctuations, it can only help us test results for small fluctuations. To get large density fluctuations, only the massless case and the massive case with attractive interactions are adequate. Indeed, as can be seen in fig. 3, the massless case allows for either 0, 1 or 2 particles at a single position, whereas the massive case with attractive interchain coupling presents either 0 or 2 particles at a given location, see fig. 5. It

is interesting to note that, even if both of these regimes allow for large density fluctuations, one of them (massless regime) presents non-universal exponents at the transition between conducting and insulating behavior, whereas the other one (massive regime) has universal exponents. Moreover, we know there is a phase transition between both of them, namely the sine-Gordon transition, meaning that, in our model, there is in fact a critical point in the conductor/insulator transition that separates a regime with universal exponents from a regime with non-universal exponents. This result points in the direction of the hypothesis that there might be a multicritical point separating regimes with universal and non-universal exponents in the superfluid/Bose glass transition.

Some other considerations must nonetheless be taken into account, once again regarding indistinguishability. In the ladder, there are two types of interactions: the interchain ones, mediated by the potential V, and the intrachain ones, which define K. If we want to identify the sum of the density of both legs with the density of a single chain, particle indistinguishability means that both of these interactions must in fact be the same. V is therefore no longer a free parameter but a function of K, and the initial conditions for K and V should be restricted in order to map the ladder to a single chain in this way. As an example, let's consider the following interaction

$$H_{int} = \frac{U}{2} \int dx \; n(x)(n(x) - 1) \tag{107}$$

We want to identify $n(x) = \rho_1(x) + \rho_2(x)$, replacing this term in H_{int}. Once again, the terms linear in $\nabla\phi_i$ can be reabsorbed into the definition of ϕ_i, and we neglect constant shifts in energy and keep only the most relevant terms. In that case, we obtain

$$\begin{aligned} H_{int} &= \frac{U}{2} \int dx \; \rho_1^2(x) + \rho_2^2(x) + 2\rho_1(x)\rho_2(x) - (\rho_1(x) + \rho_2(x)) \\ &\approx \frac{U}{2\pi^2} \int dx \; \left[(\nabla\phi_1(x))^2 + (\nabla\phi_2(x))^2\right] + U \int dx \; \rho_1(x)\rho_2(x) \end{aligned} \tag{108}$$

The kinetic part can be described by (11). Using the same approximations, we get

$$H_{kin} \approx \frac{\rho_0}{2m} \int dx \; \left[(\nabla\theta_1(x))^2 + (\nabla\theta_2(x))^2\right] \tag{109}$$

Comparing these results with eq. (35), we can recognize the parameters of the theory, having

$$\begin{aligned} V &= U \\ K &= \sqrt{\frac{\rho_0\pi}{mU}} = \sqrt{\frac{\rho_0\pi}{mV}} \\ u &= \sqrt{\frac{\rho_0 U}{m\pi}} = \sqrt{\frac{\rho_0 V}{m\pi}} \end{aligned} \tag{110}$$

These considerations serve to exemplify how the initial conditions are restrained if the intrachain and interchain interactions are forced to be the same. There are some problems regarding this particular example, specially for repulsive interactions since K and u should be real. Another flaw of this model is that u_a is zero and K_a diverges for any value of the interaction. However, these illnesses might be cured by the renormalization of parameters due to less relevant operators that we have simply neglected in the precedent development. In any case, having a constraint in the initial conditions defines which regimes are accessible for the system. An additional consideration is also necessary, and it regards the assumptions used to solve the sine-Gordon Hamiltonian. If V is of the same order of magnitude as the

other interactions, the expansion in powers of V leading to eq. (25) should be treated with more care.

5 Conclusions and Perspectives

We have found a quite rich phase diagram for our problem, arising from the interplay between impurity and interactions, even if we have considered a relatively simple situation. In all the possible cases we have found a transition from a conductor to an insulator. The conditions that determine the transition depend only on the interactions, both intrachain and interchain, and not on the impurity strength, as long as it remains much smaller than the interactions, allowing us to treat it as a small perturbation on the clean problem. As a consequence, the RG transformation gives vertical flows for the impurity potential, Figs. 4, 7 and 9. In particular this means that regardless of how big or small the impurity potential is (as long as it remains in the perturbative regime) it will flow to zero in the conducting regimes, and towards strong coupling in the insulating ones. The interchain coupling can render the antisymmetric modes massive, in which case the conductor/insulator transition takes place at either $K_s = 2$ or $K_s = 1/2$, depending on whether the chains attract or repel each other, and thus the critical exponents at the transition take universal values. When the antisymmetric mode is massless, the transition takes place at $K_s + K_a^* = 2$, and therefore the critical exponents at the transition are non-universal, since neither K_s nor K_a^* are individually fixed.

In a bosonization scheme, and neglecting irrelevant operators, the system of a bosonic ladder with a single correlated impurity that we have studied can be related to the physics of a single chain of spin 1/2 fermions, and in the regimes in which the antisymmetric modes become massive it can be related to the properties of a single chain of spinless fermions or a chain of bosons [34, 35, 39], although for quite different typical values of the Luttinger liquid parameters K. Mappings towards other systems can be proposed, but are not yet perfect. Further modifications need to be done to the model in order to attain better mappings. An important example is the comparison with a single bosonic chain with large density fluctuations, a regime in which a simple bosonization approach is not applicable. Alternative bosonization schemes are being analyzed to treat such situations, see for example [33].

Besides the mappings, the problem is interesting by itself, in particular the fact that the impurity is correlated across the rungs. One of the future perspectives for continuing this study is the addition of an interchain hopping term. With this term, the phase diagram will become more complex, and present phases with the competition of different orders. Another possible direction of future research is the study of the phase diagram of a bosonic ladder with extended disorder correlated across the rungs. A first approach to this problem was done in [32]. The addition of the interchain hopping for the problem with extended disorder is also worth considering. Finally, one further modification interesting to look at is to allow the impurity to be dynamical, which would bring the problem into the field of non-equilibrium physics. This to name just a few of the theoretical research possibilities.

Moreover, experimental research related to the topic is conceivable, in particular with cold atomic gases. The control over the range of parameters for such systems allows to explore a significative portion of the phase diagram, as well as having a good control over impurities and disorder. The experimental realization of our model without interchain hopping could be possible in the near future, using polar molecules and thus allowing for long-range interactions. Two chains slightly far apart would interact without tunneling from one to the other. Another possibility is using single chains with two species of particles, given the mappings mentioned before. A peculiar and very interesting way of realizing the system would be in the lines of [43], where a system of fermions with a (tunable) number N of spin components has proven to show, in the large N limit, some properties of bosonic systems.

A Appendix

A.1 Conventions, notation and units

Throughout this work, we use natural units, setting the reduced Planck constant and the Boltzmann constant to unity, $\hbar = k_b = 1$.

The space-time coordinates are expressed as the (1+1)-d vectors

$$r = (x, u\tau) = (x, y) \tag{A.1}$$

with u a velocity. The reciprocal space vectors are

$$\mathbf{q} = (k, \omega_n/u) \tag{A.2}$$

where k is the momentum, and ω_n the Matsubara frequency. We define as well a scalar product between space-time and reciprocal space vectors.

$$\mathbf{q} \cdot r = kx - \omega_n \tau \tag{A.3}$$

For Fourier transforms, we use the following convention

$$f(r) = \frac{1}{\beta\Omega} \sum_{\mathbf{q}} f(\mathbf{q}) e^{i\mathbf{q}\cdot r} \tag{A.4}$$

$$f(\mathbf{q}) = \int_0^\beta d\tau \int dx \; f(r) e^{-i\mathbf{q}\cdot r} \tag{A.5}$$

where $\beta = 1/T$ is the inverse temperature, and Ω the length of the system. The sums can be replaced by integrals in the thermodynamic limit and the zero temperature limit respectively as

$$\frac{1}{\Omega} \sum_k \to \frac{1}{2\pi} \int dk \tag{A.6}$$

$$\frac{1}{\beta} \sum_{\omega_n} \to \frac{1}{2\pi} \int d\omega_n \tag{A.7}$$

A.2 Some explicit calculations

Here we will show the solution of the integral (84) that appears in section 3.4

$$\begin{aligned} F(\tau, \Delta, \Lambda) &\coloneqq \int_0^\infty d\omega_n \; \frac{\cos(\omega_n \tau) e^{-\omega_n/\Lambda}}{\sqrt{\omega_n^2 + \Delta^2}} \\ &= \frac{1}{2} \int_0^\infty d\omega_n \; \frac{1}{\sqrt{\omega_n^2 + \Delta^2}} \left[e^{-\omega_n(1/\Lambda + i\tau)} + e^{-\omega_n(1/\Lambda - i\tau)} \right] \end{aligned} \tag{A.8}$$

We can use the following result

$$\int_0^\infty dx \; \frac{\left(x + \sqrt{x^2+1}\right)^n e^{-\mu x}}{\sqrt{x^2+1}} = \frac{\pi}{2} \left[\frac{1}{\pi} S_n(\mu) - E_n(\mu) - Y_n(\mu) \right] \tag{A.9}$$

where $S_n(\mu)$ is the n-th Schläfli polynomial, $E_n(\mu)$ the Weber function, and $Y_n(\mu)$ the second type Bessel function. We are interested only in the case with $n = 0$, for which $S_0(\mu) = 0$,

$$Y_0(\mu) = -\frac{2}{\pi}\int_0^\infty dx\ \cos(\mu\cosh(x)) \tag{A.10}$$

if $\Re(\mu) > 0$, and

$$E_0(\mu) = -\frac{1}{\pi}\int_0^\pi dx\ \sin(\mu\sin(x)) \tag{A.11}$$

With the substitutions $x = \omega_n/\Delta$ and $\mu = \Delta/\Lambda \pm i\Delta\tau$ our integral becomes

$$F(\tau,\Delta,\Lambda) = -\frac{\pi}{4}\left[E_0(\Delta/\Lambda + i\Delta\tau) + Y_0(\Delta/\Lambda + i\Delta\tau) + E_0(\Delta/\Lambda - i\Delta\tau) + Y_0(\Delta/\Lambda - i\Delta\tau)\right] \tag{A.12}$$

Since $E_0(z^*) = (E_0(z))^*$, and $Y_0(z^*) = (Y_0(z))^*$, this quantity is real.

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