# Single and Attractive: Uniqueness and Stability of Economic Equilibria under Monotonicity Assumptions


Patrizio Bifulco
*University of Hagen*

Jochen Glück
*University of Wuppertal*

Oliver Krebs
*ETH Zürich*

Bohdan Kukharskyy
*City University of New York*


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> *Multiple equilibria are not necessarily useless but, from the standpoint of any exact science, the existence of 'uniquely determined equilibrium [...]' is, of course of the utmost importance [...]; without any possibility of proving the existence of uniquely determined equilibrium—or at all events, of a small number of possible equilibria—at however high a level of abstraction, a field of phenomena is really a chaos that is not under analytic control.*

(Schumpeter, 1954, p. 969)

# Motivation

- Growing interdependency of global economy increases the need to study the propagation of shocks through networks
- Brexit, sanctions, covid-19, environmental policies, hurricane Katrina, infrastructure projects, RTAs etc.
- Past decade saw the rise of quantitative general equilibrium models to analyze such effects
  - Data and code for several 'of the shelf' models readily available
  - Derive new shock $\rightarrow$ run code $\rightarrow$ interpret output
- Important aspect often gets lost: Is the output really the correct/only solution?
- Workhorse Caliendo-Parro model with many countries, many sectors and intermediates

$\Rightarrow$ Even for this widely used, relatively simple, perfect competition model: no proof of uniqueness of the simulated outcome

# This paper / Literature

- Here we:
  1. Provide a new mathematical approach to determine uniqueness of a solution in a broad class of equation systems
  2. Discuss its application to quantitative trade models
- We focus on trade models but approach is very general
- Further possible applications: Economic Geography, Urban Economics, Social Networks, ...
- Several recent papers also tackle uniqueness questions (Allen et al., 2020, 2022, and Kucheryavyy et al., 2021)
- Our Theorem is most closely related to Allen, Arkolakis, Li (2022) although building on a different mathematical background for the proof
- Regarding uniqueness our proof can cover additional models, including several highly relevant extensions (see below)

# Disclaimer

- We discuss uniqueness of *interior* equilibria only (for now)
- E.g. Multi-sector Melitz model has potential for corner solutions
- Intuition:
  - Ex-post: (marginal) firm with infinite productivity draw would always produce
  - Ex-ante: entry only if *expected* profits are large enough to cover entry costs
  - But:
    - One sector: demand$\rightarrow 0$, labor demand$\rightarrow 0$, wage$\rightarrow 0 \Rightarrow$ at the margin firm entry is costless
    - Multi-sector: sector $s$ demand$\rightarrow 0$, wage determined through aggregate demand $\Rightarrow$ entry costs can become prohibitive
- However: in many cases corner solutions are unrealistic/unobserved implying that for quantifications it would be better to adopt the model
- E.g. use across-country CES nest (Kucheryavyy et al. 2021, Caliendo et al. 2020) to avoid 0 production

# Main Theorem

We consider fix-point solutions

$$x^* = F(x^*)$$

of the continuously differentiable function $F : \mathbb{R}^N_{++} \to \mathbb{R}^N_{++}$. If...

(a) function $F$ *connects all variables*, i.e. the modulus of its Jacobian matrix, $|DF(x)|$, is irreducible for each $x \in \mathbb{R}^N_{++}$

(b) function $F$ *exhibits self-interaction*, i.e. for each $x \in \mathbb{R}^N_{++}$ there exists an index $j \in \{1, \ldots, N\}$ such that $\frac{\partial F_j(x)}{\partial x_j} \neq 0$

(c) function $F$ *scales with exponent* $u$, i.e.

$$F(c^u x) = c^u F(x)$$

for all $x \in \mathbb{R}^N_{++}$ and all $c \in \mathbb{R}_{++}$ and for some $u \in \mathbb{R}^N$ with at least one $u \neq 0$

# Main Theorem

(d) *the monotonicity behavior* of $F$ is *consistent with* $u$, i.e. the set $\{1, \ldots, N\}$ can be partitioned into two disjoint subsets $\zeta_+$ and $\zeta_-$ such that $u_j \geq 0$ for all $j \in \zeta_+$ and $u_j \leq 0$ for all $j \in \zeta_-$ and that for all $x \in \mathbb{R}^N_{++}$ and all indices $j, k \in \{1, \ldots, N\}$:

$$\frac{\partial F_j(x)}{\partial x_k} \geq 0 \quad \text{if both } j \text{ and } k \text{ are in the same of the sets } \zeta_+, \zeta_-$$

$$\frac{\partial F_j(x)}{\partial x_k} \leq 0 \quad \text{if } j \text{ and } k \text{are not in the same of the sets } \zeta_+, \zeta_-$$

and there exists a solution $x^* \in \mathbb{R}^N_{++}$ then one has...

(i) Up-to-scale uniqueness: the solutions to the fixed point equation are precisely the vectors in $\mathbb{R}^N_{++}$ given by $c^u x^*$ for some $c \in \mathbb{R}_{++}$.

(ii) Lyapunov stability and attractivity: For every $x \in \mathbb{R}^N_{++}$ the iterates $F^n(x)$ converge to one of the solutions from the solution vector as $n \to \infty$.

# Simple example 1/3

- Consider a simple one sector Eaton-Kortum model with $J$ countries and consumers with CES utility $C_i$ over varieties $\nu$

$$C_i = \left( \int_0^1 \left( C_i \left( \nu \right) \right)^{\frac{\sigma-1}{\sigma}} d\nu \right)^{\frac{\sigma}{\sigma-1}}$$

- Firms use labor (with Cobb-Douglas share $\gamma_i$) and intermediates (combined with the same CES aggregator) to produce

$$p_i \left( \nu \right) = \frac{w_i^{\gamma_i} P_i^{(1-\gamma_i)}}{z_i \left( \nu \right)},$$

  where $z_i$'s are drawn from a Frechet disribution with shape parameter $\theta$ and scale parameter $A_i$
- With iceberg trade costs $\tau_{ij}$ the price index becomes

$$P_i = \Gamma \left( \frac{\theta + 1 - \sigma}{\theta} \right)^{\frac{1}{1-\sigma}} \left( \sum_{j=1}^{J} A_j \left( w_j^{\gamma_j} P_j^{(1-\gamma_j)} \tau_{ji} \right)^{-\theta} \right)^{-\frac{1}{\theta}}$$

# Simple example 2/3

- Firms and consumers source from the cheapest supplier of each variety and in equilibrium markets must clear

$$R_i = \sum_{j=1}^{J} \frac{A_i \left( w_i^{\gamma_i} P_i^{(1-\gamma_i)} \tau_{ij} \right)^{-\theta}}{\sum_{k=1}^{J} A_k \left( w_k^{\gamma_k} P_k^{(1-\gamma_k)} \tau_{kj} \right)^{-\theta}} R_j,$$

  with $R_i = w_i L_i / \gamma_i$ denoting the production value in $i$.
- Rewrite the equilibrium using multilateral resistance terms $\mathbb{P}_i \equiv P_i^{-\theta}$ and $\Omega_i = R_i \left( w_i^{\gamma_i} P_i^{(1-\gamma_i)} \right)^{\theta}$ yields:

$$\Omega_i = \sum_{j=1}^{J} \Gamma \left( \frac{\theta + 1 - \sigma}{\theta} \right)^{-\frac{\theta}{1-\sigma}} A_i \tau_{ij}^{-\theta} \left( \frac{\gamma_j}{L_j} \right)^{-\frac{\theta \gamma_j}{1+\theta\gamma_j}} \Omega_j^{\frac{1}{1+\theta\gamma_j}} \mathbb{P}_j^{\frac{1-\gamma_j}{1+\theta\gamma_j}-1}$$

$$\mathbb{P}_i = \sum_{j=1}^{J} \Gamma \left( \frac{\theta + 1 - \sigma}{\theta} \right)^{-\frac{\theta}{1-\sigma}} A_j \tau_{ji}^{-\theta} \left( \frac{\gamma_j}{L_j} \right)^{-\frac{\theta \gamma_j}{1+\theta\gamma_j}} \Omega_j^{\frac{1}{1+\theta\gamma_j}-1} \mathbb{P}_j^{\frac{1-\gamma_j}{1+\theta\gamma_j}}$$

# Simple example 3/3

- The scaling and self-interaction properties are trivially fulfilled, the connection property by assumption
- Clearly all $\Omega$ influence all $\Omega$ positively and all $\mathbb{P}$ negatively, and all $\mathbb{P}$ influence all $\mathbb{P}$ positively and all $\Omega$ negatively
- Scaling all $\Omega$ by $c^{u_\Omega}$ immediately implies that all $\mathbb{P}$ must be scaled by $c^{u_\mathbb{P}}$ with $u_\mathbb{P} = -\frac{1+\theta}{\theta} u_\Omega$

$\rightarrow$ $u_\mathbb{P}$ is negative whenever $u_\Omega$ is positive and this is consistent with the groups $\zeta_+$ and $\zeta_-$ consisting of all $\Omega$ and all $\mathbb{P}$ respectively

$\Rightarrow$ Solution is (up-to-scale) unique and iterative procedure converges to the solution

# Universal Gravity

- Allen, Arkolakis, Takahashi show that the equilibrium of several one-sector seminal trade and economic geography models can be written as:

$$p_i^{1+\phi}\bar{c}_i\left(\frac{p_i}{P_i}\right)^{\psi}=\sum_j \tau_{ij}^{-\phi}P_j^{\phi}p_j\bar{c}_j\left(\frac{p_j}{P_j}\right)^{\psi}$$

$$P_i^{-\phi}=\sum_j \tau_{ji}^{-\phi}p_j^{-\phi}$$

- Unique solution if $\psi \geq 0, \phi \geq 0$ or $\psi \leq -1, \phi \leq -1$
- Our theorem allows to extend this proof to *i*-specific exponents with unique solutions if for all *i*, $\psi_i \geq 0, \phi_i \geq 0$ or $\psi_i \leq -1, \phi_i \leq -1$
- Small but non-trivial and important extension: country-specific labor/intermediate/capital shares, housing-expenditure shares, etc.

# Multisector - no intermediates 1/2

- Eaton-Kortum type model as in the simple setup above but with multiple sectors (1:1 link between (sectoral) revenue and wages is lost)

$$\Omega_{is} = \sum_j \Gamma\left(\frac{\theta_s + 1 - \sigma_s}{\theta_s}\right)^{-\frac{\theta_s}{1-\sigma_s}} A_{is} \left(\tau_{ijs}\right)^{-\theta_s} \alpha_{js} \mathbb{P}_{js}^{-1} L_j w_j$$

$$\mathbb{P}_{is} = \sum_j \Gamma\left(\frac{\theta_s + 1 - \sigma_s}{\theta_s}\right)^{\frac{-\theta_s}{1-\sigma_s}} A_{js} \tau_{jis}^{-\theta_s} w_j^{-\theta_s}$$

$$w_i = \sum_r \frac{1}{L_i} \Omega_{ir} w_i^{-\theta_r}$$

- Define $\Theta = \sum_s \theta_s$ and $W_i = w_i^{1+\Theta}$ to get

# Multisector - no intermediates 2/2

$$\Omega_{is} = \sum_j \Gamma \left( \frac{\theta_s + 1 - \sigma_s}{\theta_s} \right)^{-\frac{\theta_s}{1-\sigma_s}} A_{is} \left( \tau_{ijs} \right)^{-\theta_s} \alpha_{js} \mathbb{P}_{js}^{-1} L_j W_j^{\frac{1}{1+\Theta}}$$

$$\mathbb{P}_{is} = \sum_j \Gamma \left( \frac{\theta_s + 1 - \sigma_s}{\theta_s} \right)^{\frac{-\theta_s}{1-\sigma_s}} A_{js} \tau_{jis}^{-\theta_s} W_j^{\frac{-\theta_s}{1+\Theta}}$$

$$W_i = \sum_r \frac{1}{L_i} \Omega_{ir} W_i^{\frac{\Theta - \theta_r}{1+\Theta}}$$

- Allows to separate variables into the two groups $\Omega$, $W$ and $\mathbb{P}$ and that groups are consistent with the exponent vector $u$ from scaling.

# Conclusion

- We use a new mathematical approach to develop a theorem stating sufficient conditions for uniqueness of a general fix-point system
- We show how this theorem can be applied to proof uniqueness in models under the universal gravity framework with country specific elasticities, i.e. varying labor/intermediate shares across countries
- We show uniqueness in the case of a multi-sector, multi-country Eaton-Kortum type model without intermediates
- Future: combine both worlds $\rightarrow$ Caliend-Parro model

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# Consistent Changes

$$c^{u_{\Omega_i}}\Omega_i = \sum_{j=1}^{J} K_{ij} c^{\frac{1}{1+\theta\gamma_j}u_{\Omega_j}+\left(\frac{1-\gamma_j}{1+\theta\gamma_j}-1\right)u_{\mathbb{P}_j}} \Omega_j^{\frac{1}{1+\theta\gamma_j}} \mathbb{P}_j^{\frac{1-\gamma_j}{1+\theta\gamma_j}-1}$$

$$c^{u_{\mathbb{P}_i}}\mathbb{P}_i = \sum_{j=1}^{J} K_{ji} c^{\left(\frac{1}{1+\theta\gamma_j}-1\right)u_{\Omega_j}+\frac{1-\gamma_j}{1+\theta\gamma_j}u_{\mathbb{P}_j}} \Omega_j^{\frac{1}{1+\theta\gamma_j}-1} \mathbb{P}_j^{\frac{1-\gamma_j}{1+\theta\gamma_j}}$$

$$u_{\Omega_i} = \frac{1}{1+\theta\gamma_j}u_{\Omega_j} + \left(\frac{1-\gamma_j}{1+\theta\gamma_j}-1\right)u_{\mathbb{P}_j}$$

$$u_{\mathbb{P}_i} = \left(\frac{1}{1+\theta\gamma_j}-1\right)u_{\Omega_j} + \frac{1-\gamma_j}{1+\theta\gamma_j}u_{\mathbb{P}_j}$$

$$u_{\Omega} = -\left(\frac{1}{\theta}+1\right)u_{\mathbb{P}}$$

$$u_{\Omega} = -\left(\frac{1}{\theta}+1\right)u_{\mathbb{P}}$$

return

# General to EK 1/2

$$\mathcal{N}_{ijsu} = N_{ijsu} \tilde{\varphi}_{ijsu}^{\sigma_s - 1}$$

$$= \frac{A_{is} p_{ijsu}^{-\theta_s}}{\sum_k A_{ks} p_{kjsu}^{-\theta_s}} \left( p_{ijsu}^{\theta_s} \sum_k A_{ks} p_{kjsu}^{-\theta_s} \right)^{\frac{\sigma_s - 1}{\theta_s}} \Gamma \left( \frac{\theta_s + 1 - \sigma_s}{\theta_s} \right)$$

$$= \underbrace{\left( \sum_k A_{ks} p_{kjsu}^{-\theta_s} \right)^{-\frac{\theta_s - \sigma_s + 1}{\theta_s}}}_{\mathcal{N}_{\mathbb{P},jsu}} \underbrace{\Gamma \left( \frac{\theta_s + 1 - \sigma_s}{\theta_s} \right) A_{is} p_{is}^{\sigma_s - 1 - \theta_s}}_{\mathcal{N}_{\Omega,is}}$$

$$\underbrace{\left( \tau_{ijsu} \left( 1 + t_{ijsu} \right) \right)^{\sigma_s - 1 - \theta_s}}_{\mathcal{N}_{\phi,ijsu}} .$$

# General to EK 2/2

Assume no outer CES-nest, i.e. $\omega_s = \sigma_s$, as in standard EK:

$$\mathbb{P}_{jsu} = \left( \sum_k A_{ks} p_{kjsu}^{-\theta_s} \right)^{\frac{\sigma_s - 1}{\theta_s}} \Gamma \left( \frac{\theta_s + 1 - \sigma_s}{\theta_s} \right)$$

$$\mathcal{N}_{\mathbb{P},jsu} = \Gamma \left( \frac{\theta_s + 1 - \sigma_s}{\theta_s} \right)^{-1} \left( \sum_k A_{ks} p_{kjsu}^{-\theta_s} \right)^{-1} \mathbb{P}_{jsu}$$

$$\Rightarrow \tilde{\mathbb{P}}_{jsu} = \mathbb{P}_{jsu} \mathcal{N}_{\mathbb{P},jsu}^{-\frac{1-\omega_s}{1-\sigma_s}} = \Gamma \left( \frac{\theta_s + 1 - \sigma_s}{\theta_s} \right) \sum_i A_{is} p_{ijsu}^{-\theta_s}$$

▸ return

# General Trade Model 1/5

- (possibly) heterogeneous agents (workers, producers, landlords...) have the following aggregate demand systems

$$C_{ju} = \prod_{s=1}^{U} \left( \frac{C_{jsu}}{\alpha_{jsu}} \right)^{\alpha_{jsu}}$$

$$C_{jsu} = \left( \sum_{i=1}^{J} C_{ijsu}^{\frac{\omega_s - 1}{\omega_s}} \right)^{\frac{\omega_s}{\omega_s - 1}}$$

$$C_{ijsu} = \left( \int_{M_{ijsu}} \left( C_{ijsu} \left( \nu \right) \right)^{\frac{\sigma_s - 1}{\sigma_s}} d\nu \right)^{\frac{\sigma_s}{\sigma_s - 1}}$$

- $\alpha_{jsu}$ Cobb-Douglas share of user group $u$ in country $j$ for the output/endowments of $s$
- $M_{ijsu}$ is the set of varieties country $j$'s agents $u$ buys from agents $s$ and country $i$

# General Trade Model 2/5

- The price index for what *ju* buys from *is*, i.e. $C_{ijsu}$, becomes

$$P_{ijsu} = \left( \int_{M_{ijsu}} \left( p_{ijsu} \left( \nu \right) \right)^{1-\sigma_s} d\nu \right)^{\frac{1}{1-\sigma_s}} =$$

$$\left( \int_0^\infty \left( p_{ijsu} \left( \varphi \right) \right)^{1-\sigma_s} N_{ijsu} u_{ijsu} \left( \varphi \right) d\varphi \right)^{\frac{1}{1-\sigma_s}} =$$

$$= N_{ijsu}^{\frac{1}{1-\sigma_s}} p_{is} \tau_{ijsu} \tilde{\varphi}_{ijsu}^{-1} = p_{is} \tau_{ijsu} \mathcal{N}_{ijsu}^{\frac{1}{1-\sigma_s}}$$

- $N_{ijsu}$ is the number of varieties *ju* buys from *is* in equilibrium (1 for homogeneous agents)
- $u_{ijsu} \left( \varphi \right)$ their conditional productivity distribution
- $\tilde{\varphi}_{ijsu}$ their 'average' productivity (1 for endowments)
- $\mathcal{N}_{ijsu}$ an 'aggregate productivity' measure
- $p_{is}$ is the part of the mill price independent of productivity (i.e. $\sigma_s / \left( \sigma_s - 1 \right) p_{is}$ in Melitz, $c_{is}$ in perfect competition, $w_i$ for labor)

# General Trade Model 3/5

- Denote expenditure of $ju$ on $s$ by $E_{jsu}$, define multilateral resistance terms $\mathbb{P}_{jsu} \equiv P_{jsu}^{1-\omega_s}$, trade freeness $\phi_{ijsu} \equiv \tau_{ijsu}^{1-\omega_s}$, and use the demand structure to write trade flows $X_{ijsu}$ as

$$X_{ijsu} = p_{is}^{1-\omega_s} \frac{E_{jsu}}{\mathbb{P}_{jsu}} \phi_{ijsu} \mathcal{N}_{ijsu}^{\frac{1-\omega_s}{1-\sigma_s}} \quad .$$

- Define the outward multilateral resistance terms $\Omega_{is} \equiv \frac{Y_{is}}{p_{is}^{1-\omega_s}}$ and use market clearing $Y_{is} = \sum_u \sum_j X_{ijsu}$ and expenditure $E_{jsu} = \sum_i X_{ijsu}$ (together with the assumption of no corner solution) to write

$$\Omega_{is} = \sum_u \sum_j \frac{E_{jsu}}{\mathbb{P}_{jsu}} \phi_{ijsu} \mathcal{N}_{ijsu}^{\frac{1-\omega_s}{1-\sigma_s}}$$

$$\mathbb{P}_{jsu} = \sum_i p_{is}^{1-\omega_s} \phi_{ijsu} \mathcal{N}_{ijsu}^{\frac{1-\omega_s}{1-\sigma_s}}$$

# General Trade Model 4/5

- We make two further assumptions on the production side (that are met by all the most common frameworks):
  - Each agent group $ju$ receives constant shares $\gamma_{ijru}$ of the production value from (potentially) around the world
  - $\mathcal{N}_{ijsu}$ can be multiplicatively split into $\mathcal{N}_{\Omega,is}$, $\mathcal{N}_{\mathbb{P},jsu}$, and $\mathcal{N}_{\phi,ijsu}$, with the last term exogenous
- (Re)define $\tilde{\Omega}_{is} = \Omega_{is}\mathcal{N}_{\Omega,is}^{-\frac{1-\omega_s}{1-\sigma_s}}$, $\tilde{\mathbb{P}}_{isu} = \mathbb{P}_{isu}\mathcal{N}_{\mathbb{P},isu}^{-\frac{1-\omega_s}{1-\sigma_s}}$, and $\tilde{p}_{is}^{1-\omega_s} = p_{is}^{1-\omega_s}\mathcal{N}_{\Omega,is}^{\frac{1-\omega_s}{1-\sigma_s}}$. ▸ example

# General Trade Model 5/5

- The full (internal) equilibrium then becomes

$$\tilde{\Omega}_{is} = \sum_{u} \sum_{j} E_{jsu} \tilde{\mathbb{P}}_{jsu}^{-1} \tilde{\phi}_{ijsu}$$

$$\tilde{\mathbb{P}}_{jsu} = \sum_{i} \tilde{p}_{is}^{1-\omega_s} \tilde{\phi}_{ijsu}$$

$$E_{jsu} = \alpha_{jsu} \sum_{i} \sum_{r} \gamma_{ijru} \tilde{\Omega}_{ir} \tilde{p}_{ir}^{1-\omega_r}$$

$$\tilde{p}_{is} = F_{\tilde{p}}(\cdot)$$

- Plug III and II into I so that the equations depend only on multilateral resistance terms and 'prices' $\tilde{p}_{is}$

→ Use our approach to find conditions on $F_{\tilde{p}}(\cdot)$ for the internal equilibrium to be unique