

Second order explicit stabilized multirate method for stiff differential equations with error control

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Based on joint work with Gilles Vilmart (University of Geneva)

Plan of the talk

Main Objective

Construct a **second-order** explicit stabilized multirate method for stiff differential equations.

Reference: M. B. and G. Vilmart, “Second order explicit stabilized multirate method for stiff differential equations with error control,” submitted for publication (2025).

- ➊ **Explicit Stabilized Methods:** Review of the Chebyshev methods [van der Houwen and Sommeijer, '80s] and ROCK2 schemes [Abdulle and Medovikov, 2002] for stiff ordinary differential equations.
- ➋ **Local Stiffness & Multirate Integration:** Severe localized stiffness from spatial discretizations of PDEs is addressed via fast-slow system splitting, such as the first-order mRKC method [Abdulle, R. De Souza and Grote 2022].
- ➌ **The New Second-Order Multirate Scheme:** New explicit stabilized **mROCK2** method, achieving second-order accuracy via a modified averaged force.

Explicit Stabilized methods for stiff problems

Diffusion PDE after spatial discretization

$$\dot{y}(t) = Ay(t) + g(y(t), t)$$

- ① A has very large negative eigenvalues (**stiff system**)!
 - ② A is an $N \times N$ matrix, with $N \gg 1$.
-
- **Standard Explicit Methods:** Cheap per step, but suffer from **severe step-size restrictions** (for Euler explicit: $\tau < \frac{2}{\rho(A)}$).
 - **Implicit Methods:** A-stable, but solving large linear systems at each step may incur **prohibitive computational and memory costs**.

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Efficient approach: Explicit Stabilized Methods

By using multiple stages, these methods **quadratically increase** the maximum allowed time step ($\tau < C \cdot s^2$), providing a fully explicit and efficient way to integrate stiff problems.

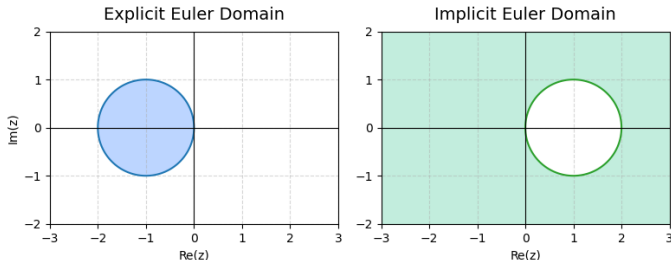
Explicit Stabilized (Chebyshev) Methods

The Stability Function

For Dahlquist's test equation $\dot{y} = \lambda y$ ($\lambda < 0$), a Runge-Kutta method with step-size τ yields:

$$y_{n+1} = R(\tau\lambda)y_n \implies y_n = [R(\tau\lambda)]^n y_0$$

Bounded solution if the stability function $R(z)$ satisfies $|R(z)| \leq 1$ for $z = \tau\lambda$. The stability domain is $\mathcal{S} := \{z \in \mathbb{C} \mid |R(z)| \leq 1\}$.



Stability domains for explicit Euler ($R(z) = 1 + z$) and implicit Euler ($R(z) = \frac{1}{1-z}$).

Explicit stabilized methods: first-order Chebyshev methods

For fixed integer s , and a first order stability function of degree at most s of the form

$$R_s(z) = 1 + z + a_2 z^2 + \cdots + a_s z^s,$$

find the coefficients $a_2, a_3, \dots, a_s$ that maximize ℓ_s where $\ell_s := \sup l \geq 0$ such that $(-l, 0) \subset \mathcal{S} = \{z \in \mathbb{C} \mid |R(z)| \leq 1\}$.

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Theorem (Markoff, 1892; ... Guillou, Lago, 1960)

For fixed integer s , the optimal solution that maximizes ℓ_s is unique and given by

$$R_s(z) = T_s\left(1 + \frac{z}{s^2}\right)$$

where $T_s(\cos x) = \cos(sx)$ are the Chebyshev polynomials.

It satisfies $|R_s(z)| \leq 1$ for all real $z \in (-\ell_s, 0)$, with:

$$\ell_s = 2s^2.$$

First-order Chebyshev methods

An s -stage Runge-Kutta method $y_0 \mapsto y_1$ implemented for the equation $\dot{y}(t) = f(y(t))$ using the Chebyshev recurrence relation

$T_j(x) = 2xT_{j-1}(x) - T_{j-2}(x)$, $j \geq 2$ (v. der Houwen and Sommeijer, '80s):

$$K_0 = y_0,$$

$$K_1 = y_0 + \mu_1 hf(y_0),$$

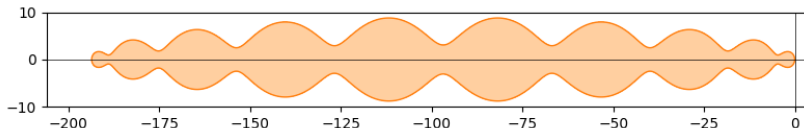
$$K_i = \mu_i hf(K_{i-1}) + \nu_i K_{i-1} + \kappa_i K_{i-2}, \quad i = 2, \dots, s,$$

$$y_1 = K_s.$$

Stability function:

$$R_s(z) = \frac{T_s(\omega_0 + \omega_1 z)}{T_s(\omega_0)} \quad \text{where} \quad \omega_0 = 1 + \frac{\varepsilon}{s^2}, \quad \omega_1 = \frac{T_s(\omega_0)}{T'_s(\omega_0)}.$$

It satisfies $|R_s(z)| \leq 1$ for all $z \in (-L_s, 0)$, with $L_s \approx (2 - \frac{4}{3}\varepsilon)s^2$.



Second-order explicit stabilized method: ROCK2 method

For a second-order stability function of degree at most s of the form

$$R_s(z) = 1 + z + \frac{1}{2}z^2 + a_3z^3 + \cdots + a_s z^s,$$

find the coefficients $a_3, a_4, \dots, a_s$ that maximize ℓ_s where $\ell_s := \sup l \geq 0$ such that $(-l, 0) \subset \mathcal{S}$.

ROCK2 method (Abdulle and Medovikov, 2002)

For a given number of stages s , the method provides **second-order** accuracy. It constructs a stability polynomial that **converges to the optimal real stability interval** using a three-term **recurrence relation** of orthogonal polynomials.

It satisfies stability $|R_s(z)| \leq 1$ for all real $z \in (-\ell_s, 0)$, with approximately:

$$\ell_s \simeq 0.81 \cdot s^2.$$

ROCK2 method (Abdulle and Medovikov, 2002)

An s-stage ROCK2 $y_0 \mapsto y_1$ method is given by:

$$k_0 = y_0$$

$$k_1 = k_0 + h\mu_1 f(k_0)$$

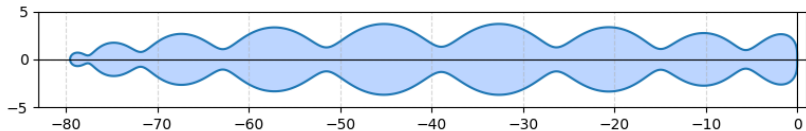
$$k_j = h\mu_j f(k_{j-1}) - \nu_j k_{j-1} - \kappa_j k_{j-2}, \quad j = 2, \dots, s-2$$

$$k_{s-1} = k_{s-2} + h\sigma_1 f(k_{s-2})$$

$$k_s^* = k_{s-1} + h\sigma_1 f(k_{s-1})$$

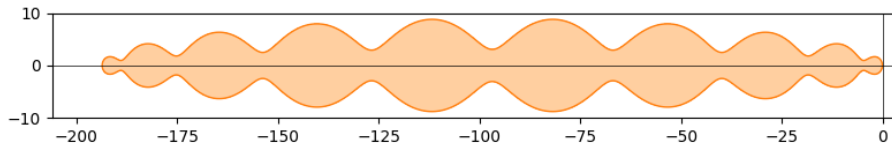
$$k_s = k_s^* - h\sigma_1(1 - \frac{\sigma_1}{\sigma_2})(f(k_{s-1}) - f(k_{s-2})),$$

The stability function is $R_s(z) = \omega_2(z)P_{s-2}(z)$. Here, $\omega_2(z) = 1 + \sigma_1 z + \sigma_2 z^2$ is a polynomial chosen to guarantee the second-order convergence and P_{s-2} the orthogonal polynomial to construct the stability domain (with coefficients μ_i, ν_i and κ_i)

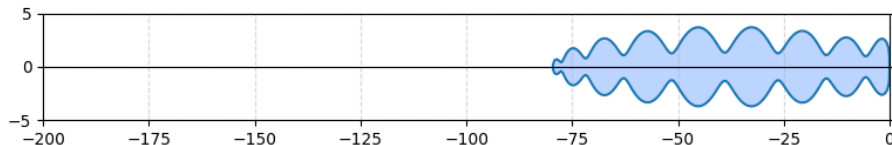


Stability comparison of the two methods

Case: $s = 10$, first-order Chebyshev method



Case: $s = 10$, ROCK2 method

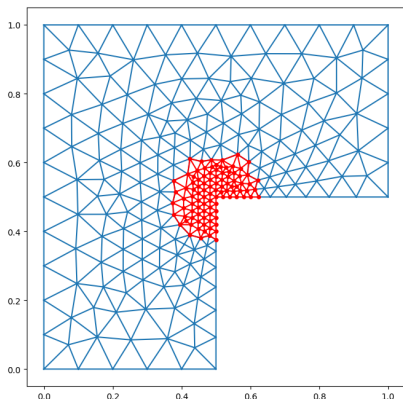


Applications of explicit stabilized methods

Explicit stabilized methods are highly efficient for very large systems of ODEs!

- **Simulations of the solar atmosphere:** Q. M. Wagnier, G. Vilmart, J. Martínez-Sykora, V. H. Hansteen, and B. De Pontieu, *Time-Adaptive PIROCK Method with Error Control for Multi-Fluid and Single-Fluid MHD Systems*, Astron. Astrophys. (2025), in collaboration with NASA.
- **Cardiology:** G. Rosilho de Souza, S. Pezzuto, and R. Krause, *High-order parallel-in-time method for the monodomain equation in cardiac electrophysiology*, arXiv preprint (2024).
- **Metal 3D Printing:** S. Essongue, B. Diarra, and E. Lacoste, *Runge-Kutta-Chebyshev Schemes to Accelerate Thermal Modelling of Additive Manufacturing Processes* (2025).
- etc...

Motivation: Stiff PDEs and Local Refinement



When solving PDEs (like the heat equation) with Finite Elements, local mesh refinement is highly efficient for singularities but creates **severe, localized stiffness**.

$$\frac{dy(t)}{dt} = \underbrace{A_F y(t)}_{f_F} + \underbrace{A_S y(t)}_{f_S} + G(t).$$

- Coarse mesh: $\rho(A_S) = 10000$
- Fine mesh: $\rho(A_F) = 500000$

Multiscale Problem & mRKC Method

The Fast-Slow Splitting

We partition our stiff system into two distinct scales:

$$\dot{y}(t) = \underbrace{f_F(y)}_{\text{Fast/Stiff}} + \underbrace{f_S(y)}_{\text{Slow/Mild}}$$

- f_F (**Fast**): **Severely stiff**, but **cheap** to compute (e.g., highly localized).
- f_S (**Slow**): **Mildly stiff**, but **expensive** to compute (e.g., global domain).

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2. The Multirate Solution: mRKC (Abdulle, de Souza, Grote, 2022)

mRKC solves the problem by splitting the work:

- **Micro-steps**: Takes many small steps of f_F to stabilize.
- **Macro-steps**: Evaluates f_S as few times as possible per large step.
- **Mechanism**: Uses an averaged vector field $f_{\eta,1}(y) = \frac{1}{\eta}(u(\eta) - y)$.

Only drawback: mRKC is only **first-order accurate**.

mRKC (Abdulle, de Souza, Grote, 2022)

1. The Modified Equation & Averaged Force To isolate the severe stiffness, we freeze the slow dynamics and solve a **micro-problem** over a small time-step: η :

$$\dot{u}(t) = \underbrace{f_F(u(t))}_{\text{Evolve fast}} + \underbrace{f_S(y)}_{\text{Freeze slow}}, \quad u(0) = y_0, \quad t \in [0, \eta]$$

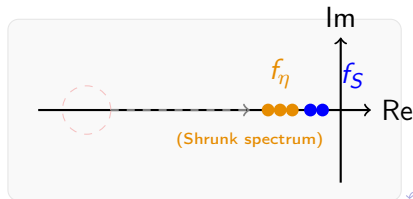
The first-order averaged force is defined as:

$$f_{\eta,1}(y) := \frac{1}{\eta}(u(\eta) - y)$$

1. Original Stiff System



2. Modified System



mRKC (Abdulle, de Souza, Grote, 2022)

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2. The mRKC Method

Instead of solving the original differential equation $\dot{y} = f(y)$, the mRKC method applies RKC scheme to the **modified system**:

$$\dot{y}_{\eta,1} = f_{\eta,1}(y_{\eta,1})$$

and for the micro-problem $\dot{u}(t) = f_F(u(t)) + f_S(y)$ with a time step η This allows the method to bypass fine-mesh restrictions and take massive **macro-steps** $\tau \gg \eta$.

Constructing a Second-Order Averaged Force

In the mRKC method, Taylor expansion of the modified equation reveals:

$$f_{\eta,1}(y_0) = f(y_0) + \frac{\eta}{2} f'_F(y_0) f(y_0) + \mathcal{O}(\eta^2)$$

Because the modified force $f_{\eta,1}$ only matches the true force $f(y)$ up to $\mathcal{O}(\eta)$, the overall method is restricted to first-order accuracy.

New modified averaged force for high-order multirate method

We introduce a newly modified averaged force $f_{\eta,2}$ which satisfies:

$$f_{\eta,2}(y) = f(y) + \mathcal{O}(\eta^2)$$

This allows the macro-step integrator to achieve genuine second-order accuracy.

mROCK2: The New Averaged Force (B., Vilmart, 2025)

Definition of the Second-Order Averaged Force

Introduce the modified differential equation $\dot{y}_{\eta,2} = f_{\eta,2}(y_{\eta,2})$ where:

$$f_{\eta,2}(y) = \frac{1}{\eta}(v(\eta) - y)$$

and $v(t)$ is the solution to the **modified** micro-problem:

$$\dot{v} = f_F\left(v - \frac{\eta}{2}f_{\eta,1}(y)\right) + f_S(y), \quad v(0) = y.$$

(Here, $f_{\eta,1}(y)$ is the classical first-order averaged force from mRKC).

By shifting the evaluation of the fast field f_F , we cancel the first-order error term entirely.

Second-order mROCK2 multirate algorithm (B., Vilmart, 2025)

Micro-macro method combining RKC and ROCK2

1. Micro-step (New Averaged Force): Compute averaged force using two steps of RKC.

```
1: function AVERAGED_FORCE( $t, y, \eta, f_F, f_S, m$ )  
2:    $f_u(u) \leftarrow f_F(u) + f_S(y)$  ▷ Freeze slow dynamics  
3:    $u_\eta \leftarrow \text{RKC}(t, y, \eta, f_u, m, \epsilon)$  ▷ First RKC step  
4:    $\bar{f}_{\eta,1} \leftarrow \frac{1}{\eta}(u_\eta - y)$  ▷ Classical averaged force  
5:    $f_v(v) \leftarrow f_F(v - \frac{\alpha m \eta}{2} \bar{f}_{\eta,1}) + f_S(y)$  ▷ Modified fast field  
6:    $v_\eta \leftarrow \text{RKC}(t, y, \eta, f_v, m, \epsilon)$  ▷ Second RKC step  
7:   return  $\bar{f}_{\eta,2} = \frac{1}{\eta}(v_\eta - y)$  ▷ New second-order averaged force  
8: end function
```

2. Macro-step: Advance y_n using ROCK2 with the new averaged force.

```
1: function mROCK2_STEP( $t_n, y_n, \tau, f_F, f_S, \rho_F, \rho_S$ )  
2:    $s, m, \eta \leftarrow \text{Get\_Stages}(\tau, \rho_S, \rho_F, \epsilon)$  ▷ Optimal parameters  
3:    $\bar{f}_{\eta,2}(t, y) \leftarrow \text{Averaged\_Force}(t, y, \eta, f_F, f_S, m)$   
4:    $y_{n+1} \leftarrow \text{ROCK2}(t_n, y_n, \tau, \bar{f}_{\eta,2}, s)$  ▷ Macro-step with ROCK2  
5:   return  $y_{n+1}$   
6: end function
```

Theoretical Properties of mROCK2

Theorem (Stability of mROCK2, B. & Vilmart, 2025)

Let $\lambda \leq 0$ and $\zeta < 0$ for the multirate linear test equation:

$$\dot{y}(t) = \lambda y(t) + \zeta y(t).$$

We define $\alpha_m = P_m''(0)$ (the second derivative of the RKC stability function at the origin), β_s the stability coefficient of the ROCK2 method. Then, for all $\tau > 0, s, m$ and η such that

$$(1 + \alpha_m)\tau|\zeta| \leq \tilde{\beta}_s s^2, \quad \eta|\lambda| \leq 1.92m^2 \quad \text{with} \quad \eta \geq \frac{2\tau(1 + \alpha_m)}{\alpha_m \tilde{\beta}_s s^2},$$

$|R_{s,m}(\lambda, \zeta, \tau, \eta)| \leq 1$, i.e. the mROCK2 scheme is stable.

Theorem (Accuracy, B. & Vilmart, 2025)

The mROCK2 scheme is second-order accurate.

Comparison of RKC, mRKC, ROCK2, mROCK2

Method	Order	$\#f_S$	$\#f_F$	ℓ_s	ℓ_m
RKC (1960s)	1	m	m	$\approx 2m^2$	$\approx 2m^2$
ROCK2 (Abdulle & Medovikov 2001)	2	m	m	$\approx 0.81m^2$	$\approx 0.81m^2$
mRKC (Abdulle, R. De Souza and Grote 2022)	1	s	ms	$\approx 2s^2$	$\approx 0.66s^2m^2$
New mROCK2 (B., Vilmart 2025)	2	s	$2ms$	$\approx 0.6s^2$	$\approx 0.27s^2m^2$

Table: Number of function evaluations per time step and stability domain sizes.

Numerical Results: Robertson's stiff non-linear problem

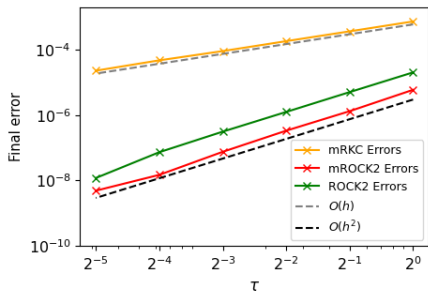
We consider Robertson's non-linear chemical reaction model ($t \in [0, 100]$):

$$y_1' = -0.04y_1 + 10^4 y_2 y_3,$$

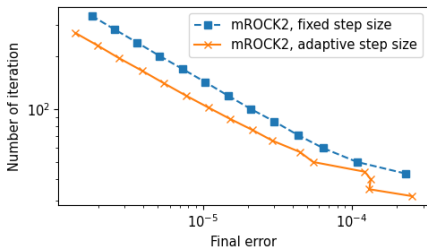
$$y_2' = 0.04y_1 - 10^4 y_2 y_3 - 3 \cdot 10^7 y_2^2,$$

$$y_3' = 3 \cdot 10^7 y_2^2.$$

Initial conditions: $y(0) = (1, 2 \cdot 10^{-5}, 10^{-1})^T$.

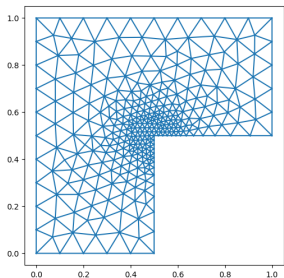


(a) Convergence: mROCK2 successfully achieves $\mathcal{O}(\tau^2)$.

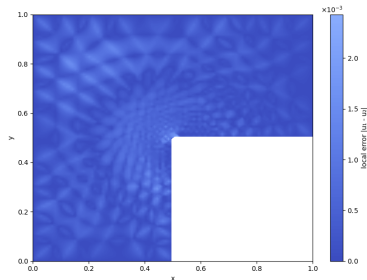


(b) Efficiency: Error vs. Function Evaluations. Adaptive mROCK2 wins.

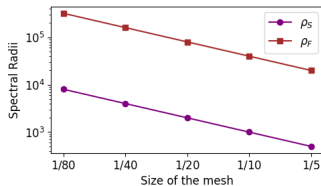
Efficiency of mROCK2 on the L-shape problem



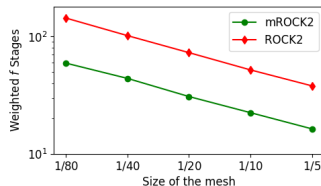
The refined mesh



The precision of the solution



Spectral radius of A_F and A_S vs refinement.



Theoretical stages: mROCK2 vs ROCK2 ($\#f = 0.95\#f_S + 0.05\#f_F$).

Conclusion and Future Work

Summary

- Developed **mROCK2**, a second-order explicit stabilized multirate method.
- Constructed a new averaged force $\bar{f}_{\eta,2}$ that eliminates the first-order error constraint.
- Proved stability bounds and second-order convergence.

Ongoing & Future Work

- Adapting mROCK2 to large-scale, high-dimensional stochastic S(P)DE's .
- Developing explicit stabilized preconditioning techniques for invariant measure sampling (joint work with G. Vilmart and A. Busnot Laurent).

Thank you for your attention!