EXERCISES ON LORENTZ SPACETIMES LES DIABLERETS, 30 JUNE – 4 JULY 2014

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In the following exercises, $\mathbb{R}^{2,1}$ denotes the 3-dimensional *Minkowski space*, namely $\mathbb{R}^3$ with the translation-invariant Lorentzian metric induced by the bilinear symmetric form

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3.$$

The *Minkowski model* of the hyperbolic plane $\mathbb{H}^2$ is the hyperboloid of $\mathbb{R}^{2,1}$ defined by $\langle x, x\rangle = -1$ and $x_3 > 0$, with the metric induced by $\langle \cdot, \cdot\rangle$.

Exercise 1. Recall that a Killing field is a vector field whose flow is isometric.

a) Prove that the set of Killing fields on $\mathbb{H}^2$ identifies with the Lie algebra $\mathfrak{psl}_2(\mathbb{R})$ of $\mathrm{PSL}_2(\mathbb{R})$ (i.e. the set of traceless real 2×2 matrices). In particular, one can define *hyperbolic*, *parabolic*, *elliptic* Killing fields; draw a picture in each situation.

b) Prove that for any Killing field X on $\mathbb{H}^2$ and any oriented geodesic line ℓ in $\mathbb{H}^2$, the component[1] of X along ℓ is the same at all points x of ℓ.

c) Prove that any vector $u \in \mathbb{R}^{2,1}$ gives rise to a Killing field X_u on $\mathbb{H}^2$, which is expressed in the Minkowski model via a Lorentzian cross-product:

$$X_u(x) = u \wedge x := (u_2x_3 - u_3x_2, u_3x_1 - u_1x_3, -u_1x_2 + u_2x_1)$$

for all $x \in \mathbb{H}^2$. Show that any Killing field arises uniquely in this way. This gives an explicit isomorphism $\mathfrak{psl}_2(\mathbb{R}) \simeq \mathbb{R}^{2,1}$, compatible with the adjoint action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathfrak{psl}_2(\mathbb{R})$ and the natural action of $\mathrm{SO}(2,1)_0 \simeq \mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{R}^{2,1}$.

Exercise 2. Consider an isometry h of $\mathbb{R}^{2,1}$ of the form $v \mapsto Av + w$ where $A \in \mathrm{SO}(2,1)^0$ and $w \in \mathbb{R}^{2,1}$, and assume that A is hyperbolic.

a) Show that A has three eigenvalues $t > 1 > t^{-1}$, and that the eigenvectors of A are lightlike for the eigenvalues t, t^{-1}, and spacelike for the eigenvalue 1.

Let v_+, v_- be future-pointing eigenvectors for the respective eigenvalues t, t^{-1}, and v_0 an eigenvector for the eigenvalue 1, such that $\langle v_0, v_0\rangle = 1$. Assume that (v_+, v_0, v_-) is a positively oriented basis of $\mathbb{R}^3$.

[1]The component of X along ℓ at x is by definition $\mathrm{pr}_{\ell,x}(X(x))$, where $\mathrm{pr}_{\ell,x} : T_x\mathbb{H}^2 \to \mathbb{R}$ is the linear form giving the signed length of the projection to ℓ.

b) Show that h preserves a unique affine line of $\mathbb{R}^{2,1}$ parallel to v_0, and that it acts on this line by a translation. By definition, the *Margulis invariant* of h is the signed length $\alpha(h) \in \mathbb{R}$ of this translation.

c) Show that $\alpha(h) = \langle w, v_0\rangle = \langle h(v) - v, v_0\rangle$ for all $v \in \mathbb{R}^{2,1}$.

d) Show that $\alpha(h') = \alpha(h)$ for any isometry h' of $\mathbb{R}^{2,1}$ conjugate to h.

e) Show that $\alpha(h^n) = |n|\, \alpha(h)$ for all $n \in \mathbb{Z}$.

f) For $g \in \mathrm{SO}(2,1)$, let $\lambda(g) := \inf_{p\in\mathbb{H}^2} d_{\mathbb{H}^2}(p, g\cdot p) \geq 0$ be the translation length of g in $\mathbb{H}^2$. Show that

$$\alpha(h) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \lambda(e^{tX_w} A),$$

where X_w is the Killing field associated with w (defined in Exercise 1.c). (Hint: observe that $A = e^{\lambda(A)v_0}$, so that $e^{tX_u}A = e^{\lambda(A)v_0 + tX_u + O(t^2)}$. Then show that $\lambda(e^{X_v}) = \sqrt{\langle v, v\rangle}$ for any spacelike $v \in \mathbb{R}^{2,1}$.)

Exercise 3. Let Γ be a discrete group and $(j_t)_{t\in\mathbb{R}}$ a smooth 1-parameter family of injective and discrete representations of Γ into $\mathrm{PSL}_2(\mathbb{R})$. Define a map $u : \Gamma \to \mathfrak{psl}_2(\mathbb{R})$ by $j_t(\gamma) = e^{tu(\gamma)+o(t)} j_0(\gamma)$ for all $\gamma \in \Gamma$.

a) Prove that u is a j_0-*cocycle*, in the sense that for all $\gamma, \gamma' \in \Gamma$,

$$u(\gamma\gamma') = u(\gamma) + \mathrm{Ad}(j_0(\gamma))\, u(\gamma').$$

b) Check that this means that

$$\gamma \cdot v := \mathrm{Ad}(j_0(\gamma))\, v + u(\gamma)$$

defines a group action of Γ on $\mathfrak{psl}_2(\mathbb{R}) \simeq \mathbb{R}^{2,1}$ by affine transformations.

c) Prove that conjugating the family $(j_t)_{t\in\mathbb{R}}$ by some element $g \in \mathrm{PSL}_2(\mathbb{R})$ corresponds to adding a j_0-*coboundary* to u, i.e. a map $\Gamma \to \mathfrak{psl}_2(\mathbb{R})$ of the form $\gamma \mapsto v - \mathrm{Ad}(j_0(\gamma))\, v$ where $v \in \mathfrak{psl}_2(\mathbb{R})$.

d) Let $(f_t)_{t\in\mathbb{R}}$ be a smooth family of smooth maps from $\mathbb{H}^2$ to itself, such that each f_t is (j_0, j_t)-*equivariant* (i.e. $f_t(j_0(\gamma)\cdot x) = j_t(\gamma)\cdot f(x)$ for all $x \in X$ and $\gamma \in \Gamma$) and $f_0 = \mathrm{Id}_{\mathbb{H}^2}$. Prove that the vector field $X : p \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\, f_t(p)$ on $\mathbb{H}^2$ is (j_0, u)-*equivariant*, in the sense that

$$X(j_0(\gamma)\cdot p) = j_0(\gamma)_*(X(p)) + X_{u(\gamma)}(j_0(\gamma)\cdot p)$$

for all $p \in \mathbb{H}^2$ and $\gamma \in \Gamma$. Here $X_{u(\gamma)}$ is the Killing field on $\mathbb{H}^2$ associated with $u(\gamma) \in \mathfrak{psl}_2(\mathbb{R}) \simeq \mathbb{R}^{2,1}$, as defined in Exercise 1.c.

Exercise 4. Let $\mathcal{X}$ be a complete metric space with isometry group G. Let Γ be a discrete group and $j, \rho \in \mathrm{Hom}(\Gamma, G)$ two representations. Let $f : \mathcal{X} \to \mathcal{X}$ be a (j,ρ)-equivariant map which is *uniformly contracting*, in the sense that there exists $C < 1$ such that $d(f(x), f(x')) \leq C\, d(x, x')$ for all $x \neq x'$ in $\mathcal{X}$.

a) Check that for any $g \in G$, the map $g^{-1} \circ f$ has a unique fixed point in $\mathcal{X}$.

b) Prove that the map $\pi_f : G \to \mathcal{X}$ taking $g \in G$ to the fixed point of $g^{-1} \circ f$ is continuous and $((j,\rho), j)$-equivariant, where the (j,ρ)-action of Γ on G is given by $\gamma \cdot g := \rho(\gamma) g j(\gamma)^{-1}$ for all $\gamma \in \Gamma$ and $g \in G$.

c) Suppose that $\mathcal{X} = \mathbb{H}^2$ and j is injective and discrete. Show that if f is constant, then the group $\rho(\Gamma)$ has a global fixed point in $\mathbb{H}^2$, and describe the fibers of π_f in terms of the Lorentzian geometry of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{AdS}^3$.

Exercise 5. Consider $p, q, r, s \in \mathbb{H}^2$ such that s belongs to the interior of the triangle pqr. Suppose $f : \{p, q, r\} \to \mathbb{H}^2$ is C-Lipschitz.

a) If $C \geq 1$, prove that f has a C-Lipschitz extension to $\{p, q, r, s\}$. (Hint: use the convexity of the distance function $t \mapsto d_{\mathbb{H}^2}(\exp_x(tv), \exp_x(tv'))$ for all $x \in \mathbb{H}^2$ and $v, v' \in T_x\mathbb{H}^2$.)

b) If $C < 1$, prove that f has a C'-Lipschitz extension to $\{p, q, r, s\}$ with $C' < 1$, but generally not with $C' = C$.

Exercise 6. Let Γ be a discrete group and $j, \rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ two representations with j injective and discrete. Show that for any $C, c \geq 0$, there exists $D > 0$ with the following property: any two C-Lipschitz, (j,ρ)-equivariant maps $f, f' : \mathbb{H}^2 \to \mathbb{H}^2$ satisfy $d(f(x), f'(x)) \leq D$ for all $x \in \mathbb{H}^2$ projecting to within distance c from the convex core.

Exercise 7. A *convex field* on $\mathbb{H}^2$ is a closed subset X of $T\mathbb{H}^2$ such that $X_x := X \cap T_x\mathbb{H}^2$ is convex for all $x \in \mathbb{H}^2$.

a) Prove that the closedness property amounts to requesting that the convex set $X_x \subset T_x\mathbb{H}^2$ depend upper semicontinuously on x for the Hausdorff topology.

b) Prove that a contracting convex field X on $\mathbb{H}^2$ always has a unique *zero* (i.e. a point $x \in \mathbb{H}^2$ such that $0_x \in X(x)$). By definition, contracting means that there exists $c < 0$ such that for any distinct $x, y \in \mathbb{H}^2$ and $(v, w) \in X_x \times X_y$, the *infinitesimal change in distance*

$$d'(v, w) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d_{\mathbb{H}^2}(\exp_x(tv), \exp_y(tw))$$

satisfies $d'(v, w) \leq c\, d_{\mathbb{H}^2}(x, y)$.

Note: $d'(v, w)$ can be computed directly in terms of the projections of v and w onto the direction of the geodesic connecting x to y.

Exercise 8. Let ℓ be a geodesic in $\mathbb{H}^2$ and $\overline{\ell}$ its closure in $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$. Consider the set $\mathcal{C}(\ell) \subset \mathfrak{psl}_2(\mathbb{R})$ of Killing fields on $\mathbb{H}^2$ (see Exercise 1.a) that have a nonrepelling fixed point in $\overline{\ell}$.

a) Show that $\mathcal{C}(\ell)$ is a piecewise linear surface that divides $\mathfrak{psl}_2(\mathbb{R})$ into two connected components.

b) Let ℓ' be another geodesic of $\mathbb{H}^2$, whose closure in $\overline{\mathbb{H}^2}$ is disjoint from $\overline{\ell}$. Show that $\mathcal{C}(\ell) \cap \mathcal{C}(\ell') = \{0\}$.

c) Show that for any $X \in \mathcal{C}(\ell)$, any $Y \in \mathcal{C}(\ell')$, and any $\varepsilon > 0$, there exists $p \in \ell$ and $q \in \ell'$ such that $d'(X(p), Y(q)) > -\varepsilon$.

d) Let $V \in \mathfrak{psl}_2(\mathbb{R})$ be an infinitesimal translation of $\mathbb{H}^2$ along an axis orthogonal to ℓ, translating away from ℓ'. Show that there exists $k > 0$ such that

$$d'(V(p), \underline{0}(q)) \geq k$$

for all $p \in \ell$ and $q \in \ell'$, where $\underline{0}$ denotes the zero vector field on $\mathbb{H}^2$.

e) Show that $\mathcal{C}(\ell) + V$ and $\mathcal{C}(\ell')$ are disjoint.

The surfaces $\mathcal{C}(\ell)$ are called *crooked planes*. They were originally defined (in a different way) by T. Drumm and used to build fundamental domains for proper actions on $\mathbb{R}^{2,1}$.