

BOUNDED COHOMOLOGY AND VOLUME RIGIDITY

EXERCISE 1. Use the continuous maps

$$C_b(G^n, \mathbb{R}) \begin{matrix} \xrightarrow{\rho^{n-1}} \\ \xleftarrow{\tau^n} \end{matrix} C_b(G^{n-1}, \mathbb{R})$$

defined by

$$\begin{aligned} (\rho^{n-1}f)(g_1, \dots, g_{n-1}) &:= f(e, g_1, g_1g_2, \dots, g_1g_2 \cdots g_{n-1}) \\ (\tau^n f')(g_1, \dots, g_n) &:= \pi(g_1)f'(g_1^{-1}g_2, g_2^{-1}g_3, \dots, g_{n-1}^{-1}g_n) \end{aligned}$$

for $f \in C_b(G^n, \mathbb{R})^G$ and $f' \in C_b(G^{n-1}, \mathbb{R})$ to give a non-homogeneous definition of continuous bounded cohomology.

EXERCISE 2. Compute

- (1) $H_{cb}^0(G, \mathbb{R})$
- (2) $H_{cb}^1(G, \mathbb{R})$

EXERCISE 3. Let $\mathcal{QM}(G, \mathbb{R})$ be the space of *quasimorphisms* of G , where $f : G \rightarrow \mathbb{R}$ is quasimorphism if it satisfies

$$\sup_{g_1, g_2 \in G} |f(g_1g_2) - f(g_1) - f(g_2)| < +\infty.$$

- (1) Show that

$$\ker\{H_b^2(G, \mathbb{R}) \rightarrow H^2(G, \mathbb{R})\} \cong \mathcal{QM}(G, \mathbb{R})/(\ell^\infty(G) + \text{Hom}(G, \mathbb{R}))$$

is an isomorphism of vector spaces.

- (2) Let \mathbb{F}_2 be the free group in two generators. Show that $H^2(\mathbb{F}_2, \mathbb{R}) = 0$ but try to show that $H_b^2(\mathbb{F}_2, \mathbb{R})$ is infinite dimensional.

EXERCISE 4. Show that if G is amenable then $H_{cb}^n(G, \mathbb{R}) = 0$ for all $n \geq 1$.