LECTURES ON KLEINIAN GROUPS

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1. Hyperbolic n-space as the projectivization of the set of time-like vectors in $R^{n,1}$. The isometry group is $G = PO(n,1) \cong SO(n,1)$, it is a matrix group.

Note on other symmetric spaces of rank 1 (negative curvature): CH^n, HH^n, OH^2.

Basic negative curvature property:

a. Unique geodesic between any pair of points.

b. Distance function is convex (convex along any pair of geodesics).

c. Geodesic triangles are δ-thin for some δ.

2. Upper half-space Riemannian model of $X = H^n$ and unit ball model. Balls, horoballs and hyperballs.

Compactification $\bar{X} = X \cup \partial_\infty X$. Stabilizier of point at infinity $\cong Sym(E^{n-1})$, group of Euclidean similarities.

Classification of isometries: Hyperbolic/loxodromic, parabolic, elliptic.

Exercise: For an isometry g define

$$U_{g,r} = \{x \in X : d(gx, x) < r\}.$$

Show that U is convex. Work out examples in the case of orientation-preserving isometries of H^3; orientation-reversing parabolic isometry of H^3.

Exercise: If α, β are parabolic and hyperbolic, fix the same point then $\langle \alpha, \beta \rangle$ is non-discrete.

Moral: Hyperbolic and parabolic isometries are like dogs and cats: They do not get along.

3. Cartan decomposition $G = KAK$, where $A \cong R_+$, $K = O(n)$.

4. Convergence property for sequences of isometries g_n of $X = H^n$: (g_n) contains a subsequence which either converges to an isometry or converges to a constant map to a point in $\partial_\infty X$ (away from one point in $\partial_\infty X$).

5. Kleinian groups: Discrete subgroups $\Gamma < G$: Properly discontinuous groups of isometries of X. Classical Kleinian groups: $n = 3$; Fuchsian groups: $n = 2$.

6. Dynamical decomposition of S^{n-1}:

$$S^{n-1} = \Omega \sqcup \Lambda$$

where Λ is the limit set (locus of chaotic dynamics) and Ω is the domain of discontinuity.

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Elementary groups: $|\Lambda| \leq 2 \iff \Gamma$ is virtually free abelian of rank $\leq n-1$. Each is either of hyperbolic type (contains no parabolics but has a hyperbolic element, preserves its axis) or of parabolic type (contains no hyperbolics, but some parabolics), elliptic type (is finite). In the first case, it is virtually Z, in the second case, can be of any virtual rank between 1 and $n-1$.

Exercise: Each elementary subgroup is contained in a maximal one.

Note: For other negatively curved symmetric spaces: Instead of free abelian - virtually nilpotent.

Convex hull of Γ: $C_\Gamma \subset X$ is the closed convex hull of the union of all geodesics connecting points of Λ. It is nonempty unless Γ is elementary of elliptic/parabolic type. Is clearly Γ-invariant; its ideal boundary is Λ (unless Γ is elementary of parabolic type).

7. Four faces of Kleinian groups:

a. Algebraic (as abstract group).

b. Geometric (geometry of the hyperbolic manifold/orbifold $M = X/\Gamma$.

c. Topological (topology of M).

d. Dynamical: Dynamics of Γ on Λ.

Questions: Which groups are isomorphic to Kleinian groups? Which manifolds are homeomorphic/diffeomorphic to their quotient-manifolds (orbifolds)? How does algebraic structure of Γ interact with geometry of the hyperbolic manifold/orbifold X/Γ and dynamics of Γ on Λ?

8. Algebraic restrictions:

a. Malcev/Selberg for f.g. matrix groups. Allow reduction to torsion-free groups and quotient hyperbolic manifolds. In particular f.g. Kleinian groups have finite virtual cohomological dimension.

b. Infinite Kleinian groups are never simple. (Easier for f.g. groups, do not need discreteness.) Contrast with $G(\mathbb{Q})$, e,g. $PO(2,1;\mathbb{Q})$ - simple!

Open problem: Discrete subgroups of $GL(n,\mathbb{R})$ are never simple.

c. Torsion-free Kleinian groups are virtually CT (commutation-transitive): If a commutes with b, b commutes with c then a^N commutes with c^N for large N.

d. Kleinian Tits alternative: Either Γ contains F_2 or Γ is elementary.

In particular, if Γ is amenable then it is virtually abelian.

e*. [Carlson and Toledo] If a Kleinian group Γ is a Kähler group then Γ is virtually a surface group or a virtually abelian group (of virtually even rank).

f*. [Joulg-?] Kleinian groups satisfy Property H (Haagerup): Each Γ admits a proper isometric (affine) action on a separable Hilbert space.

Note: Parts e* and f* are much harder and fail for other negatively curved symmetric spaces: e* fails for CH^n and f* fails for HH^n, OH^2.

Challenge: Find other algebraic restrictions on Kleinian groups.

Open Problem 0.1. *Suppose that Γ is a f.g. Coxeter group which contains no $\mathbb{Z} \times (\mathbb{Z} \star \mathbb{Z})$. Then Γ is isomorphic to a Kleinian group.*

9. Dirichlet fundamental domain: Pick $x \in X$ not fixed by any nontrivial element of Γ. Define

$$D = D_x = \{y \in X : d(y,x) \leq d(y, \gamma x) \forall \gamma \in \Gamma\}.$$

Note: D is intersection of geodesic half-spaces bounded by bisectors. In particular, D is convex. D tiles X under Γ: (i)

$$\gamma D \cap D \subset \partial D$$

unless $\gamma = 1$.

(ii)

$$\bigcup_{\gamma \in \Gamma} \gamma D = X.$$

(iii) The tiling is locally finite.

Therefore, $M = D/\sim$.

Same at infinity when using $D_\infty = \partial_\infty D \cap \Omega$.

Note on other symmetric spaces; D is no longer convex, but D is *starlike from* x. Prove that half-space defined by bisector is starlike.

Corollary 0.2. *D is (uniformly) quasiconvex — the correct generalization of convexity in δ-hyperbolic setting.*

Classical Geometric Finiteness: Γ has finitely-sided domain D.

Exercise: Work out D in the case of a skew-motion in R^2; R^3. Infinitely-sided in the case of irrational rotation and x not on axis.

Open Problem 0.3. *Let $\Gamma < PO(3,1)$ be torsion-free and $x \in \mathbb{H}^3$ is generic. Then the associated tiling on $\mathbb{H}^3$ by Γ-orbit of D_x is* simple*: Each vertex is adjacent to exactly four* tiles.

10. Zassenhaus-Kazhdan-Margulis Lemma/Theorem. Thick-thin decomposition.

Theorem 0.4. *There exists $\mu = \mu_n$ such that for every $x \in X$ and discrete $\Gamma < G$, the group $\Gamma_{\mu,x}$ generated by "short" elements of Γ:*

$$\{\gamma \in \Gamma : x \in U_{\gamma,\mu}\}$$

is elementary.

Define

$$X_{thin} = U_\Gamma = \{x \in X : |\Gamma_{\mu,x}| = \infty\},$$

the *thin part* of X with respect to Γ. This is a closed Γ-invariant subset of X. Accordingly, we get a thick-thin decomposition of M:

$$M_{thin} = U_\Gamma/\Gamma,$$
$$M_{thick} = U_\gamma^c/\Gamma$$

Point of this decomposition: M_{thin} has simple topology, while M_{thick} has simple (local) geometry since open $\mu/2$-balls are isometric to balls in X.

Example. $X = H^2$. Thin regions are (pairwise disjoint) horoballs and "hyperballs": R-neighborhoods of geodesics in X. Picture!

What do we know about μ_n? Not much. One can compute it for small dimensions; $\mu_n \to 0$ as $n \to \infty$.

Geometry/topology of thin parts. Define $E(\Gamma)$ as the set of maximal infinite elementary subgroups of Γ. Then

$$X_{thin} = \bigcup_{\Gamma_e \in E(\Gamma)} U_{\Gamma_e}$$

In view of ZKM-lemma, this union is disjoint. For each Γ_e:

$$U_{\Gamma_e} = \bigcup_{\gamma\in\Gamma, o(\gamma)=\infty} U_{\gamma,\mu},$$

(nondisjoint!) union of closed convex subsets.

Exercise: If $n \le 3$ and Γ_e preserves orientation (for $n = 3$) then U_{Γ_e} is convex, equals either to a horoball or to a hyperball.

General dimension (true for all symmetric spaces):

(1) If Γ_e is "hyperbolic", i.e. has an invariant axis A then $U = U_{\Gamma_e}$ is starlike with respect to A: If $x \in U$ and ρ is a perpendicular geodesic from x to A then $\rho \subset U$. Only finitely many elements of Γ_e contribute to the definition of U. The quotient is compact; in torsion-free case, homeomorphic to the solid torus $D^{n-1} \times S^1$.

(2) If Γ_e is "parabolic", i.e. has a (unique) fixed point ξ at infinity and no invariant geodesics (hyperbolic elements) then U is starlike with respect to ξ: If $x \in U$ then geodesic ray $x\xi$ is also in U. The quotient by Γ_e is noncompact compact. Infinitely many elements can contribute.

Corollary: U_{Γ_e} is contractible (actually, is homeomorphic to X) and is quasiconvex.

Hence,

$$X_{thick} \cup X_{thin,hyperbolic}$$

is also contractible.

Exercise: Work out the case of irrational (parabolic) skew motion in H^4.

Definition 0.5. Cusps of M are projections to M of parabolic thin components.

11. Geometric finiteness. Art of doable. Life is good.

Motivation: Γ is a lattice if X/Γ has finite volume $\iff$ M_{thick} is compact.

Several equivalent definitions of Geometric Finiteness (GF):

The first two definitions are geometric:

Definition 0.6 (GF1)**.** Γ is GF if its Dirichlet domain is finitely-sided away from parabolic thin components of X_{thin}.

In particular, each GF group is finitely presented.

Definition 0.7 (GF2)**.** Γ is GF if its is virtually torsion-free and

$$mes(C/\Gamma) < \infty$$

where mes is the hyperbolic measure whose dimension is the (toplogical) dimension of C.

Note: Fails without virtually torsion free assumption (for $n \geq 4$): E. Hamilton. This assumption is not needed if $n \leq 3$ and if Γ is a lattice.

The next two definitions are topological:

Definition 0.8 (GF3)**.** Γ is GF if C_{thick}/Γ is compact.

Corollary 0.9. *1. Thin part of M consists only of finitely many components. $E_{par}(\Gamma)/\Gamma$ is finite; Γ contains only finitely many conjugacy classes of finite subgroups.*

2. M is topologically tame, i.e. is homeomorphic to the interior of manifold/orbifold with boundary.

3. If $n = 2$ then GF if equivalent to finite generation.

Retraction $\pi : X \cup \Omega \to C$.

Definition 0.10 (GF4)**.** Γ is GF if $\pi^{-1}(C_{thin})/\Gamma$ is compact.

We now get to dynamical interpretation of geometric finiteness.

Definition 0.11 (Conical limit points)**.** Conical limit point:

a. From hyperbolic viewpoint.

b. From the topological dynamics viewpoint (dynamics on the limit set).

Conical limit set is denoted Λ_c.

Definition 0.12 (Bounded parabolic limit point)**.** Bounded parabolic limit point:

$$(\Lambda \setminus \{\xi\})/\Gamma_\xi$$

is compact.

Exercise. Parabolic fixed points are never conical.

Definition 0.13 (GF5)**.** Γ is GF if every limit point of Γ is either conical or bounded parabolic.

Definition 0.14 (GF6)**.** (Bishop) Γ is GF if $\Lambda \setminus \Lambda_c$ is at most countable.

Note: In Bishop's paper the proof is only for $n = 3$ but it works in all dimensions (and even variable curvature).

Convex-cocompactness (CC): Life is even better without parabolics.

Definition 0.15 (CC1)**.** Γ is CC (convex-cocompact) if Γ is GF w/o parabolics.

Definition 0.16 (CC2)**.** Γ is CC iff it has no parabolics and has a finitely sided Dirichlet domain.

Definition 0.17 (CC3)**.** γ is CC if it has no parabolics and every convex fundamental polyhedron of Γ is finitely-sided.

Definition 0.18 (CC4)**.** Γ is CC if it has only conical limit points.

Definition 0.19 (CC5)**.** Γ is CC if C/Γ is compact.

Definition 0.20 (CC6)**.** Γ is CC if $(X \cup \Omega)/\Gamma$ is compact.

12. **Critical exponent.** This is a single most important number associated with a Kleinian group; akin to dimension of a space.

Poincare series of a discrete group:

$$P_{\Gamma,s,x} = \sum_{\gamma\in\gamma} e^{-sd(x,\gamma x)}$$

Convergence of PS is independent of x.

Definition 0.21. Critical exponent $\delta = \delta_\Gamma$ is defined as

$$\inf\{s : P_{\Gamma,s} < \infty\}.$$

Equivalently, it is the exponential growth rate of

$$card\,(B(x,R)\cap\Gamma\cdot x) \subset \mathbb{H}^n.$$

Theorem 0.22. *(Sullivan, Tukia, et al.) For nonelementary* Γ*,*

$$\delta = dim_{Hau}(\Lambda_c),$$

Hausdorff dimension of Λ_c*.*

In particular, $\delta \le n-1$.

13. Groups with small limit sets.

Recall: $dim_{Hau} \ge dim_{Leb}$ (Lebesgue covering dimension). If $dim_{Hau} < 1$ then Λ is totally disconnected (Cantor set or finite).

Theorem 0.23. *If* $dim(\Lambda) = 0$ *then* Γ *is GF iff it is finitely generated.*

Conjecture 0.24. *If* $\delta < 2$ *then* Γ *is GF iff it is finitely generated.*

References

[1] C. Bishop, *On a theorem on Beardon and Maskit*, Ann. Acad. Sci. Fenn., vol. 21, (1996) p. 383–388.
[2] B. Bowditch, *Geometrical finiteness for hyperbolic groups*, Journal of Functional Anal., **113** (1993), p. 245–317.
[3] B. Bowditch, *Geometrical finiteness with variable negative curvature*, Duke Math. J., **77** (1995) p. 229–274.
[4] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Birkhauser, 2000.
[5] M. Kapovich, *Kleinian groups in higher dimensions*, In "Geometry and Dynamics of Groups and Spaces. In memory of Alexander Reznikov", M.Kapranov et al (eds). Birkhauser, Progress in Mathematics, Vol. 265, 2007, p. 485–562.
[6] P. Nicholls, *The ergodic theory of discrete groups*, Cambridge University Press, 1989.