ALGEBRAIC TRANSVERSALITY

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What is algebraic transversality?

- Geometric transversality is one of the most important properties of manifolds, dealing with the construction of submanifolds.
- Easy to establish for smooth manifolds (Thom, 1954)
- Hard to establish for topological manifolds (Kirby-Siebenmann, 1970), and that only for dimensions $\geqslant 5$.
- Algebraic transversality deals with the construction of subcomplexes of chain complexes over group rings.
- Algebraic transversality is needed to quantify geometric transversality, to control the algebraic topology of the submanifolds created by the geometric construction.

Codimension k subspaces

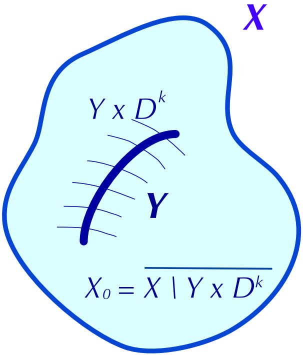
- **Definition** A **framed codimension k subspace** of a space X is a closed subspace $Y \subset X$ such that X has a decomposition

$$X \;=\; X_0 \cup_{Y \times S^{k-1}} Y \times D^k \;,$$

with the complement

$$X_0 \;=\; \mathrm{cl.}(X \backslash Y \times D^k) \subset X$$

a closed subspace homotopy equivalent to $X \backslash Y$.



Geometric transversality

- **Theorem** (Thom, 1954) Every map $f : M^m \to X$ from a smooth m-dimensional manifold to a space X with a framed codimension k subspace $Y \subset X$ is homotopic to a smooth map (also denoted f) which is transverse regular at $Y \subset X$, so that
$$N^{m-k} \; = \; f^{-1}(Y) \subset M$$
is a framed codimension k submanifold with
$$f \; = \; f_0 \cup g \times 1_{D^k} \; : \; M \; = \; M_0 \cup N \times D^k \to X \; = \; X_0 \cup Y \times D^k \; .$$

- Algebraic transversality studies analogous decompositions of chain complexes! Particularly concerned with homotopy equivalences and contractible chain complexes.

The infinite cyclic cover example of geometric transversality I.

- $X = S^1$ has framed codimension 1 subspace $Y = \{*\} \subset S^1$ with complement $X_0 = I$

$$S^1 \ = \ I \cup_{\{*\} \times S^0} \{*\} \times D^1 \ .$$

- By geometric transversality every map $f : M^m \to S^1$ is homotopic to a map transverse regular at $\{*\} \subset S^1$, with

$$N^{m-1} = f^{-1}(*) \subset M$$

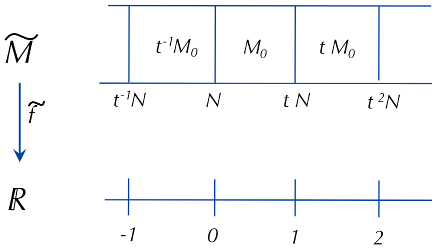
a framed codimension 1 submanifold with complement $M_0 = f^{-1}(I)$

$$M \ = \ M_0 \cup_{N \times S^0} N \times D^1 \ .$$

The infinite cyclic cover example of geometric transversality II.

- The pullback infinite cyclic cover of M has fundamental domain $(M_0; N, tN)$

$$\widetilde{M} = f^*\mathbb{R} = \bigcup_{j=-\infty}^{\infty} t^j(M_0; N, tN) .$$



The infinite cyclic cover example of algebraic transversality

- **Proposition** (Higman, Waldhausen, R.)
  For every finite f.g. free $\mathbb{Z}[t, t^{-1}]$-module chain complex C there exists a finite f.g. free $\mathbb{Z}$-module subcomplex $C_0 \subset C$ with $D = C_0 \cap tC_0$ a finite f.g. free $\mathbb{Z}$-module chain complex, and the $\mathbb{Z}$-module chain maps

$$\begin{aligned} i_0 &: D \to C_0 \ ; \ x \mapsto x \ , \\ i_1 &: D \to C_0 \ ; \ x \mapsto t^{-1}x \end{aligned}$$

  such that there is defined a short exact sequence of finite f.g. free $\mathbb{Z}[t, t^{-1}]$-module chain complexes

$$0 \longrightarrow D[t, t^{-1}] \xrightarrow{i_0 - ti_1} C_0[t, t^{-1}] \longrightarrow C \longrightarrow 0$$

- Note that if C is contractible then C_0, D need not be contractible.
- Can replace $\mathbb{Z}$ by any ring A.

Split homotopy equivalences

- **Definition** A homotopy equivalence $f : M \to X$ from a smooth m-dimensional manifold **splits** at a framed codimension k subspace $Y \subset X$ if f is transverse regular at $Y \subset X$, and the restrictions

$$g \;=\; f| \;:\; N \;=\; f^{-1}(Y) \to Y \;,$$
$$f_0 \;=\; f| \;:\; M_0 \;=\; M\backslash N \to X_0 \;=\; X\backslash Y$$

  also homotopy equivalences.
- **Definition** f **splits up to homotopy** if it is homotopic to a homotopy equivalence (also denoted by f) which is split.
- In general, homotopy equivalences do not split up to homotopy. Surgery theory provides splitting obstructions.

The uniqueness of smooth manifold structures

- **Surgery Theory Question** Is a homotopy equivalence $f : M \to X$ of smooth m-dimensional manifolds homotopic to a diffeomorphism?
- **Answer** No, in general. The Browder-Novikov-Sullivan-Wall theory (1960's) provides obstructions in homotopy theory and algebra, and systematic construction of counterexamples. For $X = S^m$ this is the Kervaire-Milnor classification of exotic spheres.
- **Example** Diffeomorphisms are split. If f is homotopic to a diffeomorphism then f splits up to homotopy at every submanifold $Y \subset X$.
- **Contrapositive** If f does not split up to homotopy at a submanifold $Y \subset X$ then f is not homotopic to a diffeomorphism.

The uniqueness of topological manifold structures

- **Surgery Theory Question** Is a homotopy equivalence $f : M \to X$ of topological m-dimensional manifolds homotopic to a homeomorphism?
- **Answer** No, in general. As in the smooth case, surgery theory provides systematic obstruction theory for $m \geqslant 5$. Need Kirby-Siebenmann (1970) structure theory for topological manifolds.
- **Example** Homeomorphisms are split. If f is homotopic to a homeomorphism then f splits up to homotopy at every submanifold $Y \subset X$.
- **Contrapositive** If f does not split up to homotopy at a submanifold $Y \subset X$ then f is not homotopic to a homeomorphism.

The Borel Conjecture

- **BC** (1953) Every homotopy equivalence $f : M \to X$ of smooth m-dimensional aspherical manifolds is homotopic to a homeomorphism.
- http://www.maths.ed.ac.uk/ aar/surgery/borel.pdf
  Birth of the Borel rigidity conjecture.
- In the last 30 years the conjecture has been verified in many cases, using surgery theory, geometric group theory and differential geometry (Farrell-Jones, Lück).

The existence of smooth manifold structures

- A smooth m-dimensional manifold M is a finite CW complex with m-dimensional Poincaré duality $H^{m-*}(M) \cong H_*(M)$
- **Surgery Theory Question** If X is a finite CW complex with m-dimensional Poincaré duality isomorphisms

  $$H^{m-*}(X) \ \cong \ H_*(X) \text{ (with } \mathbb{Z}[\pi_1(X)]\text{-coefficients)}$$

  is X homotopy equivalent to a smooth m-dimensional manifold?
- The Browder-Novikov-Sullivan-Wall surgery theory deals with both existence and uniqueness, providing obstructions in terms of homotopy theory and algebra.
- Various examples of Poincaré duality spaces not of the homotopy type of smooth manifolds

The existence of topological manifold structures

- **Surgery Theory Question** If X is a finite CW complex with m-dimensional Poincaré duality isomorphisms is X homotopy equivalent to a topological m-dimensional manifold?
- For $m \geqslant 5$ the Browder-Novikov-Sullivan-Wall surgery theory provides algebraic obstructions. The reduction to pure algebra makes use of algebraic transversality and codimension 1 splitting obstructions (R.,1992).

Obstructions to splitting homotopy equivalences up to homotopy

- In general, homotopy equivalences of manifolds are not split up to homotopy, in both the smooth and topological categories.
- There are algebraic K and L-theory obstructions to splitting homotopy equivalences up to homotopy for $m - k \geqslant 5$ (Browder, Wall, Cappell 1960's and 1970's).
- Waldhausen (1970's) dealt with the case $m = 3$, $k = 1$, motivated by the Haken theory of 3-manifolds.
- Cappell (1974) constructed homotopy equivalences

  $$f \ : \ M^m \to X \ = \ \mathbb{R}\mathbb{P}^m \# \mathbb{R}\mathbb{P}^m$$

  which cannot be split up to homotopy, for $m \equiv 1 (\bmod 4)$ with $m \geqslant 5$, and $Y = S^{m-1} \subset X$.
- Same algebraic K- and L-theory obstructions to decomposing Poincaré duality space as $X = X_0 \cup Y \times D^k$, with Y codimension k Poincaré. (R.)

CW complexes and chain complexes I.

- Given a *CW* complex X and a regular cover $\widetilde{X}$ with group of covering translations π let $C(\widetilde{X})$ be the cellular chain complex, a free $\mathbb{Z}[\pi]$-module chain complex with one generator for each cell of X.subcomplex
- A map $f : M \to X$ from a *CW* complex induces a π-equivariant map $\widetilde{f} : \widetilde{M} = f^*\widetilde{X} \to \widetilde{X}$ of the covers, and hence a $\mathbb{Z}[\pi]$-module chain map $\widetilde{f} : C(\widetilde{M}) \to C(\widetilde{X})$.
- **Theorem** (J.H.C. Whitehead) A map $f : M \to X$ is a homotopy equivalence if and only if $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism and the algebraic mapping cone $\mathcal{C}(\widetilde{f})$ is chain contractible, with $\widetilde{X}$ the universal cover of X and $\pi = \pi_1(X)$.

CW complexes and chain complexes II.

- If $i : Y \subset X$ is the inclusion of a framed codimension k subcomplex the decomposition $X = X_0 \cup_{Y \times S^{k-1}} Y \times D^k$ lifts to a π-equivariant decomposition

$$\widetilde{X} \;=\; \widetilde{X}_0 \cup_{\widetilde{Y} \times S^{k-1}} \widetilde{Y} \times D^k$$

  with $\widetilde{Y} = i^*\widetilde{X}$ the pullback cover of Y, a framed codimension k subcomplex of $\widetilde{X}$.
- The $\mathbb{Z}[\pi]$-module chain complex of $\widetilde{X}$ has an algebraic decomposition

$$C(\widetilde{X}) \;=\; C(\widetilde{X}_0) \cup_{C(\widetilde{Y}) \otimes C(S^{k-1})} C(\widetilde{Y}) \otimes C(D^k)\ .$$

- Algebraic transversality studies $\mathbb{Z}[\pi]$-module chain complexes with such decompositions.
- If $f : M \to X$ is transverse at $Y \subset X$ the algebraic mapping cone of $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ has such a decomposition

$$\mathcal{C}(\widetilde{f}) \;=\; C(\widetilde{f}_0) \cup_{C(\widetilde{g}) \otimes C(S^{k-1})} C(\widetilde{g}) \otimes C(D^k)\ .$$

The fundamental groups in codimension 1

- If X is a connected CW complex and $Y \subset X$ is a connected framed codimension 1 subcomplex then

$$X = \begin{cases} X_1 \cup_{Y \times D^1} X_2 & \text{if } X_0 = X_1 \sqcup X_2 \text{ is disconnected} \\ X_0 \cup_{Y \times S^0} Y \times D^1 & \text{if } X_0 \text{ is connected} \end{cases}$$

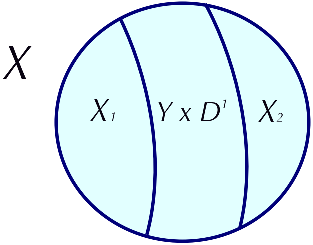
  according as to whether Y separates X or not.

- The fundamental group $\pi_1(X)$ is given by the Seifert-van Kampen theorem to be the $\begin{cases} \text{amalgamated free product} \\ HNN \text{ extension} \end{cases}$

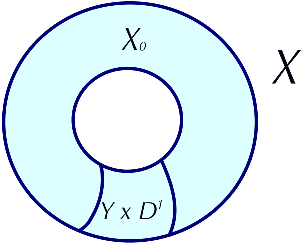
$$\pi_1(X) = \begin{cases} \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \\ \pi_1(X_0) *_{\pi_1(Y)} \{t\}. \end{cases} \quad \text{determined by the morphisms}$$

$$\begin{cases} \pi_1(Y) \to \pi_1(X_1)\ ,\ \pi_1(Y) \to \pi_1(X_2) \\ \pi_1(Y \times \{0\}) \to \pi_1(X_0)\ ,\ \pi_1(Y \times \{1\}) \to \pi_1(X_0)\ . \end{cases}$$

Separating and non-separating codimension 1 subspaces



Y separates X



Y does not separate X

Handle exchanges I.

- Will only deal with the separating case.
- Let M be an m-dimensional manifold with a separating framed codimension 1 submanifold $N^{m-1} \subset M$, so that

$$M = M_1 \cup_N M_2 .$$

- A **handle exchange** uses an embedding

$$(D^r \times D^{m-r}, S^{r-1} \times D^{m-r}) \subset (M_i, N) \ (i = 1 \text{ or } 2)$$

  to obtain a new decomposition

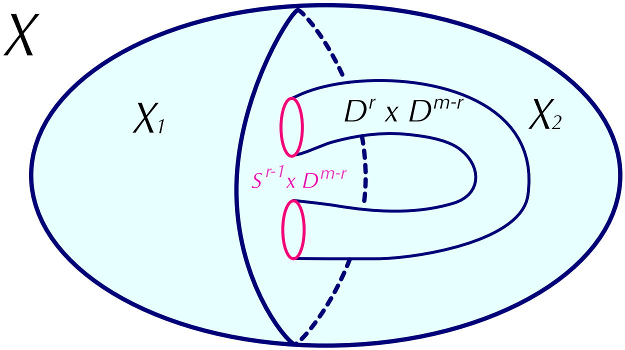
$$M = M'_1 \cup_{N'} M'_2$$

  with

$$\begin{aligned} N' &= \text{cl.}(N \backslash S^{r-1} \times D^{m-r}) \cup D^r \times S^{m-r-1} , \\ M'_i &= \text{cl.}(M_i \backslash D^r \times D^{m-r}) , \\ M'_{2-i} &= M_{2-i} \cup D^r \times D^{m-r} . \end{aligned}$$

- Initiated by Stallings ($m = 3$) and Levine in the 1960's.

Handle exchanges II.



$$X'_1 \;=\; X_1 \cup D^r \times D^{m-r} \;,\; X'_2 \;=\; \text{cl.}(X_2 \backslash D^r \times D^{m-r}) \;.$$

Codimension 1 geometric transversality I.

- Let $X = X_1 \cup_Y X_2$ be a connected CW complex with a separating connected framed codimension 1 subspace $Y \subset X$ such that $\pi_1(Y) \to \pi_1(X)$ is injective. Then

$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \;=\; \pi$$

  with injective morphisms

$$\pi_1(Y) \;=\; \rho \to \pi_1(X_1) \;=\; \pi_1\ ,\ \pi_1(Y) \;=\; \rho \to \pi_1(X_2) \;=\; \pi_2\ .$$

- The Bass-Serre tree T is a contractible non-free π-space with

$$T^{(0)} \;=\; [\pi:\pi_1] \cup [\pi:\pi_2]\ ,\ T^{(1)} \;=\; [\pi:\rho]\ ,\ T/\pi \;=\; I\ .$$

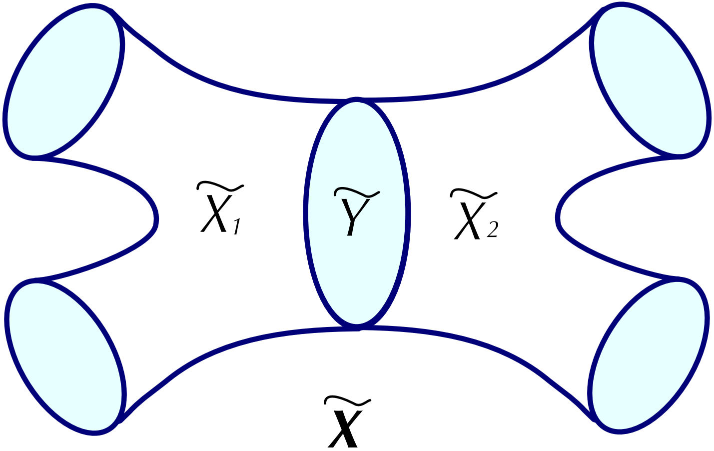
- The universal cover of X decomposes as

$$\widetilde{X} \;=\; [\pi:\pi_1] \times \widetilde{X}_1 \cup_{[\pi:\rho]\times\widetilde{Y}} [\pi:\pi_2] \times \widetilde{X}_2$$

  with $\widetilde{X}_1, \widetilde{X}_2, \widetilde{Y}$ the universal covers of X_1, X_2, Y, and

$$\widetilde{Y} \;=\; \widetilde{X}_1 \cap \widetilde{X}_2 \subset \widetilde{X}\ .$$

The universal cover $\widetilde{X}$ of $X = X_1 \cup_Y X_2$



Codimension 1 geometric transversality II.

- If X is finite the cellular f.g. free $\mathbb{Z}[\pi]$-module chain complex $C(\widetilde{X})$ has f.g. free $\mathbb{Z}[\pi_i]$-module chain subcomplexes $C(\widetilde{X}_i) \subset C(\widetilde{X})$ and a f.g. free $\mathbb{Z}[\rho]$-module chain subcomplex

$$C(\widetilde{Y}) \ = \ C(\widetilde{X}_1) \cap C(\widetilde{X}_2) \subset C(\widetilde{X})$$

  with a short exact Mayer-Vietoris sequence of f.g. free $\mathbb{Z}[\pi]$-module chain complexes

$$0 \longrightarrow \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\rho]} C(\widetilde{Y}) \longrightarrow \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1]} C(\widetilde{X}_1) \oplus \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_2]} C(\widetilde{X}_2) \longrightarrow C(\widetilde{X}) \longrightarrow 0 \ .$$

- If $f : M \to X$ is a homotopy equivalence of m-dimensional manifolds there is no obstruction to making f transverse regular at $Y \subset X$, but there are algebraic K- and L-theory obstructions to splitting f up to homotopy, involving the MV sequence of the contractible $\mathbb{Z}[\pi]$-module chain complex $\mathcal{C}(\widetilde{f} : \widetilde{M} \to \widetilde{X})$ and algebraic handle exchanges.

Codimension 1 algebraic transversality

- Let $\pi = \pi_1 *_\rho \pi_2$ be an injective amalgamated free product.
- **Proposition** (Waldhausen, R.) For any f.g. free $\mathbb{Z}[\pi]$-module chain complex C there exist f.g. free $\mathbb{Z}[\pi_i]$-module chain subcomplexes $D_i \subset C$ and a f.g. free $\mathbb{Z}[\rho]$-module chain subcomplex $E = D_1 \cap D_2 \subset C$ with a short exact MV sequence of f.g. free $\mathbb{Z}[\pi]$-module chain complexes

$$0 \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\rho]} E \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1]} D_1 \oplus \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_2]} D_2 \to C \to 0 \quad .$$

  Any two such choices (D_1, D_2, E) are related by a finite sequence of algebraic handle exchanges. If C is contractible there is an algebraic K-theory obstruction to choosing D_1, D_2, E to be contractible.
- **Corollary** (Cappell, R.) If C has m-dimensional Poincaré duality there is an algebraic L-theory obstruction to choosing $(D_i, \mathbb{Z}[\pi_i] \otimes_{\mathbb{Z}[\rho]} E)$ to have m-dimensional Poincaré-Lefschetz duality and E to have $(m-1)$-dimensional Poincaré duality.

Universal transversality

- Let X be a finite simplicial complex, with barycentric subdivision X' and dual cells

$$D(\sigma) = \{\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_r \,|\, \sigma \leqslant \sigma_0 < \sigma_1 < \dots < \sigma_r\} \ .$$

- A map $f : M \to |X| = |X'|$ from an m-dimensional manifold is **universally transverse** if each inverse image

$$M(\sigma) = f^{-1}(D(\sigma)) \subset M$$

is a framed codimension $|\sigma|$ submanifold with boundary

$$\partial M(\sigma) = \bigcup_{\tau > \sigma} M(\tau) \ .$$

- The algebraic obstruction theory for the existence and uniqueness of topological manifold structures in a homotopy type uses algebraic universal tranvsersality.

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