LINK FLOER HOMOLOGY CATEGORIFIES
THE CONWAY FUNCTION

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Abstract   Given an oriented link in the 3-sphere, the Euler characteristic of its link Floer homology is
known to coincide with its multi-variable Alexander polynomial, an invariant only defined up to a sign and
powers of the variables. In this paper we remove this ambiguity by proving that this Euler characteristic
is equal to the so-called Conway function, the representative of the multi-variable Alexander polynomial
introduced by Conway in 1970 and explicitly constructed by Hartley in 1983. This is achieved by creating
a model of the Conway function adapted to rectangular diagrams, which is then compared to the Euler
characteristic of the combinatorial version of link Floer homology.

1. Introduction

The Alexander polynomial is probably the most celebrated invariant of knots and links.
Both the single-variable and the multi-variable versions were introduced by Alexander in
his seminal 1928 paper [**1**]. The latter invariant associates with a μ-component oriented
link L in the standard 3-sphere S^3 a μ-variable Laurent polynomial $\Delta_L(t_1, \ldots, t_\mu) \in
\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ defined up to units of this ring, that is, up to a sign and powers of the
variables. The indetermination in powers of the variables is not problematic. Indeed, this
polynomial is palindromic, i.e.

$$\Delta_L(t_1^{-1}, \ldots, t_\mu^{-1}) \doteq \Delta_L(t_1, \ldots, t_\mu),$$

where $\doteq$ denotes equality up to multiplication by units; hence, a natural representative in
the ring $\mathbb{Z}[t_1^{\pm 1/2}, \ldots, t_\mu^{\pm 1/2}]$ can be chosen up to a sign. If L is a knot, then the Alexander
polynomial furthermore satisfies $\Delta_L(1) = \pm 1$, so it can be normalized by setting this
value to be $+1$. In the link case, however, this sign issue is far from trivial.

In 1970, Conway [**6**] suggested a natural representative of the multi-variable Alexander
polynomial, later called the *Conway function*. Given a μ-component oriented link L, its

Conway function is a rational function $\nabla_L(t_1, \ldots, t_\mu)$ such that

$$\nabla_L(t_1, \ldots, t_\mu) \doteq \begin{cases} (t_1 - t_1^{-1})\Delta_L(t_1^2) & \text{if } \mu = 1, \\ \Delta_L(t_1^2, \ldots, t_\mu^2) & \text{if } \mu > 1. \end{cases}$$

Conway also noticed several properties of this invariant via local transformations of the link, including the celebrated *skein relation*. However, a precise definition of this invariant together with a proof of its various properties only appeared more than a decade later. In 1983 Hartley [9] gave a model for the Conway function as a well-chosen normalization of the determinant of a Fox matrix obtained from a Wirtinger presentation of the link group via Fox free calculus. Other constructions were later given by Turaev [19] using sign-refined Reidemeister torsion, and by Cimasoni [3] using generalized Seifert surfaces.

In 2004 Ozsváth and Szabó [16] introduced Heegaard–Floer homology as an invariant for closed oriented 3-manifolds. They later extended this theory to give an invariant for null-homologous oriented knots in such manifolds [15], a construction further generalized to oriented links [17]. In its most basic form, this *link Floer homology* is a finite-dimensional bi-graded vector space over $\mathbb{Z}_2$,

$$\widehat{HL}(L) = \bigoplus_{d,s} \widehat{HL}_d(L, s),$$

the direct sum ranging over all $d \in \mathbb{Z}$ and $s = (s_1, \ldots, s_\mu) \in (\frac{1}{2}\mathbb{Z})^\mu$. All these invariants were originally defined using Heegaard diagrams and the count of holomorphic discs in the symmetric product of a Riemann surface. In [11,12] a purely combinatorial description of these invariants was provided in the case of links in S^3, relying on the count of rectangles in so-called *grid diagrams* for these links.

One of the most fundamental properties of link Floer homology is that it *categorifies* the multi-variable Alexander polynomial. More precisely, its Euler characteristic

$$\chi(\widehat{HL}(L); t_1, \ldots, t_\mu) := \sum_{d,s} (-1)^d \dim(\widehat{HL}_d(L, s)) t_1^{s_1} \cdots t_\mu^{s_\mu}$$

satisfies the equality

$$\chi(\widehat{HL}(L); t_1, \ldots, t_\mu) \doteq \begin{cases} \prod_{k=1}^{\mu} (t_k^{1/2} - t_k^{-1/2}) \Delta_L(t_1, \ldots, t_\mu) & \text{if } \mu > 1, \\ \Delta_L(t_1) & \text{if } \mu = 1. \end{cases}$$

Therefore, the Euler characteristic of link Floer homology provides a natural representative for the multi-variable Alexander polynomial, and it is natural to ask whether it coincides with the Conway function.

In the present paper we give a positive answer to this question.

Theorem 1.1. *Given a μ-component oriented link L in S^3, the equation*

$$\chi(\widehat{HL}(L); t_1, \ldots, t_\mu) = \prod_{k=1}^{\mu} (t_k^{1/2} - t_k^{-1/2}) \nabla_L(t_1^{1/2}, \ldots, t_\mu^{1/2})$$

holds in $\mathbb{Z}[t_1^{\pm 1/2}, \ldots, t_\mu^{\pm 1/2}]$.

We also sketch a proof of the fact that Turaev's model coincides with Cimasoni's geometric model, which was shown to agree with Hartley's model in [**3**]. Therefore, we are in the presence of four different constructions of the same invariant. The main interest in identifying them is that some properties of the Conway function are totally transparent in one model and quite surprising in others. For example, the geometric model is very well suited for discovering skein-type relations for the Conway function, whose translation in terms of link Floer homology is not always obvious. Also, the surgery formula obtained by Boyer and Lines in [**2**] using Hartley's model does not appear to be known in link Floer homology.

It should be mentioned that even though Hartley's model and the combinatorial version of link Floer homology are only defined for links in the standard 3-sphere, the models of Turaev and of Cimasoni are defined for links in an arbitrary integral homology 3-sphere, and so is link Floer homology. The question of whether these models coincide for links in integral homology spheres is addressed in a slightly informal way in the last section of the present paper.

This paper is organized as follows. In §2 we briefly recall Hartley's construction of the Conway function ∇_L and amend it to define a model Γ_L adapted to rectangular diagrams. We then show that Γ_L satisfies Jiang's five characterizing relations [**10**] and therefore coincides with ∇_L. In §3 we define grid diagrams, recall their use in the combinatorial definition of link Floer homology, and identify $\chi(\widehat{HL}(L))$ with Γ_L. Finally, §4 discusses the identification with other models as well as the extension to homology spheres.

2. A model for the Conway function using rectangular diagrams

This section focuses on adapting Hartley's construction to a special type of diagrams, called rectangular diagrams. We first recall Hartley's definition [**9**], and introduce rectangular diagrams as well as Neuwirth's associated presentation of the link group. We then turn to the definition of our model Γ_L, show its invariance, and identify it with Hartley's model.

2.1. Hartley's construction of ∇_L

Let $L = L_1 \cup \cdots \cup L_\mu$ be a μ-component oriented link in S^3, and let $W = \langle x_1, \ldots, x_n | r_1, \ldots, r_n \rangle$ be a Wirtinger presentation of $\pi_1(S^3 \backslash L)$. Recall that the generators (respectively, the relations) are in one-to-one correspondence with the arcs (respectively, the crossings) of the diagram. Let $\theta \colon \mathbb{Z}[\pi_1(S^3 \backslash L)] \to \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the extension of the abelianization morphism, i.e. the unique morphism of rings mapping x_j to t_k if the arc corresponding to x_j belongs to L_k. For all $1 \leqslant i \leqslant n$, consider a path from a fixed point outside the diagram to a point lying at the right of both the over-crossing and under-crossing arcs of the ith crossing; this path intersects some arcs of the diagram and thus gives a word u_i in the x_js. Define

$$D(t_1, \ldots, t_\mu) = \frac{(-1)^{i+j} \det(M^{(ij)})}{\theta(u_i)(\theta(x_j) - 1)},$$

where $M = (\theta(\partial r_i/\partial x_j))_{i,j}$ is the image under θ of the Jacobian matrix of W with respect to free differential calculus (see [8]), and $M^{(ij)}$ is M with the ith row and jth column removed. This function D is a rational function that does not depend on the choice of i and j, but still needs to be normalized to give a well-defined invariant. This is done as follows.

For each component L_k of the link, consider the projection of that component and resolve every crossing as illustrated by
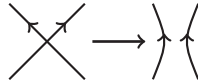
Once this process is completed, one obtains a collection of oriented circles. The *curvature* κ_k of L_k is defined as the number of anticlockwise circuits minus the number of clockwise circuits. Finally, let η_k be the number of crossings where the over-crossing arc is L_k.

The *Conway function* of L is the rational function given by

$$\nabla_L(t_1,\ldots,t_\mu) := D(t_1^2,\ldots,t_\mu^2)t_1^{\kappa_1-\eta_1}\cdots t_\mu^{\kappa_\mu-\eta_\mu}.$$

As checked by Hartley [9], it is a well-defined link invariant. Furthermore, it was recently established by Jiang [10] that it is characterized by a relatively simple set of five relations (see § 2.4).

2.2. Rectangular diagrams

Definition 2.1. A *rectangular diagram* is a link diagram D with the conditions that

(i) the diagram is composed of horizontal and vertical line segments only,

(ii) at each crossing the vertical segment is the over-crossing and the horizontal one the under-crossing,

(iii) no two segments are collinear.

Clearly, any link can be represented by a rectangular diagram. For example, the transformation
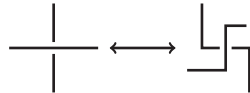
can be used to ensure that condition (ii) is satisfied. Note that because of this condition, there is no need to differentiate the over-crossings and under-crossings in such a diagram. Also, condition (iii) allows for the diagram to be included in a grid whose rows and columns contain, respectively, a unique horizontal and vertical segment. We shall call the turning points of segments *corners*; there are four types of corners, all of which appear in the example for the trivial knot given in Figure 1.

Cromwell [7] proved that any two rectangular diagrams describing the same link can be connected through a finite sequence of the following elementary moves and their inverses.

(G1) *Cyclic permutation of vertical (respectively, horizontal) segments*: the leftmost (respectively, topmost) segment is moved to the rightmost (respectively, bottommost) position.

(G2) *Commutation of horizontal or vertical segments*: two adjacent columns or rows can be permuted if, when one is projected onto the other, their images are either disjoint or one is completely contained in the other.

(G3) *Stabilization*: at a selected corner of the diagram, add a row and a column next to the corner. There is a choice of adding the new column to the right or left of the selected column and a choice of over or under the selected row for the new one, which yields four different stabilizations for each type of corner. Using the previous moves, it is enough, however, to consider that operation for one type of corner only.

There exists a presentation of the link group suited for rectangular diagrams, introduced by Neuwirth in [**14**]. The generators are in one-to-one correspondence with the vertical segments of the diagram (numbered from left to right). Suppose that there are n vertical segments, then there are also n horizontal segments. This presentation has $n-1$ relations, one for each *line* in the diagram, i.e. for each position between two consecutive horizontal segments. Consider a path crossing the diagram at constant height between the ith and $(i+1)$th horizontal segments (counted from the top). In terms of generators it corresponds to the product of the vertical segments crossed. This product is the ith relation r_i.

This presentation is well behaved under the action of Fox's free derivatives: indeed, any relation is of the form $r = x_{j_1} \cdots x_{j_m}$ with different j_ℓs, so

$$\frac{\partial r}{\partial x_j} = \begin{cases} x_{j_1} \cdots x_{j_{\ell-1}} & \text{if } j = j_\ell,\ 1 \leqslant \ell \leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

Given a rectangular diagram D, define the associated Fox matrix as the $(n-1) \times n$ matrix F_D with (i,j)-coefficient $\theta(\partial r_i / \partial x_j)$, with θ the abelianization morphism as above. Note that the non-zero entries of F_D correspond to the intersections of the vertical segments of the diagram with the lines of the diagram. Furthermore, such a non-zero coefficient at position (i,j) is given by $t_1^{\varphi_1} \cdots t_\mu^{\varphi_\mu}$, where φ_k is the algebraic intersection number of L_k with the ith line left of the jth vertical segment. (Here, an intersection point is counted positively if L_k is oriented upwards.) Note the equality $\sum_j (\partial r_i / \partial x_j)(x_j - 1) = r_i - 1$, which implies that

$$\sum_{j=1}^n (\theta(x_j) - 1)\theta\left(\frac{\partial r_i}{\partial x_j}\right) = 0 \tag{2.1}$$

for all $1 \leqslant i \leqslant n$.

Example 2.2. Examples of rectangular diagrams for the unknot K, the Hopf link H and the trefoil T are illustrated in Figure 1. The corresponding Fox matrices are given
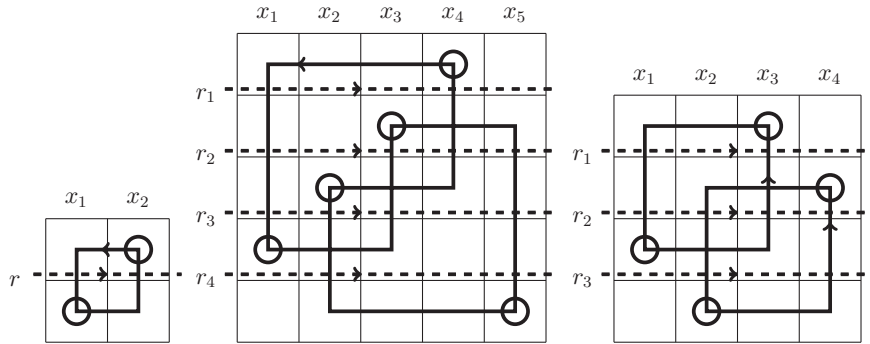
Figure 1. Three examples of rectangular diagrams for the trivial knot, the trefoil and the Hopf link. The corresponding bases are given by the permutations $x_0 = (12)$, $x_0 = (14)(23)$ and $x_0 = (13)(24)$.

by

$$F_K = \begin{pmatrix} 1 & t^{-1} \end{pmatrix},$$

$$F_T = \begin{pmatrix} 1 & 0 & 0 & t^{-1} & 0 \\ 1 & 0 & t^{-1} & 1 & t \\ 1 & t^{-1} & 1 & 0 & t \\ 0 & 1 & 0 & 0 & t \end{pmatrix} \quad \text{and} \quad F_H = \begin{pmatrix} 1 & 0 & t_1^{-1} & 0 \\ 1 & t_1^{-1} & t_1^{-1}t_2^{-1} & t_2^{-1} \\ 0 & 1 & 0 & t_2^{-1} \end{pmatrix}.$$

2.3. Definition and invariance of the model Γ_L

From this section onwards we will use the notation $t = (t_1, \ldots, t_\mu)$, and $t^s = \prod_{k=1}^{\mu} t_k^{s_k}$ for $s = (s_1, \ldots, s_\mu) \in (\frac{1}{2}\mathbb{Z})^\mu$. Additionally, the vector $(b, \ldots, b) \in (\frac{1}{2}\mathbb{Z})^\mu$ will be denoted by $\boldsymbol{b}$.

Given an oriented link L given by a rectangular diagram D, let us write

$$M(t) = \frac{(-1)^j}{\theta(x_j) - 1} \det(F_D^j),$$

where F_D^j is F_D with the jth column removed. The fact that $M(t)$ is independent of the choice of j follows from (2.1).

Each corner is either the beginning or the end of a horizontal segment; the former ones will be emphasized on the diagrams by small circles. These n circles define a permutation $x_0 \in S_n$, called the *base of the diagram*, in the following way. Number the rows top to bottom and the columns left to right from 1 to n. The rows represent the domain of the permutation, the columns the co-domain, and the pairs $(i, x_0(i))$ are given by the position of the circles. Examples are given in Figure 1.

Note that if a corner is the intersection of the ith horizontal and jth vertical segments, then only one of $(F_D)_{i-1,j}$ or $(F_D)_{i,j}$ is non-zero. We will call this coefficient the *weight* of that corner. The *total weight* ω of the diagram is defined as the product of the weights of all the corners of the diagram.

Definition 2.3. Given an oriented rectangular diagram D, define

$$\Gamma_D(t) := \frac{\mathrm{sgn}(x_0)(-1)^u}{\omega t^\kappa} M(t^2),$$

where $\mathrm{sgn}(x_0) \in \{\pm 1\}$ is the signature of the base $x_0 \in S_n$, u is the number of upwards segments, ω is the total weight, $\kappa = (\kappa_1, \ldots, \kappa_\mu)$ is the curvature, and $M(t)$ is the above normalization of the determinant of the Fox matrix associated with D.

Example 2.4. Building on Example 2.2, let us compute the value of Γ_D for the diagrams K, H and T of Figure 1. For the trivial knot K, $\mathrm{sgn}(x_0) = -1$. Using F_K, one computes $\omega = t^{-2}$. From the diagram we see that $u = \kappa = 1$. Removing the first column of F_K yields $M(t) = -t^{-1}/(t^{-1} - 1)$, and hence

$$\Gamma_K(t) = \frac{(-1)(-1)}{t^{-2}t^1} \frac{-t^{-2}}{t^{-2} - 1} = \frac{1}{t - t^{-1}}.$$

For the trefoil T, $\mathrm{sgn}(x_0) = 1$, $\omega = t^{-1}$, $u = 3$ and $\kappa = 0$. The determinant of F_T^1 equals $-t^{-2} + t^{-1} - 1$, and therefore

$$\Gamma_T(t) = (t - t^{-1})^{-1}(t^2 - 1 + t^{-2}).$$

For the Hopf link H, $\mathrm{sgn}(x_0) = 1$, $\omega = t^{-3}$, $u = 3$ and $\kappa = t^1$. The determinant of F_H^1 is given by $-t_1^{-1}t_2^{-1}(t_1^{-1} - 1)$, so

$$\Gamma_H(t) = 1.$$

Proposition 2.5. *The function Γ_D is an invariant of oriented links.*

Proof. By Cromwell's theorem, we only need to check that Γ_D is left unchanged by the moves G1, G2 and G3 described above. Throughout the proof, the unprimed quantities refer to the diagrams on the left ('before the move') and the primed ones to those on the right ('after the move'). The first pictured line of a diagram is always the ℓth, and the first column the qth. Set $\varepsilon_j = 1$ if x_j is oriented upwards and $\varepsilon_j = -1$ otherwise.

(G1) Consider first the case of vertical cyclic permutation, illustrated by
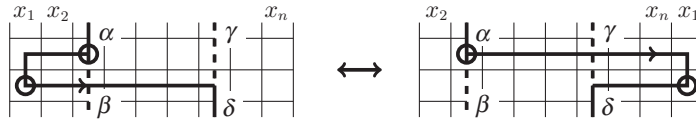
Without loss of generality, assume that the moved segment belongs to L_1 and set $\varepsilon = \varepsilon_1$. If there are m rows of the grid affected by the move, then there are $m - 1$ relations that change. (In the illustration above, $m = 2$.) This leads to the equality $\theta(\partial r_i'/\partial x_j) = t_1^{-\varepsilon}\theta(\partial r_i/\partial x_j)$ for these $m - 1$ indices i and for all js. Computing $M(t)$ by removing the first column of F, and $M'(t)$ by removing the last column of F', we obtain

$$M'(t) = (-1)^{n-1} t_1^{-(m-1)\varepsilon} M(t).$$

Note that the two bases x_0 and x_0' are related by a cyclic permutation of order n, so $\mathrm{sgn}(x_0') = (-1)^{n-1} \mathrm{sgn}(x_0)$. Also, the number of upwards segments is left unchanged by this move.

Let us now analyse how the corner weights and curvature are modified. There are $2m$ corners whose weight might change. The $2m - 2$ 'interior corners' will have their weight multiplied by $t_1^{-\varepsilon}$. As for the two remaining ones, the result will depend on the position of the vertical segment at each of these two corners. There are two possible configurations for the left one (denoted by α and β), and two for the right one (γ and δ), which are indicated in the picture above: α and δ as solid lines, β and γ as dashed lines. In the configuration α (respectively, δ) the weight of the left (respectively, right) corner does not change, while in the configuration β (respectively, γ) the weight of the left (respectively, right) corner is multiplied by a factor $t_1^{-\varepsilon}$. Note finally that the curvature is multiplied by t_1^{ε} (respectively, 1, $t_1^{2\varepsilon}$, t_1^{ε}) in the configuration $\alpha\gamma$ (respectively, $\alpha\delta$, $\beta\gamma$, $\beta\delta$). It appears that in every case, the factor ωt^{κ} changes by a factor $t_1^{-(2m-2)\varepsilon}$. Therefore, all the changes cancel out and $\Gamma_{D'} = \Gamma_D$.

Now consider the horizontal cyclic permutation, with m columns of the grid affected, illustrated in the case $m = 2$:
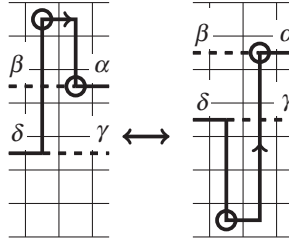

In this case, the Fox matrix only changes between the columns q and $q+m-1$. Let A be the matrix obtained from F by removing the first line and those two columns. A quick computation gives $\det(F^q) = (-1)^q t_1^{\varepsilon} \det(A)$ and $\det(F'^{q+1}) = (-1)^{q+n} t_1^{-(m-2)\varepsilon} \det(A)$, so

$$M'(t) = (-1)^{n-1} t_1^{-(m-1)\varepsilon} M(t).$$

Here again we have $\mathrm{sgn}(x_0') = (-1)^{n-1} \mathrm{sgn}(x_0)$ and $u' = u$. For the total weight and curvature, one easily checks that in each of the four possible configurations, ωt^{κ} is multiplied by a factor $t_1^{-(2m-2)\varepsilon}$. The equality $\Gamma_{D'} = \Gamma_D$ follows.

(G2) Let us first consider the case of a vertical commutation with disjoint segments. The only quantities that change are the signature of the basis, multiplied by -1, and the determinant, also multiplied by -1, so $\Gamma_{D'} = \Gamma_D$.

Let us now assume that the segments are not disjoint, as in the following illustration:
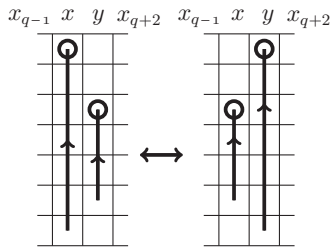
Clearly, the signature of the base is multiplied by -1, while $u' = u$ and $\kappa' = \kappa$. Let us denote by L_k (respectively, L_l) the component of L to which the vertical segment x (respectively, y) belongs. The parts of the Fox matrices affected by the move are of the form

$$F = \begin{pmatrix} * & 0 \\ \alpha_i & \alpha_i t_k^{\varepsilon_x} \\ \vdots & \vdots \\ \alpha_{i+p} & \alpha_{i+p} t_x^{\varepsilon_x} \\ * & 0 \end{pmatrix} \quad \text{and} \quad F' = \begin{pmatrix} 0 & * \\ \alpha_i & \alpha_i t_l^{\varepsilon_y} \\ \vdots & \vdots \\ \alpha_{i+p} & \alpha_{i+p} t_l^{\varepsilon_y} \\ 0 & * \end{pmatrix}.$$

Subtracting from the qth column of F its $(q+1)$th column multiplied by $t_k^{-\varepsilon_x}$, and from the $(q+1)$th column of F' its qth column multiplied by $t_l^{\varepsilon_y}$ leads to the relation $\det(F'^q) = -t_k^{-\varepsilon_x} \det(F^q)$. As for the corner weights, only the two corners of the small segment change their contribution, yielding $\omega' = \omega t_k^{-2\varepsilon_x}$. The equality $\Gamma_{D'} = \Gamma_D$ follows.

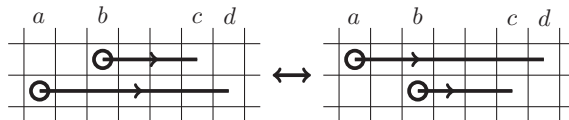Consider now the horizontal commutation with disjoint segments, pictured as
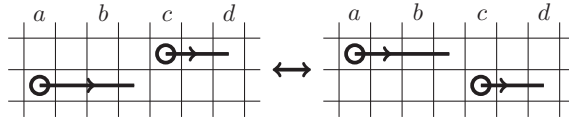
In this case, the signature of the base is multiplied by -1, while $u' = u$ and $\kappa' = \kappa$. Furthermore, one easily sees that

$$-\theta\left(\frac{\partial r_\ell}{\partial x_j}\right) = \theta\left(\frac{\partial r'_\ell}{\partial x_j}\right) - \theta\left(\frac{\partial r'_{\ell-1}}{\partial x_j}\right) - \theta\left(\frac{\partial r'_{\ell+1}}{\partial x_j}\right)$$

for all js. This implies that $\det(F'^1) = -\det(F^1)$, leading to the expected invariance.

Let us now turn to the case in which the segments are not disjoint, illustrated as

In this case, only the base and the matrix change. Let a, b, c, d be the vertical segments at the extremities of the horizontal ones as pictured above. We obtain

$$-t_k^{\varepsilon_a} \theta\left(\frac{\partial r_\ell}{\partial x_j}\right) = \theta\left(\frac{\partial r'_\ell}{\partial x_j}\right) - t_k^{\varepsilon_a} \theta\left(\frac{\partial r'_{\ell-1}}{\partial x_j}\right) - \theta\left(\frac{\partial r'_{\ell+1}}{\partial x_j}\right)$$

for all js, where a belongs to L_k. This leads to the equality $\det(F'^1) = -t_k^{-\varepsilon_a} \det(F^1)$. The only corners affected by the change in the matrix are those of the small segment: they are both multiplied by $t_k^{\varepsilon_a}$, so all the changes cancel out once again.

(G3) Consider the first stabilization pictured as

$$\varepsilon_q = -1 \qquad\qquad \varepsilon_q = 1$$
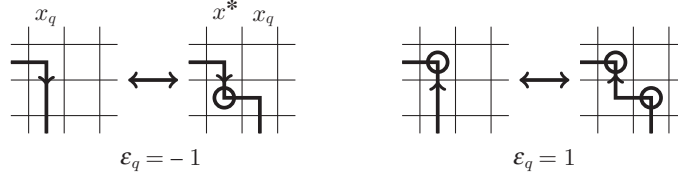
In the $\varepsilon_q = -1$ case the move adds a circle at the $(\ell + 1, q)$ position of the grid, so $\mathrm{sgn}(x_0') = (-1)^{(\ell+q+1)} \mathrm{sgn}(x_0)$, while the number of upwards segments does not change. In the $\varepsilon_q = 1$ case the move adds a circle at the $(\ell + 1, q + 1)$ position of the grid, so $\mathrm{sgn}(x_0') = (-1)^{(l+q)} \mathrm{sgn}(x_0)$, while $u' = u + 1$. Thus, in both cases the overall sign changes by a factor $(-1)^{(\ell+q+1)}$. There are two new corners, each with a weight $\alpha :=$ $\theta(\partial r^*/\partial x^*)$, where x^* and r^* denote the new generator and relation created by the stabilization. This implies that $\omega' = \omega\alpha^2$. The matrix F' is obtained from F by adding a line and a column; developing the determinant with respect to that column yields $\det(F'^1) = (-1)^{(\ell+q+1)} \alpha \det(F^1)$, so all the changes cancel out and $\Gamma_{D'} = \Gamma_D$.

The three remaining types of stabilizations are illustrated as

$$\varepsilon_q = -1 \qquad\qquad \varepsilon_q = 1$$

$$\varepsilon_q = -1 \qquad\qquad \varepsilon_q = 1$$

$$\varepsilon_q = -1 \qquad\qquad \varepsilon_q = 1$$
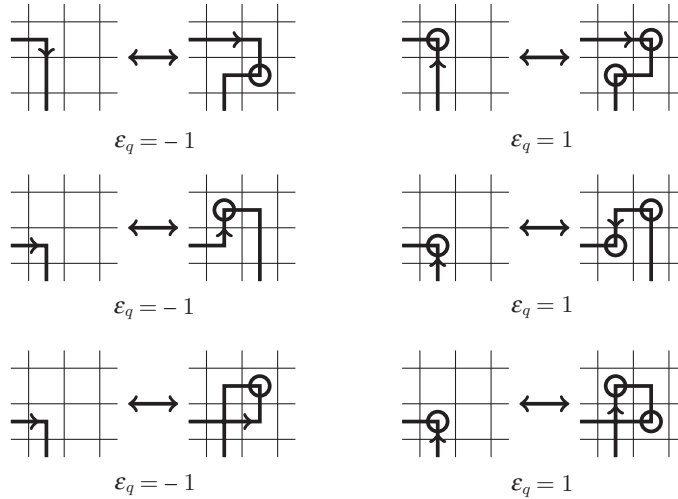
They can be treated in the same way. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.4. Identification of Γ_L and ∇_L

By Proposition 2.5, Γ_D is an invariant and can therefore be denoted by Γ_L. The aim of this paragraph is to prove the following result.

Proposition 2.6. *For any oriented link L, the invariants Γ_L and ∇_L coincide. In particular, Γ_L satisfies the equality*

$$\Gamma_L(t^{-1}) = (-1)^\mu \Gamma_L(t). \tag{2.2}$$

To verify this statement, we first used Murakami's characterization theorem [**13**], which states that ∇_L is determined uniquely by a system of six local relations. However, Jiang [**10**] recently showed that Conway's function is characterized by the following (simpler) set of five relations, which we will use in our proof:
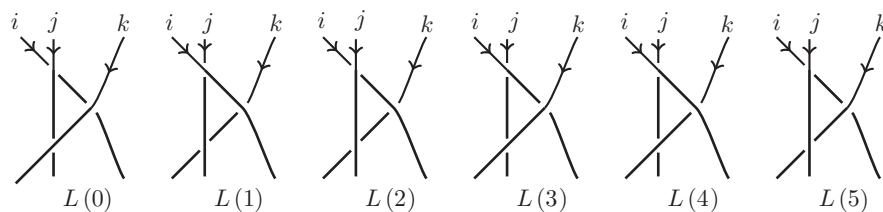
(R1) $\nabla_H = 1$, where H is the positive Hopf link;

(R2) $\nabla_{L \sqcup K} = 0$, where $L \sqcup K$ denotes the disjoint union of L and a trivial knot K;

(R3) $\nabla_{L'} = (t_i - t_i^{-1})\nabla_{L_0}$, where L' is obtained from L_0 by the local operation given by

(R4) $\nabla_{L_{++}} + \nabla_{L_{--}} = (t_i t_j - t_i^{-1} t_j^{-1})\nabla_{L_0}$, where L_{++}, L_{--} and L_0 differ by the local operation
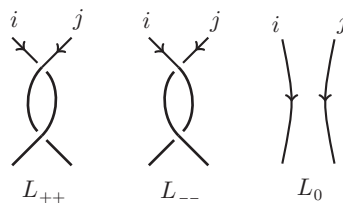
(R5) $(t_i t_k^{-1} - t_i^{-1} t_k^1)(\nabla_{L(0)} + \nabla_{L(1)}) + (t_j t_k - t_j^{-1} t_k^{-1})(\nabla_{L(2)} + \nabla_{L(3)})$
$$+ (t_i^{-1} t_j^{-1} - t_i t_j)(\nabla_{L(4)} + \nabla_{L(5)}) = 0,$$
where $L(0)$, $L(1)$, $L(2)$, $L(3)$, $L(4)$ and $L(5)$ differ by the local operation

Proof of Proposition 2.6. By Jiang's theorem, we only need to check that Γ_L satisfies the five relations described above. Let us first fix some notation. In the grid diagrams illustrated below, the first horizontal pictured segment will always be the ℓth and the first vertical one the qth. Only the 'local' Fox matrices will be written, i.e. the parts where the Fox matrices differ. The simplest diagram will be called L_0 and used as

a reference for the weight, curvature, base, matrices and number of upwards segments, with the associated quantities indexed by a zero. The quantities for the other diagrams will be indexed according to their names.
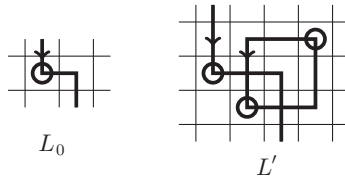
(R1) This property was verified in Example 2.4.

(R2) A possible rectangular diagram for $L \sqcup K$ is
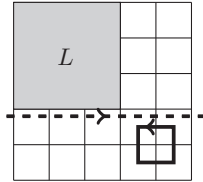
The line in the Fox matrix corresponding to the dashed line illustrated above is identically zero. Therefore, any $(n-1)$-minor of the matrix vanishes and so does $\Gamma_{L \sqcup K}$.

(R3) This relation compares the value of Γ_L for the following two diagrams:

$$L_0 \qquad\qquad L'$$

Fix $i = 1$ and $j = 2$. The (local) Fox matrices are of the form

$$F_0^q = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \qquad F'^q = \begin{pmatrix} 0 & 0 & 0 \\ \alpha t_1^{-1} & 0 & \alpha t_1^{-1} t_2^{-1} \\ \alpha & \alpha t_2^{-1} & \alpha t_2^{-1} t_1^{-1} \\ 0 & \alpha & 0 \end{pmatrix}.$$

This yields the equality $\det(F'^q) = \alpha^2 t_1^{-1} t_2^{-1}(1 - t_1^{-1}) \det(F^q)$. Clearly, $\kappa' = \kappa + (0, 1)$ and $u' = u_0 + 1$, and one easily checks that $\operatorname{sgn}(x') = -\operatorname{sgn}(x_0)$. The total weights for L' and L are related by $\omega' = \omega_0 \alpha^4 t_1^{-3} t_2^{-3}$. Combining all these equalities, we obtain

$$\Gamma' = \frac{(-1)(-1)}{\alpha^4 t_1^{-3} t_2^{-3} t_2} \alpha^4 t_1^{-2} t_2^{-2}(1 - t_1^{-2})\Gamma_0 = (t_1 - t_1^{-1})\Gamma_0.$$

The remaining two relations require more care. Given a line of a matrix, consider the determinant-preserving operation given by subtracting from this line an adjacent line. The resulting line is zero everywhere except for the entries separated by a horizontal segment of the diagram. Therefore, one can expand the determinant with respect to that line and all the information is contained in the local matrix.

(R4) Fix $i = 1$ and $j = 2$. Possible rectangular diagrams for this local operation are

$$L_0 \qquad\qquad L_{++} \qquad\qquad\qquad L_{--}$$
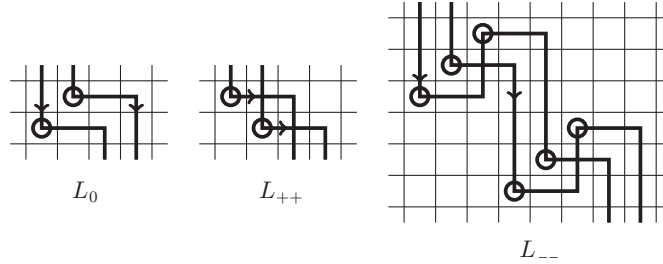
The corresponding Fox matrices are of the following form:

$$F_0^q = \begin{pmatrix} c_1 & 0 & 0 \\ \alpha t_1^{-1} & 0 & 0 \\ 0 & 0 & \alpha t_1^{-1} \\ 0 & \alpha & \alpha t_1^{-1} \\ 0 & c_2 & c_3 \end{pmatrix}, \qquad F_{++}^q = \begin{pmatrix} c_1 & 0 & 0 \\ \alpha t_1^{-1} & 0 & 0 \\ \alpha & \alpha t_2^{-1} & 0 \\ 0 & \alpha & \alpha t_1^{-1} \\ 0 & c_2 & c_3 \end{pmatrix}$$

and

$$F_{--}^q = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha t_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha t_1^{-1} & \alpha t_1^{-1} t_2^{-1} & 0 & \alpha t_2^{-1} & 0 & 0 & 0 \\ 0 & \alpha t_1^{-1} & \alpha & \alpha t_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha t_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha t_2^{-1} & \alpha t_2^{-1} t_1^{-1} & 0 & \alpha t_1^{-1} \\ 0 & 0 & \alpha & 0 & \alpha t_2^{-1} & \alpha & \alpha t_1^{-1} \\ 0 & 0 & 0 & 0 & 0 & \alpha & \alpha t_1^{-1} \\ 0 & 0 & 0 & 0 & 0 & c_2 & c_3 \end{pmatrix},$$

where c_1, c_2 and c_3 are column vectors. For $1 \leqslant i \leqslant 3$, let A_i be F_0^q with the third (local) line and the column containing c_i removed and set $B_i(t) = \det(A_i(t^2))$. Finally, set

$$\beta_0 = \frac{\mathrm{sgn}(x_0)(-1)^{u_0}}{\omega_0 t^{\kappa_0}} \frac{1}{t_1^2 - 1}.$$

The values of the functions Γ will be computed by using row operations such that the last column of each local matrix is zero, except for its two lowest entries.

Clearly, κ_0, κ_{++} and κ_{--} coincide, u_0, u_{++} and u_{--} all have the same parity, while $\mathrm{sgn}(x_0) = -\mathrm{sgn}(x_{++}) = \mathrm{sgn}(x_{--})$. Moreover, the corresponding weights satisfy the equalities $\omega_{++} = \omega_0 t_1 t_2^{-1}$ and $\omega_{--} = \omega_0 \alpha^8 t_1^{-3} t_2^{-5}$. Finally, for F_0^q, replace the third (local) line by its difference with the second (local) line, and expand with respect to the

third (local) line to yield $\Gamma_0 = (-1)^\ell \beta_0 \alpha^2 B_2$. The others are obtained similarly as

$$\Gamma_{++} = (-1)^\ell \beta_0 \alpha^2 t_1^{-1} t_2 (-(1 - t_1^{-2})B_1 + t_2^{-2}B_2),$$
$$\Gamma_{--} = (-1)^\ell \beta_0 \alpha^2 t_1 t_2 (-(t_1^{-4} - t_1^{-2})B_1 + B_2).$$

It follows easily that $\Gamma_{++} + \Gamma_{--} = (t_1 t_2 + t_1^{-1} t_2^{-1})\Gamma_0$.

(R5) Let us suppose that $i = 1$, $j = 2$ and $k = 3$. The diagrams considered are

$$L(0) \qquad\qquad L(1) \qquad\qquad L(2)$$

$$L(3) \qquad\qquad L(4) \qquad\qquad L(5)$$
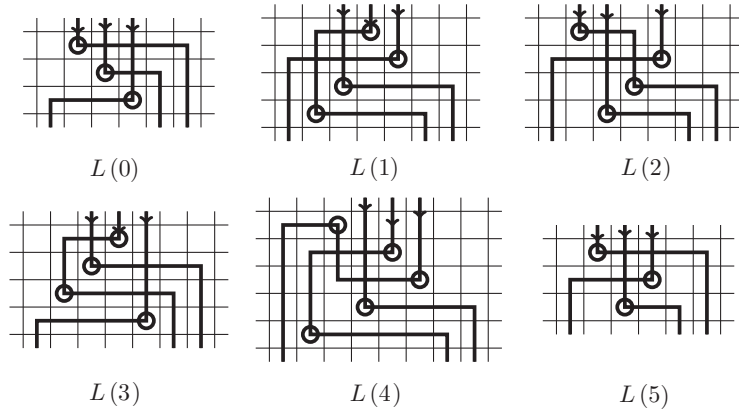
defining local Fox matrices of the form

$$F_0^{q+5} = \begin{pmatrix} 0 & c_2 & c_3 & c_4 & 0 \\ 0 & \alpha & \alpha t_1^{-1} & \alpha t_1^{-1} t_2^{-1} & 0 \\ 0 & 0 & \alpha & \alpha t_2^{-1} & 0 \\ 0 & 0 & 0 & \alpha & \alpha t_3^{-1} \\ \alpha & 0 & 0 & 0 & \alpha t_3^{-1} \\ c_1 & 0 & 0 & 0 & c_5 \end{pmatrix},$$

$$F_1^{q+6} = \begin{pmatrix} 0 & 0 & c_2 & c_3 & c_4 & 0 \\ 0 & 0 & \alpha & \alpha t_1^{-1} & \alpha t_1^{-1} t_2^{-1} & 0 \\ 0 & \alpha & \alpha t_2^{-1} & 0 & \alpha t_2^{-1} t_1^{-1} & 0 \\ \alpha & \alpha t_3^{-1} & \alpha t_3^{-1} t_2^{-1} & 0 & 0 & 0 \\ \alpha & \alpha t_3^{-1} & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & \alpha t_3^{-1} \\ c_1 & 0 & 0 & 0 & 0 & c_5 \end{pmatrix},$$

$$F_2^{q+6} = \begin{pmatrix} 0 & c_1 & c_2 & 0 & c_3 & 0 \\ 0 & 0 & \alpha & \alpha t_2^{-1} & \alpha t_2^{-1} t_1^{-1} & 0 \\ \alpha & 0 & \alpha t_3^{-1} & \alpha t_3^{-1} t_2^{-1} & 0 & 0 \\ \alpha & 0 & \alpha t_3^{-1} & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & \alpha t_3^{-1} \\ c_1 & 0 & 0 & 0 & 0 & c_5 \end{pmatrix},$$

$$F_3^{q+6} = \begin{pmatrix} 0 & 0 & c_2 & c_3 & c_4 & 0 \\ 0 & 0 & \alpha & \alpha t_1^{-1} & \alpha t_1^{-1} t_2^{-1} & 0 \\ 0 & \alpha & \alpha t_2^{-1} & 0 & \alpha t_2^{-1} t_1^{-1} & 0 \\ 0 & \alpha & 0 & 0 & \alpha t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & \alpha & \alpha t_3^{-1} \\ \alpha & 0 & 0 & 0 & 0 & \alpha t_3^{-1} \\ c_1 & 0 & 0 & 0 & 0 & c_5 \end{pmatrix},$$

$$F_4^{q+7} = \begin{pmatrix} 0 & 0 & 0 & c_2 & c_3 & c_4 & 0 \\ 0 & 0 & 0 & \alpha & \alpha t_1^{-1} & \alpha t_1^{-1} t_2^{-1} & 0 \\ \alpha & 0 & \alpha t_3^{-1} & \alpha & \alpha t_1^{-1} & \alpha t_1^{-1} t_2^{-1} & 0 \\ \alpha & \alpha t_3^{-1} & \alpha t_3^{-1} t_2^{-1} & \alpha t_2^{-1} & 0 & \alpha t_2^{-1} t_1^{-1} & 0 \\ \alpha & \alpha t_3^{-1} & 0 & \alpha t_3^{-1} t_2^{-1} & 0 & 0 & 0 \\ \alpha & \alpha t_3^{-1} & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & \alpha t_3^{-1} \\ c_1 & 0 & 0 & 0 & 0 & 0 & c_5 \end{pmatrix},$$

$$F_5^{q+5} = \begin{pmatrix} 0 & c_2 & c_3 & c_4 & 0 \\ 0 & \alpha & \alpha t_1^{-1} & \alpha t_1^{-1} t_2^{-1} & 0 \\ 0 & 0 & \alpha & \alpha t_2^{-1} & 0 \\ \alpha & 0 & \alpha t_3^{-1} & 0 & 0 \\ \alpha & 0 & 0 & 0 & \alpha t_3^{-1} \\ c_1 & 0 & 0 & 0 & c_5 \end{pmatrix}.$$

For $1 \leqslant i, j \leqslant 5$, let A_{ij} be F_0^{q+5} with the third and fourth (local) lines as well as the columns containing c_i and c_j removed, and set $B_{ij}(t) = \det(A_{ij}(t^2))$. Finally, set

$$\beta_0 = \frac{\operatorname{sgn}(x_0)(-1)^{u_0}}{\omega_0 t^{\kappa_0}} \frac{(-1)^{q+5}}{t_1^2 - 1}.$$

The values of the Γ functions are computed by using row operations such that the first column of each local matrix is zero, except for its two lowest entries.

First note that neither the curvature nor the number of upwards segments change, except for $L(4)$, where $u_4 = u_0 + 1$. Moreover, the sign of the base changes as follows: $\operatorname{sgn}(x_0) = (-1)^{\ell+q} \operatorname{sgn}(x_1) = (-1)^{\ell+q} \operatorname{sgn}(x_2) = (-1)^{\ell+q} \operatorname{sgn}(x_3) = -\operatorname{sgn}(x_4) = -\operatorname{sgn}(x_5)$. Note that there is an additional sign for $L(1)$, $L(2)$, $L(3)$ due to the fact that the removed columns do not have the same parity as $q + 5$. The weights satisfy $\omega_1 = \omega_0 \alpha^2 t_1^{-2} t_2^{-2} t_3^{-2}$, $\omega_2 = \omega_0 \alpha^2 t_1^{-1} t_2^{-3} t_3^{-2}$, $\omega_3 = \omega_0 \alpha^2 t_1^{-1} t_2^{-1}$, $\omega_4 = \omega_0 \alpha^4 t_1^{-2} t_2^{-3} t_3^{-5}$ and $\omega_5 = \omega_0 t_2^{-1} t_3^{-1}$. For F_0^{q+5}, replace the fourth (local) line by its difference with the third (local) line, then the third (local) line by its difference with the second (local) line, leading to

$$\Gamma_{L_0} = \beta_0 \alpha^4 (B_{23} - (t_2^{-2} - 1) B_{24} - t_3^{-2} B_{25} - (t_1^{-2} - 1) B_{34}$$
$$+ t_3^{-2}(t_1^{-2} - 1) B_{35} - t_2^{-2} t_3^{-2}(t_1^{-1} - 1) B_{45}).$$

The others are computed similarly:

$$\Gamma_{L(1)} = \beta_0 \alpha^4 t_1^2 t_3^{-2} (t_1^{-2} B_{23} - B_{25}),$$
$$\Gamma_{L(2)} = \beta_0 \alpha^4 t_1 t_2 t_3^{-2} (B_{12} - B_{25} + (t_1^{-2} - 1) B_{35}),$$
$$\Gamma_{L(3)} = \beta_0 \alpha^4 t_1 t_2^{-1} (t_1^{-2} B_{23} - t_1^{-2} (t_2^{-2} - 1) B_{24} - t_3^{-2} B_{25} - t_1^{-2} (t_1^{-2} - 1) B_{34}$$
$$- t_3^{-2} (t_1^{-2} - 1) B_{45}),$$
$$\Gamma_{L(4)} = \beta_0 \alpha^4 t_1^2 t_2^{-1} t_3^{-1} (t_1^{-2} B_{23} - t_1^{-2} (t_2^{-2} - 1) B_{24} - t_3^{-2} B_{25}),$$
$$\Gamma_{L(5)} = \beta_0 \alpha^4 t_2 t_3^{-1} (B_{23} - B_{25} - t_2^{-2} (t_1^{-2} - 1) B_{34} + (t_1^{-2} - 1) B_{35} - t_2^{-2} (t_1^{-2} - 1) B_{45}).$$

Letting c_{ij}^k be the coefficient of B_{ij} in $\Gamma_{L(k)}$ (divided by $\beta_0 \alpha^4$), it suffices to show that

$$\Gamma_{ij} := (t_1 t_3^{-1} - t_1^{-1} t_3)(c_{ij}^0 + c_{ij}^1) + (t_2 t_3 - t_2^{-1} t_3^{-1})(c_{ij}^2 + c_{ij}^3) + (t_1^{-1} t_2^{-1} - t_1 t_2)(c_{ij}^4 + c_{ij}^5)$$

vanishes for all $i, j \in \{2, 3, 4, 5\}$. For example, B_{24} only appears in $L(0)$, $L(3)$ and $L(4)$, and we compute

$$\Gamma_{24} = (t_1 t_3^{-1} - t_1^{-1} t_3)(-(t_2^{-2} - 1)) + (t_2 t_3 - t_2^{-1} t_3^{-1})(-t_1^{-1} t_2^{-1} (t_2^{-2} - 1))$$
$$+ (t_1^{-1} t_2^{-1} - t_1 t_2)(-t_2^{-1} t_3^{-1} (t_2^{-2} - 1)) = 0.$$

The other equations can be verified in the same way through direct computation. $\square$

3. Link Floer homology and Euler characteristic

This section gives a very quick review of the combinatorial version of link Floer homology with $\mathbb{Z}_2$ coefficients following [**11**,**12**], and mentions selected properties. We then use these properties to show that link Floer homology categorifies the function Γ_L introduced in the previous section. Theorem 1.1 follows.

3.1. Grid diagrams and gradings

Given an oriented rectangular diagram, consider each corner labelled by an O or an X with the following convention: for each horizontal segment we travel from an 'O' to an 'X', whereas for each vertical segment we travel from an 'X' to an 'O'. One obtains a *grid diagram* by fitting the data of those Xs and Os in a grid. Due to condition (iii), each row and column of the grid contain exactly one X and one O. From a grid diagram, one can construct a rectangular diagram by drawing horizontal segments oriented from O to X, and vertical segments oriented from X to O. Rectangular and grid diagrams are thus equivalent, and the elementary moves for grid diagrams are the same as for rectangular ones. Examples are given in Figure 2.

Note that the Os lie in the exact same place as the circles defining the base of a rectangular diagram. We denote by $\mathbb{X}$ the set of all Xs and by $\mathbb{O}$ the set of all Os of the diagram.

Given a grid diagram G of size n, number the horizontal lines from 1 to n, starting with the second line, and the vertical lines from 1 to n, starting with the leftmost line (see
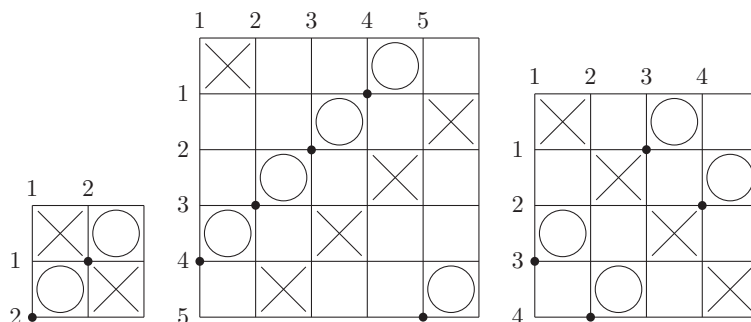
Figure 2. The three grid diagrams corresponding to the three rectangular diagrams of Figure 1.

Figure 2). Let $S(G)$ (or simply S) be the set of all n-tuples of intersection points between horizontal and vertical lines, with the property that no intersection point appears on more than one horizontal or vertical line. There is an element of $S(G)$ that corresponds exactly to the base x_0 of the diagram: it is given by the n-tuple that occupies the lower left corner of each square containing an O (indicated in Figure 2 as a collection of black dots).

The set S is equipped with two gradings. The *Maslov grading* $M \colon S \to \mathbb{Z}$ is defined as follows. Given two collections A, B of finitely many points in the plane, let $I(A, B)$ be the number of pairs $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$ such that $a_1 < b_1$ and $a_2 < b_2$, and set $J(A, B) = (I(A, B) + I(B, A))/2$. Given an element $x \in S$, we view x as a collection of points with integer coordinates. Similarly, the sets $\mathbb{O} = \{O_i\}_{i=1}^{n}$ and $\mathbb{X} = \{X_i\}_{i=1}^{n}$ are viewed as a collection of points with half-integer coordinates. Define

$$M(x) = J(x - \mathbb{O}, x - \mathbb{O}) + 1,$$

where J is extended bilinearly over formal sums and differences of subsets. Note that the Maslov index of the base equals $1 - n$.

For a link of multiplicity μ the *Alexander grading* is the function $A \colon S \to (\frac{1}{2}\mathbb{Z})^{\mu}$ defined as a μ-tuple $A(x) = (A_1(x), \ldots, A_\mu(x))$ by the formula

$$A_k(x) = J(x - \tfrac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X}_k - \mathbb{O}_k) - \left(\frac{n_k - 1}{2} \right),$$

where $\mathbb{O}_k \subset \mathbb{O}$ and $\mathbb{X}_k \subset \mathbb{X}$ are the subsets corresponding to the kth component of the link, n_k is the cardinal of $\mathbb{O}_k$, and J is again extended bilinearly.

Lemma 3.1. *For any two elements $x, y \in S$ we have*

$$(-1)^{M(x)} \operatorname{sgn}(x) = (-1)^{M(y)} \operatorname{sgn}(y).$$

Proof. By additivity of the signature, we only need to check that if the permutations x and y differ by a transposition, then $M(x) - M(y)$ is odd. Using the terminology of [**11**], this means that there is an empty rectangle r connecting x and y (or y and x). By [**11**, Lemma 2.5], we have

$$\pm(M(x) - M(y)) = 1 - 2\#(\mathbb{O} \cap r),$$

and the lemma follows. $\qquad\square$

3.2. Combinatorial link Floer homology

In this paragraph, we recall the main properties of the combinatorial version of link Floer homology over $\mathbb{Z}_2$ following [11], displaying the ones that will be needed in what follows.

Let G be a grid diagram. Let R denote the polynomial algebra over $\mathbb{Z}_2$ generated by variables $\{U_i\}_{i=1}^n$, which are in one-to-one correspondence with the elements of $\mathbb{O}$. This ring is endowed with a *Maslov grading*, defined such that the constant terms are in Maslov grading zero and the U_is are in grading -2. It is also endowed with an *Alexander multi-filtration*, defined so that the constants are in filtration level zero while the variables U_j corresponding to the ith component drop the ith multi-filtration level by one and preserve the others.

Let $C^-(G)$ be the free R-module with generating set S. It has a Maslov grading and an Alexander filtration induced by the ones on S and R. It turns out that one can define an endomorphism $\partial^- \colon C^-(G) \to C^-(G)$ that is a differential, drops the Maslov grading by one, and preserves the Alexander filtration level. Furthermore, this complex provides a link invariant as follows. Let L be an oriented link given by a grid diagram G. Number the elements of $\mathbb{O} = \{O_i\}_{i=1}^n$ so that $O_1, \ldots, O_\mu$ correspond to the different components of the link. Then the filtered quasi-isomorphism type of the complex $(C^-(G), \partial^-)$ seen as a module over $\mathbb{Z}_2[U_1, \ldots, U_\mu]$ is a link invariant [11, Theorem 1.2]. In particular, the homology of the associated graded object is a link invariant. (We refer the reader to [11, §2.1] for an explanation of this algebraic terminology.)

There are two variations on this construction. With an ordering of the Os as before, consider the chain complex $\hat{C}(G)$ obtained from $C^-(G)$ by setting $U_i = 0$ for $i = 1, \ldots, \mu$. Let $\widehat{CL}(G)$ denote the graded object associated with the Alexander filtration and let $\widehat{HL}(G)$ be its homology. Also, let $\widetilde{C}(G)$ be the chain complex obtained from $C^-(G)$ by setting all the U_is to 0. Let $\widetilde{CL}(G)$ be the associated graded object and let $\widetilde{HL}(G)$ be its homology. These two complexes can be related using standard tools of homological algebra [11, Proposition 2.15].

Proposition 3.2. *The homology groups* $\widehat{HL}(G)$ *determine* $\widetilde{HL}(G)$ *as follows:*

$$\widetilde{HL}(G) \cong \widehat{HL}(G) \otimes \bigotimes_{k=1}^\mu V_k^{\otimes(n_k-1)},$$

where V_k *is the two-dimensional vector space spanned by two generators, one in zero Maslov and Alexander multi-gradings, and the other in Maslov grading minus 1 and Alexander multi-grading corresponding to minus the kth basis vector, and n_k is the number of Os corresponding to L_k.*

We shall need one last property of link Floer homology [11, Proposition 5.3].

Proposition 3.3. *Let* $\widehat{HL}_d(L, s)$ *denote the subspace of* $\widehat{HL}(L)$ *with Maslov grading d and Alexander grading s. Then,* $\widehat{HL}_d(L, s)$ *and* $\widehat{HL}_{d-\sum_k s_k}(L, -s)$ *are isomorphic for all d, s.*

3.3. Euler characteristic and Γ_L

Given a multi-graded vector space $C = \bigoplus_{d,s} C_d(s)$ with Maslov grading d and Alexander grading s, define its *(graded) Euler characteristic* as

$$\chi(\widehat{HL}(L); t) := \sum_{d,s} (-1)^d t^s \dim C_d(s).$$

Propositions 3.2 and 3.3 immediately imply the following corollary.

Corollary 3.4. *For any oriented link L, we have the equalities in $\mathbb{Z}[t_1^{\pm 1/2}, \ldots, t_\mu^{\pm 1/2}]$:*

$$\chi(\widetilde{HL}(L); t) = \prod_{k=1}^{\mu} (1 - t_k^{-1})^{n_k - 1} \chi(\widehat{HL}(L); t),$$

$$\chi(\widehat{HL}(L); t) = \chi(\widehat{HL}(L); t^{-1}).$$

Theorem 1.1 now follows from Proposition 2.6 and the following result.

Proposition 3.5. *For any oriented link L we have the equality*

$$\chi(\widehat{HL}(L); t) = \prod_{k=1}^{\mu} (t_k^{1/2} - t_k^{-1/2}) \Gamma_L(t^{1/2}).$$

Before giving the proof, let us define one more matrix, following [**11**, §6]. Given a rectangular diagram D of size n (or equivalently, a grid diagram G), define an $n \times n$ matrix $W(G)$ by

$$W(G)_{i,j} = t^{a(i,j)},$$

where $a(i, j) \in \mathbb{Z}^\mu$ is the vector whose kth component is minus the winding number of L_k around the point (i, j), counted from the second horizontal line and first vertical line. Note that this definition implies that $W(G)_{n,j} = W(G)_{i,1} = 1$ for all i, j, as these points lie outside the link. An easy example for the trivial knot is given by
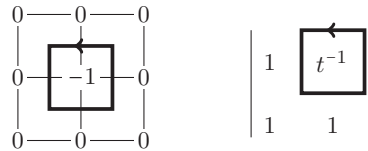
where the function $a(i, j)$ is shown on the left and the matrix $W(G)$ on the right.

Proof of Proposition 3.5. Let D be a rectangular diagram representing L. Let us first compare the determinant of $W(G)$ with $\Gamma_L = \Gamma_D$. In $W(G)$, subtract each column from the next one. In every column but the first of the resulting matrix there is a zero where the vertical segment does not intervene, and where a vertical segment is present the coefficient is divisible by $\theta(x_j) - 1$. Therefore, for each column but the first, we can factor out $\theta(x_j) - 1$. The last vertical segment, which corresponds to the generator x_n,

does not contribute to these factors as it does not lie in between two columns of the matrix. Moreover, the last line is identically zero except for the first coefficient, which is 1. Therefore,

$$\det(W(G)) = (-1)^{n+1} \prod_{j=1}^{n} (\theta(x_j) - 1) \frac{\det(A)}{\theta(x_n) - 1},$$

where A is the $(n, 1)$-minor of the resulting matrix, a minor that is nothing but F_D^n (recall §2.2). Define u_k as the number of upwards segments of L_k, and let d be the total number of downward segments. Transforming each term $\theta(x_j) - 1$ into $1 - t_k^{-1}$ by factoring out a sign if x_j is downwards and t_k if it is upwards, we obtain

$$\det(W(G)) = (-1)^{1+d} \prod_{k=1}^{\mu} t_k^{u_k} \prod_{k=1}^{\mu} (1 - t_k^{-1})^{n_k} \frac{(-1)^n \det(F_D^n)}{\theta(x_n) - 1}.$$

Set $\gamma \in (\frac{1}{2}\mathbb{Z})^{\mu}$ such that $t^{2\gamma} = \omega t^{\kappa}$ and $\beta = \gamma + (u_1, \dots, u_{\mu})$. Multiplying both sides by the sign $(-1)^{M(x_0)} \operatorname{sgn}(x_0)$ and using the equalities $M(x_0) = 1 - n = 1 - d - u$, we eventually get

$$(-1)^{M(x_0)} \operatorname{sgn}(x_0) \det(W(G)) = t^{\beta} \prod_{k=1}^{\mu} (1 - t_k^{-1})^{n_k} \Gamma_D(t^{1/2}). \qquad (3.1)$$

Let us now compare the determinant of $W(G)$ with $\chi(\widehat{HL}(L); t)$. By Corollary 3.4,

$$\prod_{k=1}^{\mu} (1 - t_k^{-1})^{n_k-1} \chi(\widehat{HL}(L); t) = \chi(\widetilde{HL}(L); t) = \chi(\tilde{C}(G); t).$$

Furthermore, using the definitions of $\tilde{C}(G)$ and of $A(x)$ together with Lemma 3.1, we have

$$\begin{aligned}
\chi(\tilde{C}(G); t) &= \sum_{d,s} (-1)^d t^s \dim(\tilde{C}_d(s)) \\
&= \sum_{d,s} (-1)^d t^s \#\{x \in S \mid M(x) = d \text{ and } A(x) = s\} \\
&= \sum_{x \in S} (-1)^{M(x)} t^{A(x)} \\
&= (-1)^{M(x_0)} \operatorname{sgn}(x_0) \sum_{x \in S} \operatorname{sgn}(x) t^{A(x)} \\
&= (-1)^{M(x_0)} \operatorname{sgn}(x_0) t^{\nu+1/2} \sum_{x \in S} \operatorname{sgn}(x) \prod_{k=1}^{\mu} t_k^{J(x, \mathbb{X}_k - \mathbb{O}_k)},
\end{aligned}$$

where $\nu_k = J(-\frac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X}_k - \mathbb{O}_k) - \frac{1}{2}n_k$, $\nu = (\nu_1, \dots, \nu_{\mu})$ and $\mathbb{X}_k$ and $\mathbb{O}_k$ are the markings corresponding to L_k. It is straightforward to see that $J(x, \mathbb{X}_k - \mathbb{O}_k)$ is minus the sum of the winding numbers of L_k around the points in x. Therefore,

$$\sum_{x \in S} \operatorname{sgn}(x) \prod_{k=1}^{\mu} t_k^{J(x, \mathbb{X}_k - \mathbb{O}_k)} = \det(W(G)),$$

and we eventually obtain

$$(-1)^{M(x_0)}\operatorname{sgn}(x_0)\det(W(G)) = t^{-\nu-1/2}\prod_{k=1}^{\mu}(1-t_k^{-1})^{n_k-1}\chi(\widehat{HL}(L);t). \qquad (3.2)$$

Equations (3.1) and (3.2) lead to

$$\chi(\widehat{HL}(L);t) = t^{\beta+\nu}\prod_{k=1}^{\mu}(t_k^{1/2}-t_k^{-1/2})\Gamma_D(t^{1/2}).$$

The fact that $\beta+\nu$ vanishes now follows from (2.2) and the second part of Corollary 3.4. This concludes the proof. $\square$

4. Other models and homology spheres

As mentioned in the introduction, there are four different constructions of a well-defined representative of the multi-variable Alexander polynomial. In this slightly informal section, we discuss the identification of these models and of their extension to integral homology spheres.

Let us start by recalling these four models.

(1) As explained in §2.1, Hartley's model ∇_L is based on Fox free differential calculus and the Wirtinger presentation of the link group [9].

(2) Turaev's model τ_L is constructed using sign-refined Reidemeister torsion [19].

(3) Cimasoni's model Ω_L is defined via generalized Seifert surfaces [3].

(4) Finally, Ozsváth and Szabó's link Floer homology [16] provides one last model in the form of its renormalized Euler characteristic

$$\chi_L(t) := \prod_k(t_k - t_k^{-1})^{-1}\chi(\widehat{HL}(L);t^2).$$

All these models being symmetric representatives of the Alexander polynomial, they agree up to sign. Theorem 1.1 states that ∇_L and χ_L coincide, while the equality $\Omega_L = \nabla_L$ is checked in [3]. There is no published proof of the fact that τ_L agrees with ∇_L, so let us explain one way of verifying this fact. In [19, §4], Turaev gave a list of six axioms satisfied by τ_L that characterize this invariant for links in S^3. It is not difficult to check that the geometric model Ω_L satisfies these axioms: the first five follow from [3] (together with the proof of [5, Proposition 2.5]), while the last one—the so-called *doubling axiom*— can be verified using the same elementary homological techniques. Hence, all four models coincide for links in S^3.

Now note that while Hartley's model ∇_L is only defined for links in the standard sphere S^3, the definition of the other three models can be extended *verbatim* to links in arbitrary integral homology spheres. Do they coincide in this more general setting? To address this question, let us recall two properties of ∇_L.

The Torres formula [9, 18]. *If $L = L_1 \cup \cdots \cup L_\mu$ and $L^* = L \cup L_{\mu+1} \cup \cdots \cup L_n$, then*

$$\nabla_{L^*}(t_1, \ldots, t_\mu, 1, \ldots, 1) = f_{L^*,L}(t_1, \ldots, t_\mu)\nabla_L(t_1, \ldots, t_\mu),$$

where $f_{L^*,L}(t_1, \ldots, t_\mu) = \prod_{i=\mu+1}^{n}(t_1^{\ell_{1i}} \cdots t_\mu^{\ell_{\mu i}} - t_1^{-\ell_{1i}} \cdots t_\mu^{-\ell_{\mu i}})$ and $\ell_{ij} = \mathrm{lk}(L_i, L_j)$.

Variance under surgery [2]. *Let L be a framed link in S^3 such that the manifold obtained by surgery along L is again S^3. Then*

$$\nabla_{\hat{L}}(t) = \det(B)^{-1}\nabla_L(t \cdot B^{-1}),$$

where $\hat{L}$ is the link given by the cores of the surgery tori, B is the framing matrix associated with L, and $t \cdot A$ stands for $(t_1^{a_{11}}t_2^{a_{21}} \cdots t_\mu^{a_{\mu 1}}, \ldots, t_1^{a_{1\mu}}t_2^{a_{2\mu}} \cdots t_\mu^{a_{\mu\mu}})$ if $A = (a_{ij})$.

The following proposition is an easy consequence of the work of Boyer and Lines [2].

Proposition 4.1. *Let ∇ and ∇' be two invariants of links in homology spheres such that*

(1) $\nabla_L = \nabla'_L$ *for any link L in S^3,*

(2) ∇ *and ∇' satisfy the Torres formula for any link in a homology sphere, and*

(3) ∇ *and ∇' satisfy the variance under surgery formula for any framed link in S^3 such that the surgery manifold is a homology sphere.*

Then ∇ and ∇' coincide for any link in a homology sphere.

Proof. Let L be a link in a homology sphere Σ. By [2, Lemma 2.3], there exists a link $L^* \subset \Sigma$, containing L as a sublink, such that $f_{L^*,L}(t)$ does not vanish and $\Sigma \setminus L^*$ is homeomorphic to $S^3 \setminus L^0$ for some link L^0 in S^3. The image of the meridians under this homeomorphism defines a framing of L^0 such that the resulting surgery manifold is Σ and the resulting link $\widehat{L^0}$ is L^*. By the first and third assumption, we have

$$\nabla_{L^*}(t^*) = \det(B)^{-1}\nabla_{L^0}(t^* \cdot B^{-1}) = \det(B)^{-1}\nabla'_{L^0}(t^* \cdot B^{-1}) = \nabla'_{L^*}(t^*).$$

Using this equality together with the second assumption, we obtain

$$f_{L^*,L}(t)\nabla(t) = \nabla_{L^*}(t, 1) = \nabla'_{L^*}(t, 1) = f_{L^*,L}(t)\nabla'_L(t),$$

and the conclusion follows since $f_{L^*,L}(t)$ does not vanish. $\qquad\square$

This result can be used to identify Ω_L and τ_L. Indeed, we already know that they coincide for links in S^3. The Torres formula for τ_L is a special case of [20, Chapter VII, Theorem 1.4], while it can be checked for Ω_L using standard homological computations. As for the variance of τ_L under surgery, we have found no proof in the literature but several related results are known (see, for example, [20, Chapter VIII] and [4]) and the same methods apply to give the desired result. Unlike τ_L, the geometrical model Ω_L is unfortunately not suited for surgery considerations. However, the variance under surgery

can be deduced from several simpler properties (S. Boyer, personal communication, 2004) that can be checked for Ω_L. In conclusion, Ω_L and τ_L coincide for any link in any homology sphere.

We have not been able to find in the literature a result in link Floer homology that categorifies the variance of χ_L under surgery. For this reason, we do not know if this method applies to identify χ_L and τ_L in the general setting of homology spheres. Alternatively, one could try to identify these two models working directly from the definition.

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