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The dimer and Ising models on non-orientable surfaces

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To my wife and my little daughter

Résumé de la thèse en français

Le but de cette thèse est d'étudier le modèle des dimères, le modèle des monomères-dimères, et le modèle d'Ising, plus particulièrement les fonctions de partition de ces modèles pour des graphes plongés dans des surfaces (possiblement) non-orientables. Nous avons obtenu trois résultats nouveaux, un pour chacun des modèles ci-dessus; ils constituent les chapitres 2, 3 et 4 de ce mémoire, le chapitre 1 rappelant les bases de la théorie.

Il avait été observé par plusieurs auteurs que les fonctions de partition du *modèle des dimères* pour le réseau carré avec diverses conditions au bord satisfont de "curieuses relations". Dans le chapitre 2, nous expliquons ces relations et les étendons à une grande famille de graphes pondérés. Plus précisément, nous utilisons l'extension aux surfaces non-orientables de la méthode Pfaffienne de Kasteleyn pour obtenir une relation entre les fonctions de partition "tordues" pour n'importe quel graphe pondéré G plongé dans une surface non-orientable Σ, et pour le graphe pondéré $\widetilde{G}$ plongé dans le revêtement d'orientation $\widetilde{\Sigma}$ de Σ. Lorsque G satisfait certaines propriétés combinatoires, ces fonctions de partition tordues coïncident avec les fonctions de partition standard, et l'on obtient les égalités voulues.

Le *modèle des monomères-dimères* peut être interprété comme une généralisation naturelle du modèles des dimères. Une différence notable entre ces deux modèles est que contrairement au modèle des dimères, la fonction de partition du modèle des monomères-dimères n'est pas calculable en temps polynomial même pour les graphes planaires. Néanmoins, il a récemment été observé par Giuliani, Jauslin et Lieb que dans le cas d'un graphe plongé dans le disque avec des monomères uniquement sur le bord, la fonction de partition peut s'exprimer comme le Pfaffien d'une matrice d'adjacence associée. Dans le chapitre 3, nous étendons ce résultat à n'importe quel graphe pondéré G plongé dans une surface Σ de topologie arbitraire – orientable ou non, de genre quelconque, avec un nombre quelconque de composantes de bord. L'idée est de considérer un graphe associé G', lui aussi plongé dans Σ, tel la fonction de partition pour les dimères sur G' est reliée de façon simple à la fonction de partition pour les monomères-dimères sur G. On peut ensuite appliquer la formule Pfaffienne de Cimasoni-Reshetikhin à $G' \subset \Sigma$ et obtenir le résultat à l'aide de manipulations algébriques des matrices associées. La formule finale est une somme alternée de $H_1(\Sigma; \mathbb{Z}_2)$ Pfaffiens. Dans le cas du disque, il y a un seul Pfaffien, et l'on retrouve la formule de Giuliani, Jauslin et Lieb.

Le dernier modèle étudié est le célèbre *modèle d'Ising*. Il y a une multitude de façons

de calculer la fonction de partition de ce modèle pour un graphe pondéré G plongé dans une surface Σ, et ces méthodes sont toutes essentiellement équivalentes: il est possible d'utiliser des matrices de Kac-Ward pour G, ou alors des matrices de Kasteleyn pour un graphe associé. Notons néanmoins que pour chacune de ces méthodes, la démonstration de la formule Pfaffienne correspondante est relativement difficile: ces preuves nécessitent en effet des arguments combinatoires pénibles pour certaines d'entre elles (Tesler, Gallucio-Loebl), et des outils géométriques non-triviaux pour d'autres (Cimasoni-Reshetikhin). Pour cette raison, nous avons été surpris d'en découvrir une preuve tout à fait élémentaire et relativement courte, valide pour un graphe pondéré plongé dans une surface de topologie quelconque. Cette preuve ne repose sur aucune idée fondamentalement nouvelle, mais utilise de manière subtile des idées de Kasteleyn, Tesler, Masbaum-Loebl, Cimasoni, et Chelkak-Cimasoni-Kassel. Elle constitue les premières sections du chapitre 4.

Finalement, dans les dernières sections du chapitre 4, nous utilisons cette formule pour étudier le comportement asymptotique de la fonction de partition du modèle d'Ising pour des graphes plongés dans la bouteille de Klein. Nous montrons que, à la température critique, elle admet le développement asymptotique

$$Z = \exp\{A\,\mathrm{f}_0 + \mathrm{fsc} + o(1)\}\,,$$

où A est l'aire du domaine fondamental, f_0 l'énergie libre, et fsc un terme correctif qui ne dépend essentiellement que du paramètre conforme du domaine. Un tel développement a été conjecturé par certains physiciens, tels Cardy, dans le contexte très général des modèles de dimension 2 au point critique. Dans un article récent, Kenyon-Sun-Wilson l'ont démontré pour le modèle des dimères sur un graphe planaire périodique-périodique (i.e. plongé dans le tore), ce qui implique facilement le résultat correspondant pour le modèle d'Ising. Mais notre calcul dans le cas d'un graphe planaire périodique-antipériodique (i.e. plongé dans la bouteille de Klein), est nouveau. Il repose sur un travail non-publié de Cimasoni-Kassel, et sur un résultat technique de Kenyon-Sun-Wilson. Notons pour finir que l'énergie libre f_0 est la même que dans le cas du tore, comme il se doit, mais que le terme correctif fsc est différent.

Acknowledgement

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Last but not least, I would like to thank my wife, my little daughter and my family members for supporting me throughout my PhD study.

Contents

Introduction

In this part, we shall introduce the main objects of the thesis, namely the dimer model, the monomer-dimer model and the Ising model. In parallel with reviewing each model, we will also summarise some principal results that were already obtained before, and explain what we achieve in our work.

The dimer model

The dimer model, whose name "dimer" means polymer with only two atoms, is a model that describes the absorption of di-atomic molecules on a crystal surface. It was first introduced to the physics and chemists communities in a paper by Fowler-Rushbrooke [21] in 1937.

This model can be simply defined on an arbitrary finite graph G as follows. A *dimer covering*, or a *dimer configuration* (a.k.a *perfect matching*) of G, is a collection of edges that cover all the vertices exactly once. One then can think of vertices of G as univalent atoms, and each bonding to exactly one neighbour. A well-known example is when the graph is a square lattice, then a dimer covering is equivalent to a domino tiling of the corresponding chessboard, via a simple duality (see Figure 1). Endowing G with an edge weight system $x = (x_e)_e$, we can define the corresponding *Gibbs measure* on the set of all dimer coverings by

$$\mathbb{P}(D) = \frac{x(D)}{Z(G, x)}.$$

In this equality $x(D) = \prod_{e \in D} x_e$, and

$$Z(G, x) = \sum_D x(D)$$

(where the sum is over all dimer coverings of G) is called the *dimer partition function.* The *dimer model* is the study of the measure $\mathbb{P}$, and particularly of the dimer partition function.

The first triumph in the study of the dimer model, due to Kasteleyn [35] and to Temperley-Fisher [62], is to compute precisely the partition function for a square lattice of arbitrary size. This computation was then extended by Kasteleyn [37] to any planar graph. Inspired by this remarkable result of Kasteleyn, a lot of progresses were achieved in understanding the model, see for example Kenyon [38, 39], Kuperberg [47], Galluccio-Loebl [22], Tesler [63], Kenyon-Propp-Wilson [43] and Cimasoni-Reshetikhin [14, 8]. Furthermore

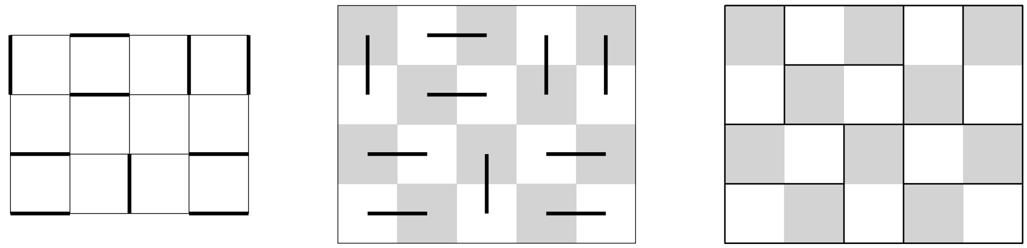


Figure 1: A dimer covering of a square lattice and the equivalent domino tiling of the corresponding chessboard.

Kenyon-Okounkov-Sheffield [42], followed by Kenyon-Okounkov [41, 40] had a big success to fully understand the model on infinite, periodic, bipartite graphs. Another recent contribution to the study of long-range properties of the model is due to Kenyon-Sun-Wilson [44]. We will discuss some results obtained by these authors in the following paragraphs.

Before discussing principal results of interest, let us first answer some natural questions. For example let us examine whether a given graph admits a dimer covering. Obviously the graph needs to have an even number of vertices, however this condition is not enough: the Y-shaped graph has four vertices but does not have any dimer configuration. One of the first characterisations for a graph G admitting a perfect matching is Tutte's theorem [50]: "G admits a perfect matching if and only if for every subset S of vertices, the graph $G \setminus S$ has at most $\#S$ connected components with an odd number of vertices." Moreover there is an efficient algorithm, called the *Hungarian method* [50], to find a perfect matching of any bipartite graph. This algorithm was then extended to arbitrary graphs by Edmonds [18]. Interested readers are referred to [50, 18] for combinatorial aspect of matching theory.

Let us now answer a very basic question: how to compute the dimer partition function efficiently? It was proved by Valient [64] that even when an arbitrary finite graph is (bipartite) unweighted, i.e. all the edge weights are equal to 1, the dimer partition function (which is now the number of perfect matchings of the graph) is "$\#P$-complete". It means that in general, there does not exist a polynomial time algorithm to compute the dimer partition function of an arbitrary graph. On the other hand, if the graph is planar, or more generally is embedded in a surface of a fixed genus, there does exist such an algorithm. Next we will discuss how to compute the dimer partition function in such cases.

The first exact computation of the dimer partition function is for the $m \times n$ square lattice, which was achieved independently by Kasteleyn [35] and Temperley-Fisher [62]. This result, being extended to any planar graph later by Kasteleyn [36, 37], became a major breakthrough in the study of the dimer model. Let us sketch the method, which is called *Pfaffian method*, to prove this remarkable result. The idea is to relate the dimer partition function of a weighted graph (G, x) to the Pfaffian of an adjacency matrix of G with respect to a suitable orientation K on the edges of G. By definition, the Pfaffian of this matrix counts all weighted perfect matchings of G with signs, and these signs are determined by the orientation K. Therefore, to have (the absolute value of) this Pfaffian to be equal to the dimer partition function, one needs to choose K so that all the signs are equal. It is observed by Kasteleyn that these signs can be given by the number of edges of certain cycles of G (which are called *superposition cycles*) on which some fixed orientation of each cycle is different from K. It turns out that working with superposition cycles on the plane is quite simple (recall that G is planar), and hence we end up with a desired orientation K, called *Kasteleyn orientation*, so that the dimer partition function of G is equal to the absolute value of the Pfaffian of the corresponding adjacency matrix, called *Kasteleyn matrix*. This equality is now called *Pfaffian formula*.

Let us emphasise that in the proof of Kasteleyn described above, the graph being planar is very important. For instance, if the graph is embedded in a higher genus surface, studying superposition cycles mentioned above is much more difficult. Actually Kasteleyn showed that for the square lattice embedded in the torus, the dimer partition function can be expressed as a linear combination of four Pfaffians. He also stated without proof that for graphs embedded in an orientable surface of genus g, the Pfaffian formula to compute the corresponding partition function requires 2^{2g} Pfaffians with signs [36, 37]. This fact was then proved combinatorially by Galluccio-Loebl [22] using particular polygonal representations of surfaces. Independently, Tesler [63] also gave a proof to this fact using *crossing orientations*, which generalised Kasteleyn orientations to graphs drawn in the plane with crossings. It should be remarked that by the method of Tesler, we can have Pfaffian formula even when the graph is embedded in a non-orientable surface. However Tesler's final result consists of an algorithmic way to compute the partition function, but not in a closed formula. Finally Cimasoni-Reshetikhin [14, 8] was successful to give such a formula for graphs embedded in (possibly) non-orientable surfaces, using geometric tools called spin and pin$^-$ structures.

As we just mentioned, the Pfaffian method for planar graphs can be demonstrated in a straightforward way, however the extension of this method to non-planar graphs in general involves either tedious combinatorial examinations or additional geometric tools. Nevertheless, in certain near-planar case one can prove the Pfaffian formula in an elementary way. For instance, Kuperberg [47] gave such a formula for locally but not globally bipartite graphs embedded in the projective plane.

One of the goals of the present work is to give a straightforward extension of the Pfaffian method for graphs embedded in the Möbius band. The proof, using only obvious properties of simple closed curves in the Möbius strip, is very elementary and readable. It should be also mentioned that, inspired by this proof, one can explain and generalise some "curious identities" relating the dimer partition functions of square lattices. More clearly, by explicit computations Lu-Wu [51] (see also [3]) found

$$Z^{\mathscr{C}}_{2m,4n}(x,y) = \left(Z^{\mathscr{M}}_{2m,2n}(x,y)\right)^2,$$
$$Z^{\mathscr{C}}_{2m-1,4n}(x,y) = \frac{1}{2}\left(Z^{\mathscr{M}}_{2m-1,2n}(x,y)\right)^2,$$

while Izmailian-Oganesyan-Hu [32] established the formula

$$Z^{\mathbb{T}}_{2m-1,4n}(x,y) = \frac{1}{2}\left(Z^{\mathscr{K}}_{2m-1,2n}(x,y)\right)^2$$

where $Z^{\mathscr{C}}_{m,n}$ (resp. $Z^{\mathscr{M}}_{m,n}$, $Z^{\mathbb{T}}_{m,n}$ and $Z^{\mathscr{K}}_{m,n}$) is the dimer partition function of the $m \times n$ square lattice weighted x, y embedded in the cylinder (resp. the Möbius strip, the torus and the Klein bottle). Let us mention once again that these identities are obtained by comparing explicit values of different partition functions, and these authors can not give an underlying principle that explains these identities. In this thesis, we will give such a principle that allows us, not only to explain these identities, but also to generalise them. More precisely, one of our main results is the following theorem.

Theorem 2.1.1. *Let $G \subset \mathscr{M}$ be a weighted graph with an even number of vertices embedded in the Möbius strip, and let $\widetilde{G} \subset \mathscr{C}$ denote its orientation cover.*

(i) If G is locally bipartite but not bipartite, then $Z(\widetilde{G}) = Z(G)^2$.

(ii) Assume that $G \subset \mathscr{M}$ is invariant by a horizontal translation τ of the Möbius strip. If there is some $D_0 \in \mathcal{D}(G)$ such that the symmetric difference $D_0 \Delta \tau(D_0)$ winds around $\mathscr{M}$ an odd number of time, then $Z(\widetilde{G}) = \frac{1}{2} Z(G)^2$.

Let us emphasise that the result stated in part (i) of this theorem was obtained by Kuperberg [47, Theorem 7] in 1998. However in this work we present a new proof of this result, which allows us to prove the identity in part (ii), and also extend it to the case of graphs embedded in the Klein bottle. The latter is the content of our second main theorem.

Theorem 2.1.6. *Let $G \subset \mathscr{K}$ be a weighted graph with an even number of vertices embedded in the Klein bottle, and let $\widetilde{G} \subset \mathbb{T}$ denote its orientation cover. Let us assume that G is locally bipartite but not bipartite, and invariant by a horizontal translation τ. If there is some $D_0 \in \mathcal{D}(G)$ such that the symmetric difference $D_0 \Delta \tau(D_0)$ winds an odd number of times around $\mathscr{K}$ in the horizontal direction (and an arbitrary number of times in the vertical direction), then $Z(\widetilde{G}) = \frac{1}{2} Z(G)^2$.*

Note that the identities stated in Theorems 2.1.1 and 2.1.6, which generalise the results for the square lattice obtained before, can be extended similarly to arbitrary non-orientable surfaces. Interested readers are referred to Theorem 2.3.2 for more details.

Let us terminate this section about the dimer model by reviewing some long-range properties of the model, which can be derived from the exact computation of the partition function. Denote by Z_{mn} the dimer partition function of the $m \times n$ square lattice. With mn even, it was obtained by Kasteleyn [35] that when the lattice is unweighted and embedded in the plane or in the torus, it has asymptotic *free energy*

$$\mathrm{f}_0 := \lim_{m,n \to \infty} \frac{1}{mn} \log Z_{mn} = \frac{\mathrm{G}}{\pi}$$

where $\mathrm{G} = \sum_{j \geq 0} \dfrac{(-1)^j}{(2j+1)^2}$ is Catalan's constant. If m, n are both even, Fisher [20] made a more precisely calculation and found that the free energy of the lattice embedded in the plane can be given by

$$\log Z_{mn} = mn\,\mathrm{f}_0 - 2(m+n)\,\mathrm{f}_1 + O(1)$$

with $\mathrm{f}_1 = \frac{1}{4}\log(1+2^{1/2}) - \mathrm{G}/\pi$. Finally Ferdinand [19] refined the calculation of Kasteleyn and Fisher, finding a universal correction term which depends on the "shape" of the region (i.e. the choice of the plane or the torus in which the lattice is embedded in, as well as the aspect ratio n/m) and on the parities of m, n. More precisely, for mn even Ferdinand found

$$\log Z_{mn} = \begin{cases} mn\,\mathrm{f}_0 + \mathrm{fsc}^{\mathrm{torus}}_{(-1)^{m+n}}(\frac{n}{m}) + o(1) & \text{(embedding in the torus);} \\ mn\,\mathrm{f}_0 + 2(m+n)\mathrm{f}_1 + 4\mathrm{f}_\square + \mathrm{fsc}^{\mathrm{plane}}_{(-1)^{m+n}}(\frac{n}{m}) + o(1) & \text{(embedding in the plane),} \end{cases}$$

where $\mathrm{f}_\square$ can be understood as the free energy per corner, and $\mathrm{fsc}^{\mathrm{torus}}_{\pm 1}, \mathrm{fsc}^{\mathrm{plane}}_{\pm 1}$ are explicit analytic functions of the ratio n/m. These functions are called the *finite-size corrections* to the free energy, and contain information about long-range properties of the dimer system (see [5] for example).

Note also that Kasteleyn, Fisher and Ferdinand remade these calculations for weighted square lattice with horizontal edge weight a and vertical edge weight b and obtained similar asymptotic expansions for $\log Z_{mn}$. In this case the free energy coefficients $\mathrm{f}_0, \mathrm{f}_1, \mathrm{f}_\square$ depend on the weights a, b in a complicated way, but the finite-size corrections are the same functions appearing in the unweighted case, now applied to the aspect ratio $\frac{nb}{ma}$.

Remark that finite-size corrections for the dimer model on a square lattice have also been calculated explicitly under various embeddings of the lattice: it was obtained by Brankov-Priezzhev [3] and by Izmailian-Oganesyan-Hu [32] for the lattice embedded in the cylinder and the Möbius band, and also by Izmailian-Oganesyan-Hu [32] for the Klein bottle. In each of these surfaces, for each given choice of parities of m, n, the finite-size correction is an analytic function of the aspect ratio $\frac{nb}{ma}$. Additionally, the finite-size correction was also computed precisely by Boutilier and de Tilière [2] for the hexagonal lattice embedded in the torus.

Furthermore, as pointed out by Cardy-Peschel [6], such an expansion for $\log Z_{mn}$ is expected to hold for any lattice, however the lattice must be in surfaces with Euler characteristic equal to zero, that is, the cylinder, the Möbius band, the torus and the Klein bottle. These are precisely all the cases mentioned above.

The most general result to compute finite-size corrections for the dimer model is only obtained recently by Kenyon-Sun-Wilson [44] for toroidal graphs. For instance, on bipartite lattices embedded in the torus, the associated finite-size correction can be given by certain Jacobi theta functions parameterised by a number τ. This parameter, belonging to the complex upper half-plane, describes the *conformal shape* of the domain: $\frac{\tau}{i}$ generalises the aspect ratio $\frac{nb}{ma}$ appearing in the case of the square lattice. Actually the parameter τ can be computed from the Hessian of a certain polynomial called the *dimer characteristic polynomial*. Furthermore, it turns out that many large-scale quantities of interest can be computed from this characteristic polynomial, and we refer the reader to [41, 42] for the study of this polynomial, as well as related results.

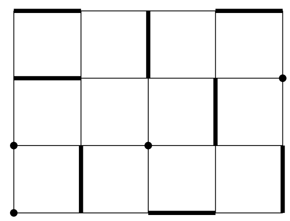


Figure 2: An MD covering on a square lattice.

Last but not least, finite-size corrections are conjectured to exist and be universal not only for the dimer model, but also for a lot of models. For example similar corrections can be also obtained for toroidal graphs in the case of the Ising model [10, Corollary 3.5]. Inspired by the work of Kenyon-Sun-Wilson [44], and using an unpublished work of Cimasoni-Kassel [12], we can compute these correction terms efficiently for the Ising model on graphs embedded in the Klein bottle. This will be explained more precisely in the last section of the introduction.

The monomer-dimer model

We now turn to another model that is more general than the dimer one, the monomer-dimer model. Similarly to the dimer model, the monomer-dimer model is a model that describes the absorption of molecules of different sizes on a lattice [21].

The monomer-dimer model can be defined on a finite abstract graph G as follows. A *monomer-dimer covering*, or simply an MD covering is a set of edges (called dimers) and of vertices (called monomers) of G so that each vertex of G is covered by exactly one element of this set (see Figure 2 for example). Given an edge weight system $x = (x_e)_e$ and a vertex weight system $y = (y_v)_v$ one can define the *monomer-dimer partition function* (or simply *MD partition function*) by

$$Z(G,x,y) = \sum_{\tau} \prod_{e_\tau} x(e_\tau) \prod_{v_\tau} y(v_\tau),$$

where the sum is over all monomer-dimer coverings τ with dimers e_τ and monomers v_τ. The study of the model, as explained before, is based on the study of the corresponding partition function.

Let us now discuss how to compute the MD partition function efficiently. On one hand, it is obvious that there always exists (many) MD coverings of a given graph. On the other hand, it is intractable to compute the MD partition function, even for planar graphs in general [33]. However it turns out that there are some particular cases in which we can compute the MD partition function efficiently. The easiest case perhaps is when monomers are not allowed to appear, then we go back to the case of the dimer model that is discussed before. A more interesting case is when monomers are allowed to appear. Then the most general result in this case, to the best of our knowledge, is obtained by Giuliani-Jauslin-Lieb

[24] for graphs embedded in a disc with monomers restricted to graph boundary of the disc. Their result basically states that the MD partition function of a given graph G in this case is equal to the Pfaffian of a general adjacency matrix of G with respect to a well-chosen orientation and a well-chosen vertex labelling. The proof of this result is based either on the expansion of the Pfaffian of the general adjacency matrix [24, Theorem 2.7], or on the bijection between MD coverings of G and dimer coverings of an auxiliary graph [24, Appendix E]. Nonetheless, a similar formula for non-planar graphs does not seem to be known. In this thesis, inspired by the work of Giuliani-Jauslin-Lieb, we will give such a formula to compute the MD partition function of graphs embedded in (possibly) non-orientable surfaces with boundary, in which monomer are restricted to the boundary of the surface.

Let us be more precise. For G a weighted graph embedded in a surface Σ with boundary, denote by $Z_{\mathcal{MD}}(G)$ the boundary MD partition function of G, that is, the one counting all weighted MD coverings whose monomers are allowed to appear only on the boundary of the surface. First assume that Σ is orientable of genus g and has b boundary components. Denote by $\overline{\Sigma}$ the closed surface obtained by gluing b disks along the boundary of Σ. Let $\{\alpha_i\}_{1\leq i\leq 2g}$ be a set of closed curves that are transverse to G and avoid ∂G so that their homology classes form a basis of $H_1(\overline{\Sigma};\mathbb{Z}_2)$, the first homology group over $\mathbb{Z}_2$ of $\overline{\Sigma}$. Then for each $\beta=(\beta_1,\dots,\beta_b)\in\mathbb{Z}_2^b$ so that $\sum_{k=1}^b\beta_k=0$ and for each $\epsilon\in\mathbb{Z}_2^{2g}$, let us denote by K_β^ϵ a well-chosen orientation on G with respect to $\{\alpha_i\}$, parameterised by ϵ,β (see subsection 3.2.3 for more details). Setting $M_\beta^K(G)$ to be the modified adjacency matrix of G with respect to an orientation K and a parameter β (see Definition 3.2.1), we have the following result.

Theorem 3.2.2. *Let (G,x,y) be a weighted graph embedded in an orientable surface Σ of genus g with b boundary components. Then the boundary monomer-dimer partition function of G is given by*

$$Z_{\mathcal{MD}}(G)=\frac{1}{2^g}\sum_{\substack{\beta\in\mathbb{Z}_2^b\\ \sum_k\beta_k\ \mathit{even}}}\left|\sum_{\epsilon\in\mathbb{Z}_2^{2g}}(-1)^{\sum_{i<j}\epsilon_i\epsilon_j\alpha_i\cdot\alpha_j}Pf(M_\beta^{K_\beta^\epsilon}(G))\right|.$$

In this formula the first sum is over all $\beta=(\beta_1,\dots,\beta_b)\in\mathbb{Z}_2^b$ such that $\sum_{k=1}^b\beta_k$ is even, while in the second sum, $\alpha_i\cdot\alpha_j$ denotes the intersection number of α_i and α_j.

Note that when Σ is non-orientable, we also have the similar formula to compute the boundary MD partition function, see Theorem 3.4.3. Additionally, other versions of Theorems 3.2.2 and 3.4.3, namely Theorems 3.2.4 and 3.4.2 can be also found in Chapter 3.

The Ising model

We now put an end to the introduction by reviewing the third model of interest, the Ising model. This model was invented by the physicist Wilhelm Lenz in 1920 as an attempt to understand the para/ferromagnet transition observed in Curie's experiment of magnetic material. However, the model was named after his PhD student Ernst Ising, who proved that there is no phase transition in the one dimensional case, and incorrectly conjectured that it is also the case in any dimension. (This is perhaps the only case in the history of science that the work was named after the student, not after the professor!)

Before giving the definition of the model, let us explain more about phase transitions in some examples. One of the most obvious phenomena that present a phase transition is the transformation from ice to water, and from water to vapour under the increase of temperature. This means that some macroscopic properties of a large system of H_2O molecules (here it is the density) vary when the temperature rises. Another more subtle phenomenon of phase transitions can be described by the behaviour of a lattice of magnetic moments under the effect of an external magnetic field. To be more precise, let us assume that the local magnetic moment is represented by a spin, and that each spin has the tendency to be aligned with other spins in its neighbour. Assume also that the spins at the beginning are disordered: they are randomly oriented. Introducing an external magnetic field, then the spins tend to align with this field. If the external field is slowly decreased to zero, there are two cases that can happen. In the first case, the spins gradually lose their global alignment, and become disordered again when the field is zero. In the second case, the interactions between the spins are strong enough to keep them still aligned, even if the magnetic field decreases and reaches zero. The behaviour of spins is called *paramagnetism* in the first case, and *ferromagnetism* in the second case. These two behaviours also happen if we replace the external magnetic field by a source of heat, and vary the temperature of the source: if the temperature is high, we have paramagnetic phenomenon, and if the temperature is low, we have ferromagnetic one. Intuitively there should be a point, called *critical point* (or *critical temperature*), so that when the temperature is varied through this point, we have a switch from one phenomenon to the other. This switch of phenomena represents the phase transition in the experiment of magnetic moments.

A major question of theoretical physicists at the beginning of the 20^{th} century was to determine whether phase transitions could be described using the framework of statistical physics. The answer to this question was positive, using the Ising model that we now define. Although the model can be defined for a lot of material, in the following we will only consider this model for ferromagnet, which corresponds to the second example described above. Also, this model can be defined on any graph, however for simplicity we will restrict ourselves to an $m \times n$ square lattice in the plane. Assume that at each vertex of this square lattice, a local magnetic moment is represented by a spin with two states $\{\pm 1\}$, then a magnetic moment on the whole lattice is equivalent to a *spin configuration* $\sigma = (\sigma_i)_i$ where i represents lattice vertices. The energy of a spin configuration σ includes two contributions: the interactions between neighbouring spins, and the effect of an external field on each individual one. The interaction between neighbouring spins tends to induce alignment of the neighbours, therefore it should be small when the neighbours are both $+1$ or both -1, and big when the neighbours are $+1$ next to -1. So the interaction energy for each pair of neighbours i and j can be given by $-\beta\sigma_i\sigma_j$, where $\beta > 0$ measures the strength of the interaction, and is also interpreted as the inverse temperature. The interaction of a constant magnetic field $h \in \mathbb{R}$ with the spin at i contributes to the energy of the configuration by an amount $-h\sigma_i$. To sum up, the *energy* of a spin configuration σ is given by

$$\mathscr{H}(\sigma) = -\beta \sum_{i \sim j} \sigma_i \sigma_j - h \sum_i \sigma_i,$$

where the first sum is over all pairs of neighbouring vertices. This energy gives rise to the *Ising partition function* of the lattice

$$Z^\beta_{mn} = \sum_\sigma \exp(-\mathscr{H}(\sigma)),$$

where the sum is taken over all spin configurations.

As in the dimer case, there is a lot of data that arises from this partition function. For example, the Gibbs measure μ^β_{mn} defined on the set of all spin configurations by

$$\mu^\beta_{mn}(\sigma) = \frac{1}{Z^\beta_{mn}} \exp(-\mathscr{H}(\sigma))$$

gives the precise probability of observing the system. It is also possible to extend the lattice by letting m, n tend to infinity, and consider the limit

$$\mathrm{f}_0(\beta) := \lim_{m,n \to \infty} \frac{1}{mn} \log Z^\beta_{mn},$$

which is called the free energy of the model. It turns out that if the external field h vanishes (and we shall always assume so), then the free energy function is analytic for every $\beta > 0$ except at $\beta = \beta_c$, the inverse critical temperature. Moreover, similar to the case of the dimer model, it is proved that if the square lattice is embedded in the torus or the Klein bottle, we have an asymptotic expansion

$$Z^\beta_{mn} = \begin{cases} \exp\big(mn\,\mathrm{f}_0(\beta_c) + \mathrm{fsc} + o(1)\big) & \text{if } \beta = \beta_c; \\ \exp\big(mn\,\mathrm{f}_0(\beta) + o(1)\big) & \text{if } \beta \neq \beta_c, \end{cases}$$

where the finite-size correction fsc, as mentioned in the case of the dimer model, can be given by Jacobi theta functions and only depends on the conformal shape of the lattice, on the parity type of m, n and on the surface we embed the square lattice in (a.k.a boundary conditions). To be precise, this correction term is calculated explicitly by Lu-Wu [52] when the square lattice is embedded in the Klein bottle (i.e. periodic-antiperiodic boundary conditions). Similar expansion is also obtained by the same authors when the lattice is embedded in the Möbius strip (i.e. free-antiperiodic boundary conditions), and the finite-size correction fsc is also computed precisely in this case. This correction is then calculated completely by Izmailian [30] with all possible boundary conditions. As will be clarified later, the Ising partition function of a graph can be related to the dimer partition function of an associated one. Therefore using the result of Kenyon-Sun-Wilson [44] for the dimer model mentioned before, one can compute finite-size corrections in the case of the Ising model for arbitrary toroidal graphs, that corresponds to the periodic-periodic boundary conditions [10, Corollary 3.5].

One of the main goals of this thesis is to compute finite-size corrections for the Ising model under periodic-antiperiodic boundary conditions, that is, for graphs embedded in the Klein bottle. This is based on the work of Kenyon-Sun-Wilson [44], on an in-progress work of Cimasoni-Kassel [12], and on a Pfaffian formula to compute the Ising partition function of graphs embedded in a non-orientable surface. This will be explained with further details in the following.

Let us first review some notable results to compute the Ising partition function of arbitrary graphs, and then explain how to adapt them to the case of graphs embedded in non-orientable surfaces. In spite of the fact that the Ising model can be defined on any graph, it turns out that the corresponding partition function is only computable for planar graphs, or more generally for graphs embedded in surfaces. Let us be more precise. The very first solution in 1944 by Onsager [54] to compute the Ising partition function of a square lattice was considered to be very difficult, that left open the possibility of a simpler approach. With this motivation, and based on van der Waerden's reformulation of the Ising partition function [65], Kac and Ward [34] announced a new solution in 1952, in which the Ising partition function of a square lattice can be expressed as the determinant of a relatively small matrix. More specifically, they defined a so-called Kac-Ward martix whose rows and columns are indexed by oriented edges of the lattice, and proved that the determinant of this matrix is equal to the square of the Ising *high-temperature expansion*. Unfortunately the proof of this Kac-Ward formula contained a serious combinatorial error. This error was then replaced by a conjecture by Feynman (in an unpublished work), that was proved later by Sherman [59] in 1960. The first rigorous direct combinatorial proof of Kac-Ward formula, to the best of our knowledge, was achieved much later by Dolbilin *et al.* [17] in 1999 for any planar graph with straight edges.

In the mean time, independently of Sherman, Hurst-Green [28] found another method to compute the Ising partition function. They proved that the Ising partition function of a square lattice is equal to the Pfaffian of a skew-symmetric incidence matrix of a new weighted graph. This work inspired Fisher-Temperley [62], and independently Kasteleyn [35] to enumerate perfect matchings on a square lattice, leading to the study of the dimer partition function as we just mentioned before.

As a summary, there are basically two approaches to compute the Ising partition function of a planar graph: either using Kac-Ward matrices associated to the original graph, or using Kasteleyn matrices associated to a new one. As discovered by Kasteleyn [35] and Fisher [20], the Ising model can be interpreted as a special type of the dimer model. For example, the correspondence $G \mapsto G^T$ with G^T the *terminal graph* associated to G brings the Ising model on G to the dimer model on Γ_G. Another example is the *Fisher correspondence* $G \mapsto \Gamma_G$, which also has the same property (cf. Figure 3). Therefore it turns out that the two above approaches are equivalent (see [9] for a proof of such an equivalence). Recall that since the dimer partition function can be computed using Pfaffian formula even for surface graphs [22, 63, 14, 8], it is possible to have a similar formula for the Ising partition function

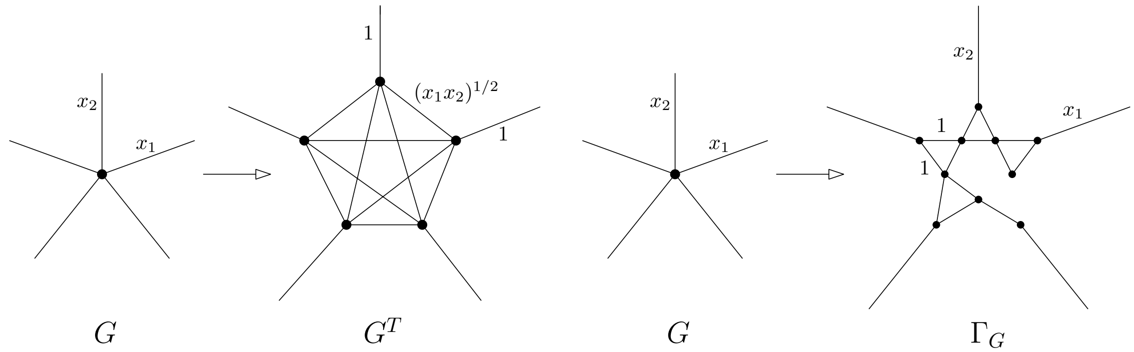


Figure 3: The correspondences $G \mapsto G^T$ and $G \mapsto \Gamma_G$.

of the same type of graphs, using these Ising-dimer relations. For instance, Cimasoni [9] used the Fisher correspondence to establish such a formula (called generalised Kac-Ward formula) for graphs embedded in an orientable surface. This formula was also obtained later by Chelkak-Cimasoni-Kassel [7] using the terminal graph. Remark that both of these formulas and their proofs require an additional geometric tool called spin structures.

Inspired by the work of Chelkak-Cimasoni-Kassel [7] together with ideas of Tesler [63] and Loebl-Masbaum [48], as well as of Cimasoni-Reshetikhin [14, 8], we found a Pfaffian formula to compute the Ising partition function of arbitrary graphs embedded in any closed surface. Roughly speaking, our formula expresses the Ising partition function of a graph G embedded in a surface as a linear sum of Pfaffians of adjacency matrices of the terminal graph G^T with respect to certain well-chosen orientations. We would like to emphasise that, although such a formula was already known to exist, our proof of this formula is completely elementary and requires neither geometric tools as in [14, 8], nor tedious combinatorial considerations as in [63]. The simplest form of our Pfaffian formula can be stated as follows.

Assume that (G, x) is a weighted graph embedded in an orientable surface Σ of genus g. Denote by (G^T, x^T) the weighted terminal graph associated to (G, x). With a particular drawing of G and *good* orientations $K_{\epsilon,\epsilon'}$ for $(\epsilon, \epsilon') \in \mathbb{Z}_2^{2g}$ (cf. Definition 4.2.1), the Ising partition function of G is given by the following theorem.

Theorem 4.2.2. *Let (G, x) be a weighted graph embedded in the orientable surface Σ of genus g. Then the Ising partition function of G is given by*

$$Z_{\mathcal{I}}(G, x) = \frac{1}{2^g}\left| \sum_{(\epsilon,\epsilon')\in\mathbb{Z}_2^{2g}} (-1)^{\sum\limits_{i=1}^{g} \epsilon_i \epsilon_i'} Pf(A^{K_{\epsilon,\epsilon'}}(G^T, x^T))\right|,$$

where $A^{K_{\epsilon,\epsilon'}}(G^T, x^T)$ is the adjacency matrix of the terminal graph (G^T, x^T) with respect to the orientation $K_{\epsilon,\epsilon'}$.

An alternative version of the stated formula in Theorem 4.2.2 can be found in Theorem 4.2.3. Similarly, when Σ is non-orientable we obtain two versions of the Pfaffian formula, namely Theorem 4.4.1 and Theorem 4.4.4.

Having an exact formula to compute the Ising partition function of graphs embedded in a non-orientable surface, and particularly in the Klein bottle $\mathscr{K}$ enables us to compute finite-size corrections in the latter case. Let us be more precise. Inspired by the work of Kenyon-Sun-Wilson [44], we can define the *Ising characteristic polynomial* $P(z,w)$ of $G \subset \mathscr{K}$ as the determinant of the *twisted adjacency matrix* of the terminal graph G^T (see Definition 4.5.1). Denote also by $P_{MN}(z,w)$ the characteristic polynomial of G_{MN}, the MN-sheeted covering of G, and write $\widetilde{P}$ for P_{12}. Then by our Pfaffian formula, the Ising partition function of G_{MN} can be written as a linear sum of $P_{MN}(\pm 1, \pm 1)^{1/2}$. Our key result, which is based on an unpublished work of Cimasoni-Kassel [12], is to relate P_{MN} to $P, \widetilde{P}$. Setting $z_k := \exp(k\pi i/M), w_l := \exp(2l\pi i/N)$ this result can be stated as follows.

Theorem 4.5.14. *Given G a graph embedded in the Klein bottle $\mathscr{K}$ and G_{MN} its MN-sheeted covering. For N odd we have*

$$P_{MN}(1,\pm 1) = \prod_{l=1}^{N} P(1, \pm w_l) \prod_{l=1}^{N} \prod_{k=1}^{(M-1)/2} \widetilde{P}(z_{2k}, w_l^2),$$

$$P_{MN}(-1,\pm 1) = \prod_{l=1}^{N} P(-1, \pm w_l) \prod_{l=1}^{N} \prod_{k=0}^{(M-3)/2} \widetilde{P}(z_{2k+1}, w_l^2)$$

if M is odd, and

$$P_{MN}(1,\pm 1) = \prod_{l=1}^{N} P(1, \pm w_l) \prod_{l=1}^{N} P(-1, \mp w_l) \prod_{l=1}^{N} \prod_{k=1}^{(M-2)/2} \widetilde{P}(z_{2k}, w_l^2),$$

$$P_{MN}(-1,\pm 1) = \prod_{l=1}^{N} \prod_{k=0}^{(M-2)/2} \widetilde{P}(z_{2k+1}, w_l^2)$$

if M is even.

Let us denote by $\widetilde{G}$ the lift of G to the torus, the orientation cover of the Klein bottle. Also denote by $\widetilde{G}_{MN}$ the MN-sheeted covering of $\widetilde{G}$, and by $\widetilde{P}_{MN}(z,w)$ the corresponding characteristic polynomial. By the product formulas presented above, one can relate $P_{MN}(\pm 1, \pm 1)$ with $\widetilde{P}_{MN}(\pm 1, 1)$. As $\widetilde{G}$ is in the torus, the latter quantity can be evaluated asymptotically using the result of Kenyon-Sun-Wilson [44]. Then by some analysis we end up with the following result.

Theorem 4.5.21. *The finite-size correction terms to the free energy of the Ising model on a weighted graph $G \subset \mathscr{K}$ at criticality is given by $\log\left(\iota\,\Xi + \Xi'\right)$ for M even, and by $\log\left(\iota_1\,\Xi + \iota_2\,\Xi'\right)$ for M odd. The numbers ι, ι_1, ι_2 belong to $\{0, 1, \sqrt{2}/2\}$ and depend only on N modulo 8, while the functions Ξ, Ξ' are given by*

$$\Xi = \left|\frac{\vartheta_{00}(0|\tau)}{\eta(\tau)}\right|^{1/2}, \quad \Xi' = \left|\frac{\vartheta_{10}(0|\tau)}{\eta(\tau)}\right|^{1/2}$$

with ϑ_{rs} Jacobi theta functions, η the Dedekind η function, and τ being computable from the Hessian of $\widetilde{P}(z,w)$ at $(-1,-1)$.

Chapter 1

Background

In this chapter we will introduce our three models which are the main objects studied in the rest of the thesis. We will also give some preliminaries that are needed to understand our results stated in the next three chapters. From now on, without stating explicitly, G will be understood as a finite graph.

1.1 The three models

Let us start with the monomer-dimer model which can be defined as follows. Given a finite graph G with vertex set $V(G)$ and edge set $E(G)$, a *monomer-dimer covering*, or shortly, an *MD covering* of G is a pair $\tau = (\tau_D, \tau_M) \in E(G) \times V(G)$ so that each vertex of G is covered by exactly one element of τ_D (which we shall call a *dimer*) or one element of τ_M (which we shall call a *monomer*). Let us denote by $\mathcal{MD}(G)$ the set of all MD coverings of G. If G is endowed with an edge weight system $x = (x_e)_{e\in E(G)}$ and a vertex weight system $y = (y_v)_{v\in V(G)}$, then the monomer-dimer partition function (or simply the MD partition function) of the weighted graph (G, x, y) is defined by

$$Z_{\mathcal{MD}}(G) := Z_{\mathcal{MD}}(G, x, y) = \sum_{\tau\in\mathcal{MD}(G)} \prod_{e\in\tau_D} x_e \prod_{v\in\tau_M} y_v.$$

In Chapter 3 we will show how to compute this partition function efficiently when the graph is embedded in a surface with boundary, and monomers are allowed to appear only on the boundary of the surface.

Let us now continue with the dimer model. In short, the dimer model is a special case of the monomer-dimer model when monomers are not allowed to appear. More precisely, if $\tau = (\tau_D, \tau_M) \in E(G) \times V(G)$ is an MD covering of G with $\tau_M = \emptyset$, then we simply say that $\tau = \tau_D$ is a *dimer covering* (a.k.a *perfect matching*) of G. We denote by $\mathcal{D}(G)$ the set of all dimer coverings of G. Similarly to the MD partition function, the dimer partition function of the edge-weighted graph (G, x) is defined by

$$Z_{\mathcal{D}}(G) := Z_{\mathcal{D}}(G, x) = \sum_{\tau\in\mathcal{D}(G)} \prod_{e\in\tau_D} x_e.$$

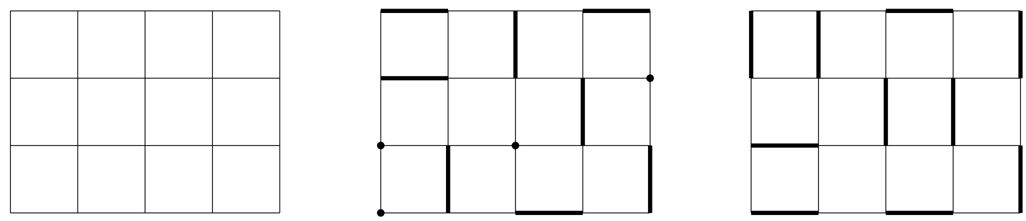


Figure 1.1: From left to right: the original graph, an MD covering, a dimer covering.

Note that the dimer partition function of (G, x) is equal to the MD partition function of (G, x, y) if we set all the vertex weights $y = (y_v)_{v \in V(G)}$ equal to 0. In Section 1.4 of this chapter, we will review a general theory to compute the dimer partition function $Z_{\mathcal{D}}(G)$ when G is embedded in a surface of arbitrary topology. Moreover, based on this general theory, in the next chapter we shall prove some identities between the dimer partition functions of graphs embedded in different surfaces.

We now terminate this section by introducing the last model, the Ising model. Recall that G is a finite weighted graph with vertex set $V(G)$ and edge set $E(G)$. We define a *spin configuration* on G as an element $\sigma \in \{-1, 1\}^{V(G)}$. Given a positive edge weight system $J = (J_e)_{e \in E(G)}$, we define the *energy* (or the *Hamiltonian*) of such a σ by

$$H(\sigma) := -\sum_{e=(u,v)} J_e \sigma_u \sigma_v,$$

where the sum is taken over all the edges $e = (u, v)$ of G (note that we always consider the case of external field $h \equiv 0$). Fixing an *inverse temperature* $\beta > 0$, we can define a probability measure on the set $\Omega(G)$ of all the spin configurations on G by

$$\mu_{G,\beta}(\sigma) = \frac{\exp(-\beta H(\sigma))}{Z^J_\beta(G)},$$

where $Z^J_\beta(G) := \sum_{\sigma \in \Omega(G)} \exp(-\beta H(\sigma))$ is the *partition function* of the Ising model on the weighted graph (G, x) with the inverse temperature β. Moreover, by the observation of van der Waerden [65], calculating the partition function $Z(G)$ of the Ising model on the graph G with fixed $x = (x_e)_{e \in E(G)}$ and β can be transformed to calculating its *high-temperature expansion*

$$Z_{\mathcal{I}}(G) = \sum_{G' \subset G} \prod_{e \in E(G')} \tanh(\beta J_e),$$

where the sum is over all *even subgraphs* G' of G, i.e. subgraphs such that each vertex of G is met by an even number of edges of G'. From now on, by saying the Ising partition function we mean this high-temperature expansion. In Chapter 4 we will give a formula to compute the Ising partition function of graphs embedded in surfaces. We will also derive some long-range properties of the Ising model from its partition function, such as the free energy and the finite-size correction terms.

1.2 Algebraic preliminaries

In this section we will recall the definitions of quadratic forms and quadratic enhancements. We will also define the Arf and Brown invariants, and give some of their properties that are needed for our purpose. Note that in this section we will only consider these objects in the algebraic setting, while their geometric interpretations and applications will be reviewed later in subsection 1.4.3.

1.2.1 Quadratic forms and Arf invariant

Let us first begin with quadratic forms and their Arf invariant. In this subsection we only give some basic properties of them, and refer the reader to [58, Chapter 9] for another approach and more results.

Definition 1.2.1. *Given $(H, \cdot)$ a vector space of finite dimension over $\mathbb{Z}_2$ together with a bilinear form $(\cdot)$, a function $q : H \to \mathbb{Z}_2$ is called a* quadratic form *on $(H, \cdot)$ if for every $x, y \in H$ we have*

$$q(x + y) = q(x) + q(y) + x \cdot y.$$

Let us now spend a while to study a few basic properties of quadratic forms before giving the definition of their Arf invariant. By definition, in order to have the set $\mathcal{Q}(H, \cdot)$ of all quadratic forms on $(H, \cdot)$ to be nonempty, it is necessary that the bilinear form must be symmetric (i.e. $x \cdot y = y \cdot x$ for every $x, y \in H$), which we shall always assume from now on. It is also clear that if q is a quadratic form, then $q(0) = 0$, and hence for every $x \in H$ we have

$$x \cdot x = 2q(x) + x \cdot x = q(2x) = q(0) = 0.$$

Additionally, one can verify easily by definition that every quadratic form is completely determined by its value on a basis of H, and these values can be chosen freely. As a consequence, there are exactly $2^{\dim H} = |H|$ quadratic forms on H. Another way to see that there are precisely $|H|$ quadratic forms on H is to look at the action of $H^* := Hom(H, \mathbb{Z}_2)$ on the set $\mathcal{Q}(H, \cdot)$ given by

$$H^* \times \mathcal{Q}(H, \cdot) \to \mathcal{Q}(H, \cdot) \quad (\varphi, q) \mapsto \varphi + q.$$

By definition of quadratic forms, this action is well-defined. Moreover for any $q, q' \in \mathcal{Q}(H, \cdot)$, there is a unique element $\varphi = q' - q \in H^*$ so that $\varphi + q = q'$. Hence this action is free and transitive, and so $|\mathcal{Q}(H, \cdot)| = |H^*| = |H|$.

Next we will define the Arf invariant of quadratic forms on $(H, \cdot)$. To do so we shall assume that the bilinear form is non-degenerate.

Lemma and Definition 1.2.2. *Let q be a quadratic form on $(H, \cdot)$ where $(\cdot)$ is a non-degenerate symmetric bilinear form. Then there exists a number $Arf(q) \in \mathbb{Z}_2$ so that*

$$(-1)^{Arf(q)} = \frac{1}{\sqrt{|H|}} \sum_{x \in H} (-1)^{q(x)}.$$

This number $Arf(q)$ is called the Arf invariant *of the quadratic form q.*

Proof. We need to prove that the square of the quantity on the right hand side is equal to 1. By definition of quadratic forms and using the fact that $x \cdot x = 0$ we get

$$\begin{aligned}
\Big(\frac{1}{\sqrt{|H|}}\sum_{x\in H}(-1)^{q(x)}\Big)^2 &= \frac{1}{|H|}\sum_{x,y\in H}(-1)^{q(x)+q(y)} = \frac{1}{|H|}\sum_{x,y\in H}(-1)^{q(x+y)+x\cdot y} \\
&= \frac{1}{|H|}\sum_{x,z\in H}(-1)^{q(z)+x\cdot(z-x)} = \frac{1}{|H|}\sum_{x,z\in H}(-1)^{q(z)+x\cdot z} \\
&= 1+\frac{1}{|H|}\sum_{z\neq 0}(-1)^{q(z)}\sum_{x\in H}(-1)^{x\cdot z}.
\end{aligned}$$

To conclude the proof, it is sufficient to prove that $\sum_{x\in H}(-1)^{x\cdot z}=0$ for every fixed non-zero element $z\in H$. Since the bilinear form is non-degenerate and since $z\neq 0$, there exists an element $z_1\in H$ so that $z_1\cdot z=1$. Then it is obvious that the map $x\mapsto x+z_1$ is a bijection between two sets $\{x\in H : x\cdot z=0\}$ and $\{x\in H : x\cdot z=1\}$. It follows that $x\cdot z$ takes both values 0 and 1 equally often when x varies in H, and so $\sum_{x\in H}(-1)^{x\cdot z}=0$. The proof is concluded. $\square$

It is a well-known fact that the Arf invariant is an invariant which classifies quadratic forms up to isomorphism. Additionally one can prove that $\mathrm{Arf}(q)$ is equal to the value taken the most often by q. As we will not use these facts in our work, their proofs will not be presented here, however interested readers can look at [58, Chapter 9] for details. On the other hand we will need the two following properties of the Arf invariant which can be found in [48, Lemma 2.10] and in [14, Lemma 1]. Alternatively, the reader can wait until Lemma 1.2.6 from which we can get back these two properties.

Lemma 1.2.3. *Let q be a quadratic form on $(H,\cdot)$, then we have:*

(i) The equality $\frac{1}{\sqrt{|H|}}\sum_{q\in\mathcal{Q}(H,\cdot)}(-1)^{Arf(q)+q(x)}=1$ holds for every $x\in H$, where the sum is taken over all the set of quadratic forms.

(ii) If q' is also a quadratic form, then $Arf(q)+Arf(q')=q(\Delta)=q'(\Delta)$, where $\Delta\in H$ satisfies $(q+q')(x)=\Delta\cdot x$ for every $x\in H$. $\square$

1.2.2 Quadratic enhancements and Brown invariant

In this subsection we will give the definition of quadratic enhancements and their Brown invariant, that generalise quadratic forms and the Arf invariant defined in the previous part. Interested readers can look at [4] for other related results. Recall that in the following, $(H,\cdot)$ is still a finite dimensional vector space over $\mathbb{Z}_2$ together with a symmetric bilinear form $(\cdot): H\times H\to\mathbb{Z}_2$.

Definition 1.2.4. *Given a function $w_1 : H \to \mathbb{Z}_2$, a function $q : H \to \mathbb{Z}_4$ is called a* quadratic enhancement *of $(H, \cdot)$ with respect to w_1 if for every $x, y \in H$ we have $q(x) - w_1(x) \in 2\mathbb{Z}_2$, and*

$$q(x + y) = q(x) + q(y) + 2(x \cdot y),$$

where $2 : \mathbb{Z}_2 \to \mathbb{Z}_4$ denotes the inclusion homomorphism.

Let us denote by $\mathcal{Q}(H, \cdot, w_1)$ the set of all quadratic enhancements on $(H, \cdot)$ with respect to w_1. As one can see by definition, it is necessary to assume that w_1 is linear and that $(\cdot)$ is symmetric (and we will always do so) so that the set $\mathcal{Q}(H, \cdot, w_1)$ is non-empty. Unlike in the case of quadratic forms, it is not necessary to have $x \cdot x = 0$ for every $x \in H$. On the other hand, it is still true that there are precisely $|H|$ quadratic enhancements on $(H, \cdot)$ with respect to any fixed linear function w_1. The reason is that the action of $H^* = Hom(H, \mathbb{Z}_2)$ on the set $\mathcal{Q}(H, \cdot, w_1)$ given by

$$H^* \times \mathcal{Q}(H, \cdot, w_1) \to \mathcal{Q}(H, \cdot, w_1) \quad (\varphi, q) \mapsto 2\varphi + q$$

is still free and transitive. It is also obvious that if $w_1 \equiv 0$, a quadratic enhancement q induces a quadratic form $\frac{q}{2} \in \mathcal{Q}(H, \cdot)$, and vice versa. Hence quadratic enhancements are generalisations of quadratic forms.

Let us now define the Brown invariant of quadratic enhancements. As in the case of quadratic forms, we shall assume that the bilinear form is non-degenerate.

Lemma and Definition 1.2.5. *For any quadratic enhancement $q \in \mathcal{Q}(H, \cdot, w_1)$ where the bilinear form is non-degenerate, there exists an integer modulo 8 depending on q, denoted by $Br(q)$, so that*

$$\exp\left(\frac{i\pi}{4}\right)^{Br(q)} = \frac{1}{\sqrt{|H|}} \sum_{x \in H} i^{q(x)}.$$

This number $Br(q)$ is called the Brown invariant *of the quadratic enhancement q.*

Proof. It is sufficient to show that the square of the quantity on the right hand side is equal to ± 1 or $\pm i$. To do so, first note that since the bilinear form is symmetric we have

$$(u + v) \cdot (u + v) = u \cdot u + v \cdot v + u \cdot v + v \cdot u = u \cdot u + v \cdot v.$$

This means that the $\mathbb{Z}_2$-valued function on H defined by $u \mapsto u \cdot u$ is linear. As the bilinear form is non-degenerate, there exists $t \in H$ satisfying $t \cdot u = u \cdot t = u \cdot u$ for every $u \in H$. Thus we obtain

$$0 = q(2x) = 2q(x) + 2\, x \cdot x = 2q(x) + 2(x \cdot t),$$

which leads to $2q(x) = 2(x \cdot t) \in \mathbb{Z}_4$ for every $x \in H$. Using this fact and the definition of quadratic enhancements we have

$$(\frac{1}{\sqrt{|H|}} \sum_{x \in H} i^{q(x)})^2 = \frac{1}{|H|} \sum_{x,y \in H} i^{q(x)+q(x+y)} = \frac{1}{|H|} \sum_{x,y \in H} i^{2q(x)+q(y)+2(x \cdot y)}$$

$$= \frac{1}{|H|} \sum_{x,y \in H} i^{q(y)+2(x\cdot(t+y))} = \frac{1}{|H|} \sum_{y \in H} i^{q(y)} \sum_{x\in H} (-1)^{x\cdot(t+y)}.$$

As shown in the proof of Lemma 1.2.2, the sum $\sum_{x\in H}(-1)^{x\cdot(t+y)}$ is equal to 0 if $t+y \neq 0$. Thus we obtain

$$\Big(\frac{1}{\sqrt{|H|}} \sum_{x\in H} i^{q(x)}\Big)^2 = \frac{1}{|H|} i^{q(t)} \sum_{x\in H} 1 = i^{q(t)},$$

which concludes our proof. □

Observe that in the case $w_1 \equiv 0$, the quadratic form $\frac{q}{2}$ induced from q has the Arf invariant equal to $\frac{1}{4}\,\mathrm{Br}(q)$. Hence the following lemma implies the two properties of the Arf invariant stated in Lemma 1.2.3.

Lemma 1.2.6. *Given $(H, \cdot, w_1)$ a finite dimensional vector space over $\mathbb{Z}_2$ together with a non-degenerate bilinear form $(\cdot)$ and a fixed function $w_1 : H \to \mathbb{Z}_2$, then we have:*

(i) The equality $\frac{1}{\sqrt{|H|}} \sum_{q\in\mathcal{Q}(H,\cdot,w_1)} \exp(i\pi/4)^{-Br(q)} i^{q(x)} = 1$ holds for every $x \in H$, where the sum is taken over all the set of quadratic enhancements.

(ii) If q_1, q_2 are quadratic enhancements on $(H, \cdot, w_1)$, then $Br(q_1) - Br(q_2) = 2q_1(\Delta) = 2q_2(\Delta)$, where $\Delta \in H$ satisfies $q_1(x) + 2(\Delta \cdot x) = q_2(x)$ for every $x \in H$.

Proof. For the first part, let us fix a quadratic enhancement q_*. For each $z \in H$, we define a function $q_z : H \to \mathbb{Z}_4$ by $q_z(x) := q_*(x) + 2(z \cdot x)$. It is clear by definition that q_z is also a quadratic enhancement. With this notation we have

$$\sum_{x\in H} i^{q_z(x)} = \sum_{x\in H} i^{q_*(x)+2(z\cdot x)} = \sum_{x\in H} i^{q_*(x+z)-q_*(z)} = i^{-q_*(z)} \sum_{x\in H} i^{q_*(x+z)} = i^{-q_*(z)} \sum_{x\in H} i^{q_*(x)},$$

which means that

$$\exp\left(\frac{i\pi}{4}\right)^{\mathrm{Br}(q_z)} = i^{-q_*(z)} \exp\left(\frac{i\pi}{4}\right)^{\mathrm{Br}(q_*)},$$

or equivalently, $\mathrm{Br}(q_z) + 2q_*(z) = \mathrm{Br}(q_*)$. Therefore, observing that $z \mapsto q_z$ is a bijection between H and the set of quadratic enhancements on it, we get

$$\begin{aligned}
\sum_{q\in\mathcal{Q}(H,\cdot,f)} \exp\left(\frac{i\pi}{4}\right)^{-\mathrm{Br}(q)} i^{q(x)} &= \sum_{z\in H} \exp\left(\frac{i\pi}{4}\right)^{-\mathrm{Br}(q_z)} i^{q_z(x)} \\
&= \sum_{z\in H} \exp\left(\frac{i\pi}{4}\right)^{-\mathrm{Br}(q_*)+2q_*(z)} i^{q_z(x)} \\
&= \exp\left(\frac{i\pi}{4}\right)^{-\mathrm{Br}(q_*)} \sum_{z\in H} i^{q_z(x)+q_*(z)}
\end{aligned}$$

$$= \exp\left(\frac{i\pi}{4}\right)^{-\mathrm{Br}(q_*)} \sum_{z\in H} i^{q_*(x+z)}$$
$$= \exp\left(\frac{i\pi}{4}\right)^{-\mathrm{Br}(q_*)} \sum_{z\in H} i^{q_*(z)}$$
$$= \sqrt{|H|}.$$

This concludes the proof of part (i). Let us move on to the second part. By definition $q_2 - q_1$ is a linear form taking even values, so there exists an element $\Delta \in H$ such that $q_2(x) - q_1(x) = 2\,\Delta\cdot x$ for every $x \in H$. Thus we get $2q_1(\Delta) = 2q_2(\Delta)$. Now we can write

$$\begin{aligned} \exp\left(\frac{i\pi}{4}\right)^{\mathrm{Br}(q_1)} &= \frac{1}{2^{b_1/2}}\sum_{x\in H} i^{q_1(x)} = \frac{1}{2^{b_1/2}}\sum_{x\in H} i^{q_1(x+\Delta)} = \frac{1}{2^{b_1/2}}\sum_{x\in H} i^{q_1(x)+q_1(\Delta)+2(\Delta\cdot x)} \\ &= \frac{1}{2^{b_1/2}}\sum_{x\in H} i^{q_2(x)+q_1(\Delta)} = i^{q_1(\Delta)}\exp\left(\frac{i\pi}{4}\right)^{\mathrm{Br}(q_2)}, \end{aligned}$$

which implies that $\mathrm{Br}(q_1) - \mathrm{Br}(q_2) = 2q_1(\Delta)$. This completes the proof of part (ii). □

1.2.3 Representations of finite groups

To prepare for the last chapter, let us quickly review some facts about linear representations of finite groups. Readers are referred to [60] for more details. Throughout this subsection, Γ will denote a finite group.

We begin with some basic definitions. A *linear representation* of Γ is a homomorphism $\rho : \Gamma \to GL(V)$ where V is a vector space of finite dimension over $\mathbb{C}$. The dimension of V is called the *degree* of ρ. In particular the *trivial representation* of Γ is the homomorphism $\rho : \Gamma \to GL(\mathbb{C}) \simeq \mathbb{C}^*$ given by $\rho(g) = 1$ for all $g \in \Gamma$.

We say that two representations $\rho_1 : \Gamma \to GL(V)$ and $\rho_2 : \Gamma \to GL(W)$ are *equivalent* if there exists an isomorphism $f : V \to W$ such that $f\rho_1(g) = \rho_2(g)f$ for every $g \in \Gamma$. In this case we write $\rho_1 \sim \rho_2$. One of the classical problems in representation theory is to classify all representations of a given group up to equivalence. In order to do that we will need some additional terminology.

Given $\rho : \Gamma \to GL(V)$ a representation, a subspace W of V is Γ-*invariant* if for every $g \in \Gamma$ and $w \in W$ we have $\rho(g)(w) \in W$. We shall say that ρ is *irreducible* if the only Γ-invariant subspaces of V are $\{0\}$ and V. Moreover if $\rho_1 : \Gamma \to GL(V_1)$ and $\rho_2 : \Gamma \to GL(V_2)$ are two representations, then one can define their *direct sum*

$$\rho_1 \oplus \rho_2 : \Gamma \to GL(V_1 \oplus V_2)$$

given by

$$(\rho_1 \oplus \rho_2)(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2))$$

for $g \in \Gamma$, $v_1 \in V_1$, $v_2 \in V_2$. We then have the following important theorem.

Theorem (Maschke's theorem)**.** *Given a representation $\rho : \Gamma \to GL(V)$, there exists irreducible representations $\rho_k : \Gamma \to GL(V_k)$ for $1 \le k \le n$ such that $V \simeq V_1 \oplus \cdots \oplus V_n$ and that $\rho \sim \rho_1 \oplus \cdots \oplus \rho_n$.* $\square$

Furthermore, one can show that the number of equivalence classes of irreducible representations of Γ is precisely the number s of conjugacy classes of Γ (see [60, Corollary 4.4.8] for example). Denoting by $\{\rho_1 \cdots \rho_s\}$ a complete set of irreducible representations of Γ up to equivalence, then by Maschke's theorem, for each representation ρ there exists non-negative integers m_i's so that

$$\rho \sim \rho_1^{\oplus m_1} \oplus \cdots \oplus \rho_s^{\oplus m_s}.$$

The number m_i, only depending on ρ (but not on the way ρ is decomposed), is called the *multiplicity* of ρ_i in ρ. If $m_i \ge 1$ we shall call ρ_i an *irreducible constituent* of ρ. Actually, this multiplicity can be determined by characters of ρ and ρ_i, as we now describe.

Definition 1.2.7. *Let $\rho : \Gamma \to GL(V)$ be a representation. The* character *$\chi_\rho : \Gamma \to \mathbb{C}$ of ρ is defined by setting $\chi_\rho(g) = Tr\,\rho(g)$, where $Tr\,\rho(g)$ is the trace of $\rho(g)$.*

The following result, which can be found in [60, Theorem 4.3.14], is useful for computing the decomposition of a given representation into irreducible ones.

Theorem 1.2.8. *Let $\{\rho_1 \cdots \rho_s\}$ be a complete set of representatives of the equivalence classes of irreducible representations of Γ, and ρ be any representation. Then the multiplicity m_i of ρ_i in ρ is given by*

$$m_i = \langle \chi_\rho, \chi_{\rho_i} \rangle := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_\rho(g) \overline{\chi_{\rho_i}(g)}. \quad \square$$

Before concluding this subsection, let us recall the definition of induced representations. Let Γ_1 be a subgroup of Γ and $\rho : \Gamma_1 \to GL(V)$ be a representation. Let $n = [\Gamma : \Gamma_1]$ be the index of Γ_1 in Γ, and let $\{g_1, \ldots, g_n\}$ be a complete set of representatives in Γ of the left cosets in Γ/Γ_1. Set

$$W = \bigoplus_{i=1}^{n} g_i V$$

where each $g_i V$ is a copy of V whose elements are written as $g_i v$ with $v \in V$. Note that $\{g_i \Gamma_1 : 1 \le i \le n\}$ is a partition of Γ, so for each $g \in \Gamma$ and for each $1 \le i \le n$, there exists a unique number $j(i) \in \{1, \ldots, n\}$ such that $g g_i \in g_{j(i)} \Gamma_1$. With these notations we have the following definition.

Definition 1.2.9. *The* induced representation *of ρ from Γ_1 to Γ is the homomorphism $\rho^{\#} : \Gamma \to GL(W)$ given by*

$$\rho^{\#}(g)\Big(\sum_{i=1}^{n} g_i v_i\Big) = \sum_{i=1}^{n} g_{j(i)} \rho(g_{j(i)}^{-1} g g_i) v_i$$

for every $g \in \Gamma$.

Finally we terminate this subsection by giving an example of irreducible representations of dihedral groups, that will be used later in our work.

Example 1.2.10. Recall that the dihedral group D_M of order $2M$ is given by

$$D_M = \langle r, s \mid r^M = s^2 = (sr)^2 = 1\rangle.$$

It is a well-known fact that if M is odd, D_M admits precisely $\frac{M+3}{2}$ irreducible representations: there are 2 of degree 1, and $\frac{M-1}{2}$ of degree 2. Two degree one representations $\tau_1, \tau_2 : D_M \to \mathbb{C}^*$ are given by $\tau_1(r) = \tau_1(s) = 1$ and $\tau_2(r) = -\tau_2(s) = 1$, while $\frac{M-1}{2}$ degree two irreducible representations are $\rho_{2k} : D_M \to GL(\mathbb{C}^2)$ for $1 \le k \le \frac{M-1}{2}$ with

$$\rho_k(r) = \begin{pmatrix} e^{k\pi i/M} & 0 \\ 0 & e^{-k\pi i/M} \end{pmatrix} \quad \text{and} \quad \rho_k(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On the other hand, if M is even then D_M admits exactly $\frac{M}{2}+3$ irreducible representations in which there are 4 of degree 1, and $\frac{M}{2}-1$ of degree 2. Four degree one representations are τ_1, τ_2 mentioned above together with τ_3, τ_4 given by $\tau_3(r) = \tau_3(s) = -1$ and $\tau_4(r) = -\tau_4(s) = -1$, while $\frac{M}{2} - 1$ degree two irreducible representations are still ρ_{2k}'s for $1 \le k \le \frac{M}{2} - 1$ with ρ_k as in the previous case.

1.3 Geometric preliminaries

In this section we shall recall some necessary terminology related to connected closed surfaces. To be more precise, we will recall the definitions of surfaces and their orientation covers, as well as the classification of closed surfaces together with their polygonal representations. We also recall the first (co)homology group of surfaces, the intersection form on these groups and define a class which characterises the orientability of surfaces.

1.3.1 Surfaces and orientation covers

Let us begin with a very basic notion. An *n-manifold* (or a *manifold* of dimension n) is a second countable Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^n$. And a *surface* is simply a 2-manifold. For example the sphere S^2, the torus $\mathbb{T}$, the Klein bottle $\mathscr{K}$ and the real projective plane $\mathbb{R}P^2$ are familiar surfaces. In this section, we shall focus on connected closed surfaces.

We now continue with the orientability of surfaces, and their orientation covers. The interested reader is referred to [25, Section 3.3] for the case of general manifolds. Given a connected closed surface Σ, one can observe that each simple closed curve (or simply a loop) γ on the surface Σ (i.e. an injective continuous function $\gamma : S^1 \to \Sigma$) admits a neighbourhood in Σ which is either a cylinder or a Möbius strip. Equivalently, in the first case any local orientation of Σ is preserved along γ, while in the second case it is not. One can also verify that in the first case, the complement of γ in Σ is disconnected, while in the latter this complement is connected. We shall say that Σ is *orientable* if every loop on Σ

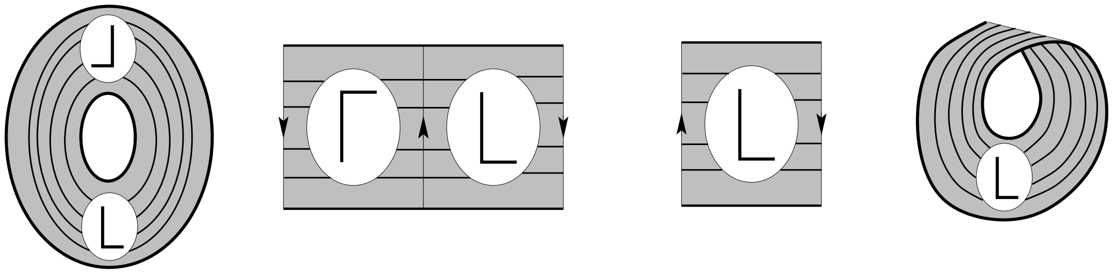


Figure 1.2: A schematic description of the orientation cover of the Möbius strip by the cylinder.

admits a cylinder neighbourhood, and that Σ is *non-orientable* if there exists a loop whose neighbourhood is a Möbius strip. For example, the sphere and the torus are orientable surfaces, while the Klein bottle and the projective plane are non-orientable.

Next let us define orientation covers of surfaces. Recalling that Σ is a connected closed surface, the orientation cover $\widetilde{\Sigma}$ of Σ is the two fold cover $\widetilde{\Sigma} \xrightarrow{\pi} \Sigma$ determined by the following property: a loop in Σ lifts to a loop in $\widetilde{\Sigma}$ if and only if it admits a neighbourhood in Σ which is a cylinder. For example, the orientation cover of the real projective plane is the sphere, and the orientation cover of the Klein bottle is the torus. If Σ is a surface with boundary, that is, Σ is obtained from a closed surface by removing some disks, the orientation cover of Σ can be defined similarly. An example is given in Figure 1.2 when Σ is the Möbius band. Note that the orientation cover $\widetilde{\Sigma}$ is always orientable. Moreover, if Σ is non-orientable then $\widetilde{\Sigma}$ is connected, otherwise $\widetilde{\Sigma}$ is simply the disjoint union of two copies of Σ.

Now let us recall the classification theorem of closed surfaces, and describe certain surface polygonal representations. In short, the classification theorem asserts that a closed connected surface is determined, up to homeomorphism, by its Euler characteristic and the orientability. More precisely we have the following result (cf. [45, Chapter 4]).

Theorem (The classification theorem of closed surfaces)**.** *Any connected closed surface is homeomorphic to either the connected sum of $g \geq 1$ tori (the sphere if $g = 0$) if the surface is orientable, or the connected sum of $h \geq 1$ real projective planes if the surface is non-orientable.* □

We shall say that g (reps. h) is the orientable (resp. non-orientable) *genus* of the surface. Note that the *Euler characteristic* of the surface is then given by $\chi = 2 - 2g$ (resp. $\chi = 2 - h$). As a consequence, since the Euler characteristic of the orientation cover $\widetilde{\Sigma}$ is twice as big as that of Σ, if Σ is non-orientable then we have

$$g(\widetilde{\Sigma}) = 1 - \frac{\chi(\widetilde{\Sigma})}{2} = 1 - \chi(\Sigma) = h(\Sigma) - 1.$$

For instance, the Klein bottle $\mathscr{K}$ has (non-orientable) genus $h = 2$ and the Euler characteristic $\chi = 0$, while the torus $\mathbb{T}$ has (orientable) genus $g = 1$ and the same Euler characteristic.

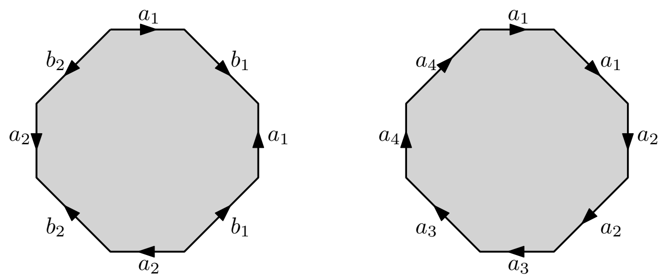


Figure 1.3: Two representations corresponding to $g = 2$ and $h = 4$.

We now turn to polygonal representations of surfaces. In fact one can always represent a connected closed surface Σ by a polygon with some sides identified. More specifically, if Σ is orientable and has genus g, a standard representation of Σ is a $4g$-gon with $2g$ pairs of sides identified following the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ (if $g = 0$, then Σ is the sphere and can be represented by the word aa^{-1}). On the other hand if Σ is non-orientable and has genus h, Σ can be represented by a $2h$-gon with sides identified following the word $a_1 a_1 \cdots a_h a_h$ (see Figure 1.3). Moreover, in this case one can prove that Σ is homeomorphic to the connected sum of an orientable surface of genus $g = \frac{h}{2} - 1$ with the Klein bottle $\mathscr{K}$ if h is even, and is homeomorphic to the connected sum of an orientable surface of genus $g = \frac{h-1}{2}$ with the real projective plane $\mathbb{R}P^2$ if h is odd. Therefore one can represent Σ as a polygon with sides identified following the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} aba^{-1}b$ in the former case, and following the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} cc$ in the latter. We will use these two representations very often in the rest of the thesis.

1.3.2 Homology and the first Stiefel-Whitney class

Let us start this subsection with the definition of the first homology group and the intersection form. The reader is referred to [25, Chapter 2] for details. Given a surface Σ and a graph $G \subset \Sigma$ whose complement consists of topological discs, let C_0 (resp. C_1, C_2) denote the $\mathbb{Z}_2$-vector space with basis the set of vertices (resp. edges, faces) of $G \subset \Sigma$. Also, let

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

denote the boundary operators defined on the basis of these sets as follows. For a face f of G, one defines

$$\partial_2(f) := \sum_{e \in \partial f} e$$

where the sum is over all the boundary edges of f, while for an edge $e = (u, v)$ of G we define

$$\partial_1(e) := u + v.$$

Note that for every face f we get

$$\partial_1 \circ \partial_2(f) = \sum_{e \in \partial f} \partial_1(e) = \sum_{e \in \partial f} (u_e + v_e) = 0$$

since in the second sum each boundary vertex of f appears exactly twice, so $\partial_1 \circ \partial_2$ vanishes. Therefore the space of 1-cycles $\ker(\partial_1)$ contains the space $\partial_2(C_2)$ of 1-boundaries, and thus one can define the *first homology space* $H_1(\Sigma; \mathbb{Z}_2)$ as the quotient $\ker(\partial_1)/\partial_2(C_2)$. Note that $H_1(\Sigma; \mathbb{Z}_2)$ is also a group, and this group turns out not to depend on G, but only on Σ. Similarly one can define the first homology space $H_1(\Sigma; \mathbb{Z})$ of Σ over $\mathbb{Z}$. The Hurewicz theorem states that $H_1(\Sigma; \mathbb{Z})$ is the abelianization of $\pi_1(\Sigma)$, the fundamental group of Σ. For further purpose, let us set $H^1(\Sigma; \mathbb{Z}_2) := Hom(H_1(\Sigma; \mathbb{Z}_2); \mathbb{Z}_2)$, which is called the *first cohomology space (group)* of Σ.

Remark that the intersection of curves on Σ induces a symmetric bilinear non-degenerate form on $H_1(\Sigma, \mathbb{Z}_2)$, which will be denoted by $(\alpha, \beta) \mapsto \alpha \cdot \beta$ from now on. Let us be more specific. If Σ is orientable of genus g, then Σ can be represented by the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. In this case, the space $H_1(\Sigma; \mathbb{Z}_2)$ has dimension $2g$ with a basis given by $\{[\alpha_1], [\alpha_2], \ldots, [\alpha_{2g-1}], [\alpha_{2g}]\}$, where α_i $(1 \leq i \leq 2g)$ are the cycles in Σ obtained from the $4g$ sides a_i, b_i of the polygon after identifications. Then the intersection form with respect to this basis is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus g}.$$

On the other hand, if Σ is non-orientable of genus h then $H_1(\Sigma; \mathbb{Z}_2)$ has dimension h. Moreover, recall that if h is even, Σ can be represented by $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} a b a^{-1} b$, and $H_1(\Sigma, \mathbb{Z}_2)$ admits the set $\{[\alpha_1], [\alpha_2], \ldots, [\alpha_{2g-1}], [\alpha_{2g}], [\beta_1], [\beta_2]\}$ as its basis. Here α_i's are defined as above, and β_1, β_2 are the cycles obtained from the sides a, b of the polygon after identifications. The intersection form with respect to this basis is then given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus h/2-1} \oplus \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Similarly if h is odd, using the representation $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} cc$ of Σ, the intersection form with respect to the basis $\{[\alpha_1], [\alpha_2], \ldots, [\alpha_{2g-1}], [\alpha_{2g}], [\gamma]\}$ of $H_1(\Sigma, \mathbb{Z}_2)$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus (h-1)/2} \oplus (1).$$

To conclude this section, let us recall the first Stiefel-Whitney class $w_1 := w_1(\Sigma) \in H^1(\Sigma; \mathbb{Z}_2)$ of the surface Σ, which characterises the orientability of Σ. More precisely, the orientability of the surface Σ is detected by the homomorphism $\pi_1(\Sigma) \to \mathbb{Z}_2$ that assigns 0 or 1 to each loop according to whether local orientations of Σ are preserved or reserved when we travel along this loop. Since $\mathbb{Z}_2$ is abelian, this homomorphism factors through the abelianization $H_1(\Sigma; \mathbb{Z})$ of $\pi_1(\Sigma)$. As $\mathbb{Z}_2$ has order 2, this factorisation can be considered as a homomorphism from $H_1(\Sigma; \mathbb{Z}_2)$ to $\mathbb{Z}_2$, and therefore we obtain an element $w_1 \in H^1(\Sigma; \mathbb{Z}_2)$ associated to Σ which is zero exactly when Σ is orientable.

For example, if Σ is orientable and represented by the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ then $H_1(\Sigma; \mathbb{Z}_2)$ has a basis given by $\{[\alpha_1], [\alpha_2], \ldots, [\alpha_{2g-1}], [\alpha_{2g}]\}$. In this case the first Stiefel-

Whitney class w_1 is the trivial function $H_1(\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_2$ sending all $[\alpha_i]$ to 0. On the other hand if Σ is non-orientable and represented by the word $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}aba^{-1}b$ (resp. $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}cc$), w_1 is the function sending all $[\alpha_i]$ to 0, and sending $[\beta_1]$ to 1, $[\beta_2]$ to 0 (resp. sending $[\alpha_i]$ to 0 and $[\gamma]$ to 1).

Furthermore, if the graph G is embedded in Σ, one can define a function $\omega : E(G) \to \mathbb{Z}_2$ which represents w_1 as follows. Represent Σ by a planar polygon $\mathcal{P}$ with some pairs of sides identified, and assume that G is drawn inside $\mathcal{P}$ so that its edges intersect the sides of $\mathcal{P}$ transversely. Endowing $\mathcal{P}$ with a fixed orientation, a side of $\mathcal{P}$ is called a *1-side* if the identification along this side does not preserve the orientation of $\mathcal{P}$, and *0-side* otherwise. For an edge e of G, we define $\omega(e)$ to be the parity of the number of intersections of e with all the 1-sides of $\mathcal{P}$. Then it can be verified easily that local orientations of Σ is preserved along a cycle C of G if and only if $\omega(C) = 0$, and therefore ω does represent w_1. This function ω, called the *standard representation* of w_1 (with respect to a fixed polygonal representation of Σ), will be used very often in the rest of the thesis.

Before moving to the next section, let us mention that if the triple $(H, \cdot, w_1)$ in Definition 1.2.4 is given by $H_1(\Sigma;\mathbb{Z}_2)$, the intersection form, and the first Stiefel-Whitney class, then we simply speak of a *quadratic enhancement on* Σ (and of a *quadratic form on* Σ in the orientable case). We shall also denote by $\mathcal{Q}(\Sigma, w_1)$ or sometimes by $\mathcal{Q}(\Sigma,\omega)$ (resp. $\mathcal{Q}(\Sigma)$) the corresponding set of quadratic enhancements (resp. quadratic forms). By the discussion right after Definition 1.2.4, there are exactly $|H_1(\Sigma;\mathbb{Z}_2)|$ quadratic enhancements (quadratic forms) on Σ, i.e. 2^{2g} if Σ is orientable of genus g, and 2^h if Σ is non-orientable of genus h.

1.4 Kasteleyn theory for the dimer model

In this section we will review the method firstly obtained by Kasteleyn-Temperley-Fisher [35, 36, 62] to compute the dimer partition function of any planar graph. Their method being extended later by Cimasoni-Reshetikhin to graphs embedded in orientable surfaces [14], and to graphs embedded in non-orientable surfaces [8] will also be summarised in this section.

1.4.1 Dimers and Pfaffians

To begin let us first recall Pfaffians of adjacency matrices, and then explain how Pfaffians are related to dimer partition functions. By definition the *Pfaffian* of a skew-symmetric matrix $A = (a_{ij})_{1\le i,j\le 2n}$ is given by

$$\mathrm{Pf}(A) = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathrm{sign}(\sigma) a_{\sigma(1)\sigma(2)}\cdots a_{\sigma(2n-1)\sigma(2n)}.$$

As A is skew-symmetric, each term in the right-hand side corresponding to σ only depends on the matching of $\{1,\ldots,2n\}$ into n unordered pairs $\{\sigma(1),\sigma(2)\},\ldots,\{\sigma(2n-1),\sigma(2n)\}$.

Since there are exactly $2^n n!$ permutations representing a same matching, one can write

$$\mathrm{Pf}(A) = \sum_{[\sigma]} \mathrm{sign}(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)},$$

where the sum is over the set of matchings of $\{1, \ldots, 2n\}$. A well-known fact of Pfaffians is that the square of the Pfaffian of any skew-symmetric matrix A is equal to the determinant of A. Also, similarly to the determinant, adding a multiple of a row to another row and doing the same for corresponding columns do not change the Pfaffian. Furthermore, we also have the Laplace expansion for the Pfaffian. This expansion, which is the content of [29, Proposition 2.3] can be stated as follows, while other basic properties of Pfaffians can be found in [49, 1].

Lemma 1.4.1. *If $A = (a_{ij})$ is a skew-symmetric matrix of size $2n$, then for any $1 \leq i \leq 2n$ we have*

$$Pf(A) = \sum_{\substack{j=1 \\ j \neq i}}^{2n} (-1)^{i+j+1+\theta(i-j)} a_{ij} Pf(A_{\hat{i}\hat{j}}),$$

where $A_{\hat{i}\hat{j}}$ is the matrix obtained from A by removing both i^{th} and j^{th} rows and columns, while θ is the Heaviside step function*, that is, $\theta(l)$ is equal to 1 if $l \geq 0$ and 0 otherwise.* □

We are now interested in the Pfaffian of adjacency matrices of the graph G, which can be related to the dimer partition function of G. Recall that if (G, x) is an edge-weighted graph of $2n$ vertices labelled by $\{1, \ldots, 2n\}$ and K is an arbitrary orientation on its edges, then the adjacency matrix of G with respect to K, denoted by $A^K(G) := A^K(G, x)$, has entries defined by

$$a_{ij} = \sum_{e=(i,j)} \epsilon_{ij}^K(e) x(e). \tag{1.1}$$

In this equality the sum is taken over all the edge e of G between two vertices i, j, and

$$\epsilon_{ij}^K(e) = \begin{cases} +1 & \text{if } e \text{ is oriented by } K \text{ from } i \text{ to } j; \\ -1 & \text{if not.} \end{cases}$$

Looking at the Pfaffian of $A^K(G)$, it is clear that a matching of the vertices of G contributes to this Pfaffian if and only if it is realised by a dimer configuration of G. Therefore we can write

$$\mathrm{Pf}(A^K(G)) = \sum_{D \in \mathcal{D}(G)} \epsilon^K(D) x(D), \tag{1.2}$$

where the sum is taken over the set $\mathcal{D}(G)$ of dimer configurations of G, $x(D) := \prod_{e \in D} x(e)$ while the sign $\epsilon^K(D) = \pm 1$ can be computed as follows. If the dimer configuration D is

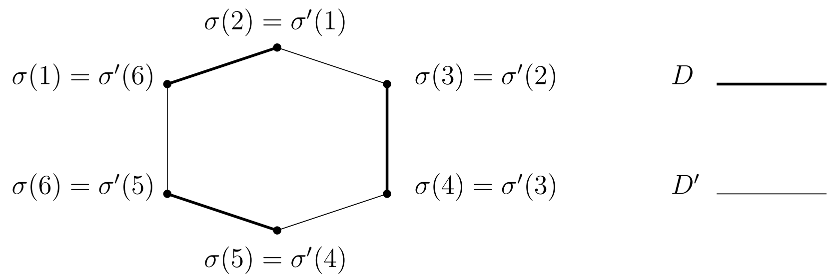


Figure 1.4: Two permutations σ, σ' representing D, D' on a superposition cycle.

given by edges e_l matching vertices i_l and j_l for $1 \le l \le n$, let σ denote the permutation mapping the set $\{1, 2, \dots, 2n-1, 2n\}$ to $\{i_1, j_1, \dots, i_n, j_n\}$. The sign then is equal to

$$\epsilon^K(D) = \mathrm{sign}(\sigma) \prod_{l=1}^{n} \epsilon^K_{i_l j_l}(e_l). \tag{1.3}$$

By Equation (1.2), it is clear that $|\mathrm{Pf}(A^K(G))| = Z_{\mathcal{D}}(G)$ if and only if $\epsilon^K(D) \in \{\pm 1\}$ is constant for every $D \in \mathcal{D}(G)$, i.e. if and only if $\epsilon^K(D)\epsilon^K(D') = 1$ for every $D, D' \in \mathcal{D}(G)$. The product $\epsilon^K(D)\epsilon^K(D')$ can be interpreted as follows. Observe that if D and D' are two dimer configurations of G, then their symmetric difference $D\Delta D'$ is a vertex-disjoint union of simple cycles C_j's called *superposition cycles*; each of them has edges alternated between D and D' and thus has an even length. Denoting by $n^K(C_j)$ the number of edges on C_j on which a fixed orientation of C_j is different from K (since C_j is of even length, the parity of this number is independent of the orientation of C_j), we have the following result.

Lemma 1.4.2. *For any dimer configurations $D, D' \in \mathcal{D}(G)$, setting $D\Delta D' = \bigsqcup_j C_j$ we get*

$$\epsilon^K(D)\epsilon^K(D') = \prod_j (-1)^{n^K(C_j)+1}. \tag{1.4}$$

Proof. Let us order and orient the superposition cycles C_j's arbitrarily. Since one has the freedom to choose permutations representing D and D', let us consider the two permutations $\sigma, \sigma' \in S_{2n}$ representing D, D' as in Figure 1.4. Then it is clear that $\sigma' \circ \sigma^{-1}$ restricted on each C_j is the rotation of this cycle by one edge. Since C_j is of even length, we get

$$\mathrm{sign}(\sigma)\mathrm{sign}(\sigma') = \mathrm{sign}(\sigma' \circ \sigma^{-1}) = \prod_j (-1).$$

Using this particular choice of representatives of D, D' together with Equation (1.3), we obtain the equality as stated. □

Note that a superposition cycle can be realised as a cycle C of an even length such that $G \setminus C$ admits a dimer configuration. Therefore, in order to have $Z_{\mathcal{D}}(G) = |\mathrm{Pf}(A^K(G))|$, by Equation (1.4) we need to find an orientation K with the following property: for any cycle C of even length so that $G \setminus C$ admits a dimer configuration, $n^K(C)$ is odd. Such an orientation K is called a *Pfaffian orientation*. In the next part, we will recall Kasteleyn's theorem which justifies the existence of Pfaffian orientations on any planar graph.

1.4.2 Planar graphs

Kasteleyn's remarkable theorem asserts that every planar graph admits a Pfaffian orientation. More precisely, an orientation K on $G \subset \mathbb{R}^2$ is called *Kasteleyn* if for each face f of G, $n^K(\partial f)$ is odd, where ∂f is the boundary of f oriented counterclockwise. Then the following important result can be found in [36, 37]. For the sake of completeness, we also include a short proof of this result.

Theorem (Kasteleyn's theorem)**.** *Given a planar graph G, there always exists a Kasteleyn orientation K on G. Any such orientation is Pfaffian, and hence we have $Z_{\mathcal{D}}(G) = |Pf(A^K(G))|$.*

Proof. Let K be an arbitrary orientation on G. If a face f of $G \subset \mathbb{R}^2$ is such that $n^K(\partial f)$ is even, draw a path from the interior of f to the outer face, transverse to G, and invert K on each edge crossed by this path. Repeating this procedure for each face f with $n^K(\partial f)$ even, we obtain a Kasteleyn orientation. To prove that such a Kasteleyn orientation K on G is Pfaffian, let us fix a cycle $C \subset G$ of even length such that $G \setminus C$ admits a dimer configuration. Since C is a simple closed plane curve, it bounds a closed disc, with V vertices, E edges, and F faces. Let us write $V = V_{\text{int}} + V_{\text{ext}}$ and $E = E_{\text{int}} + E_{\text{ext}}$, where V_{int} (resp. E_{int}) denotes the number of vertices (resp. edges) in the interior of the disc. Note that since C is simple and closed, V_{ext} and E_{ext} coincide with the length of C, and are therefore even. Also, since $G \setminus C$ admits a dimer configuration, V_{int} is even as these interior vertices are matched by this dimer configuration. Summing over all the faces of this disc, and computing modulo 2, we therefore have

$$0 = \sum_f (n^K(\partial f) + 1) = n^K(C) + E_{\text{int}} + F = n^K(C) + V + E + F = n^K(C) + 1\,,$$

since the Euler characteristic of the disc is $V - E + F = 1$. This completes the proof. □

Remark 1.4.3. As we can see in the above proof, one manages to control the quantity $n^K(C)$ for any superposition cycle C by controlling $n^K(\partial f)$ for any face f. The reason is that every cycle C in the plane bounds a face. This fact is nothing but the first homology group of the plane $H_1(\mathbb{R}^2; \mathbb{Z}_2)$ being trivial. It is of course not the case when the graph is embedded in a surface, because then there are some cycles that do not bound any face. We will see in the next part that, in this case, controlling the quantity $n^K(C)$ is not as easy as doing so in the planar case.

For further purpose, let us conclude this subsection with a new terminology. First note that if K is a Kasteleyn orientation, then any orientation K' obtained by inverting K on all edges adjacent to a fixed vertex is also a Kasteleyn orientation: for every face f of G, it is clear that $n^{K'}(\partial f)$ and $n^K(\partial f)$ have a same parity. We shall say that two orientations are *equivalent* if they can be related by a sequence of such moves. Note that the effect on the corresponding Pfaffian is very simple: each such vertex flip changes the sign of $\mathrm{Pf}(A^K(G))$.

In the plane, any two Kasteleyn orientations K, K' are equivalent. Indeed, consider the set C^* of edges of the graph $G^* \subset \mathbb{R}^2$ dual to $G \subset \mathbb{R}^2$ corresponding to the edges of G

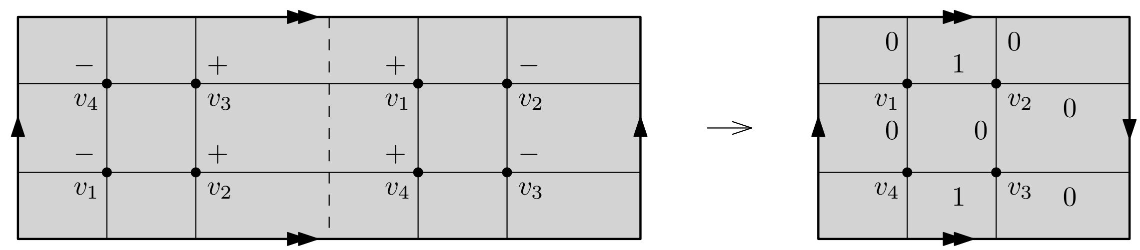


Figure 1.5: A graph $G \subset \mathscr{K}$ with a given ω (right) and its covering $\widetilde{G} \subset \mathbb{T}$ together with a vertex labelling induced from ω (left).

where K and K' differ. Since both K and K' are Kasteleyn orientations, for each face f of G, the number of edges of G on ∂f where K and K' differ is always even. Equivalently, C^* is of even degree at each vertex of G^*. In other words, C^* is a union of cycles in $G^* \subset \mathbb{R}^2$, which bound a set of faces of G^*, i.e. a set of vertices of G. Then, K' is obtained from K by making a vertex flip around each vertex of this set, so K and K' are indeed equivalent.

1.4.3 Graphs embedded in surfaces

As mentioned at the beginning of the section, the method of Kasteleyn-Temperley-Fisher shown in the previous subsection to compute the dimer partition function of any planar graph was generalised to graphs embedded in surfaces by Cimasoni-Reshetikhin. In this subsection, we will review this generalisation and state some main results that are needed for our purpose. Interested readers are referred to [14, 8] for more details.

Throughout this subsection, we shall assume that the graph G is embedded in a possibly non-orientable closed connected surface Σ. To begin, let us define the right notion of Kasteleyn orientations in this setting, which involves the orientation cover and the first Stiefel-Whitney class.

Let us first recall that the orientation cover $\widetilde{\Sigma}$ of Σ is the two fold cover $\widetilde{\Sigma} \xrightarrow{\pi} \Sigma$ determined by the property that a loop in Σ lifts to a loop in $\widetilde{\Sigma}$ if and only if it admits a cylinder neighbourhood in Σ. We also recall that the first Stiefel-Whitney class $w_1 \in H^1(\Sigma; \mathbb{Z}_2)$ of Σ can be represented by a function $\omega : E(G) \to \mathbb{Z}_2$ which assigns to each edge of G a value 0 or 1 so that the local orientation of Σ is preserved along a cycle of G if and only if this cycle contains an even number of edges with value 1. This function determines a labelling of the vertices of $\widetilde{G} := \pi^{-1}(G)$ with signs $\pm$'s such that each vertex of G lifts to two vertices of $\widetilde{G}$ with different signs, and $\omega(e) = 0$ if and only if the two endpoints of a lift of e have the same sign (see Figure 1.5 for example). Clearly, this labelling is determined uniquely by ω up to changing signs of all the vertices of G. Remember that if Σ is orientable, then $\widetilde{G}$ is the disjoint union of two copies of G, and $\omega \equiv 0$ provides a natural choice of labelling of vertices of $\widetilde{G}$, that is, we put all the $+$ in one copy of G, and all the $-$ in the other.

Now let us define Kasteleyn orientations on the graph G embedded in the surface Σ. Fix an ω representing the first Stiefel-Whitney class of Σ, and pick a vertex labelling of $\widetilde{G}$ induces by ω. Let us also fix an orientation (pictured counterclockwise) on $\widetilde{\Sigma}$. For an orientation K on the edges of G, let us denote by $\widetilde{K}$ the orientation obtained by lifting K to all the edges of $\widetilde{G}$. Given a face f of G and let $\widetilde{f}$ be one of its two lifts to $\widetilde{G}$, then $\widetilde{f}$ inherits the orientation from $\widetilde{\Sigma}$, which also induces an orientation to its boundary $\partial\widetilde{f}$. With these notations, consider the quantity

$$c^K(\widetilde{f}) := n^{\widetilde{K}}(\partial\widetilde{f}) + \sharp\{\text{edges in } \partial\widetilde{f} \text{ joining two vertices labelled -}\} + 1,$$

where $n^{\widetilde{K}}(\partial\widetilde{f})$ is understood as in the planar case. By [8, Section 4], one can prove that the parity of this quantity does not depend either on the choice of the lift $\widetilde{f}$ of f or on the swapping of the vertex labelling of $\widetilde{G}$. So the parity $c^K(f)$ of $c^K(\widetilde{f})$ only depends on the face f, the orientation K and the fixed function ω. With ω fixed, we shall say that K is a *(general) Kasteleyn orientation* (with respect to ω) if $c^K(f)$ is even for every face f of G.

As a special case, if Σ is orientable then one can fix $\omega \equiv 0$ which induces a vertex labelling of $\widetilde{G}$ whose restriction to a copy of G consists of all $+$. Hence an orientation K on the edges of G is Kasteleyn if $n^k(\partial f)$ is odd for every face f of G (note that in this case f is oriented according to a fixed orientation of Σ, and its boundary ∂f inherits its orientation). Thus we obtain a similar definition of Kasteleyn orientations as in the planar case.

The following result [8, Theorem 4.3] tells us whether Kasteleyn orientations exist when the graph G is embedded in the surface Σ. Note that we always fix a function ω representing the first Stiefel-Whitney class of Σ, and Kasteleyn orientations are understood to be with respect to this function.

Proposition 1.4.4. *There exists a Kasteleyn orientation on $G \subset \Sigma$ if and only if G has an even number of vertices.* □

Now let us analyse the relation between Kasteleyn orientations, dimer configurations and quadratic enhancements in the surface case. To state this relation explicitly, we refer the reader to [8, Section 5], however in the scope of this subsection, we only quote some important results. Firstly the definition of equivalent Kasteleyn orientations is still valid as in the planar case, so let us denote by $\mathcal{K}(G, \omega)$ the set of all equivalence classes of Kasteleyn orientations on $G \subset \Sigma$ with respect to ω. Secondly, fixing a dimer configuration of G, this set of equivalence classes can be mapped bijectively to the set $\mathcal{Q}(\Sigma, w_1)$ of quadratic enhancements on Σ. The following result [8, Proposition 5.2] gives us a characterisation for each of such a quadratic enhancement.

Theorem 1.4.5. *Given a fixed dimer configuration D of $G \subset \Sigma$ and a fixed function ω representing the first Stiefel-Whitney class of Σ, one can assign to each equivalence class $[K]$ of Kasteleyn orientations a quadratic enhancement $q_D^{K,\omega} : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_4$. This assignment is bijective between $\mathcal{K}(G, \omega)$ and $\mathcal{Q}(\Sigma, w_1)$, and the enhancement $q_D^{K,\omega}$ can be characterised by the following property: if C is an oriented simple cycle in G, then we have*

$$q_D^{K,\omega}([C]) = 2(n^K(C) + l_D(C) + 1) + \omega(C \cap D) - \omega(C \setminus D) \in \mathbb{Z}_4.$$

In this equality, $l_D(C)$ is the number of vertices v in C such that (C, D) induces a local orientation at $v \in \Sigma$ which lifts to the positive orientation of $\widetilde{\Sigma}$ at v^+ (or equivalently, to the negative one at v^-). □

Note that when Σ is orientable one can fix $\omega \equiv 0$, and in this case the set of quadratic enhancements $\mathcal{Q}(\Sigma, w_1)$ reduces to the set of quadratic forms $\mathcal{Q}(\Sigma)$ by the map $q \mapsto \frac{q}{2}$. Endowing Σ with an orientation pictured counterclockwise, we obtain the "orientable version" of Theorem 1.4.5 as follows (cf. [8, Lemma 3.4]).

Theorem 1.4.6. *Let D be a fixed dimer configuration of $G \subset \Sigma$. If Σ is orientable, one can assign to each equivalence class of Kasteleyn orientations $[K]$ a quadratic form $q_D^K : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$. This assignment is bijective between $\mathcal{K}(G)$ and $\mathcal{Q}(\Sigma)$, and the quadratic form q_D^K is characterised by the following property: for C an oriented simple cycle in G, we have*

$$q_D^K([C]) = n^K(C) + l_D(C) + 1 \in \mathbb{Z}_2.$$

In this equality, $l_D(C)$ is the number of vertices in C at which D points to the left of C. $\square$

We conclude this chapter with two versions of a Pfaffian formula to compute the dimer partition function of surface graphs. To state this formula, we need a twisted version of adjacency matrices that can be defined as follows. Recall that (G, x) is an edge-weighted graph embedded in the surface Σ with $2n$ vertices labelled by $\{1, \ldots, 2n\}$. Fix a function ω representing the first Stiefel-Whitney class of Σ and let K be an arbitrary orientation on G, the twisted adjacency matrix of G with respect to K and ω, denoted by $A^{K,\omega}(G)$, has entries given by

$$a_{ij} = \sum_{e=(i,j)} \epsilon_{ij}^K(e) i^{\omega(e)} x(e), \tag{1.5}$$

where $\epsilon_{ij}^K(e)$ is defined as before. The first version of Pfaffian formula can be stated as follows [8, Theorem 6.1].

Theorem 1.4.7. *Let D be a fixed dimer configuration of $G \subset \Sigma$, and ω a fixed function representing the first Stiefel-Whitney class of Σ. Setting $b_1 = \dim H_1(\Sigma; \mathbb{Z}_2)$, the dimer partition function of G is given by the formula*

$$Z_{\mathcal{D}}(G) = \frac{(-i)^{\omega(D)}}{2^{b_1/2}} \sum_{[K] \in \mathcal{K}(G,\omega)} \exp(i\pi/4)^{Br(q_D^{K,\omega})} \epsilon^K(D) Pf(A^{K,\omega}(G)).$$

In particular, when Σ is orientable of genus g the formula above reduces to

$$Z_{\mathcal{D}}(G) = \frac{1}{2^g} \sum_{[K] \in \mathcal{K}(G)} (-1)^{Arf(q_D^K)} \epsilon^K(D) Pf(A^K(G)). \quad \square$$

We now turn to the second version of the formula. This version requires a particular choice of Kasteleyn orientations that can be described as follows. Recall that if Σ is the orientable surface Σ_g of genus g, then it can be represented by a polygon with sides identified following the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. Recall that the cycles α_i $(1 \le i \le 2g)$ in Σ obtained from the $4g$ sides of the polygon after identifications yield a basis of $H_1(\Sigma; \mathbb{Z}_2)$.

For $\gamma \in \{\alpha_i : 1 \leq i \leq 2g\}$ let us define C_γ as the oriented cycle of G having γ on its immediate left, meeting every vertex of G adjacent to γ on this side. Then one can choose a Kasteleyn orientation K so that $n^K(C_\gamma)$ is odd for every γ.

Similarly for the non-orientable case, if Σ is homeomorphic to $\Sigma_g \# \mathscr{K}$ (resp. $\Sigma_g \# \mathbb{R}P^2$) it can be represented by the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} aabb$ (resp. by the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} cc$). (Note that here we use a different representation from subsection 1.3.1.) The set $\{\alpha_i, \beta_1, \beta_2\}$ (resp. $\{\alpha_i, \beta_1\}$) of cycles obtained after identifications then yields a basis of $H_1(\Sigma; \mathbb{Z}_2)$. Now for $\gamma \in \{\alpha_i : 1 \leq i \leq g\}$ we define C_γ as above, while for $\gamma \in \{\beta_1, \beta_2\}$ (resp. $\{\beta_1\}$) let us define C_γ as the oriented cycle of G given by an edge e of G which crosses γ once, together with a simple path in G connecting the two endpoints of e having γ on its immediate left. Note that one can assume C_{β_1} and C_{β_2} disjoint. Then there exists a Kasteleyn orientation K so that $n^K(C_\gamma)$ is odd for every γ.

In both case, for $\epsilon = (\epsilon_1, \cdots, \epsilon_{2g}) \in \mathbb{Z}_2^{2g}$ and $\eta = (\eta_1, \eta_2) \in \mathbb{Z}_2^2$ (resp. $\eta = \eta_1 \in \mathbb{Z}_2$), let $K_{\epsilon,\eta}$ be the orientation obtained from K by inverting the orientation of an edge of G each time this edge intersects α_j with $\epsilon_j = 1$ for $1 \leq j \leq 2g$, or intersects β_k with $\eta_k = 1$ for $k \in \{1, 2\}$ (resp. intersects β_1 with $\eta_1 = 1$). We can see that all the orientations $K_{\epsilon,\eta}$'s are Kasteleyn.

Finally, we also define a function ω by $\omega(e) := e \cdot (\beta_1 + \beta_2)$ (resp. $\omega(e) := e \cdot \beta_1$) for $e \in E(G)$, which clearly represents the first Stiefel-Whitney class of Σ. With this setting, the second version of the Pfaffian formula can be stated as follows [8, Theorem 6.3].

Theorem 1.4.8. *For G a weighted graph embedded in the surface Σ, the dimer partition function of G is given by*

$$Z_{\mathcal{D}}(G) = \frac{1}{2^g} \Big| \sum_{\epsilon \in \mathbb{Z}_2^{2g}} (-1)^{\sum_{j<k} \epsilon_j \epsilon_k \alpha_j \cdot \alpha_k} \Big(Re\big(Pf(A^{K_\epsilon}(G))\big) + Im\big(Pf(A^{K_\epsilon}(G))\big) \Big) \Big|$$

if $\Sigma = \Sigma_g$ or $\Sigma = \Sigma_g \# \mathbb{R}P^2$ with $A^{K_\epsilon}(G) := A^{K_{\epsilon,0}}(G)$, and by

$$Z_{\mathcal{D}}(G) = \frac{1}{2^g} \Big| \sum_{\epsilon \in \mathbb{Z}_2^{2g}} (-1)^{\sum_{j<k} \epsilon_j \epsilon_k \alpha_j \cdot \alpha_k} \Big(Im\big(Pf(A^{K_\epsilon}(G))\big) + Re\big(Pf(A^{K_{\epsilon'}}(G))\big) \Big) \Big|$$

if $\Sigma = \Sigma_g \# \mathscr{K}$ with $A^{K_\epsilon}(G) := A^{K_{\epsilon,0,0}}(G)$ and $A^{K_{\epsilon'}}(G) := A^{K_{\epsilon,1,0}}(G)$. □

Chapter 2

Identities between dimer partition functions on different surfaces

In this chapter we study some relations between dimer partition functions of graphs embedded in different surfaces. More precisely, if G is a weighted graph embedded in a non-orientable surface Σ and $\widetilde{G}$ is the lift of G to the orientation cover $\widetilde{\Sigma}$ of Σ, we can prove identities relating twisted partition functions of the dimer model on these two graphs. When Σ is the Möbius strip or the Klein bottle, then $\widetilde{\Sigma}$ is the cylinder or the torus, respectively, and under some natural assumptions, these identities imply relations between the genuine dimer partition functions $Z_{\mathcal{D}}(G)$ and $Z_{\mathcal{D}}(\widetilde{G})$. For example, we show that if G is a locally but not globally bipartite graph embedded in the Möbius strip, then $Z_{\mathcal{D}}(\widetilde{G})$ is equal to the square of $Z_{\mathcal{D}}(G)$. This extends results for the square lattice previously obtained by various authors. It should be mentioned that one of our results presented in this chapter, namely Theorem 2.1.1 part (i), was also obtained by Kuperberg [47] before. However our approach is different and allows us to extend this result to other surfaces. The present chapter is also the content of our article [13].

2.1 Introduction

2.1.1 Background

Let us briefly recall that a dimer configuration on a finite graph G is a family of edges, called dimers, such that each vertex of G is covered by exactly one of these dimers. To each edge e of G, assign an edge weight $x_e > 0$ and for a dimer configuration D, write $x(D) = \prod_{e\in D} x_e$. The associated dimer partition function is then defined as

$$Z(G) := Z_{\mathcal{D}}(G) = \sum_{D\in\mathcal{D}(G)} x(D)$$

the sum being over the set $\mathcal{D}(G)$ of dimer configurations on G. Note that the theory is empty unless the graph has an even number of vertices, which we will always assume.

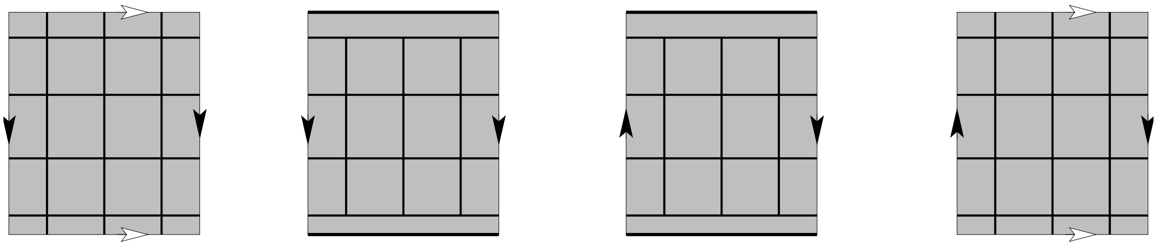


Figure 2.1: The 4×3-square lattice embedded in the torus $\mathbb{T}$, the cylinder $\mathscr{C}$, the Möbius strip $\mathscr{M}$, and the Klein bottle $\mathscr{K}$.

As reviewed in the introduction of the thesis, computing the dimer partition function was first achieved by Kasteleyn [35] and Temperley-Fisher [62] for the $m \times n$ square lattice $G_{m,n}$ with free boundary conditions (i.e. embedded in the plane), using Pfaffian method. By properties of the Pfaffian, and assuming that all horizontal (resp. vertical) edges have the same weight x (resp. y), these authors were able to give a closed formula for $Z(G_{m,n}) =: Z^{free}_{m,n}(x,y)$. This result was then extended in two directions. On the one hand, it was shown by Kasteleyn [37] that the computation of the dimer partition function using Pfaffians is valid for any planar weighted graph. General Pfaffian formulas for any surface graph were then obtained by Galluccio-Loebl [22], Tesler [63] and Cimasoni-Reshekhin [14, 8]. On the other hand, the exact value for the partition function of the square lattice (of width m, length n, and weights x, y) was computed for various boundary conditions, or in other words, for embeddings in various surfaces. This was performed by Kasteleyn [35] for periodic-periodic boundary conditions (which correspond to an embedding in the torus $\mathbb{T}$), by McCoy-Wu [53] for free-periodic ones (embedding in the cylinder $\mathscr{C}$), by Brankov-Priezzhev [3] and Tesler [63] for free-antiperiodic ones (embedding in the Möbius strip $\mathscr{M}$) and by Lu-Wu [51] for periodic-antiperiodic boundary conditions (embedding in the Klein bottle $\mathscr{K}$). These boundary conditions (and surfaces) are best described using a rectangle with identifications of pairs of sides encoded by arrows, as illustrated in Figure 2.1. We shall use the notations $Z^{\mathbb{T}}_{m,n}(x,y)$, $Z^{\mathscr{C}}_{m,n}(x,y)$, $Z^{\mathscr{M}}_{m,n}(x,y)$ and $Z^{\mathscr{K}}_{m,n}(x,y)$ for the corresponding partition functions.

Using these results, Brankov-Priezzhev [3] proposed the identity

$$Z^{\mathscr{C}}_{2m,4n}(x,y) = \left(Z^{\mathscr{M}}_{2m,2n}(x,y)\right)^2 , \tag{2.1}$$

which they established in some large m, n expansion. This equality was later proved by Lu-Wu [51], who also discovered the identity

$$Z^{\mathscr{C}}_{2m-1,4n}(x,y) = \frac{1}{2}\left(Z^{\mathscr{M}}_{2m-1,2n}(x,y)\right)^2 . \tag{2.2}$$

Finally Izmailian-Oganesyan-Hu [32] established the formula

$$Z^{\mathbb{T}}_{2m-1,4n}(x,y) = \frac{1}{2}\left(Z^{\mathscr{K}}_{2m-1,2n}(x,y)\right)^2 . \tag{2.3}$$

Let us point out once again that these results were obtained by comparing explicit values of the various partition functions, values that are often available only for the square lattice with weights x, y. In particular, these authors regret the lack of "general underlying principle" explaining such "curious identities" ([3, p.650] and [51, p.111], respectively).

In the present chapter, we provide such a general principle that allows us, not only to explain these identities, but also to generalise them in quite a significant way. We now summarise our results.

2.1.2 Main results, and examples

Recall that if G is a graph embedded in a non-orientable surface Σ of genus $h \geq 1$, one can consider its lift $\widetilde{G}$ to the orientation cover $\widetilde{\Sigma}$ of genus $g = h-1$ of Σ (cf. Subsection 1.3.1). Then the three identities displayed above all relate $Z(G)$ to $Z(\widetilde{G})$ for some specific examples of $G \subset \Sigma$.

We now explain briefly how to obtain these identities. Using the geometric approach to the Pfaffian method developed in [8], we show that for any weighted graph $G \subset \Sigma$ with an even number of vertices, there are $2^g = 2^{h-1}$ distinct identities relating twisted versions of the dimer partition functions for G and $\widetilde{G}$. In fact, there are essentially 2^{h-1} orientations that can be used in the Pfaffian method to study the dimer model on G embedded in a non-orientable surface Σ of genus h. To any such orientation K, one can associate an orientation $\widetilde{K}$ on the corresponding cover $\widetilde{G} \subset \widetilde{\Sigma}$ which can be used to study the dimer model on $\widetilde{G}$. This allows us to relate the Pfaffians of the two corresponding skew-adjacency matrices. As proved in [8], these Pfaffians are equal to twisted partition functions, so we obtain the 2^{h-1} identities.

In particular when $h = 1$, i.e. when the graph G is embedded in the Möbius strip, we obtain a single identity, namely

$$Z(\widetilde{G}) = Z_0(G)^2 + Z_1(G)^2 \,, \tag{2.4}$$

where $Z_\alpha(G)$ denotes the partial partition function given by $Z_\alpha(G) = \sum_{[D\Delta D_0]=\alpha} \exp(-\mathcal{E}(D))$, the sum being over all dimer configurations whose symmetric difference with a fixed $D_0 \in \mathcal{D}(G)$ winds around $\mathscr{M}$ an even (resp. odd) number of times if $\alpha = 0$ (resp. if $\alpha = 1$). Of course, we are interested in identities between the actual partition functions, not some twisted ones. For this purpose, observe that Equation (2.4) implies that the identity $Z(\widetilde{G}) = Z(G)^2$ (resp. $Z(\widetilde{G}) = \frac{1}{2}Z(G)^2$) holds if and only if $Z_1(G) = 0$ (resp. $Z_0(G) = Z_1(G)$). Therefore, we are left with the task of finding natural conditions on $G \subset \mathscr{M}$ for this to happen.

Recall that a graph embedded in a surface is *locally bipartite* (resp. *bipartite*) if the boundary of each face (resp. if each cycle) has even length. The proof of the following theorem, which can be found in subsection 2.2.3, follows easily from the discussion above.

Theorem 2.1.1. *Let $G \subset \mathscr{M}$ be a weighted graph with an even number of vertices embedded in the Möbius strip, and let $\widetilde{G} \subset \mathscr{C}$ denote its orientation cover.*

(i) If G is locally bipartite but not bipartite, then $Z(\widetilde{G}) = Z(G)^2$.

(ii) *Assume that $G \subset \mathscr{M}$ is invariant by a horizontal translation τ of the Möbius strip. If there is some $D_0 \in \mathcal{D}(G)$ such that the symmetric difference $D_0 \Delta \tau(D_0)$ winds around $\mathscr{M}$ an odd number of time, then $Z(\widetilde{G}) = \dfrac{1}{2} Z(G)^2$.*

Remark that part (i) of the above theorem was also proved by Kuperberg [47, Theorem 7] for unweighted graphs, using a quite elementary method. In this work we will present another one, which enables us to extend this result to other surfaces.

Examples of Theorem 2.1.1 are plentiful and include many classical regular lattices. In the context of this discussion, we shall only mention the following ones.

Example 2.1.2. Consider the case of the $2m \times 2n$ square lattice G embedded in the Möbius strip, endowed with arbitrary weights, and let $\widetilde{G}$ be the corresponding $2m \times 4n$ square lattice in the cylinder. Since G is locally bipartite but not bipartite, we have the equality $Z(\widetilde{G}) = Z(G)^2$ which extends Equation (2.1) to arbitrary weights.

Example 2.1.3. Consider a portion of arbitrary size of the hexagonal lattice, embedded in the Möbius strip as illustrated in the left-hand side of Figure 2.2, and endowed with arbitrary weights. This graph G is easily seen not to be bipartite, so we have the equality $Z(\widetilde{G}) = Z(G)^2$.

Example 2.1.4. Let G be a portion of arbitrary size of the square-octagon lattice embedded in the Möbius strip as illustrated in the center of Figure 2.2. This graph is locally bipartite but not bipartite, so the equality $Z(\widetilde{G}) = Z(G)^2$ holds for any weights on G.

Example 2.1.5. Consider the $(2m-1) \times 2n$ square lattice G embedded in the Möbius strip, and endowed with any weight system that is invariant under horizontal translation by one edge. Choosing for D_0 any dimer configuration made of horizontal edges (this is possible since the lattice has even length), we find that $D_0 \Delta \tau(D_0)$ consists of all horizontal edges of G, and hence winds around the Möbius strip $2m-1$ times. Therefore, we have the equality $Z(\widetilde{G}) = \frac{1}{2} Z(G)^2$ which extends Equation (2.2).

This example can be generalised as follows: consider planar weighted graph $G_1, \dots, G_{m-1}$ that admit a dimer configuration, and let $\overline{G}_1, \dots, \overline{G}_{m-1}$ denote the weighted graphs obtained from these planar graphs via a horizontal reflexion. In each column of the square lattice, add a copy of G_1 in the first face, a copy of G_2 in the second, and so on until a copy of G_{m-1} in the $(m-1)^{\text{th}}$ face, and then a copy of $\overline{G}_{m-1}$, of $\overline{G}_{m-2}$, until a copy of $\overline{G}_1$ in the last face. Finally, join each of these graphs to the adjacent vertices of G in an arbitrary but fixed way for each G_i. (The case $m = 2$ is illustrated in the right-hand side of Figure 2.2.) By the second part of Theorem 2.1.1, the resulting graph G' will also satisfy the equality $Z(\widetilde{G}') = \dfrac{1}{2} Z(G')^2$. Note that this class of graphs contains the triangular lattice as a particularly natural example.

Let us now consider the case of genus $h = 2$, i.e. of a graph G embedded in the Klein bottle. In this case, we obtain two identities, see Equation (2.10) below. They easily lead to the following result, whose proof can be found in Section 2.3.

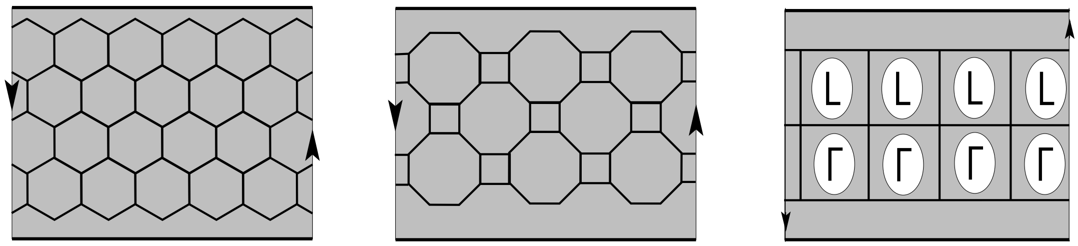


Figure 2.2: The graphs in Examples 2.1.3, 2.1.4 and 2.1.5.

Theorem 2.1.6. *Let $G \subset \mathscr{K}$ be a weighted graph with an even number of vertices embedded in the Klein bottle, and let $\widetilde{G} \subset \mathbb{T}$ denote its orientation cover. Let us assume that G is locally bipartite but not bipartite, and invariant by a horizontal translation τ. If there is some $D_0 \in \mathcal{D}(G)$ such that the symmetric difference $D_0 \Delta \tau(D_0)$ winds an odd number of times around $\mathscr{K}$ in the horizontal direction (and an arbitrary number of times in the vertical direction), then $Z(\widetilde{G}) = \frac{1}{2} Z(G)^2$.*

Example 2.1.7. Consider the $(2m-1) \times 2n$ square lattice embedded in the Klein bottle, and endowed with any weight system that is invariant under translation by one edge along the horizontal direction in $\mathscr{K}$. By the same arguments as in Examples 2.1.2 and 2.1.5, one easily checks that it satisfies the hypothesis of Theorem 2.1.6, thus extending Equation (2.3) to more general weights. (This can be further generalised as described at the end of Example 2.1.5.)

2.1.3 Consequences

Let us begin with the combinatorial consequences of our results. Simply observe that if the energy of each edge is zero, then the dimer partition function $Z(G)$ is equal to the number of perfect matchings on G. Therefore, each of the statements above implies an identity between the number of perfect matchings on G and on $\widetilde{G}$. For example, Theorem 2.1.1 implies the following combinatorial statement: *If G is a non-bipartite graph with an even number of vertices embedded in $\mathscr{M}$ in a locally bipartite way, then the number of perfect matchings on its orientation cover $\widetilde{G} \subset \mathscr{C}$ is equal to the square of the number of perfect matchings on G.* It should be mentioned that by Kuperberg [47], such a statement can be proved by purely combinatorial means, however the proof, left as an exercise, was not presented in his article [47].

We now come to the physical consequences of our results. Consider a two-dimensional system at criticality, with area A and boundary length L. The total free energy $\log Z$ of such a system is commonly assumed [57] to have a large A expansion at fixed shape of the form

$$F = Af_b + Lf_s + D + o(1)\,,$$

where f_b is the bulk free energy (per unit surface area), which is independent of the boundary conditions, f_s is the surface free energy (per unit boundary length), and D is a dimensionless coefficient. Furthermore, the coefficients f_b and f_s depend on the lattice, but D is expected to be universal, depending only on the shape of the system, and possibly on the boundary conditions. (Analogies with conformal field theory lead several authors to conjecture the explicit form of this coefficient D, see e.g. [16] and references therein.) As pointed out by Cardy-Peschel [6], such an expansion is actually valid only for surfaces with vanishing Euler characteristic, that is, precisely the four surfaces $\mathscr{M}, \mathscr{C}, \mathscr{K}$ and $\mathbb{T}$ considered above.

Applying the first part of Theorem 2.1.1, we find that such an expansion holds for a system on a locally bipartite non-bipartite graph $G \subset \mathscr{M}$ if and only if it holds for its orientation cover $\widetilde{G} \subset \mathscr{C}$; in such a case both expansions have the same coefficients f_b and f_s (observe that the area and the boundary length are doubled in $\mathscr{C}$), while the corresponding universal coefficients are simply related by $\widetilde{D} = 2D$. (This was the precise result obtained by Brankov-Priezzhev [3] for the square lattice with x, y weights.) Also, such an expansion holds for a graph G satisfying the assumptions of the second part of Theorem 2.1.1, or of Theorem 2.1.6, if and only if it holds for its orientation cover $\widetilde{G}$; in such a case both expansions have the same bulk free energy, as they should, the same surface free energy (which vanishes in the setting of Theorem 2.1.6), while the universal coefficients are related by $\widetilde{D} = 2D - \log(2)$.

2.1.4 Discussion, and organisation of the chapter

As is well-known since the work of Kasteleyn [37], the validity of the Pfaffian method for planar graphs can be demonstrated in a straightforward way. However, the extension of this method to graphs embedded in non-planar surfaces requires either tedious combinatorial considerations [63, 22], or additional geometric tools known as spin or pin$^-$ structures [14, 8]. It would have been possible to write a slightly shorter version of this chapter relying heavily on [8] as a *black box* – and the most general form of our identities actually still does hold. However, in the course of our investigations, we discovered that the special geometry of simple closed curves in the Möbius strip allows an elementary proof of the Pfaffian method in this case as well. It thus became possible to give a simple self-contained proof of the identities in the genus $h = 1$ case, leading in particular to an elementary demonstration of Theorem 2.1.1. Although an elementary proof of the Pfaffian method for the Möbius strip is known before [47], our proof is interesting of its own. So we decided to include it in the present chapter, which is therefore organised in the following slightly unorthodox way: Section 2.2 dealing with the special case of the Möbius band, is entirely self-contained and meant to be readable without any knowledge of algebraic topology, while Section 2.3 deals with the general case, relying heavily on previous work [8].

To be more precise, subsection 2.2.2 contains the aforementioned proof of the corresponding statement in the Möbius strip (Theorem 2.2.2), while subsection 2.2.3 contains the proof of Equation (2.4), and of Theorem 2.1.1. We then start Section 2.3 with a result that relates quadratic enhancements on a non-orientable surface with quadratic forms on the orientation cover (Proposition 2.3.1). Finally, we state our main result (Theorem 2.3.2) and give its proof, as well as the proof of Theorem 2.1.6.

2.2 The identities for the cylinder and Möbius strip

The aim of this section is to give a self-contained treatment of the Pfaffian method for graphs in the cylinder (subsection 2.2.1) and in the Möbius strip (subsection 2.2.2), leading to the proof of Theorem 2.1.1 (subsection 2.2.3).

Throughout this section, G is a finite graph with an even number of vertices, and each edge e of G is endowed with a positive weight x_e. Note that the weights x_e can be considered as formal variables, as all of our identities hold true in the ring $\mathbb{Z}[i][\{x_e\}_e]$.

2.2.1 Graphs in the cylinder

Let us consider the case where a graph G is embedded in a cylinder $\mathscr{C}$. Since $\mathscr{C}$ embeds in the plane, G is actually planar so Kasteleyn's theorem (cf. subsection 1.4.2) applies and G admits a Pfaffian orientation. However, a new phenomenon occurs in this case, namely the existence of non-equivalent Kasteleyn orientations, that we now explain.

Recall that in the plane any two Kasteleyn orientations on $G \subset \mathbb{R}^2$ are equivalent. However the situation changes if we consider G as a graph embedded in a cylinder $\mathscr{C}$, which we now picture as embedded in the plane $\mathbb{R}^2$. Very naturally, we shall say that an orientation K is a *Kasteleyn orientation* on $G \subset \mathscr{C}$ if $n^K(\partial f)$ is odd for every face f of $G \subset \mathscr{C}$. Note that a Kasteleyn orientation on $G \subset \mathscr{C}$ is not always a Kasteleyn orientation on $G \subset \mathbb{R}^2$, since $\mathbb{R}^2 \setminus \mathscr{C}$ has a bounded component f_0, and $n^K(\partial f_0)$ might be odd, or even. This determines two distinct equivalence classes of Kasteleyn orientations on $G \subset \mathscr{C}$. The two corresponding Pfaffians count different quantities, that are best described using the following notation. Fix an arbitrary dimer configuration D_0 on $G \subset \mathscr{C}$; for $\alpha \in \mathbb{Z}_2$, set $Z_{\alpha,D_0}(G) = \sum_{[D\Delta D_0]=\alpha} x(D)$, where the sum is over all $D \in \mathcal{D}(G)$ such that the parity of the number of times $D\Delta D_0$ winds around the cylinder is given by α.

Proposition 2.2.1. *For any Kasteleyn orientation K on $G \subset \mathscr{C}$,*

$$\epsilon^K(D_0)\, Pf(A^K(G)) = Z_{0,D_0}(G) \pm Z_{1,D_0}(G)\,,$$

where the sign is $+$ for one equivalence class of Kasteleyn orientation, and $-$ for the other.

Proof. First note that this equation is left unchanged by replacing the Kasteleyn orientation K by an equivalent one, as both $\epsilon^K(D_0)$ and $\mathrm{Pf}(A^K(G))$ change sign if K is inverted around a vertex. Now, using the notation of the discussion above, Kasteleyn's theorem implies that if K is such that $n^K(\partial f_0)$ is odd, then

$$\epsilon^K(D_0)\,\mathrm{Pf}(A^K(G)) = \sum_{D\in\mathcal{D}(G)} (-1)^{\sum_j (n^K(C_j)+1)}\, x(D) = Z(G) = Z_{0,D_0}(G) + Z_{1,D_0}(G)\,.$$

Finally, consider a path transverse to G joining the two components of $\mathbb{R}^2 \setminus \mathscr{C}$, and let K' be obtained by inverting K on all the edges that intersect this path. By construction, $n^{K'}(\partial f_0)$ is even, so K' is a representative of the other equivalence class. Observe that the effect of this transformation on the central part of the equation displayed above is to change the sign of the contribution of each D such that $D\Delta D_0 = \bigsqcup_j C_j$ winds around the cylinder an odd number of times. Hence, we have the equation $\epsilon^{K'}(D_0)\,\mathrm{Pf}(A^{K'}) = Z_{0,D_0}(G) - Z_{1,D_0}(G)$ and the proposition is proved. □

2.2.2 Graphs in the Möbius strip

We now turn to the case of a graph G embedded in a Möbius strip $\mathscr{M}$. This situation is more tricky and involves the introduction of complex valued coefficients in the skew-adjacency matrices, as first noticed by Tesler [63]. Here, we give a new elementary proof of this result, adapting the geometrical treatment given in [8].

In what follows, we will always represent the Möbius strip as a rectangle with antiperiodic boundary conditions on the vertical sides, and free boundary conditions on the horizontal sides (recall Figure 2.1). The graph G is embedded in this rectangle with some specific edges intersecting transversally the vertical sides, and no edge intersecting the horizontal sides. We assign to each such specific edge e the value $\omega(e) = 1$, and to all the others the value $\omega(e) = 0$ (hence this function ω is the standard representation of the first Stiefel-Whitney class of the Möbius band). As usual, we also write $\omega(D) = \sum_{e\in D}\omega(e)$, and similarly for any set of edges.

Given an arbitrary orientation K on the edges of $G \subset \mathscr{M}$, recall that $A^{K,\omega}(G) = (a_{uv})$ is the modified weighted skew-adjacency matrix given by

$$a_{uv} = \sum_{e=(u,v)} \epsilon^K_{uv}(e)\, i^{\omega(e)} x(e)\,.$$

Following the notation and the discussion of subsection 1.4.1 up to Equation (1.4), we can easily find

$$\begin{aligned}
i^{-\omega(D_0)}\,\epsilon^K(D_0)\,\mathrm{Pf}(A^{K,\omega}(G)) &= i^{-\omega(D_0)}\,\epsilon^K(D_0) \sum_{D\in\mathcal{D}(G)} \epsilon^K(D) i^{\omega(D)} x(D)\\
&= \sum_{D\in\mathcal{D}(G)} \epsilon^K(D)\epsilon^K(D_0) i^{\omega(D)-\omega(D_0)} x(D)\\
&= \sum_{D\in\mathcal{D}(G)} (-1)^{\sum_j (n^K(C_j)+1)}\, i^{\sum_j \omega(C_j\setminus D_0)-\omega(C_j\cap D_0)}\, x(D)\,, \qquad (2.5)
\end{aligned}$$

where $D\Delta D_0 = \bigsqcup_j C_j$.

Next we define a Kasteleyn orientation on graphs embedded in the Möbius strip. Inspired by definition of Kasteleyn orientations in the most general setting (cf. subsection 1.4.3), this can be done as follows. Given a graph $G \subset \mathscr{M}$ as described above, let $\widetilde{G} \subset \mathscr{C}$ denote the graph embedded in the cylinder obtained by taking two copies of $G \subset \mathscr{M}$ and gluing them along the vertical sides, as illustrated in Figure 1.2. Also, if K is an orientation on G, let $\widetilde{K}$ denote the orientation on $\widetilde{G}$ obtained by lifting K to the edges of $\widetilde{G}$, and by inverting the orientation of all the edges that are completely contained in the second copy of the Möbius strip. We shall say that K is a *Kasteleyn orientation* on $G \subset \mathscr{M}$ if $\widetilde{K}$ is a

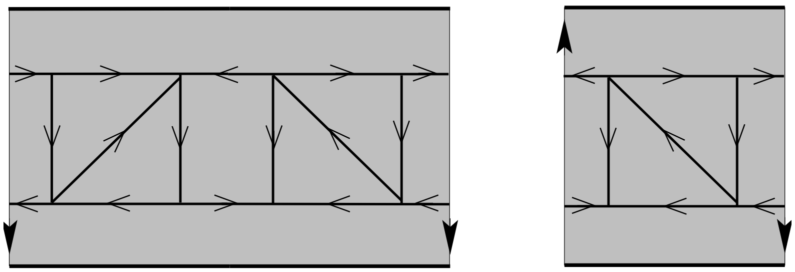


Figure 2.3: A Kasteleyn orientation K on a graph G embedded in the Möbius strip (right), and the corresponding Kasteleyn orientation $\widetilde{K}$ on $\widetilde{G} \subset \mathscr{C}$ (left).

Kasteleyn orientation on $\widetilde{G} \subset \mathscr{C}$, as defined in subsection 2.2.1. A didactic example is given in Figure 2.3. Note that each face f of $G \subset \mathscr{M}$ lifts to two faces $\widetilde{f}'$ and $\widetilde{f}''$ of $\widetilde{G} \subset \mathscr{C}$, and by definition, $\widetilde{K}$ satisfies the parity condition around $\widetilde{f}'$ if and only if it does around $\widetilde{f}''$. Then, one can check that any $G \subset \mathscr{M}$ admits a Kasteleyn orientation, as in the planar case. Note that this notion not only depends on the way G is embedded in $\mathscr{M}$, but also on the way the Möbius strip is drawn as a rectangle, which is encoded by ω. Therefore, we shall sometimes be more precise and speak of a Kasteleyn orientation on $(G \subset \mathscr{M}, \omega)$.

Let us define $Z_{\alpha,D_0}(G) = \sum_{[D\Delta D_0]=\alpha} x(D)$, where the sum is over all $D \in \mathcal{D}(G)$ such that the parity of the number of times $D\Delta D_0$ winds around the Möbius strip is given by $\alpha \in \mathbb{Z}_2$. We are now ready to state the main result of this paragraph.

Theorem 2.2.2. *Fix an arbitrary dimer configuration D_0 on a graph G embedded in a Möbius strip $\mathscr{M}$. Then, for any Kasteleyn orientation K on $(G \subset \mathscr{M}, \omega)$,*

$$i^{-\omega(D_0)}\,\epsilon^K(D_0)\,Pf(A^{K,\omega}(G)) = Z_{0,D_0}(G) \pm i\,Z_{1,D_0}(G)\,.$$

As an immediate corollary, the dimer partition function of a graph embedded in a Möbius strip can be computed using a single Pfaffian.

Corollary 2.2.3. *For any Kasteleyn orientation K, the dimer partition function of $G \subset \mathscr{M}$ is given by*

$$Z(G) = |\mathrm{Re}\,Pf(A^{K,\omega})(G)| + |\mathrm{Im}\,Pf(A^{K,\omega})(G)|. \quad \square$$

Proof of Theorem 2.2.2. We will first show that this equality does not depend on the way the Möbius strip is cut open into a rectangle, i.e. on the choice of ω. (This is evident for the right-hand side.) More precisely, by general cohomology theory, any two choices of ω are cohomologous, so they are related by a sequence of “vertex flips”, consisting in changing the value of ω at each edge adjacent to a fixed vertex v. One can also visualise geometrically that this corresponds to moving the cut across the vertex v. We claim that if ω' is obtained from ω by such a vertex flip, then there is a canonical way to transform a Kasteleyn orientation K on $(G \subset \mathscr{M}, \omega)$ into a Kasteleyn orientation K' on $(G \subset \mathscr{M}, \omega')$ so that $i^{-\omega(D_0)}\,\epsilon^K(D_0)\,\mathrm{Pf}(A^{K,\omega}(G))$ is left unchanged. Indeed, define K' as coinciding with K everywhere, except precisely on the edges e adjacent to v such that $\omega(e) = 1$ (see Figure 2.4).

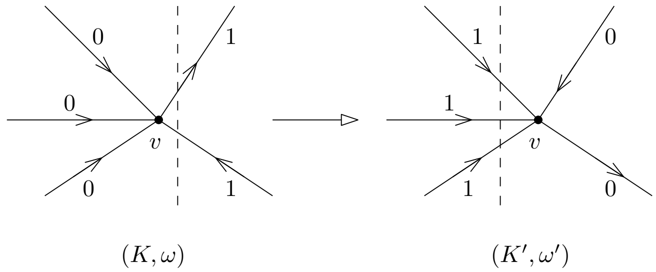


Figure 2.4: A canonical transformation from K to K' around a vertex v according to the transformation from ω to ω'.

One can verify that K' is a Kasteleyn orientation on $(G \subset \mathscr{M}, \omega')$ as follows. By moving the cut across the vertex v, an edge e adjacent to v with $\omega(e) = 1$ not lying completely in the second copy of the Möbius strip encoded by ω turns out to lie completely in the second copy of the Möbius strip encoded by ω'. Inverting K on such an edge e to obtain K' guarantees that the Kasteleyn condition holds for K' with respect to ω' as well.

We also can check that

$$\mathrm{Pf}(A^{K',\omega'}(G)) = i\,\mathrm{Pf}(A^{K,\omega}(G)). \tag{2.6}$$

Indeed from the equation

$$\mathrm{Pf}(A^{K,\omega}(G)) = \sum_{D \in \mathcal{D}(G)} \epsilon^K(D) i^{\omega(D)} x(D)$$

(and similarly for $\mathrm{Pf}(A^{K',\omega'}(G))$), it is sufficient to show that

$$\epsilon^{K'}(D) i^{\omega'(D)} = \epsilon^K(D) i^{\omega(D)+1} \tag{2.7}$$

for any dimer configuration D. Actually, for each $D \in \mathcal{D}(G)$ there exists exactly one edge e adjacent to v. If $\omega(e) = 0$ then K coincides with K' on e, and hence $\epsilon^{K'}(D) = \epsilon^K(D)$. Also in this case $\omega'(e) = 1$ leading to $\omega'(D) = \omega(D) + 1$, and thus Equation (2.7) follows as well as Equation 2.6. The case $\omega(e) = 1$ is treated the same. Additionally the argument to prove Equation (2.7) also implies that

$$i^{-\omega'(D_0)}\,\epsilon^{K'}(D_0) = (-i) i^{-\omega(D_0)}\,\epsilon^K(D_0).$$

This equation together with Equation (2.6) proves that the left-hand side of the equality stated in Theorem 2.2.2 does not depend on the choice of ω.

As a second step, let us recall a couple of well-known geometric facts. Simple closed curves in the Möbius strip $\mathscr{M}$ fall into three categories: they either bound a disc, or bound a Möbius strip isotopic to $\mathscr{M}$, or do not bound. In the last case they wind around $\mathscr{M}$ exactly once. Furthermore, any two curves that do not bound must intersect each other (an odd number of times). Therefore, given any two dimer configurations D, D_0 on $G \subset \mathscr{M}$, their symmetric difference $D\Delta D_0 = \bigsqcup_j C_j$ consists of a certain number of simple closed curves of the first two types, and at most one of the third type.

By Equation (2.5), we therefore need to show the following claim: if D is such that $D\Delta D_0$ bounds discs and Möbius strips isotopic to $\mathscr{M}$, then

$$(-1)^{\sum_j (n^K(C_j)+1)}\, i^{\sum_j \omega(C_j\setminus D_0)-\omega(C_j\cap D_0)} = 1\,.$$

Let us fix D as above. Clearly, one can cut $\mathscr{M}$ open into a rectangle without touching any of these discs, without intersecting any of the edges of D_0, and cutting each of the Möbius strips open into a rectangle. Choosing the ω which corresponds to such a cut, as permitted by the first part of the proof, the equality displayed above is now a consequence of the following two statements:

(i) if $C \subset D\Delta D_0$ bounds a disc, then $n^K(C)$ is odd (note that $\omega(C) = 0$ in this case);

(ii) if $C \subset D\Delta D_0$ bounds a Möbius strip, then $n^K(C)$ is even (note that here, $\omega(C) = 2$).

The proof of the first statement follows *verbatim* the proof of Kasteleyn's theorem given in subsection 1.4.2. To show the second statement, let us denote by V, E, and F the number of vertices, edges and faces, respectively, of the Möbius strip $\mathscr{M}'$ bounded by C, and let us write $V = V_{\text{int}} + V_{\text{ext}}$ and $E = E_{\text{int}} + E_{\text{ext}}$, where V_{int} (resp. E_{int}) denotes the number of vertices (resp. edges) in the interior of $\mathscr{M}'$. The cutting open of $\mathscr{M}$ into a rectangle defines a decomposition of the oriented simple closed curve C bounding $\mathscr{M}'$ into two oriented simple curves C' and C'' in the rectangle, as illustrated in Figure 2.5. Summing over all the faces of $\mathscr{M}'$, and using the definition of a Kasteleyn orientation in a Möbius strip, we get

$$\sum_f (n^K(\partial f)+1) = \sum_f (n^{\widetilde{K}}(\partial \widetilde{f})+1) + \ell = \ell\,,$$

with ℓ denoting the number of edges bounding faces of $\mathscr{M}$ where the orientation K lifted to the cylinder is inverted to give $\widetilde{K}$. (In Figure 2.5, there are $\ell = 3$ such edges, that are drawn with heavier lines.) On the other hand, computing modulo 2, this same sum is equal to

$$\sum_f (n^K(\partial f)+1) = n^K(C') + n^K(-C'') + \ell + E_{\text{int}} + F\,.$$

These two equations lead to the equality modulo 2

$$\begin{aligned} 0 &= n^K(C') + n^K(-C'') + E_{\text{int}} + F = n^K(C) + |C''| + E_{\text{int}} + F \\ &= n^K(C) + |C'| + E_{\text{ext}} + E_{\text{int}} + F = n^K(C) + |C'| + V\,, \end{aligned} \tag{2.8}$$

using the facts that $|C'| + |C''| = E_{\text{ext}}$ and that the Euler characteristic of the Möbius strip is equal to $V - E + F = 0$. Note that since D_0 does not meet the cut, the length of both C' and C'' is even, and so is $|C'|$. Finally, V_{int} is also even since these interior vertices are matched by D_0. By Equation (2.8), $n^K(C)$ is therefore even, and the claim is proved. In summary, we showed that for any $D_0 \in \mathcal{D}(G)$ and for any Kasteleyn orientation

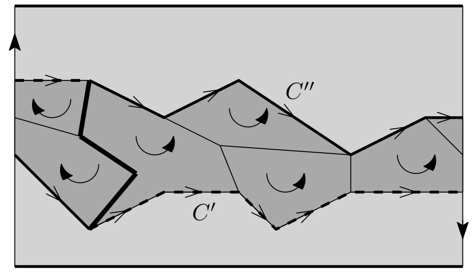


Figure 2.5: Proof of Theorem 2.2.2: the shaded Möbius strip $\mathscr{M}'$ is bounded by the oriented simple curves C' in dashed lines and C'' in solid lines, while the $l = 3$ special edges are in heavier solid lines.

on $(G \subset \mathscr{M}, \omega)$, we have the equality

$$i^{-\omega(D_0)}\, \epsilon^K(D_0)\, \mathrm{Pf}(A^{K,\omega}(G)) = \textstyle\sum_{[D\Delta D_0]=0} x(D) + i \sum_{[D\Delta D_0]=1} \pm x(D)\,. \tag{2.9}$$

If there is no D such that $[D\Delta D_0] = 1$, then the proof is complete. Otherwise, fix such a dimer configuration D_1. Note that $\omega(D_0)$ and $\omega(D_1)$ have different parity, and that for any $D \in \mathcal{D}(G)$, $[D\Delta D_0]$ and $[D\Delta D_1]$ are different. Therefore, applying Equation (2.9) to D_1, we get

$$\begin{aligned} i^{-\omega(D_0)}\, \epsilon^K(D_0)\, \mathrm{Pf}(A^{K,\omega}(G)) &= \pm i \cdot i^{-\omega(D_1)}\, \epsilon^K(D_1)\, \mathrm{Pf}(A^{K,\omega}(G)) \\ &= \pm i \left(\textstyle\sum_{[D\Delta D_1]=0} x(D) + i \sum_{[D\Delta D_1]=1} \pm x(D) \right) \\ &= \textstyle\sum_{[D\Delta D_0]=0} \pm x(D) \pm i \sum_{[D\Delta D_0]=1} x(D)\,. \end{aligned}$$

Considering the quantity $i^{-\omega(D_0)}\, \epsilon^K(D_0)\, \mathrm{Pf}(A^{K,\omega}(G))$ as a polynomial in the variables $x(e)$ with $e \in E(G)$, the equation displayed above together with Equation (2.9) yield the statement of the theorem. $\square$

2.2.3 Proof of the identities for the cylinder and Möbius strip

Our proof relies on the following easy but crucial lemma.

Lemma 2.2.4. *Let K be any Kasteleyn orientation on $G \subset \mathscr{M}$, and $\widetilde{K}$ the corresponding orientation on $\widetilde{G} \subset \mathscr{C}$. If G has an even number of vertices, then*

$$|Pf(A^{\widetilde{K}}(\widetilde{G}))| = |Pf(A^{K,\omega}(G))|^2.$$

Proof. Enumerate the vertices of $\widetilde{G}$ contained in one copy of G, followed by the vertices contained in the other copy with the same order. Recall that $\widetilde{K}$ is obtained by lifting K to the edges of $\widetilde{G}$ and by inverting the orientation of all the edges that are completely contained in the second copy of the Möbius strip. Using the definitions (1.1) and (1.5) of the corresponding skew-adjacency matrices, we obtain

$$A^{\widetilde{K}}(\widetilde{G}) = \begin{pmatrix} M_1 & M_2 \\ M_2 & -M_1 \end{pmatrix} \quad \text{and} \quad A^{K,\omega}(G) = M_1 + iM_2\,,$$

with M_1, M_2 real square matrices of even dimension. Using this fact and obvious operations, we get

$$\det A^{\widetilde{K}}(\widetilde{G}) = \begin{vmatrix} M_1 & iM_2 \\ iM_2 & M_1 \end{vmatrix} = \begin{vmatrix} M_1 + iM_2 & iM_2 \\ M_1 + iM_2 & M_1 \end{vmatrix} = \begin{vmatrix} M_1 + iM_2 & iM_2 \\ 0 & M_1 - iM_2 \end{vmatrix} = \left|\det A^{K,\omega}(G)\right|^2 .$$

Since the determinant is the square of the Pfaffian, this proves the lemma. □

Proposition 2.2.5. *Let $G \subset \mathscr{M}$ be a weighted graph embedded in the Möbius strip, and let $\widetilde{G} \subset \mathscr{C}$ denote its 2-fold cover embedded in the cylinder. If G has an even number of vertices, then*

$$Z(\widetilde{G}) = Z_{0,D_0}(G)^2 + Z_{1,D_0}(G)^2$$

for any dimer configuration D_0 on G.

Proof. Using Theorem 2.2.2 and Lemma 2.2.4, we have

$$Z_{0,D_0}(G)^2 + Z_{1,D_0}(G)^2 = \left|\mathrm{Pf}(A^{K,\omega}(G))\right|^2 = |\mathrm{Pf}(A^{\widetilde{K}}(\widetilde{G}))| .$$

By Kasteleyn's theorem, we only need to check that, given any Kasteleyn orientation K on $G \subset \mathscr{M}$, the corresponding Kasteleyn orientation $\widetilde{K}$ on $\widetilde{G} \subset \mathscr{C}$ is also a Kasteleyn orientation on $\widetilde{G} \subset \mathbb{R}^2$. In other words, writing f_0 for the bounded component of $\mathscr{C} \subset \mathbb{R}^2$, we need to verify that $n^{\widetilde{K}}(\partial f_0)$ is odd. Let C denote the simple closed curve in G bounding the Möbius strip $\mathscr{M}$. Cutting $\mathscr{M}$ open into a rectangle decomposes C into two simple curves C' and C'' (recall Figure 2.5), and by definition of $\widetilde{K}$, we have the equality $n^{\widetilde{K}}(\partial f_0) = n^K(C) + |C'| + 1$ in $\mathbb{Z}_2$. (Note that one of the lifts of C is precisely ∂f_0.) Therefore, we need to check that $n^K(C)$ and $|C'|$ have the same parity. By Equation (2.8), this is the case if and only if G contains an even number of vertices, which we assumed. □

The proof of Theorem 2.1.1 is now straightforward.

Proof of Theorem 2.1.1. If $G \subset \mathscr{M}$ is locally bipartite but not globally bipartite, then any simple closed curve in G that does not bound in $\mathscr{M}$ is of odd length. Hence, there is no dimer configuration D such that $[D\Delta D_0] = 1$, and $Z_{1,D_0}(G)$ vanishes. By Proposition 2.2.5, $Z(\widetilde{G}) = Z_{0,D_0}(G)^2 = Z(G)^2$, proving the first point.

To show the second point, let us assume that $G \subset \mathscr{M}$ in invariant by a horizontal translation τ, and that for some $D_0 \in \mathcal{D}(G)$, $[D_0\Delta\tau(D_0)] = 1$. Then, τ^{-1} defines a bijection of $\mathcal{D}(G)$ onto itself with $x(\tau^{-1}(D)) = x(D)$ for all $D \in \mathcal{D}(G)$. Furthermore, we have the equalities in $\mathbb{Z}_2$:

$$[\tau^{-1}(D)\Delta D_0] = [D\Delta\tau(D_0)] = [D\Delta D_0] + [D_0\Delta\tau(D_0)] = [D\Delta D_0] + 1 .$$

In other words, τ^{-1} maps the dimer configurations contributing to $Z_{0,D_0}(G)$ to the ones contributing to $Z_{1,D_0}(G)$, and vice-versa. This implies an equality between these two quantities. The result now follows from Proposition 2.2.5 together with the trivial equality $Z(G) = Z_{0,D_0}(G) + Z_{1,D_0}(G)$. □

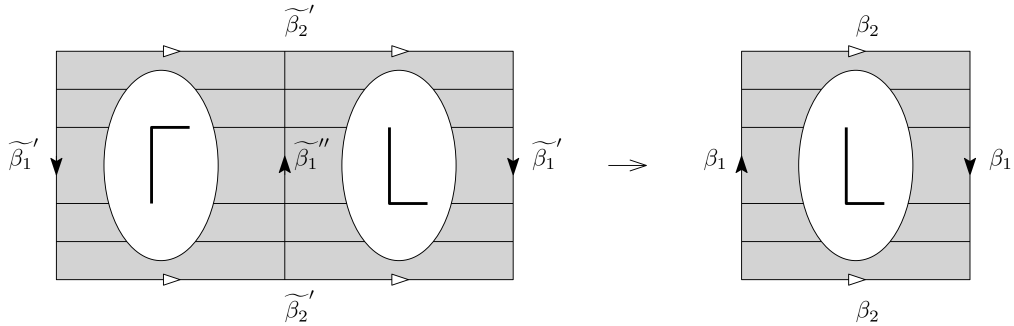


Figure 2.6: A schematic description of the orientation cover of the Klein bottle (right) by the torus (left), with explicit bases of $H_1(\mathscr{K};\mathbb{Z}_2)$ and $H_1(\mathbb{T};\mathbb{Z}_2)$.

2.3 The identities for surfaces of arbitrary genus

The aim of this section is to prove our main result in its most general form, using the results of [8]. We start this section by proving a proposition that is needed to state the main theorem. We then give the proof of the main theorem, as well as the proof of Theorem 2.1.6.

In order to state our results in the most general form, we will need the following proposition. Readers are referred to Section 1.3 for basic terminology such as homology groups, quadratic enhancements and the Stiefel-Whitney class w_1.

Proposition 2.3.1. *Let Σ be a non-orientable surface, and let $p\colon \widetilde{\Sigma} \to \Sigma$ be its orientation cover. Given any quadratic enhancement q on Σ, there exists a unique quadratic form $\widetilde{q}$ on $\widetilde{\Sigma}$ with the following properties.*

(i) If $C \subset \Sigma$ is a simple closed curve with $w_1(C) = 1$, then $\widetilde{q}(\widetilde{C}) = 0$, where $\widetilde{C} = p^{-1}(C)$.

(ii) If $C \subset \Sigma$ is a simple closed curve with $w_1(C) = 0$, then $\widetilde{q}(\widetilde{C}') = \widetilde{q}(\widetilde{C}'') = \frac{1}{2}q(C)$ with $p^{-1}(C) = \widetilde{C}' \sqcup \widetilde{C}''$.

Proof. Let us first assume that the genus h of Σ is odd. Then, Σ is homeomorphic to the connected sum of an orientable surface Σ_g of genus $g = \dfrac{h-1}{2}$ with $\mathbb{R}P^2$, so its homology splits as a direct sum $H_1(\Sigma;\mathbb{Z}_2) = H_1(\Sigma_g;\mathbb{Z}_2) \oplus H_1(\mathbb{R}P^2;\mathbb{Z}_2)$. Let us fix a collection of simple closed curves $\alpha_1, \ldots, \alpha_{h-1}$ representing a basis of $H_1(\Sigma_g;\mathbb{Z}_2)$, and a non-trivial simple closed curve $\beta \subset \mathbb{R}P^2$ (recall the polygonal representation $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}cc$ of Σ). The orientation cover $\widetilde{\Sigma}$ is obtained by gluing two copies of Σ cut along β; therefore, a basis of $H_1(\widetilde{\Sigma};\mathbb{Z}_2)$ is given by the simple closed curves $\widetilde{\alpha}_1', \widetilde{\alpha}_1'', \ldots, \widetilde{\alpha}_{h-1}', \widetilde{\alpha}_{h-1}''$. Note that $w_1(\alpha_i) = 0$ for every $1 \le i \le h-1$, and that a quadratic form is completely determined by its values on a basis, condition (ii) alone determines $\widetilde{q}$ uniquely. This $\widetilde{q}$ automatically satisfies condition (i): indeed, any cycle C with $w_1(C) = 1$ represents a homology class of the form $\alpha + \beta$ with $\alpha \in H_1(\Sigma_g;\mathbb{Z}_2)$, which lifts to $\widetilde{C}$ representing the class $\widetilde{\alpha}' + \widetilde{\alpha}'' + \widetilde{\beta}$

in $H_1(\widetilde{\Sigma};\mathbb{Z}_2)$. As the lift $\widetilde{\beta}$ represents the trivial homology class, $\widetilde{q}(\widetilde{C})$ vanishes since $\widetilde{q}(\widetilde{\alpha}')=\widetilde{q}(\widetilde{\alpha}'')$.

If h is even, then Σ is homeomorphic to the connected sum of an orientable surface Σ_g of genus $g=\dfrac{h}{2}-1$ with the Klein bottle $\mathscr{K}$, so $H_1(\Sigma;\mathbb{Z}_2)$ splits as $H_1(\Sigma_g;\mathbb{Z}_2)\oplus H_1(\mathscr{K};\mathbb{Z}_2)$ and one can fix a collection of simple closed curves $\alpha_1,\dots,\alpha_{h-2}$ representing a basis of $H_1(\Sigma_g;\mathbb{Z}_2)$ together with simple closed curve β_1,β_2 representing a basis of $H_1(\mathscr{K};\mathbb{Z}_2)$ with $\beta_1\cdot\beta_2=1$, $w_1(\beta_1)=0$ and $w_1(\beta_2)=1$ (recall the polygonal representation $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}abab^{-1}$ of Σ). Then, $\widetilde{\Sigma}$ is obtained by gluing two copies of Σ cut along β_1, so a basis of $H_1(\widetilde{\Sigma};\mathbb{Z}_2)$ is given by $\widetilde{\alpha}_1',\widetilde{\alpha}_1'',\dots,\widetilde{\alpha}_{h-2}',\widetilde{\alpha}_{h-2}'',\widetilde{\beta}_1',\widetilde{\beta}_2$. (Note that $\widetilde{\beta}_1'$ and $\widetilde{\beta}_1''$ represent the same homology class, see Figure 2.6.) Therefore, conditions (i) and (ii) uniquely determine $\widetilde{q}$. □

Let G be a weighted graph embedded in a surface Σ. Given any $\alpha\in H_1(\Sigma;\mathbb{Z}_2)$ and $D_0\in\mathcal{D}(G)$, let us define

$$Z_{\alpha,D_0}(G)=\sum_{[D\Delta D_0]=\alpha}x(D)\,,$$

where the sum is over all $D\in\mathcal{D}(G)$ such that the homology class of the symmetric difference $D\Delta D_0$ is equal to α. With $\widetilde{q}$ determined by Proposition 2.3.1, our main theorem can be stated as follows.

Theorem 2.3.2. *Let G be a weighted graph with an even number of vertices, embedded in a non-orientable surface Σ in such a way that its complement consists of topological discs, and let $\widetilde{G}\subset\widetilde{\Sigma}$ denote the weighted graph obtained by lifting G via the orientation cover $\widetilde{\Sigma}\to\Sigma$. Then, for any quadratic enhancement q on Σ and any $D_0\in\mathcal{D}(G)$, $\widetilde{D}_0\in\mathcal{D}(\widetilde{G})$, we have*

$$\Big|\sum_{\widetilde{\alpha}}(-1)^{\widetilde{q}(\widetilde{\alpha})}Z_{\widetilde{\alpha},\widetilde{D}_0}(\widetilde{G})\Big|=\Big(\sum_{w_1(\alpha)=0}(-1)^{\frac{q(\alpha)}{2}}Z_{\alpha,D_0}(G)\Big)^2+\Big(\sum_{w_1(\alpha)=1}(-1)^{\frac{q(\alpha)-1}{2}}Z_{\alpha,D_0}(G)\Big)^2,$$

where the first sum is over all $\widetilde{\alpha}\in H_1(\widetilde{\Sigma};\mathbb{Z}_2)$, the second (resp. third) over all $\alpha\in H_1(\Sigma;\mathbb{Z}_2)$ with $w_1(\alpha)=0$ (resp. $w_1(\alpha)=1$). Moreover, neither side of this equality depends on the choice of D_0 and $\widetilde{D}_0$.

Remark 2.3.3. Recall that if Σ is of (non-orientable) genus $h\geq 1$, then there are 2^h quadratic enhancements on it, and $\widetilde{\Sigma}$ is of (orientable) genus $g=h-1$. Furthermore, one easily sees from Proposition 2.3.1 that the map $q\mapsto\widetilde{q}$ is two-to-one: indeed, picking a basis of $H_1(\Sigma;\mathbb{Z}_2)$ with w_1 nonvanising on a single element, $\widetilde{q}$ is not affected by changing the value of q on this element. Moreover the right-hand side of the equality in Theorem 2.3.2 is also left unchanged by this transformation. Indeed, let us choose a basis $\{\alpha_1,\dots,\alpha_h\}$ of $H_1(\Sigma;\mathbb{Z}_2)$ with $w_1(\alpha_1)=1$ and $w_1(\alpha_i)=0$ for $2\leq i\leq h$. Let q and q' be two quadratic enhancements on Σ which coincide on α_i for $i\geq 2$ and differ on α_1. As $q'(\alpha_1)$ and $q(\alpha_1)$

have the same parity as $w_1(\alpha_1)$, we get $q'(\alpha_1) = q(\alpha_1) + 2$. Then by definition of quadratic enhancements, one can check easily that for every $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$,

$$q'(\alpha) = \begin{cases} q(\alpha) & \text{if } w_1(\alpha) = 0; \\ q(\alpha) + 2 & \text{if } w_1(\alpha) = 1. \end{cases}$$

It follows that the right-hand side of the equality in Theorem 2.3.2 is the same for q as for q'. Therefore we actually have exactly $2^{h-1} = 2^g$ distinct equalities.

Example 2.3.4. Let us consider the case where Σ is of genus $h = 1$, i.e. the projective plane $\mathbb{R}P^2$, whose orientation cover $\widetilde{\Sigma}$ is the sphere. There are two quadratic enhancements on $\mathbb{R}P^2$, assigning the value $q(\alpha) \in \{1, 3\}$ to the unique non-trivial class $\alpha = 1$ (and the value $q(\alpha) = 0$ to the trivial class $\alpha = 0$), and a single (trivial) quadratic form on the sphere. Therefore, in this case, Theorem 2.3.2 gives a single equality, namely

$$Z(\widetilde{G}) = Z_{0,D_0}(G)^2 + Z_{1,D_0}(G)^2 \,.$$

Removing a small disk from Σ and its lifts from $\widetilde{\Sigma}$, we obtain the 2-fold cover $\mathscr{C} \to \mathscr{M}$ considered in Section 2.2. Hence, the equality above is exactly Proposition 2.2.5, of which an elementary proof was given in Section 2.2.

Example 2.3.5. Let us now consider the case of genus $h = 2$, i.e. the Klein bottle $\mathscr{K}$ whose orientation cover is the torus $\mathbb{T}$. As usual, fix a basis of $H_1(\mathscr{K}; \mathbb{Z}_2)$ given by two simple closed curves β_1, β_2 with $\beta_1 \cdot \beta_2 = 1$, $w_1(\beta_1) = 0$ and $w_1(\beta_2) = 1$, which determines a basis $\widetilde{\beta}'_1, \widetilde{\beta}_2$ of $H_1(\mathbb{T}; \mathbb{Z}_2)$, as described in Figure 2.6. There are 4 quadratic enhancements on $\mathscr{K}$, depending on the values of $q(\beta_1) \in \{0, 2\}$ and of $q(\beta_2) \in \{1, 3\}$. By Proposition 2.3.1, the corresponding quadratic form $\widetilde{q}$ on $\mathbb{T}$ vanishes on $\widetilde{\beta}_2$ and takes the value $\dfrac{q(\beta_1)}{2}$ on $\widetilde{\beta}'_1$. Writing ϵ for $(-1)^{\frac{q(\beta_1)}{2}}$, using the equalities $q(\beta_1 + \beta_2) = q(\beta_1) + q(\beta_2) + 2$ and $\widetilde{q}(\widetilde{\beta}'_1 + \widetilde{\beta}_2) = \widetilde{q}(\widetilde{\beta}'_1) + 1$, and leaving $D_0, \widetilde{D}_0$ implicit, we obtain the two identities

$$|Z_{00}(\widetilde{G}) + Z_{01}(\widetilde{G}) + \epsilon Z_{10}(\widetilde{G}) - \epsilon Z_{11}(\widetilde{G})| = (Z_{00}(G) + \epsilon Z_{10}(G))^2 + (Z_{01}(G) - \epsilon Z_{11}(G))^2 \tag{2.10}$$

for $\epsilon = \pm 1$. The proof of Theorem 2.1.6 is now straightforward, as explained below.

Proof of Theorem 2.1.6. Since $G \subset \mathscr{K}$ is locally bipartite, the map assigning to each cycle its length induces a well-defined linear map $\ell\colon H_1(\mathscr{K}; \mathbb{Z}_2) \to \mathbb{Z}_2$, which is not identically zero since G is not bipartite. Note that $D \Delta D_0$ consists of cycles of even lengths, so $Z_\alpha(G)$ vanishes as soon as $\ell(\alpha) = 1$. By assumption, there exists $D_0 \in \mathcal{D}(G)$ such that the homology class of $D_0 \Delta \tau(D_0)$ is β_2 or $\beta_1 + \beta_2$. Let us first assume that $[D_0 \Delta \tau(D_0)] = \beta_2$. Since $D_0 \Delta \tau(D_0)$ is of even length, we then have $\ell(\beta_2) = 0$; and as l is not identically zero we get $\ell(\beta_1) = 1$ and $\ell(\beta_1 + \beta_2) = 1$. This implies that $Z_{10}(G)$ vanishes, as well as $Z_{11}(G)$. Using the length function $\widetilde{l}$ on the torus which is determined by ℓ, we find that $\widetilde{l}(\widetilde{\beta}'_1) = 1$, $\widetilde{l}(\widetilde{\beta}_2) = 0$ and $\widetilde{l}(\widetilde{\beta}'_1 + \widetilde{\beta}_2) = 1$, leading to $Z_{10}(\widetilde{G}) = Z_{11}(\widetilde{G}) = 0$. Hence by Equation (2.10) we get

$$Z(\widetilde{G}) = Z_{00}(\widetilde{G}) + Z_{01}(\widetilde{G}) = Z_{00}(G)^2 + Z_{01}(G)^2.$$

The facts that G is invariant by a horizontal translation τ and that $[D_0 \Delta \tau(D_0)] = \beta_2$ allows us to construct a weight-preserving bijection between the dimer configurations contributing to $Z_{00}(G)$ and those contributing to $Z_{01}(G)$, as in the proof of Theorem 2.1.1. The result follows.

The case where $[D_0 \Delta \tau(D_0)] = \beta_1 + \beta_2$ can be treated similarly: in this case we have $l(\beta_1 + \beta_2) = 0$ and $l(\beta_1) = l(\beta_2) = 1$, as well as $\widetilde{l}(\widetilde{\beta}_2) = 0$ and $\widetilde{l}(\widetilde{\beta}_1') = \widetilde{l}(\widetilde{\beta}_1' + \widetilde{\beta}_2) = 1$. This leads to $Z_{10}(G) = Z_{01}(G) = 0$ and $Z_{10}(\widetilde{G}) = Z_{11}(\widetilde{G}) = 0$, so Equation (2.10) boils down to

$$Z(\widetilde{G}) = Z_{00}(G)^2 + Z_{11}(G)^2.$$

Again by assumption, there is a weight-preserving bijection between the dimer configurations contributing to $Z_{00}(G)$ and those contributing to $Z_{11}(G)$. The result now follows. $\square$

Remark 2.3.6. Obviously, other formulas relating $Z(\widetilde{G})$ to $Z(G)$ can be derived from Theorem 2.3.2. For example, using Equation (2.10) and the notation in the discussion of this $h = 2$ case, one can show the following statement: *If $G \subset \mathscr{K}$ is locally bipartite and such that $\ell(\beta_1) = 0$, $\ell(\beta_2) = 1$ and $Z_{10}(\widetilde{G}) = 0$ or $Z_{11}(\widetilde{G}) = 0$, then $Z(\widetilde{G}) = Z(G)^2$.* However, such a statement is not as satisfactory as Theorems 2.1.1 and 2.1.6 above, as it does assume a vanishing condition that is difficult to obtain as a consequence of a natural property of the graph. Note also that the number of such conditions will grow exponentially with the genus h of Σ. For this reason, we do not expect to obtain very compelling corollaries in higher genus. Of course, Theorem 2.3.2 might nevertheless explain other "curious identities" obtained for specific examples.

We conclude this chapter with the proof of our main result.

Proof of Theorem 2.3.2. Fix a weighted graph G with an even number of vertices, embedded in a non-orientable surface Σ so that $\Sigma \setminus G$ consists of topological discs. Note that using a suitable polygonal representation of the surface, one can always find a simple closed curve β, transverse to the graph G, so that the orientation cover $\widetilde{\Sigma}$ of Σ can be constructed by gluing along their boundary two copies of Σ cut along β. This determines a map ω on the edges of G by setting $\omega(e) = 1$ if e intersects β, and $\omega(e) = 0$ else. With such a data in hand, any orientation K on the edges of G induces a modified weighted skew-adjacency matrix $A^{K,\omega}(G)$ as defined by Equation (1.5).

Let us now recall the right notion of Kasteleyn orientations in this setting (cf. subsection 1.4.3). Consider the weighted graph $\widetilde{G} \subset \widetilde{\Sigma}$ obtained taking two copies of $G \subset \Sigma$ cut along the edges with $\omega(e) = 1$, and glued back again along these edges, each such edge joining the two copies. Given an orientation K on G, let $\widetilde{K}$ be the orientation on $\widetilde{G}$ obtained by lifting K to $\widetilde{G}$, and inverting it on all edges of $\widetilde{G}$ completely contained in one of the copies of Σ cut along β. Then K is a Kasteleyn orientation on $(G \subset \Sigma, \omega)$ if $\widetilde{K}$ is a Kasteleyn orientation on $\widetilde{G} \subset \widetilde{\Sigma}$, i.e: for any face $\widetilde{f}$ of $\widetilde{G} \subset \widetilde{\Sigma}$, $n^{\widetilde{K}}(\partial \widetilde{f})$ is odd, where the boundary of $\widetilde{f}$ is oriented according to a fixed orientation on the surface $\widetilde{\Sigma}$.

By Proposition 1.4.4, $(G \subset \Sigma, \omega)$ always admits a Kasteleyn orientation since G is assumed to have an even number of vertices. As proved in [8, p.174], if K is such an orientation, then for any $D_0 \in \mathcal{D}(G)$, we have

$$i^{-\omega(D_0)}\,\epsilon^K(D_0)\,\mathrm{Pf}(A^{K,\omega}(G)) = \sum_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} i^{-q(\alpha)} Z_{\alpha,D_0}(G)\,,$$

where $q = q^{K,\omega}_{D_0}$ is the quadratic enhancement on Σ associated with K and D_0 mentioned in Theorem 1.4.5. Also by this theorem, any quadratic enhancement on Σ can be obtained in such a way by varying the orientation K. Applying the $\omega = 0$ case of this formula to $\widetilde{G} \subset \widetilde{\Sigma}$, we get

$$\epsilon^{\widetilde{K}}(\widetilde{D}_0)\,\mathrm{Pf}(A^{\widetilde{K}}(\widetilde{G})) = \sum_{\widetilde{\alpha}\in H_1(\widetilde{\Sigma};\mathbb{Z}_2)} (-1)^{\widetilde{q}(\widetilde{\alpha})} Z_{\widetilde{\alpha},\widetilde{D}_0}(\widetilde{G})\,,$$

where $\widetilde{q} = \widetilde{q}^{\widetilde{K}}_{\widetilde{D}_0}$ is a quadratic form on $\widetilde{\Sigma}$. The proof of Lemma 2.2.4 can be applied *verbatim*, leading to $|\mathrm{Pf}(A^{\widetilde{K}}(\widetilde{G}))| = |\mathrm{Pf}(A^{K,\omega}(G))|^2$. This equality, together with the two equations displayed above, and the fact that a quadratic enhancement always satisfies $q(\alpha) - w_1(\alpha) \in 2\mathbb{Z}_2$, lead to

$$\begin{aligned}\Big|\sum_{\widetilde{\alpha}}(-1)^{\widetilde{q}(\widetilde{\alpha})} Z_{\widetilde{\alpha},\widetilde{D}_0}(\widetilde{G})\Big| &= |\mathrm{Pf}(A^{\widetilde{K}}(\widetilde{G}))| = |\mathrm{Pf}(A^{K,\omega}(G))|^2 \\ &= (\mathrm{Re}\,\mathrm{Pf}(A^{K,\omega}(G)))^2 + (\mathrm{Im}\,\mathrm{Pf}(A^{K,\omega}(G)))^2 \\ &= \Big(\sum_{w_1(\alpha)=0}(-1)^{\frac{q(\alpha)}{2}} Z_{\alpha,D_0}(G)\Big)^2 + \Big(\sum_{w_1(\alpha)=1}(-1)^{\frac{q(\alpha)-1}{2}} Z_{\alpha,D_0}(G)\Big)^2.\end{aligned} \tag{2.11}$$

Also, by these equalities it is obvious that neither side of Equation (2.11) depends on the choice of D_0 and $\widetilde{D}_0$.

Therefore, we are left with the proof that the map $q = q^{K,\omega}_{D_0} \mapsto \widetilde{q}^{\widetilde{K}}_{\widetilde{D}_0} = \widetilde{q}$ corresponds to the map $q \mapsto \widetilde{q}$ described in Proposition 2.3.1. To do so, let us assume without loss of generality that $\widetilde{D}_0$ is the lift of D_0. By Theorem 1.4.6, $\widetilde{q}$ is characterised by the following property: if $\widetilde{C}$ is an oriented simple closed curve in $\widetilde{G} \subset \widetilde{\Sigma}$, then

$$\widetilde{q}(\widetilde{C}) = n^{\widetilde{K}}(\widetilde{C}) + \ell_{\widetilde{D}_0}(\widetilde{C}) + 1\,,$$

where $\ell_{\widetilde{D}_0}$ denotes the number of vertices of $\widetilde{C}$ whose adjacent dimer of $\widetilde{D}_0$ sticks out to the left of $\widetilde{C}$. (Recall that both $\widetilde{C}$ and $\widetilde{\Sigma}$ are oriented.) To check the first condition of Proposition 2.3.1, let us apply this formula to the lift of an oriented simple closed curve C in $G \subset \Sigma$ with $w_1(C) = 1$. Writing C^0 (resp. C^1) for the edges of C with $\omega(e) = 0$ (resp. $\omega(e) = 1$), and computing modulo 2, by definition of $\widetilde{K}$ we get

$$n^{\widetilde{K}}(\widetilde{C}) = n^K(C^0) + n^K(-C^0) + 2n^K(C^1) = |C^0| = |C| + w_1(C) = |C| + 1\,.$$

Furthermore, $\ell_{\widetilde{D}_0}(\widetilde{C})$ is given by the number of vertices of C whose adjacent dimer of D_0 sticks out of C. Since this number has the same parity as $|C|$, we have $\widetilde{q}(\widetilde{C}) = 0$ as expected.

To show that $\widetilde{q}$ satisfies the second condition of Proposition 2.3.1, first note that we can assume C to be contained in one copy of Σ cut along β. In such a case, Theorem 1.4.5 gives the equality modulo 4

$$q(C) = 2(n^K(C) + \ell_{D_0}(C) + 1)\,,$$

where the vertices contributing to $\ell_{D_0}(C)$ can be counted using an orientation on one of the copies of Σ cut along β. By the value of $\widetilde{q}(\widetilde{C}')$ displayed above, the second condition is verified, and the proof complete. $\square$

Chapter 3

A Pfaffian formula for the monomer-dimer model

This chapter is devoted to studying the monomer-dimer model on graphs embedded in surfaces with boundary. More specifically, if monomers are only allowed to appear on the surface boundary, one can show a Pfaffian formula for the corresponding partition function, which generalises the one obtained by Giuliani, Jauslin and Lieb for graphs embedded in the disk [24]. Our proof is based on an extension of a bijective method mentioned in [24], together with the Pfaffian formula for the dimer partition function of Cimasoni-Reshetikhin [14] reviewed in the first chapter. This chapter is the content of our paper [55].

3.1 Introduction

The monomer-dimer model is one of the important models in both statistical physics and theoretical computer science. As a classical lattice model, it was first introduced to describe the absorption of a mixture of unequal-size molecules on crystal surfaces [21]. A brief history of the study of this model in statistical mechanics, and some fundamental results can be found in [26, 27]. Moreover, the impact and the interest of the monomer-dimer model goes beyond physics: indeed the monomer-dimer problem represents a prototypical problem in computational complexity theory [23].

Let us recall the definition of the monomer-dimer model. For G a finite graph with vertex set $V(G)$ and edge set $E(G)$, a monomer-dimer covering, or shortly, an MD covering of G is a pair $\tau = (\tau_D, \tau_M) \in E(G) \times V(G)$ consisting of a set τ_D of dimers and a set τ_M of monomers so that each vertex of G is covered by exactly one dimer or one monomer. Recall that $\mathcal{MD}(G)$ denotes the set of all MD coverings of G, and when monomers are not allowed to appear, this set reduces to the set $\mathcal{D}(G)$ of dimer coverings of G. Also if G is endowed with an edge weight system $x = (x_e)_{e\in E(G)}$ and a vertex weight system $y = (y_v)_{v\in V(G)}$, then the monomer-dimer partition function (or simply the MD partition function) of the weighted graph (G, x, y) is defined by

$$Z_{\mathcal{MD}}(G) := Z_{\mathcal{MD}}(G, x, y) = \sum_{\tau \in \mathcal{MD}(G)} \prod_{e \in \tau_D} x_e \prod_{v \in \tau_M} y_v.$$

Note that if all the vertex weights $y = (y_v)_{v \in V(G)}$ are equal to 0, which means that monomers are not allowed to appear, then we get the dimer partition function of the edge-weighted graph (G, x).

A classical problem associated to the monomer-dimer model on a given weighted graph G is to compute its partition function $Z_{\mathcal{MD}}(G)$. Unfortunately, it turns out that this computation, even for planar graphs, is intractable [33]. More precisely, it is proved in [33] that computing the number of matchings (and so computing the number of MD coverings) for planar graphs is "#P complete". Nevertheless, there are some particular cases in which we can compute the MD partition function efficiently. Let us now discuss these cases.

If monomers are not allowed to appear, or if they are fixed, computing the monomer-dimer partition function boils down to computing the dimer partition function of some resulting graph. For the latter, as already reviewed in the first chapter, one can use the Pfaffian formula if the resulting graph is embedded in a surface. Furthermore, if monomers are allowed to appear, and if one restricts their locations, similar Pfaffian formulas for MD partition functions can be obtained in some cases [61, 66, 56, 67, 24]. To the best of our knowledge, the most general Pfaffian formula for the MD partition function was obtained only recently in the paper [24] of Giuliani, Jauslin and Lieb in which the authors consider graphs embedded in the disk with monomers restricted on the boundary of this disk. There are two methods to prove their Pfaffian formula. One method is to define a suitable matrix associated to a well-chosen orientation on the edges of the graph together with a good labelling of its vertices so that the Pfaffian of this matrix is equal to the MD partition function (see [24] for more details). Another method is based on a one-to-two mapping from the set of MD coverings of the original graph to the set of dimer coverings of an auxiliary planar graph, and the application of the Pfaffian formula for the dimer partition function to this auxiliary graph (see [24, Appendix E]).

The aim of the present chapter is to generalise the Pfaffian formula obtained by Giuliani, Jauslin and Lieb to graphs embedded in surfaces of arbitrary topology with any number of boundary components, where monomers are still restricted on the surface boundary. Roughly speaking, if G is embedded in an orientable surface of genus g with b boundary components, the MD partition function of G (with monomers located only on the boundary) is given by

$$Z_{\mathcal{MD}}(G) = \frac{1}{2^g} \left| \sum_K \pm \mathrm{Pf}(M^K(G)) \right|.$$

In this formula, the sum is over 2^{2g+b-1} well-chosen orientations on G, and $M^K(G)$ is a modified adjacency matrix of G with respect to the orientation K (see Definition 3.2.1 below). The more precise statement of our formula can be found in Theorem 4.2.3. As a consequence, if monomers are not allowed to appear, we get back the general Pfaffian formula for the dimer partition function obtained in [8] (cf. Theorem 1.4.8). Also, if $g = 0$ and $b = 1$ we get back the Pfaffian formula for the MD partition function given in [24].

Before concluding this introduction, let us give the idea of the proof of our formula. In short, by extending the bijective method described in [24, Appendix E] to graphs embedded in surfaces with boundary, we can define an auxiliary surface graph G_β for each $\beta = (\beta_1, \dots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even with the following property: the dimer partition function of G_β can be related to a partial MD partition function of G with the parity of the number of monomers on each boundary component given by β. The former quantity then can be computed using the Pfaffian formula for the dimer partition function given in [8]. It follows that the MD partition function of G can be given by a linear sum of the Pfaffian of skew-adjacency matrices associated to G_β. Furthermore, one can show that the latter can be related to the Pfaffian of modified adjacency matrices of G mentioned above, and hence we end up with the formula as stated.

This chapter is organised as follows. In Section 3.2 we define modified adjacency matrices associated to graphs embedded in orientable surfaces with boundary. We also describe particular orientations on edges of graphs together with a specific labelling of vertices. Finally we state our general Pfaffian formula for the monomer-dimer partition function (Theorem 4.2.3). Section 3.3 is devoted to give some preliminaries (subsection 3.3.1) before proving our formula (subsection 3.3.2). We conclude this chapter with the case of graphs embedded in non-orientable surfaces (Section 3.4).

3.2 Definitions and statement of the main result

In this section we will fix the setting for the rest of this chapter, and introduce some basic definitions and notations that are necessary to state our result. We then give our Pfaffian formula for the MD partition function together with an example at the end of the section.

3.2.1 Setting

Throughout this chapter, we shall assume that G is a finite connected graph endowed with an edge weight system $x = (x_e)_{e \in E(G)}$ and a vertex weight system $y = (y_v)_{v \in V(G)}$. Note that G is allowed to have multiple edges, while loops can always be removed since they do not play any role in our model. We also assume that G is embedded in an orientable surface Σ of genus g with boundary $\partial\Sigma$ consisting of b components. Let us define the boundary ∂G of G as the subgraph of G consisting of vertices and edges that can be connected to $\partial\Sigma$ by a path without crossing any other edge of G (cf. [24]).

Following [24], one can make the following assumptions without any loss of generality. First of all, by adding suitable edges of weights 0 and by deforming the boundary of Σ if needed, we can assume that ∂G consists of b circuits $B_1, \dots, B_b$ which coincide with the b components $C_1, \dots, C_b$ of $\partial\Sigma$ respectively (see Figure 3.1 when $g = 1$, $b = 1$). This means that one can travel along each boundary component of G (as well as that of Σ) visiting each vertex in this component exactly once. We refer the reader to [24, Section 4.1] for an

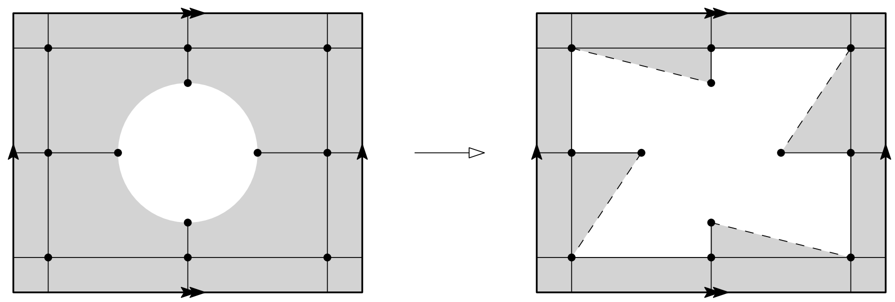


Figure 3.1: Add suitable (dashed) edges and deform $\partial\Sigma$ to get a boundary circuit.

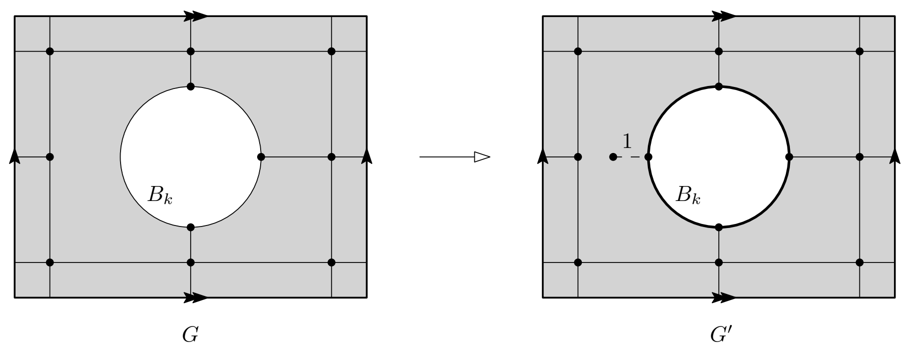


Figure 3.2: Add an edge (in dashed line) to the boundary circuit B_k of G to get the graph G' with the new boundary circuit B'_k (in heavily bolded arcs).

explicit algorithm to construct a boundary circuit for any planar graph, which is also valid for any graph embedded in surfaces with boundary. Note that by this first assumption, we can identify ∂G with $\partial\Sigma$, and hence the MD partition function with monomers restricted to $\partial\Sigma$ coincides with the "boundary MD partition function" defined in [24] for $g = 0$ and $b = 1$. For this reason we shall also call the former one (which we want to compute) the boundary MD partition function, and still denote it by $Z_{\mathcal{MD}}(G)$. Let us now continue with the second assumption. Denoting by N_k the number of vertices of G on B_k for $1 \leq k \leq b$, one can assume that N_k is even for every k. Indeed, if N_k is odd for some k, let G' be the graph obtained by transforming G as in Figure 3.2: add an edge of weight 1 with one endpoint on B_k and the other one in the interior of Σ. This added edge must be occupied by every MD covering of G' whose monomers are restricted to $\partial G'$. Then one can verify easily that $Z_{\mathcal{MD}}(G) = Z_{\mathcal{MD}}(G')$, and that G' has an even number of vertices on its newly-created boundary circuit B'_k. Finally, we can assume that $|V(G)|$ is even: otherwise, let G'' be the surface graph obtained by adding a vertex of weight 1 to an arbitrary face of G, and by connecting this vertex to all the vertices on the boundary of this face by edges of weights 0; then it is clear that $Z_{\mathcal{MD}}(G) = Z_{\mathcal{MD}}(G'')$, and that $|V(G'')|$ is even. To sum up, from now on G will be assumed to have an even total number of vertices, as well as an even number of vertices on each of its boundary circuit.

For further purpose, let us denote by $\overline{\Sigma}$ the closed surface of genus g obtained from Σ by gluing a topological 2-disc D_k along each boundary component C_k for every $1 \leq k \leq b$.

3.2.2 Modified adjacency matrices

We now define modified adjacency matrices of the graph G with respect to some labelling of its vertices and some orientation K on its edges. These matrices are actually inspired by the work of Giuliani, Jauslin and Lieb (cf. [24, Theorem 1.1]), however we slightly adjust their definition, and extend it to the general case when G has more than one boundary circuit. More precisely, our matrices, which are parametrised by elements of $\mathbb{Z}_2^b$, are defined as follows.

Definition 3.2.1. *Let (G, x, y) be a weighted graph and K an orientation on its edges. Given $\beta = (\beta_1, \ldots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even, add an isolated vertex v_k of weight 1 for each k such that $\beta_k = 1$. Labelling the vertices of G together with all such v_k's by a number set $I \subset \mathbb{N}$, we define the* modified adjacency matrix $M_\beta^K(G) = (m_{ij}^\beta)_{i,j\in I}$ *of G as the skew-symmetric matrix whose entries are given by*

$$m_{ij}^\beta = \sum_e \epsilon_{ij}^K(e) x_e + (-1)^{i+j} d_{ij} y_i y_j \quad \text{for } i < j,$$

where the sum is taken over all the edge e between two vertices i, j while $\epsilon_{ij}^K(e)$ is equal to 1 if e is oriented by K from i to j, and equal to -1 otherwise. Moreover we define

$$d_{ij} = \begin{cases} 1 & \text{if } i, j \text{ belong to the same boundary component;} \\ 0 & \text{if not,} \end{cases}$$

where the "virtual vertex" v_k, by convention, is always considered to be on the boundary component B_k.

Note that by definition, $M_\beta^K(G)$ does not only depend on the parameter β but also on the orientation K and on the way we label the vertices of G (and the v_k's). We now describe a specific labelling and an orientation of interest.

Let us begin with a specific labelling of the vertices of G. Recall that G is embedded in the orientable surface Σ with boundary $\partial\Sigma$ consisting of b circuits B_k's. Then endowing Σ with an orientation (pictured counterclockwise) induces a natural orientation (pictured clockwise) on each B_k (cf. Figure 3.3). For each parameter β, we label the vertices on each B_k (together with v_k if $\beta_k = 1$) by increasing consecutive numbers so that we see these vertices in that order when we travel along B_k with its inverse orientation (and then v_k if $\beta_k = 1$). This labelling is illustrated in Figure 3.3. Note that when travelling along B_k, the starting vertex is not important. The vertices of G that do not belong to any B_k then can be labelled in an arbitrary way.

We now continue by describing a specific orientation K of interest. Recall that the orientable surface Σ is endowed with a pictured-counterclockwise orientation which also induces an orientation on each face f of G, as well as its boundary ∂f. Firstly, we require K to be Kasteleyn (i.e. for each face f of G, the number $n^K(\partial f)$ of edges on its boundary where the orientation of ∂f is different from K is odd). Secondly, for each $\beta = (\beta_1, \ldots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even, we want K to be specific on B_k for $1 \leq k \leq b$ as follows: if $\beta_k = 0$, then

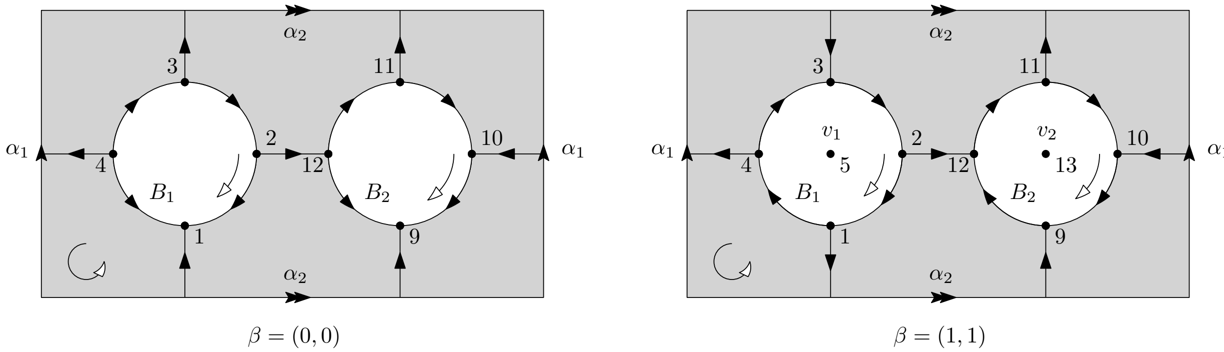


Figure 3.3: Labellings and orientations of interest for $b = 2$. $\gamma_1 = (1, 4, 3)$ and $\gamma_2 = (4, 3, 2, 12, 11, 10)$ are oriented cycles having α_1 and α_2 on their immediate left.

K is from big-labelled vertices to small-labelled vertices; if $\beta_k = 1$ then K coincides with the natural orientation induced on B_k (see Figure 3.3). Let us denote by $\mathcal{O}_\beta$ the set of all orientations which are Kasteleyn and satisfy the above condition. For $\beta = (\beta_1, \ldots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even, the set $\mathcal{O}_\beta$ (which will be shown to be nonempty in Lemma 3.3.3) consists of the orientations we are interested in.

3.2.3 Main result

There are two versions of our Pfaffian formula. To state the version which is useful for computational purposes, we need to choose a specific orientation in each $\mathcal{O}_\beta$, that can be described as follows. Let $\{\alpha_i\}_{1\le i\le 2g}$ be a set of closed curves that are transverse to G and avoid ∂G so that their homology classes $\{[\alpha_i]\}_{1\le i\le 2g}$ form a basis of $H_1(\overline{\Sigma}; \mathbb{Z}_2)$, the first homology group over $\mathbb{Z}_2$ of $\overline{\Sigma}$. For each $1 \le i \le 2g$, let γ_i be an oriented cycle of G having α_i on its immediate left, and choose $K_\beta \in \mathcal{O}_\beta$ so that $n^{K_\beta}(\gamma_i)$ is odd for every $1 \le i \le 2g$ (see Figure 3.3 for example).

However, in general it is not obvious that there exists K_β as required for each $\beta = (\beta_1, \ldots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even. Therefore, the proof of this fact will be postponed until the next section (see Lemma 3.3.3). With this specific K_β and with each $\epsilon = (\epsilon_1, \ldots, \epsilon_{2g}) \in \mathbb{Z}_2^{2g}$, let us denote by K_β^ϵ the orientation obtained by inverting K_β on every edge e each time e crosses α_i with $\epsilon_i = 1$. Then one can state our formula as follows.

Theorem 3.2.2. *Let (G, x, y) be a weighted graph embedded in an orientable surface Σ of genus g with b boundary components. Then the boundary monomer-dimer partition function of G is given by*

$$Z_{\mathcal{MD}}(G) = \frac{1}{2^g} \sum_{\substack{\beta\in\mathbb{Z}_2^b \\ \sum_k \beta_k \text{ even}}} \left| \sum_{\epsilon\in\mathbb{Z}_2^{2g}} (-1)^{\sum_{i<j} \epsilon_i\epsilon_j \alpha_i\cdot\alpha_j} Pf(M_\beta^{K_\beta^\epsilon}(G)) \right|.$$

In this formula the first sum is over all $\beta = (\beta_1, \ldots, \beta_b) \in \mathbb{Z}_2^b$ such that $\sum_{k=1}^b \beta_k$ is even, while in the second sum, $\alpha_i \cdot \alpha_j$ denotes the intersection number of α_i and α_j.

Remark 3.2.3. Note that by our formula, the number of Pfaffians is equal to 2^{2g+b-1}, which is exactly the order of $H_1(\Sigma;\mathbb{Z}_2)$. In particular, when $g=0$ and $b=1$, the MD partition function is given by a single Pfaffian: this is precisely the formula stated in [24]. Additionally, if $b=0$, i.e. Σ is a closed surface, then the modified adjacency matrices reduce to the usual adjacency matrices (recall Equation (1.1)), and we get back the Pfaffian formula for the dimer partition function (Theorem 1.4.8).

Let us now move on to the second version of our Pfaffian formula. We denote by $\mathcal{S}$ the set of all $\beta=(\beta_1,\dots,\beta_b)\in\mathbb{Z}_2^b$ with $\sum_{k=1}^b\beta_k$ even so that there exists a matching D_β of G that covers an even (resp. odd) number of vertices on B_k with $\beta_k=0$ (resp. $\beta_k=1$) for every $1\le k\le b$. Fixing such a D_β, one can show that each Kasteleyn orientation K on G corresponds to a quadratic form $q^K_{D_\beta}$ on $\overline{\Sigma}$ that can be characterised by only D_β and K (see the proof of Theorem 3.2.4 below). Recalling that $\mathrm{Arf}(q^K_{D_\beta})$ denotes the Arf invariant of this quadratic form, we have the following result.

Theorem 3.2.4. *Let (G,x,y) be a weighted graph embedded in an orientable surface Σ of genus g with b boundary components. Then the boundary monomer-dimer partition function of G is given by*

$$Z_{\mathcal{MD}}(G)=\frac{1}{2^g}\sum_{\beta\in\mathcal{S}}\sum_{[K]:K\in\mathcal{O}_\beta}(-1)^{Arf(q^K_{D_\beta})}Pf(M^K_\beta(G)).$$

In this formula, the second sum is taken over well-chosen representatives of all the equivalence classes of Kasteleyn orientations on G with aforementioned specific restrictions on ∂G.

The proof of Theorem 3.2.2 and 3.2.4 can be found in subsection 3.3.2. Let us now conclude this section by showing how the formula stated in Theorem 3.2.2 works in a concrete example.

Example 3.2.5. Let G be the weighted graph embedded in the annulus (i.e. $g=0$ and $b=2$) illustrated in Figure 3.4. Assume that the four vertices of G are weighted a,b,c,d while its edges are all weighted 1. Following the argument after Definition 3.2.1, we label the vertices of G corresponding to two parameters $(0,0)$ and $(1,1)$ and choose the specific orientations $K_{0,0}$ and $K_{1,1}$ as in Figure 3.4. Then by Theorem 3.2.2 we have

$$Z_{\mathcal{MD}}(G)=\left|\mathrm{Pf}(M_{0,0}^{K_{0,0}})+\mathrm{Pf}(M_{1,1}^{K_{1,1}})\right|$$

where the two matrices $M_{0,0}^{K_{0,0}}$ and $M_{1,1}^{K_{1,1}}$ are given by

$$M_{0,0}^{K_{0,0}}=\begin{pmatrix}0 & -2-bc & -1 & 0\\ 2+bc & 0 & 0 & 1\\ 1 & 0 & 0 & -2-ad\\ 0 & -1 & 2+ad & 0\end{pmatrix}$$

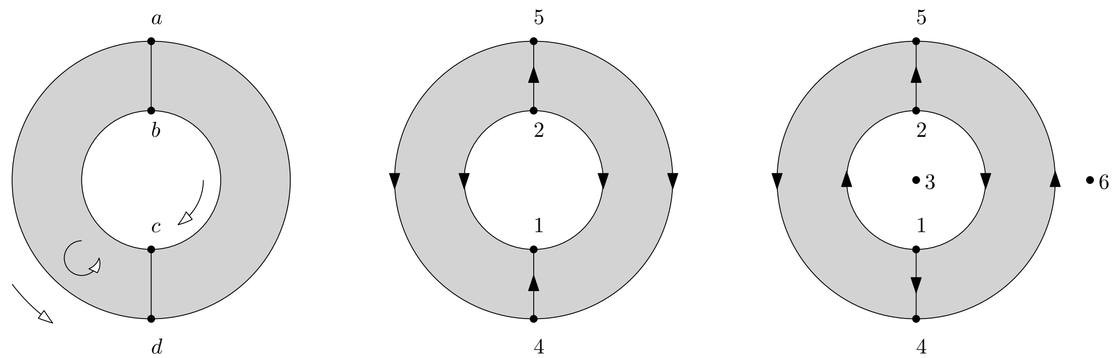


Figure 3.4: From left to right: the weighted graph G, the orientation $K_{0,0}$ and the orientation $K_{1,1}$ (together with a vertex labelling of G).

and

$$M_{1,1}^{K_{1,1}} = \begin{pmatrix} 0 & -bc & c & 1 & 0 & 0 \\ bc & 0 & -b & 0 & 1 & 0 \\ -c & b & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -ad & d \\ 0 & -1 & 0 & ad & 0 & -a \\ 0 & 0 & 0 & -d & a & 0 \end{pmatrix}.$$

A direct calculation leads to

$$\mathrm{Pf}(M_{0,0}^{K_{0,0}}) = 5 + 2ad + 2bc + abcd \qquad \mathrm{Pf}(M_{1,1}^{K_{1,1}}) = ab + cd$$

and so we obtain

$$Z_{\mathcal{MD}}(G) = 5 + 2ad + 2bc + abcd + ab + cd.$$

As a reality check, one can verify easily that $\mathrm{Pf}(M_{0,0}^{K_{0,0}})$ (resp. $\mathrm{Pf}(M_{1,1}^{K_{1,1}})$) counts all the weighted MD coverings of G with an even (resp. odd) number of monomers on each boundary component, and hence their sum is precisely equal to $Z_{\mathcal{MD}}(G)$.

3.3 Proof of the main result

This section is devoted to prove Theorems 3.2.2 and 3.2.4 stated above. The proof is based on Theorem 1.4.8 to compute the dimer partition function of graphs embedded in surfaces, together with some bijections between monomer-dimer coverings of G and dimer coverings of some auxiliary graphs. We then prove that Pfaffians of adjacency matrices of these auxiliary graphs are equal to that of modified adjacency matrices of G (recall Definition 3.2.1) by some elementary matrix transformations together with the Laplace expansion for Pfaffians. Since in the proof we need to use the Pfaffian formula to compute the dimer partition function of surface graphs, we refer readers to subsection 1.4.3 (as well as [14, 8]) for this formula for graphs embedded in (possibly) non-orientable closed surfaces, and to [15] for graphs embedded in surfaces with boundary.

3.3.1 Preliminaries

In this subsection we will prove the existence of Kasteleyn orientations K_β mentioned in the statement of Theorem 3.2.2. We also introduce some auxiliary graphs associated to the original graph G, and relate their dimer partition functions to partial MD partition functions of G.

Let us first recall that for an oriented cycle C, $n^K(C)$ is the number of times that, travelling along C by its orientation, we travel along an edge in the opposite direction to the one given by K. Recall also that an orientation K on G is called Kasteleyn if $n^K(\partial f)$ is odd for every face f of G. We will need the following result which can be found in [15, Proposition 1].

Proposition 3.3.1. *Let G be a graph embedded in an orientable surface Σ of genus g with b boundary components $C_1, \dots, C_b$, and let β_k be 0's or 1's for $1 \leq k \leq b$. Then there exists a Kasteleyn orientation K on G such that $1 + n^K(-C_k) \equiv \beta_k$ (modulo 2) for all k if and only if $\sum_{k=1}^b \beta_k$ has the same parity as $|V(G)|$.* □

Remark 3.3.2. Let us discuss equivalence classes of Kasteleyn orientations in more detail. Recall that two Kasteleyn orientations are equivalent if one can obtained from other by reversing orientations on all edges adjacent to some vertices, and that $\mathcal{K}(G)$ denotes the set of all the equivalence classes of Kasteleyn orientations on G. Now, ignoring the condition on the boundary of Σ, by Proposition 3.3.1 there always exists Kasteleyn orientations on G, and moreover we have exactly $\#H_1(\Sigma; \mathbb{Z}_2) = 2^{2g+b-1}$ equivalence classes of Kasteleyn orientations (see also [15]). Furthermore, following [8, Remark 4.5] one can construct 2^{2g+b-1} representatives of these equivalence classes as follows. Firstly, let us pick any Kasteleyn orientation K, and recall that the set $\{\alpha_i\}_{1 \leq i \leq 2g}$ consists of $2g$ closed curves transverse to G whose homology classes form a basis of $H_1(\Sigma; \mathbb{Z}_2)$. For any subset $I \subset \{1, \dots, 2g\}$, let K^I be the orientation obtained by inverting K on every edge e each time e crosses α_i with $i \in I$. The set $\{K^I\}_I$ consists of 2^{2g} non-equivalent Kasteleyn orientations. Secondly, for any $\beta = (\beta_1, \dots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even, consider the set of boundary components $\{C_k : \beta_k = 1\}$ which is of even cardinality $2m_\beta$, and partition this set into m_β pairs arbitrarily. Connect the interiors of two discs whose boundaries are boundary components in a same pair by a path transverse to G, and finally invert K^I (for every fixed $I \subset \{1, \dots, 2g\}$) on every edge e each time e is crossed by one of these m_β paths. The set $\{K_\beta^I : \sum_{k=1}^b \beta_k \text{ is even}\}$ consists of 2^{b-1} non-equivalent Kasteleyn orientations, non of which is equivalent to any of $\{K_\beta^{I'} : \sum_{k=1}^b \beta_k \text{ is even}\}$ for any $I' \neq I$. This implies that the set $\{K_\beta^I\}$ consists of 2^{2g+b-1} representatives of 2^{2g+b-1} equivalence classes.

We now apply Proposition 3.3.1 to show that the sets $\mathcal{O}_\beta$ are non-empty, and that there exists the orientations K_β as required (recall the arguments after Definition 3.2.1). Recall that G is assumed to have the boundary circuits B_k which coincide with the boundary components C_k of the surface Σ. We also recall that the number N_k of the vertices of G on B_k is even for every $1 \leq k \leq b$, as well as the total number $|V(G)|$ of the vertices of G.

Lemma 3.3.3. *For each $\beta = (\beta_1, \dots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even, the set $\mathcal{O}_\beta$ is nonempty and there exists K_β such that $n^{K_\beta}(\gamma_i)$ is odd for every $1 \le i \le 2g$. Moreover the collection $\{[K] : K \in \mathcal{O}_\beta\}_\beta$ is a partition of $\mathcal{K}(G)$.*

Proof. Let us fix a $\beta = (\beta_1, \dots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even. Note that the length of B_k is equal to N_k which is even, hence an element in $\mathcal{O}_\beta$ firstly must be some Kasteleyn orientation L_β satisfying $n^{L_\beta}(-B_k) \equiv n^{L_\beta}(B_k) \equiv \beta_k + 1$ modulo 2 for all $1 \le k \le b$. Such an orientation L_β clearly exists by Proposition 3.3.1 since both $|V(G)|$ and $\sum_{k=1}^b \beta_k$ are even. Moreover, one can obtain an orientation belonging to $\mathcal{O}_\beta$ from such L_β by the following transformation: one can transform L_β to a Kasteleyn orientation L'_β so that the restrictions of L_β and L'_β on B_k are only different on any two arbitrary edges e_1, e_2 of B_k $(1 \le k \le b)$. Then by repeating this transformation if needed, one obtains an element in $\mathcal{O}_\beta$. Now to construct L'_β we can do as follows: draw a path γ from the interior of the disk D_k (whose boundary is B_k) first crossing transversely e_1, then some edges of G and then e_2, finally coming back to the interior of D_k; invert L_β on every edge each time this edge is crossed by γ to obtain L'_β. It is easy to see that since L_β is Kasteleyn, so is L'_β. Moreover by the construction L_β and L'_β restricted to B_k are only different on e_1, e_2. We have proved the first part of the lemma.

To construct K_β as required, let us pick an element $J_\beta \in \mathcal{O}_\beta$. If $n^{J_\beta}(\gamma_i)$ is odd for every $1 \le i \le 2g$ then we are done. If there exists i so that $n^{J_\beta}(\gamma_i)$ is even, let us pick a closed curve α_i^* transverse to G and disjoint from $\partial\Sigma$ so that its homology class in $H_1(\overline{\Sigma}; \mathbb{Z}_2)$ is dual to that of α_i. Inverting J_β on every edge e each time e is crossed by α_i^* results to change the parity of $n^{J_\beta}(\gamma_i)$ but not the parity of $n^{J_\beta}(\gamma_j)$ for every $j \neq i$. Repeating this procedure for each i with $n^{J_\beta}(\gamma_i)$ even, we get the orientation K_β as expected.

Finally, to prove the last statement of the lemma, note that by Proposition 3.3.1, the set $\mathcal{O}_\beta$ is empty if $\sum_{k=1}^b \beta_k$ is odd. Additionally, any Kasteleyn orientation K on G must have the type of L_β for some β. Hence, by the argument at the beginning of this proof, each Kasteleyn orientation on G must be equivalent to an element of some $\mathcal{O}_\beta$ with $\sum_{k=1}^b \beta_k$ even. Moreover, by Remark 3.3.2, any element of $\mathcal{O}_\beta$ is not equivalent to any of $\mathcal{O}_{\beta'}$ if $\beta \neq \beta'$ so that the sets $\{[K] : K \in \mathcal{O}_\beta\}'s$ are all disjoint. Therefore, the collection $\{[K] : K \in \mathcal{O}_\beta\}_\beta$ (which can be restricted on β's with $\sum_{k=1}^b \beta_k$ even) is a partition of $\mathcal{K}(G)$. □

We now continue by introducing some auxiliary graphs G_β which depend on the parameter $\beta = (\beta_1, \dots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even. Inspired by [24, Appendix E], we define G_β as the graph obtained from G by adding a planar "shuriken" graph S_k of N_k "blades" on each boundary circuit B_k of G $(1 \le k \le b)$ so that all the "blade-tip" vertices of S_k coincide with vertices on B_k (see Figures 3.5 and 3.6). Note that there are two types of shuriken graphs, depending on whether β_k is equal to 0 or 1. Moreover, if we label vertices on some boundary circuit B_k of G by $1, \dots, N_k$ then we would like to label the remaining vertices of S_k, called *inner vertices*, by $N_k + 1, \dots, 2N_k$ (resp. by $N_k + 1, \dots, 2N_k + 1$) if β_k is even (resp. odd) consecutively following the inverse orientation of B_k. This labelling together with the specific labelling of the vertices of G gives us a labelling of the vertices of G_β that we will use from now on. Additionally, we can require that the vertex j is

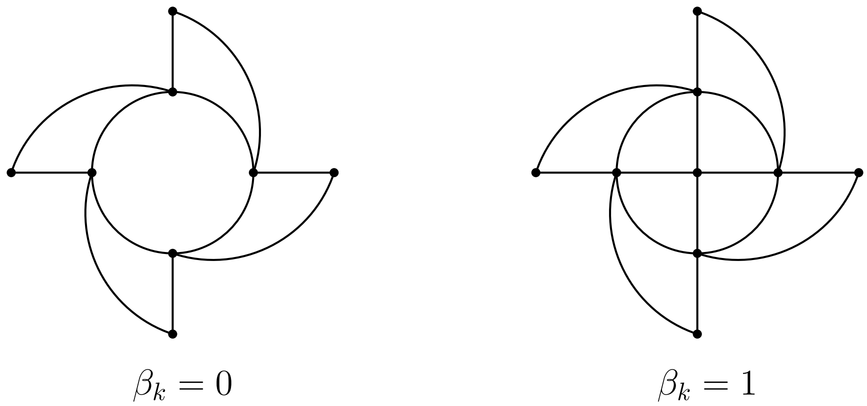


Figure 3.5: The shuriken graph S_k corresponding to $\beta_k = 0$ and $\beta_k = 1$ for N_k=4.

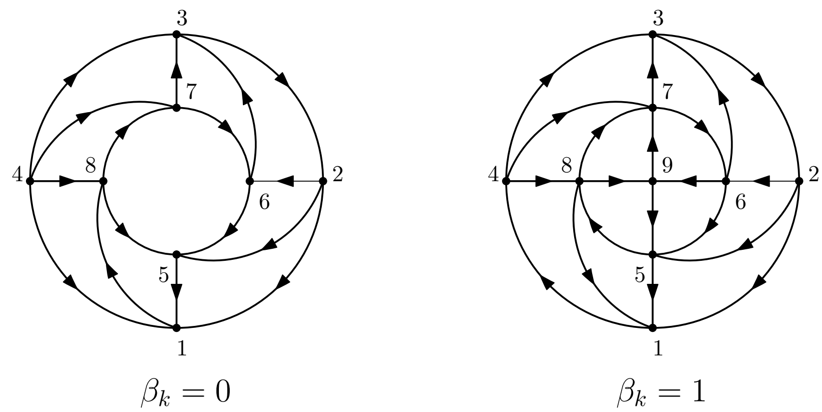


Figure 3.6: The vertex labelling and the orientation on S_k added to the boundary circuit B_k for $N_k = 4$.

adjacent to $j + N_k - 1$ and $j + N_k$ for each $2 \leq j \leq N_k$ and that the vertex 1 is adjacent to $N_k + 1$ and $2N_k$ (see Figure 3.6). The corresponding edges connecting these vertices will be called *blade edges*, while the edges whose two endpoints are inner vertices will be called *inner edges*. Furthermore we shall endow G_β with an edge weight system as follows. Let e be an edge of G_β. If e is an edge of G then it inherits the edge weight from G. If e is a blade edge, then by definition e is endowed with the vertex weight of its endpoint on the boundary circuit of G. Otherwise e is weighted 1. From now on, without stated explicitly, we will always consider parameters $\beta = (\beta_1, \ldots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^{b} \beta_k$ even so that G_β has an even number of vertices. We also consider G_β as a graph embedded in the closed surface $\overline{\Sigma}$. Note that we label the vertices of G_β by the way mentioned above.

For $\beta = (\beta_1, \ldots, \beta_b) \in \mathbb{Z}_2^b$, let us denote by $\mathcal{MD}^\beta(G)$ the set of all MD coverings of G that consist of an even (resp. odd) number of monomers on B_k with $\beta_k = 0$ (resp. $\beta_k = 1$) for every $1 \leq k \leq b$. Setting the partial MD partition function of G with respect to β by

$$Z^\beta_{\mathcal{MD}}(G) := Z^\beta_{\mathcal{MD}}(G, x, y) := \sum_{\tau \in \mathcal{MD}^\beta(G)} \prod_{e \in \tau_D} x_e \prod_{v \in \tau_M} y_v,$$

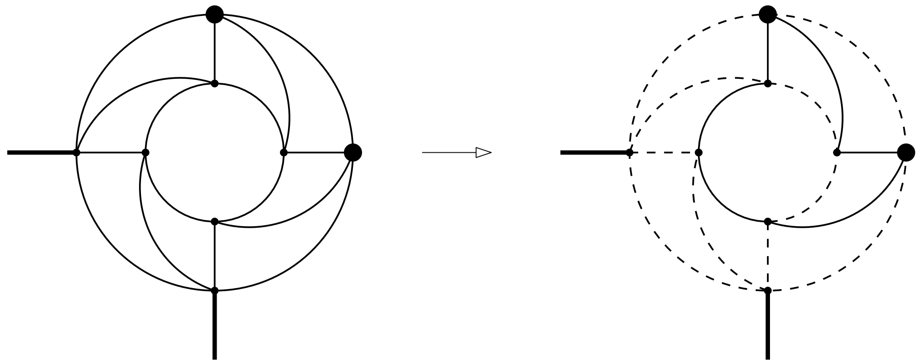


Figure 3.7: Proof of Lemma 3.3.4 for $\beta_k = 0$: an MD covering τ (left), and the closed path of even length which induces precisely 2 perfect matchings that cover $\tau_{\mathcal{M}}$ (right).

the following lemma indicates the relation between this partial partition function and the dimer partition function of G_β.

Lemma 3.3.4. *For every* $\beta = (\beta_1, \dots, \beta_b) \in \mathbb{Z}_2^b$ *we have*

$$Z_{\mathcal{D}}(G_\beta) = 2^{\#\{k:\beta_k=0\}} \prod_{k:\beta_k=1} N_k \; Z^\beta_{\mathcal{MD}}(G).$$

Proof. As mentioned before, we are interested in the case where $\sum_{k=1}^b \beta_k$ is even. Our aim is to show that each MD covering $\tau \in \mathcal{MD}^\beta(G)$ corresponds to $2^{\#\{k:\beta_k=0\}} \prod_{k:\beta_k=1} N_k$ dimer coverings τ_β of G_β whose weights are equal to the weight of τ. The idea is to keep the dimer part $\tau_{\mathcal{D}}$ of τ and match its monomer part $\tau_{\mathcal{M}}$ to vertices of $G_\beta \setminus G$ suitably. For the latter, it is clear that monomers of τ which lies on a boundary circuit B_k of G must be matched to inner vertices of the shuriken graph S_k. Therefore, we only need to prove that to each boundary circuit B_k of G, there are precisely 2 possibilities to match monomers of τ if $\beta_k = 0$, while there are precisely N_k possibilities if $\beta_k = 1$. In the former case, observe that every inner edge which is opposite to a monomer of τ can be ignored. Now if we remove from S_k all such inner edges as well as all blade edges not adjacent to monomers of τ, we are left with a closed path of even length, containing monomers of τ. This path gives exactly 2 matchings covering these monomers and its remaining vertices (see Figure 3.7). In the second case when $\beta_k = 1$, we see that there are exactly N_k possibilities to match the centre vertex of the shuriken graph S_k. Hence we only need to show that each such matching can be extended uniquely to a matching that covers monomers of τ lying on B_k and remaining vertices of S_k. Similarly to the previous case, if we remove from S_k its centre vertex and the inner vertex matched to it as well as all edges adjacent to these two, and also remove all inner edges that are opposite to monomers of τ together with all blade edges not adjacent to monomers of τ, we are left with an open path of odd length. This path, containing monomers of τ, gives an unique matching covering them (see Figure 3.8). Additionally, by definition the weight of τ and the weight of all the dimer coverings τ_β's constructed above are equal.

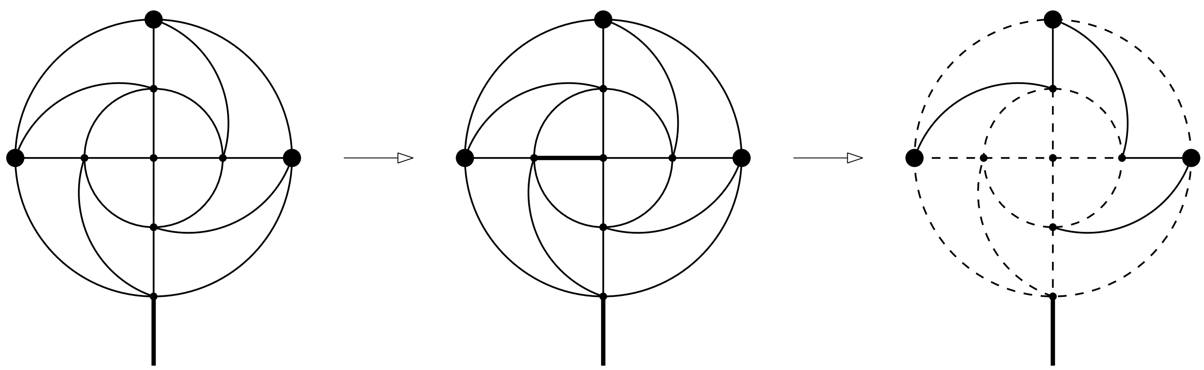


Figure 3.8: Proof of Lemma 3.3.4 for $\beta_k = 1$: an MD covering τ (left), a match of the centre vertex (middle), and the open path of odd length which induces precisely one perfect matching that covers $\tau_{\mathcal{M}}$ (right).

Finally to conclude our proof, one needs to show that the collection $\{\{\tau_\beta\} : \tau \in \mathcal{MD}^\beta(G)\}$ is a partition of $\mathcal{D}(G_\beta)$. Indeed by the construction above if $\tau \neq \tau'$, then $\tau_{\mathcal{D}} \neq \tau'_{\mathcal{D}}$ and so $\{\tau_\beta\}$ and $\{\tau'_\beta\}$ are disjoint. Furthermore, if D is a dimer covering of G_β, then D covers an even (resp. odd) number of vertices of B_k with $\beta_k = 0$ (resp. $\beta_k = 1$) for every $1 \leq k \leq b$. Denote by τ the MD covering whose dimer part $\tau_{\mathcal{D}}$ coincides with D restricted on G and the monomer part $\tau_{\mathcal{M}}$ consists of all vertices on ∂G which are matched by D with some inner vertices of shuriken graphs. Then it is obvious that τ belongs to $\mathcal{MD}^\beta(G)$. Moreover, since D matches the monomers of τ with some inner vertices of shuriken graphs, by the argument at the beginning of the proof, D must coincide with one of the τ_β's. We have completed the proof. $\square$

Remark 3.3.5. By the proof of Lemma 3.3.4, a dimer covering D of G_β gives rise to a matching D_β of G that covers an even (resp. odd) number of vertices of B_k with $\beta_k = 0$ (resp. $\beta_k = 1$) for every $1 \leq k \leq b$, and vice versa. Hence, recalling that $\mathcal{S}$ is the set of all β's so that such D_β exists, we see that $\mathcal{MD}^\beta(G)$ is empty for every $\beta \notin \mathcal{S}$.

Next we will describe some specific orientations on G_β. For an orientation $L \in \mathcal{O}_\beta$, let us define $\overline{L}$ as the orientation on G_β which coincides with L on G and is determined on the remaining edges of G_β as follows. Recalling the labelling of vertices of G_β mentioned before, if $\beta_k = 1$, on blade edges of S_k we require $\overline{L}$ to go out from j if j is even, and go toward j if j is odd for all $1 \leq j \leq N_k$. In addition, on inner edges, we require $\overline{L}$ to be from big-labelled vertices to small-labelled ones except that it goes from $N_k + 1$ to $2N_k$, whilst it goes from the centre vertex of S_k to odd-labelled inner vertices, and toward this centre vertex from even-labelled ones. If $\beta_k = 0$, we require $\overline{L}$ on blade edges to be the same as in the previous case, except that we invert the orientation on the edge between 1 and $2N_k$. On inner edges, we would like $\overline{L}$ to be from big-labelled vertices to small-labelled ones. This is illustrated in Figure 3.6. It is straightforward to verify that the orientation $\overline{L}$ defined by this way is Kasteleyn on $G_\beta \subset \overline{\Sigma}$ for every $L \in \mathcal{O}_\beta$. Moreover, by Remark 3.3.2, the set $\{[\overline{K}] : K \in \mathcal{O}_\beta\}$ consists of all equivalence classes of Kasteleyn orientations on G_β.

3.3.2 Proof of Theorems 3.2.2 and 3.2.4

In this subsection we will use the preliminaries given in the previous part to prove Theorem 3.2.2. Recall that we label vertices of G in a specific way (cf. the argument after Definition 3.2.1), that induces a vertex labelling of G_β (described after Lemma 3.3.3). With these vertex labellings we have the following result.

Proposition 3.3.6. *For $L \in \mathcal{O}_\beta$, let $A^{\overline{L}}(G_\beta)$ be the adjacency matrix of G_β with respect to the orientation $\overline{L}$. We have*

$$Pf(A^{\overline{L}}(G_\beta)) = (-1)^{\sum_{k=1}^{b} N_k/2} 2^{\#\{k:\beta_k=0\}} \prod_{k:\beta_k=1} N_k \; Pf(M_\beta^L(G)).$$

The proof of this proposition will be left until the end of this subsection. We now show how this proposition implies Theorems 3.2.2 and 3.2.4.

Proof of Theorem 3.2.2. Recall that $\{\alpha_i\}_{1\le i\le 2g}$ is a set of closed curves whose homology classes form a basis of $H_1(\overline{\Sigma};\mathbb{Z}_2)$, and that γ_i is an oriented closed cycle of G which is disjoint from ∂G and has α_i on its immediate left for every $1 \le i \le 2g$. Recall also that K_β is an element of $\mathcal{O}_\beta$ satisfying that $n^{K_\beta}(\gamma_i)$ is odd for every i. By definition, $\overline{K_\beta}$ is Kasteleyn on G_β and still satisfies that $n^{\overline{K_\beta}}(C_i)$ is odd. Moreover, since α_i is chosen to be disjoint from ∂G, it is clear that $\overline{K_\beta}^{\epsilon} = \overline{K_\beta^{\epsilon}}$. Using this fact and applying Theorem 1.4.8 for G_β and $\overline{K_\beta}$ we get

$$\begin{aligned} Z_{\mathcal{D}}(G_\beta) &= \frac{1}{2^g}\Bigg|\sum_{\epsilon\in\mathbb{Z}_2^{2g}} (-1)^{\sum_{i<j}\epsilon_i\epsilon_j\alpha_i\cdot\alpha_j}\mathrm{Pf}(A^{\overline{K_\beta}^{\epsilon}}(G_\beta))\Bigg| \\ &= \frac{1}{2^g}\Bigg|\sum_{\epsilon\in\mathbb{Z}_2^{2g}} (-1)^{\sum_{i<j}\epsilon_i\epsilon_j\alpha_i\cdot\alpha_j}\mathrm{Pf}(A^{\overline{K_\beta^{\epsilon}}}(G_\beta))\Bigg|. \end{aligned}$$

In the following, the sum over β will be understood as over $\beta = (\beta_1,\dots,\beta_b) \in \mathbb{Z}_2^b$ such that $\sum_{k=1}^b \beta_k$ is even. With this convention and by the equality above we can write

$$\begin{aligned} Z_{\mathcal{MD}}(G) &= \sum_\beta Z_{\mathcal{MD}}^\beta(G) \\ &\overset{Lem.3.3.4}{=} \sum_\beta \left(2^{\#\{k:\beta_k=0\}}\prod_{k:\beta_k=1} N_k\right)^{-1} Z_{\mathcal{D}}(G_\beta) \\ &= \sum_\beta \left(2^{\#\{k:\beta_k=0\}}\prod_{k:\beta_k=1} N_k\right)^{-1} \Bigg|\sum_{\epsilon\in\mathbb{Z}_2^{2g}} (-1)^{\sum_{i<j}\epsilon_i\epsilon_j\alpha_i\cdot\alpha_j}\mathrm{Pf}(A^{\overline{K_\beta^{\epsilon}}}(G_\beta))\Bigg| \\ &\overset{Prop.3.3.6}{=} \frac{1}{2^g}\sum_\beta \Bigg|\sum_{\epsilon\in\mathbb{Z}_2^{2g}} (-1)^{\sum_{i<j}\epsilon_i\epsilon_j\alpha_i\cdot\alpha_j}\mathrm{Pf}(M_\beta^{K_\beta^{\epsilon}}(G))\Bigg|. \end{aligned}$$

This concludes the proof of Theorem 3.2.2. ∎

Proof of Theorem 3.2.4. Firstly, recall that D_β is a dimer covering of G that covers an even (resp. odd) number of vertices on B_k if $\beta_k = 0$ (resp. if $\beta_k = 1$). Hence by the proof of Lemma 3.3.4, D_β can be extended to a dimer covering of G_β, which will be denoted by $\overline{D}_\beta$. Secondly, $\overline{K}$ is Kasteleyn on G_β for every $K \in \mathcal{O}_\beta$, and recall that the set $\{[\overline{K}] : K \in \mathcal{O}_\beta\}$ consists of all equivalence classes of Kasteleyn orientations on G_β. Finally let us apply Theorem 1.4.7 for the graph $G_\beta \subset \overline{\Sigma}$ and the dimer covering $\overline{D}_\beta$ to get

$$Z_{\mathcal{D}}(G_\beta) = \frac{1}{2^g} \sum_{[\overline{K}]:K\in\mathcal{O}_\beta} (-1)^{\mathrm{Arf}(q^{\overline{K}}_{\overline{D}_\beta})} \epsilon^{\overline{K}}(\overline{D}_\beta)\mathrm{Pf}(A^{\overline{K}}(G_\beta)),$$

where the quadratic form $q^{\overline{K}}_{\overline{D}_\beta} : H_1(\overline{\Sigma};\mathbb{Z}_2) \to \mathbb{Z}_2$ is determined by the property that, if C is an oriented simple cycle of G_β then

$$q^{\overline{K}}_{\overline{D}_\beta}([C]) = n^{\overline{K}}(C) + l_{\overline{D}_\beta}(C) + 1 \pmod 2.$$

Obviously this quadratic form can be determined by only K and D_β by the similar equation as well, where C is restricted to be an oriented cycle of G. Hence let us denote this quadratic form simply by $q^K_{D_\beta}$. Using this convention and Remark 3.3.5 we can write

$$\begin{aligned}
Z_{\mathcal{MD}}(G) \quad &= \quad \sum_{\beta\in\mathcal{S}} Z^\beta_{\mathcal{MD}}(G) \\
&\overset{Lem.3.3.4}{=} \quad \sum_{\beta\in\mathcal{S}} \left(2^{\#\{k:\beta_k=0\}} \prod_{k:\beta_k=1} N_k\right)^{-1} Z_{\mathcal{D}}(G_\beta) \\
&= \quad \frac{1}{2^g}\sum_{\beta\in\mathcal{S}} \left(2^{\#\{k:\beta_k=0\}} \prod_{k:\beta_k=1} N_k\right)^{-1} \sum_{[K]:K\in\mathcal{O}_\beta} (-1)^{\mathrm{Arf}(q^K_{D_\beta})} \epsilon^{\overline{K}}(\overline{D}_\beta)\mathrm{Pf}(A^{\overline{K}}(G_\beta)) \\
&\overset{Prop.3.3.6}{=} \quad \frac{1}{2^g}\sum_{\beta\in\mathcal{S}} \sum_{[K]:K\in\mathcal{O}_\beta} (-1)^{\mathrm{Arf}(q^K_{D_\beta})} \epsilon^{\overline{K}}(\overline{D}_\beta)\mathrm{Pf}(M^K_\beta(G)).
\end{aligned}$$

Finally, we can really set the sign $\epsilon^{\overline{K}}(\overline{D}_\beta)$ equal to 1 by choosing a good representative K: by Equation (1.3), if $\epsilon^{\overline{K}}(\overline{D}_\beta) = -1$ for some K, we only need to invert K on all edges adjacent to an arbitrary vertex of $G \setminus \partial G$. This concludes our proof. □

Now we are only left with the proof of Proposition 3.3.6. The idea of the proof is that, for each $K \in \mathcal{O}_\beta$, we can transform the matrix $A^{\overline{K}}(G_\beta)$ to a new matrix simply using elementary row-column operations so that their Pfaffians are equal, and then we relate the Pfaffian of the latter to that of $M^K_\beta(G)$ using Laplace expansions (recall Lemma 1.4.1).

Proof of Proposition 3.3.6. As mentioned above, for an orientation $K \in \mathcal{O}_\beta$, we will first transform the matrix $A^{\overline{K}}(G_\beta)$ to a new matrix using row-column operations, and then use Laplace expansions to relate the Pfaffian of the latter to that of $M^K_\beta(G)$. The point is that all these operations and expansions, as we will see, are local and only depend on boundary circuits individually. Therefore, without loss of generality, in the following we will only work with the matrices $A^{\overline{K}}(G_\beta)$ and $M^K_\beta(G)$ partially, that is, we will work with their submatrices corresponding to each of the boundary circuits. However readers should keep in mind that what we will show now works completely well in the global context. Throughout the rest of this proof, let us fix a $\beta \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even, and an orientation $K \in \mathcal{O}_\beta$.

Let us begin with a boundary circuit B_k corresponding to $\beta_k = 0$. Recalling the labelling of G_β on this boundary circuit as well as the orientation $\overline{K}$, we can write the submatrix corresponding to B_k of $A^{\overline{K}}(G_\beta)$ as a block matrix

$$A_k = \begin{pmatrix} A^\partial & A \\ -A^T & A^{\text{in}} \end{pmatrix}.$$

Here A^∂ represents the adjacencies between vertices on B_k, A^{in} represents the adjacencies between inner vertices of the shuriken graph S_k, while A represents adjacencies between vertices of these two types. Note that all these matrices are of size N_k. More precisely, denoting vertex weights of vertices on B_k by $y_1, \dots, y_{N_k}$ we can write $A = (a_{ij})_{1 \le i,j \le N_k}$ with all of entries equal to 0 except $a_{11} = -y_1 = -a_{1,N_k}$ and $a_{i,i-1} = a_{ii} = (-1)^i y_i$ for $2 \le i \le N_k$. Also we have $A^{\text{in}} = (b_{ij})_{1 \le i,j \le N_k}$ with $b_{1,N_k} = -b_{N_k,1} = -1$, $b_{i,i+1} = -b_{i+1,i} = -1$ for $1 \le i \le N_k - 1$ and other entries equal to 0. Denoting by M_k the submatrix of $M^K_\beta(G)$ corresponding to B_k and writing $N_k = 2n_k$ for further purposes, our aim now is to prove that

$$\mathrm{Pf}(A_k) = 2(-1)^{n_k} \mathrm{Pf}(M_k). \tag{3.1}$$

To do so, we will first transform the matrix A_k using row and column operations so that its Pfaffian does not change. Let us denote by $R_l(X)$ and $C_l(X)$ the l^{th} row and the l^{th} column of a matrix X. Add $(-1)^l y_l R_{N_k+1}(A_k)$ to $R_l(A_k)$ as well as $(-1)^l y_l C_{N_k+1}(A_k)$ to $C_l(A_k)$ for each $2 \le l \le N_k$ step by step. Since $b_{ij} = 0$ if both i, j are odd, one can verify easily that after each step all the odd columns of A (as well as odd rows of $-A^T$) do not change. By the same reason, for each $1 \le m \le n_k - 1$ we can add $(-1)^l y_l R_{N_k+2m+1}(A_k)$ to $R_l(A_k)$ (and similarly for corresponding columns) for every $2m+2 \le l \le N_k$. Note that these operations keep A^{in} unchanged. Moreover, by definition of $M^K_\beta(G)$, it is straightforward to check that after all these operations above, the matrix A_k becomes

$$A'_k = \begin{pmatrix} M_k & A' \\ -A'^T & A^{\text{in}} \end{pmatrix}$$

whose Pfaffian is equal to the Pfaffian of A_k. Let us now determine the entries a'_{ij} of A'. By the argument above, all the odd columns of A' are equal to those of A. We will show that all the even columns of A' are 0 except the last one. Indeed, for $1 \le j \le n_k - 1$ we have

$$a'_{2j,2j} = a_{2j,2j} + y_{2j}(b_{1,2j} + b_{3,2j} + \cdots + b_{2j-3,2j} + b_{2j-1,2j}) = 0$$

since $a_{2j,2j} = y_{2j}$, $b_{2j-1,2j} = -1$ while $b_{1,2j} = \cdots = b_{2j-1,2j} = 0$. Similarly we have

$$a'_{2j+1,2j} = a_{2j+1,2j} - y_{2j+1}(b_{1,2j} + b_{3,2j} + \cdots + b_{2j-3,2j} + b_{2j-1,2j}) = 0$$

as $a_{2j+1,2j} = -y_{2j+1}$. If $l < 2j$ then we have

$$a'_{l,2j} = a_{l,2j} + (-1)^l y_l (b_{1,2j} + b_{3,2j} + \cdots + b_{p,2j})$$

with some p odd and $p \leq 2j-3$. Since in this case $a_{l,2j} = 0$, we get $a'_{l,2j} = 0$. If $l > 2j+1$, we can write

$$a'_{l,2j} = a_{l,2j} + (-1)^l y_l (b_{1,2j} + \cdots + b_{2j-1,2j} + b_{2j+1,2j} + \cdots + b_{q,2j})$$

with some q odd and $q \geq 2j+3$. As in this case we also have $a_{l,2j} = 0$, $b_{2j-1,2j} = -b_{2j+1,2j} = -1$ while others $b_{i,2j}$'s are 0, we obtain $a'_{l,2j} = 0$ as well. Finally let us look at the last column of A'. Since all the entries of this column are determined only by the last entries of $R_{N_k+1}(A_k)$ and $R_{2N_k-1}(A_k)$ together with the last column of A, we simply get $C_{N_k}(A') = (y_1, -y_2, \ldots, y_{N_k-1}, -y_{N_k})^T$.

Let us make one more transformation. We add $\sum_{m=1}^{n_k} R_{N_k+2m-1}(A'_k)$ to $R_{2N_k}(A'_k)$ as well as $\sum_{m=1}^{n_k} C_{N_k+2m-1}(A'_k)$ to $C_{2N_k}(A'_k)$ so that we get

$$A''_k = \begin{pmatrix} M_k & A'' \\ -A''^T & A^{\text{in}} \end{pmatrix}.$$

Note that now the block A'' of A''_k has all even columns equal to 0, and we still have $\mathrm{Pf}(A''_k) = \mathrm{Pf}(A'_k) = \mathrm{Pf}(A_k)$. Applying Lemma 1.4.1 for the last row of A''_k in which there are only 2 entries different from 0, we get

$$\mathrm{Pf}(A''_k) = -2\mathrm{Pf}\begin{pmatrix} M_k & D \\ -D^T & E \end{pmatrix}.$$

Here E is the matrix obtained from A_{in} by removing its last two rows and last two columns, while D is a matrix whose even columns consist of 0's. Repeating this expansion for the new block matrix with a remark that now its last row contains only one nonzero element, and using the fact that the top right block has all even columns equal to 0, by recursion we get

$$\mathrm{Pf}(A_k) = \mathrm{Pf}(A''_k) = 2(-1)^{n_k}\mathrm{Pf}(M_k).$$

This concludes the proof of Equation (3.1).

Next we consider a boundary circuit B_k corresponding to $\beta_k = 1$. In this case the corresponding submatrix of $A^{\overline{K}}(G_\beta)$ to this circuit is given by

$$A^1_k = \begin{pmatrix} A^\partial_1 & A_1 \\ -(A_1)^T & A^{\text{in}}_1 \end{pmatrix}.$$

Due to the differences of S_k and $\overline{K}$ in this case, the block A_1 is obtained from A as follows: we change the sign of a_{1,N_k}, and then add a column of all 0's to the right of A. Similarly, the matrix A^{in}_1 is obtained from A^{in} by changing the signs of b_{1,N_k} and $b_{N_k,1}$, and by adding the column $(-1,1,\dots,-1,1,0)^T$ to its rights and the row $(1,-1,\dots,1,-1,0)$ to its bottom. Our purpose now is to prove that

$$\mathrm{Pf}(A_k^1) = (-1)^{n_k} N_k \mathrm{Pf}(M_k^1). \tag{3.2}$$

For each $0 \le m \le n_k - 1$ let us add $(-1)^l y_l R_{N_k+2m+1}(A_k^1)$ to $R_l(A_k^1)$ and $(-1)^l y_l C_{N_k+2m+1}(A_k^1)$ to $C_l(A_k^1)$ for every $2m+2 \le l \le N_k$ as in the previous case. By the same argument we obtain

$$(A_k^1)' = \begin{pmatrix} M_k & A_1' \\ -(A_1')^T & A_1^{\mathrm{in}} \end{pmatrix}$$

so that $\mathrm{Pf}(A_k^1)' = \mathrm{Pf}(A_k^1)$. However note that here M_k is only the submatrix of $M_\beta^K(G)$ corresponding to the circuit B_k without adding the vertex v_k. More precisely, M_k^1 is obtained from M_k by adding the column $(y_1, -y_2, \dots, y_{N_k-1}, -y_{N_k})^T$ to its right and the row $(-y_1, y_2, \dots, -y_{N_k-1}, y_{N_k})$ to its bottom. Also A_1' and A' coincide on the first $N_k - 1$ columns, while their N_k^{th} columns are of opposite signs. Moreover, following our operations above and by some simple calculations, we can find that the last column of A_1' is

$$(0, -y_2, y_3, \dots, -iy_{2i}, iy_{2i+1}, \dots, -(n_k-1)y_{N_k-2}, (n_k-1)y_{N_k-1}, -n_k y_{N_k})^T.$$

Now let us do some more transformations. First of all, similarly to the previous case we add $-\sum_{m=1}^{n_k} R_{N_k+2m-1}(A_k^1)'$ to $R_{N_k+2n_k}(A_k^1)'$ (and do similarly for corresponding columns) to get

$$(A_k^1)'' = \begin{pmatrix} M_k & A_1'' \\ -(A_1'')^T & (A_1^{\mathrm{in}})' \end{pmatrix}$$

so that $\mathrm{Pf}(A_k^1)'' = \mathrm{Pf}(A_k^1)'$ and A_1'' has all even columns equal to 0. Also, $(A_1^{\mathrm{in}})'$ can be obtained from A^{in} by adding the column $(-1,1,\dots,-1,n_k+1,0)^T$ to its right and the row $(1,-1,\dots,1,-n_k-1,0)$ to its bottom. Secondly we add $\sum_{m=1}^{n_k} m R_{N_k+2m-1}(A_k^1)''$ to $R_{2N_k+1}(A_k^1)''$ (and do similarly for corresponding columns) to obtain

$$(A_k^1)''' = \begin{pmatrix} M_k & A_1''' \\ -(A_1''')^T & (A_1^{\mathrm{in}})'' \end{pmatrix}$$

so that $\mathrm{Pf}(A_k^1)''' = \mathrm{Pf}(A_k^1)''$, A_1''' still has all even columns equal to 0 while $(A_1^{\mathrm{in}})''$ is obtained by adding the columns $(-1,0,-1,0,\dots,-1,N_k,0)^T$ to the right of A^{in} and adding the row $(1,0,1,0,\dots,1,-N_k,0)$ to its bottom. Finally let us add $\sum_{m=2}^{n_k} R_{N_k+2m-1}(A_k^1)'''$ to $R_{N_k+1}(A_k^1)'''$ (and similarly for columns) to get

$$A_{\mathrm{final}} = \begin{pmatrix} M_k & P \\ -P^T & Q \end{pmatrix}$$

so that $\mathrm{Pf}(A_{\mathrm{final}}) = \mathrm{Pf}(A_k^1)'''$ and the matrix P has all even columns equal to 0, while its first column is $(-y_1, y_2, \ldots, -y_{N_k-1}, y_{N_k})^T$. Also one gets

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 & * \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & * \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & N_k \\ * & 0 & * & 0 & \cdots & * & -N_k & 0 \end{pmatrix}$$

is a skew-symmetric matrix of size $N_k + 1$ (where $*$ indicates non-important entries). Now applying Lemma 1.4.1 for the $(2N_k)^{\text{th}}$ row of A_{final} in which there are exactly two entries different from 0, namely 1 and N_k, we get

$$\mathrm{Pf}(A_{\text{final}}) = N_k \mathrm{Pf}\begin{pmatrix} M_k & U \\ -U^T & X_{N_k-1} \end{pmatrix} - \mathrm{Pf}\begin{pmatrix} M_k & V \\ -V^T & Y_{N_k-1} \end{pmatrix}. \tag{3.3}$$

In this equation, U, V are matrices with all even columns equal to 0 while X_m is the matrix of size m of the following type

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Additionally Y_m is obtained by adding the column of type $(*, 0, \ldots, *, 0, 0)^T$ to the right of X_{m-1} and the row of same type to its bottom. To compute the first Pfaffian on the right hand side of (3.3), one can apply Lemma 1.4.1 for the row corresponding to the second row of X_{N_k-1} in which the only nonzero element is -1. Note that by removing rows and columns corresponding to this entry, we obtain a new matrix of the same type as before with size decreased by 2. By induction, and by changing the sign of the last column (and the last row) of the final matrix, we get the first term on the right hand side of (3.3) exactly equal to $N_k(-1)^{n_k}\mathrm{Pf}(M_k^1)$. By the same argument one can compute the second term of (3.3) equal to 0. This leads to

$$\mathrm{Pf}(A_k^1) = \mathrm{Pf}(A_k^1)' = \mathrm{Pf}(A_k^1)'' = \mathrm{Pf}(A_k^1)''' = \mathrm{Pf}(A_{\text{final}}) = N_k(-1)^{n_k}\mathrm{Pf}(M_k^1)$$

which proves Equation (3.2).

Finally, to conclude our proof, one only needs to combine Equation (3.1) and (3.2) together, and remark that all the operations and expansions we show above work completely well in the global context. This concludes the proof of Proposition 3.3.6, as well as of Theorem 3.2.2 and Theorem 3.2.4. □

3.4 The non-orientable case

Let us now study the case where the weighted graph (G, x, y) is embedded in a non-orientable surface Σ of genus h and of b boundary components. As before, without any loss of generality, one can assume that each boundary circuit B_k of G coincides with a boundary component of Σ, that each B_k contains an even number of vertices, and that G has an even number of vertices as well. We still denote by $\overline{\Sigma}$ the closed non-orientable surface obtained from Σ by gluing a disc along each of its boundary components.

Being similar to the dimer model in the non-orientable case, one needs to twist modified adjacency matrices by a function ω representing the first Stiefel-Whitney class of Σ. More precisely, fixing such an ω we have the following definition.

Definition 3.4.1. *Let (G, x, y) be a weighted graph and K an orientation on its edges. Given $\beta = (\beta_1, \dots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even, add an isolated vertex v_k of weight 1 for each k such that $\beta_k = 1$. Labelling the vertices of G together with all such v_k's by a number set $I \subset \mathbb{N}$, we define the* twisted modified adjacency matrix $M_\beta^{K,\omega}(G) = (m_{ij}^\beta)_{i,j\in I}$ *of G with respect to ω as the skew-symmetric matrix whose entries are given by*

$$m_{ij}^\beta = \sum_e \epsilon_{ij}^K(e)\sqrt{-1}^{\omega(e)} x_e + (-1)^{i+j} d_{ij} y_i y_j \quad \textit{for } i < j,$$

where the sum is taken over all the edge e between two vertices i, j while $\epsilon_{ij}^K(e)$ is equal to 1 if e is oriented by K from i to j, and equal to -1 otherwise. Moreover we define

$$d_{ij} = \begin{cases} 1 & \textit{if } i, j \textit{ belong to a same boundary component;} \\ 0 & \textit{if not,} \end{cases}$$

where the "virtual vertex" v_k is always considered to be on the boundary component B_k.

Let us now choose a specific vertex labelling and an orientation of interest as in the orientable case. Fix a polygonal representation of Σ so that its boundary components B_k's are contained completely in the interior of this polygon, and endow this polygon an orientation pictured counterclockwise. It is clear that all the conditions described right after Definition 3.2.1 still make sense, and therefore we can choose a vertex labelling of G and some orientation of interest as in the orientable setting. Without any confusion, let us still use the notations $K_\beta, \mathcal{O}_\beta, G_\beta$ as in the orientable context.

We now state the first version of our Pfaffian formula in the non-orientable case. Recall that $\mathcal{S}$ is the set of all $\beta = (\beta_1, \dots, \beta_b) \in \mathbb{Z}_2^b$ with $\sum_{k=1}^b \beta_k$ even so that there exists a matching D_β of G covering an even (resp. odd) number of vertices on B_k with β_k=0 (resp. $\beta_k = 1$) for every k. Note that by Lemma 3.3.4, such a D_β can be extended to a dimer covering $\overline{D}_\beta$ of G_β. As in the orientable case, fixing such a D_β and the standard representation ω of the first Stiefel-Whitney class of Σ (recall subsection 1.3.2), each Kasteleyn orientation K on G can be mapped bijectively to a quadratic enhancement $q_{D_\beta}^{K,\omega}$ on $\overline{\Sigma}$. Recalling that $\mathrm{Br}(q_{D_\beta}^{K,\omega})$ denotes the Brown invariant of this enhancement, we have the following result which generalises Theorem 3.2.4 to the non-orientable case.

Theorem 3.4.2. *Let (G, x, y) be a weighted graph embedded in a non-orientable surface Σ of genus h with b boundary components. Then the boundary monomer-dimer partition function of G is given by*

$$Z_{\mathcal{MD}}(G) = \frac{1}{2^{h/2}} \sum_{\beta \in \mathcal{S}} \sum_{[K]: K \in \mathcal{O}_\beta} (-i)^{\omega(\overline{D}_\beta)} \exp(i\pi/4)^{Br(q^{K,\omega}_{D_\beta})} Pf(M^{K,\omega}_\beta(G)).$$

In this formula, the second sum is taken over well-chosen representatives of all the equivalence classes of Kasteleyn orientations on G with aforementioned specific restrictions on ∂G.

We now turn to a practical version of the formula stated in Theorem 3.4.2. Recalling the setting right before Theorem 1.4.8, one can choose an orientation $K_\beta \in \mathcal{O}_\beta$ so that $n^{K_\beta}(C_\gamma)$ is odd for every γ. Also for $\epsilon = (\epsilon_1, \cdots, \epsilon_{2g}) \in \mathbb{Z}_2^{2g}$ and $\eta = (\eta_1, \eta_2) \in \mathbb{Z}_2^2$ (resp. $\eta = \eta_1 \in \mathbb{Z}_2$), let $K_\beta^{\epsilon,\eta}$ be the orientation obtained from K_β by inverting the orientation of an edge of G each time this edge intersects α_j with $\epsilon_j = 1$ for $1 \leq j \leq 2g$, or intersects β_k with $\eta_k = 1$ for $k \in \{1, 2\}$ (resp. intersects β_1 with $\eta_1 = 1$). Finally we can define a function ω by $\omega(e) := e \cdot (\beta_1 + \beta_2)$ (resp. $\omega(e) := e \cdot \beta_1$) for any edge e of G. With these notations, we have the following result.

Theorem 3.4.3. *For (G, x, y) a weighted graph embedded in the non-orientable surface Σ, the boundary monomer-dimer partition function of G is given by*

$$Z_{\mathcal{MD}}(G) = \frac{1}{2^g} \sum_{\substack{\beta \in \mathbb{Z}_2^b \\ \sum_k \beta_k \text{ even}}} \Big| \sum_{\epsilon \in \mathbb{Z}_2^{2g}} (-1)^{\sum_{j<k} \epsilon_j \epsilon_k \alpha_j \cdot \alpha_k} \Big(Re\big(Pf(M_\beta^{K_\beta^\epsilon}(G))\big) + Im\big(Pf(M_\beta^{K_\beta^\epsilon}(G))\big)\Big) \Big|$$

if $\Sigma = \Sigma_{g,b} \# \mathbb{R}P^2$ with $M^{K_\beta^\epsilon}(G) := M^{K_\beta^{\epsilon,0},\omega}(G)$, and by

$$Z_{\mathcal{MD}}(G) = \frac{1}{2^g} \sum_{\substack{\beta \in \mathbb{Z}_2^b \\ \sum_k \beta_k \text{ even}}} \Big| \sum_{\epsilon \in \mathbb{Z}_2^{2g}} (-1)^{\sum_{j<k} \epsilon_j \epsilon_k \alpha_j \cdot \alpha_k} \Big(Im\big(Pf(M_\beta^{K_\beta^\epsilon}(G))\big) + Re\big(Pf(M_\beta^{K_\beta^{\epsilon'}}(G))\big)\Big) \Big|$$

if $\Sigma = \Sigma_{g,b} \# \mathscr{K}$ with $M^{K_\beta^\epsilon}(G) := M^{K_\beta^{\epsilon,0,0},\omega}(G)$ and $M^{K_\beta^{\epsilon'}}(G) := M^{K_\beta^{\epsilon,1,0},\omega}(G)$. In both cases $\Sigma_{g,b}$ denotes an orientable surface of genus g with b boundary components.

Proof of Theorems 3.4.2 and 3.4.3. First of all, fixing a polygonal representation of Σ, the proof of Proposition 3.3.1 in the orientable case works verbatim in the non-orientable context, using Proposition 1.4.4. So Lemma 3.3.3 also holds for non-orientable surfaces. Secondly, the auxiliary graphs G_β, their vertex labelling and the extension of L_β on G to $\overline{L}_\beta$ on G_β are still well-defined when Σ is represented by a polygon, hence Lemma 3.3.4 still works completely well. Finally one can check that Proposition 3.3.6 is still valid for twisted modified adjacency matrices, therefore the proofs of Theorems 3.2.2 and 3.2.4 in the orientable case can be applied verbatim to prove Theorems 3.4.3 and 3.4.2. $\square$

Chapter 4

The Ising model on graphs embedded in non-orientable surfaces

In this final chapter, we will study the Ising model on graphs embedded in (possibly) non-orientable surfaces. This chapter consists of two parts. The first part (Sections 4.2 to 4.4) is devoted to a self-contained elementary proof of a Pfaffian formula to compute the Ising partition function of graphs embedded in an arbitrary closed surface. In the second part (Section 4.5), using the Pfaffian formula restricted to the Klein bottle, together with a result by Kenyon-Sun-Wilson [44] as well as an unpublished work of Cimasoni-Kassel [12], we study the macroscopic behaviour of the Ising model when the underlying graph is embedded in the Klein bottle and enlarged periodically. This enables us to compute the associated finite-size corrections.

4.1 Introduction

The Ising model is probably one of the most famous models in statistical physics, and one of the classical problems for this model is to compute the partition function

$$Z_\beta^J(G) := \sum_{\sigma\in\Omega(G)} \exp(-\beta H(\sigma))$$

of a graph G with an edge weight system J and an inverse temperature β (cf. Section 1.1 for precise definition). By the observation of van der Waerden [65], calculating the partition function $Z_\beta^J(G)$ of the Ising model on the graph G with fixed J and β can be transformed to calculating its high-temperature expansion

$$Z_{\mathcal{I}}(G) = \sum_{G'\subset G} \prod_{e\in E(G')} x_e,$$

where the sum is over all even subgraphs G' of G, i.e., subgraphs such that each vertex of G is met by an even number of edges of G', and $x_e = \tanh(\beta J_e)$. One of the aims of this chapter is to give a formula for computing this high-temperature expansion efficiently, when the graph G is embedded in a closed surface.

Before stating the formula, let us first quickly summarise some results that have been achieved for this computation so far. As already reviewed in the introduction of the thesis, the very first solution to the computation of the Ising partition function for a square lattice G is due to Onsager [54]. However his solution seemed to be very difficult, motivating Kac and Ward to find an easier approach [34]: they defined a so-called *Kac-Ward matrix* $M(G)$ with rows and columns indexed by oriented edges of G, and proved that $Z_{\mathcal{I}}(G) = (\det M(G))^{1/2}$. This formula is now known as *Kac-Ward formula*. Unluckily, some arguments in [34] are not correct. It was only in 1999, by Dolbilin *et al.* [17], that the first direct combinatorial proof of the Kac-Ward formula for any planar graph G with straight edges was obtained. In the meantime, there were several attempts by Sherman [59], Hurst-Green [28], Kasteleyn [35] and Fisher [20] to prove the Kac-Ward formula indirectly (still for a square lattice) by relating the high-temperature expansion with the dimer partition function of some associated graph. The latter then can be computed using the Pfaffian method that we already reviewed in Section 1.4. It should be recalled that this method was extended by Kasteleyn [36, 37] to any planar graph, by Galluccio-Loebl [22] to any graph embedded in an orientable surface with a particular drawing in the plane, by Tesler [63] to any graph drawn in the plane, and by Cimasoni-Reshetikhin [14, 8] to any graph embedded in a surface. Afterwards, inspired by the two approaches mentioned above, Cimasoni [9] gave two different proofs for a general Kac-Ward formula, which holds for any graph embedded in an orientable surface.

Let us discuss these two proofs in more detail. The first one, which is completely combinatorial, relies on the proof of Dolbilin *et al.* [17] for planar graphs. In fact these authors developed nearly all of necessary tools to obtain the general Kac-Ward formula, but only a right notion of Kac-Ward matrices for surface graphs was missing. Actually it turns out that Kac-Ward matrices in this general case can be encoded using spin structures [9, Definition 1], and the proof in [17] can be adapted. The second proof of the general Kac-Ward formula is based on the *Fisher correspondence* $G \mapsto \Gamma_G$ where Γ_G is an associated graph still embedded in the same surface as G. Then by relating the determinant of generalised Kac-Ward matrices of G with the Pfaffian of adjacency matrices of Γ_G, and using the Pfaffian formula to compute the dimer partition function of Γ_G (Theorem 1.4.7) one obtains the general Kac-Ward formula.

Following these two proofs, it is possible to extend the general Kac-Ward formula to graphs embedded in (possibly) non-orientable surfaces. Let us first consider the approach that goes through the dimer partition function of an associated graph. It was first observed by Kasteleyn [36] that one can transform the Ising model on a graph G to its associated *terminal graph* G^T. (However G^T in general is not embedded in the same surface as G, so we can not apply the Pfaffian formula, namely Theorems 1.4.7 and 1.4.8, for G^T.) Building on this idea of Kasteleyn, Chelkak-Cimasoni-Kassel [7] also obtained the general Kac-Ward formula for graphs embedded in an orientable surface as in [9]. In more detail, they multiplied a generalised Kac-Ward matrix $KW_\lambda(G)$ encoded by a spin structure λ

with suitable ones to obtain a new matrix $\widehat{K}_\lambda$ which is skew-symmetric, whose determinant is the same as that of $KW_\lambda(G)$. Since this new matrix can be thought of as a weighted adjacency matrix of the terminal graph G^T, its Pfaffian $\mathrm{Pf}\,(\widehat{K}_\lambda)$ counts weighted perfect matchings of G^T with signs. The latter quantity then can be proved combinatorially to be equal to the Ising high-temperature expansion twisted by signs, and therefore we end up with the general Kac-Ward formula.

As mentioned in [7, p.15], the signs of perfect matchings of G^T in the expansion of $\mathrm{Pf}\,(\widehat{K}_\lambda)$ amounts to saying that the spin structure λ is equivalent to a *crossing orientation* of G^T as defined by Tesler [63]. Therefore it is possible to replace the generalised Kac-Ward matrix $KW_\lambda(G)$ by an adjacency matrix of G^T with respect to the corresponding crossing orientation so that the general Kac-Ward formula for G boils down to a Pfaffian formula for G^T. Inspired by this fact, for a graph G embedded in a (possibly) non-orientable surface Σ, we define a *good orientation* K on G^T with respect to a particular drawing of G^T in the plane (cf. Definition 4.2.1). This allows us to prove the following Pfaffian formula for the Ising partition function of G

$$Z_{\mathcal{I}}(G,x) = \frac{\epsilon_0}{2^{b_1/2}} \sum_{q\in\mathcal{Q}(\Sigma,\omega)} \exp\left(\frac{i\pi}{4}\right)^{-\mathrm{Br}(q)} \mathrm{Pf}(A^{K_q,\omega}(G^T,x^T)).$$

In this formula $\epsilon_0 = \pm 1$ is a constant, $b_1 = \dim H_1(\Sigma;\mathbb{Z}_2)$, K_q is an orientation derived from a good orientation K, and $A^{K_q,\omega}(G^T,x^T)$ is the twisted adjacency matrix of the terminal graph (G^T,x^T) with respect to K_q and the standard representation ω of w_1. This formula is the content of Theorem 4.4.4, and a more practical version can be found in Theorem 4.4.1.

Let us emphasise that one can use the whole proof of Tesler [63] for crossing orientations on arbitrary graphs to prove our formula. However, it turns out that in the case of the terminal graph, the method developed by Tesler can be reduced extremely. Hence together with an idea of Loebl-Masbaum [48], we get an elementary, self-contained proof for the Pfaffian formula.

To terminate the first part of this chapter, let us mention that using the method of Dolbilin *et al.* [17] (and then developed by Cimasoni [9]) one can have a Kac-Ward formula for graphs embedded in (possibly) non-orientable surfaces. In this case generalised Kac-Ward matrices can be encoded using pin^- structures, which generalise spin structures to non-orientable surfaces [46]. However we will not present this approach here.

The second aim of this chapter is to study the Ising model when the underlying graph, which is embedded in the Klein bottle, is extended periodically. To be more precise, let G be a graph embedded in the torus $\mathbb{T}$ or the Klein bottle $\mathscr{K}$. Representing these surfaces as rectangles with sides identified suitably, one can enlarge G by gluing MN copies of the corresponding rectangles together (with M copies in one direction and N copies in the other) to obtain a new graph G_{MN}. It is conjectured by Cardy-Peschel [6] that at criticality, for M, N big we have

$$Z_{\mathcal{I}}(G_{MN}) = \exp(MN\mathrm{f}_0 + \mathrm{fsc} + o(1)),$$

where $\mathrm{f}_0 := \lim_{M,N\to\infty} \frac{1}{MN} \log Z_{\mathcal{I}}(G_{MN})$ is the *free energy* and fsc is a *finite-size correction* term, which only depends on the conformal shape of the domain, the parity of M, N and the surface that G_{MN} is embedded in (a.k.a the boundary conditions). Note that as pointed out in [6], a similar expansion is also expected to hold if G is embedded in the cylinder $\mathscr{C}$ or in the Möbius band $\mathscr{M}$ so that finite-size correction terms still fulfil the above conditions. For example, these correction terms were successfully computed by Lu-Wu [52] for any square lattice embedded in the Möbius band or the Klein bottle at criticality. Izmailian [30] then also computed these terms completely (still for a square lattice) with all possible boundary conditions. Furthermore, the above expansion of $Z_{\mathcal{I}}(G_{MN})$ is expected to be true for any graph, and for a lot of models. For instance, this expansion was obtained by Izmailian-Oganesyan-Hu [32] for the dimer model, and by Izmailian-Kenna [31] for the spanning-tree model, both on the square lattice with various boundary conditions. Additionally this expansion was also obtained by Boutilier and de Tilière [2] for the dimer model on the hexagonal lattice embedded in the torus. Moreover by Kenyon-Sun-Wilson [44], this expansion holds for the dimer model on arbitrary graphs embedded in the torus (i.e. periodic-periodic boundary conditions), under some conjectural assumptions on the zero locus of the *dimer characteristic polynomial*. Using the result in [44] and knowing precisely zeros of the Ising characteristic polynomial of toroidal graphs [11], Cimasoni [10, Corollary 3.5] also obtained the similar expansion for the Ising model on toroidal graphs without any additional assumption.

In the present work, we will study the correction term of the Ising model at criticality for periodic-antiperiodic boundary conditions, that is, when G_{MN} (and hence G) is embedded in the Klein bottle $\mathscr{K}$. Inspired by [42] we shall also define an *Ising characteristic polynomial* $P(z, w)$ for $G \subset \mathscr{K}$, which is the determinant of a twisted adjacency matrix $A(z, w)$ of the terminal graph G^T (see Definition 4.5.1). Our core result, which is based on an in-progress work of Cimasoni-Kassel [12], is a product formula which relates the characteristic polynomials of G and of G_{MN} (Theorem 4.5.14).

Let us explain how to get this product formula in more detail. In fact one can define a *general twisted adjacency matrix* A^ρ_G with respect to a representation ρ of $\pi_1(G)$ so that this general twisted matrix reduces to the usual twisted one for certain degree-one representation ρ (cf. Definition 4.5.4). It turns out that if $\rho = \oplus\rho_i$, the determinant of A^ρ_G can be written as the product of that of $A^{\rho_i}_G$. Moreover by [12], for a finite covering $G^\# \to G$, the determinant of $A^\rho_{G^\#}$ is equal to that of $A^{\rho^\#}_G$ where $\rho^\#$ is the induced representation of ρ from $\pi_1(G^\#)$ to $\pi_1(G)$. Therefore looking for a product formula for the characteristic polynomial of G_{MN}, which is equal to the determinant of $A^\rho_{G_{MN}}$ for a well-chosen ρ, boils down to looking for a decomposition of the corresponding $\rho^\#$ into irreducible representations. In particular, if the covering $G_{MN} \to G$ is an abelian cover, then $\rho^\#$ factors through $\pi_1(G)/\pi_1(G_{MN})$ which is abelian, and so $\rho^\#$ can be decomposed as a direct sum of degree-one representations. For example this fact holds when G is in the torus, and we get back the product formula in [42, Theorem 3.3]. It is not the case for G in the Klein bottle, however in this case one can show that $\rho^\#$ factors through a dihedral group, and hence it can be decomposed into irreducible representations of degree one or two. This leads to our product formula in this case, as stated in Theorem 4.5.14.

As a consequence, one can relate $P(z,w)$ with the Ising characteristic polynomial of $\widetilde{G} \subset \mathbb{T}$, the lift of $G \subset \mathscr{K}$ to the orientation cover $\mathbb{T}$, and therefore we know exactly the zero locus of $P(z,w)$. Using the product formula and [44, Theorem 1], together with some symmetric properties of P, we can show that the correction term under periodic-antiperiodic boundary conditions can be given by the leading coefficients and the roots of modulus bigger than 1 of the polynomials $P(\pm 1, w)$, together with certain functions that give the correction term of the Ising model on $\widetilde{G} \subset \mathbb{T}$ (see Theorem 4.5.21).

This chapter is organised as follows. In Section 4.2 we state a Pfaffian formula for (the high-temperature expansion of) the Ising partition function for graphs embedded in orientable surfaces (Theorems 4.2.2 and 4.2.3). The proof of this formula is presented in subsection 4.3.2. We conclude the first part of this chapter by Section 4.4, in which we show a Pfaffian formula for the Ising partition function of graphs embedded in a non-orientable surface (Theorems 4.4.1 and 4.4.4). In the second part (Section 4.5) we restrict ourselves to graphs embedded in the Klein bottle. We define characteristic polynomials in subsection 4.5.1, show a product formula in subsection 4.5.2 (Theorem 4.5.14) and study finite-size correction terms in subsection 4.5.3 (Theorem 4.5.21).

4.2 Pfaffian formula for the Ising partition function

The most general Pfaffian formula for the Ising partition function of graphs embedded in surfaces can be found in Theorems 4.4.1 and 4.4.4 where surfaces are possibly non-orientable. However for simplicity, in this section let us first consider the case of orientable surfaces as a warm-up case, and study the case of non-orientable surfaces later.

4.2.1 Graph drawing and good orientations

Before stating the formula, let us first give some basic notions and terminology that are needed. More precisely, we shall describe a particular drawing of the graph G and its terminal graph G^T in the plane, together with some particular choice of orientations. For more details and properties, we refer readers to subsection 4.3.1.

First of all, we will need a fixed drawing of G in the plane which is specified as follows. Recall that G is embedded in the orientable surface Σ of genus g. Firstly, we represent Σ as a planar $4g$-gon $\mathcal{P}$ with $2g$ pairs of sides identified following the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$, and draw G in this polygon so that all the edges of G intersect the $4g$ sides transversely. Secondly, for each pair of identified sides of $\mathcal{P}$, we add a strip to the outside of $\mathcal{P}$ connecting them (so that this strip is consistent with the side identification), and then extend all the edges of G intersecting these two sides inside this strip (still following the side identification) so that their extended parts intersect each other transversely. Finally, ignoring all the sides of $\mathcal{P}$ as well as the added strips, we are left with the desired drawing of the graph G (see Figure 4.1). For further purposes, let us call edges of G which are partly outside $\mathcal{P}$ the *outside* edges.

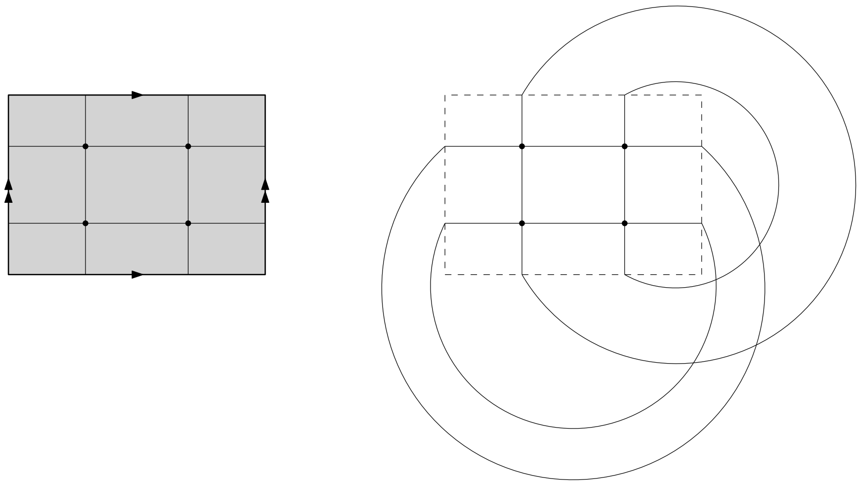


Figure 4.1: The 2×2 square lattice embedded in the torus and its drawing in the plane.

Now let us continue with the terminal graph G^T, and particular orientations of interest. The idea of using the terminal graph to treat the Ising model is due to Kasteleyn [36] for a square lattice. Then Chelkak-Cimasoni-Kassel [7] adapt this idea to graphs embedded in orientable surfaces. Here we also employ this idea to our setting. Recall that the *terminal graph* G^T (cf. [36]) associated to a given graph G is the one obtained by replacing each vertex v of degree $d(v)$ of G by a complete graph $K_{d(v)}$ (see Figure 4.2). Note that this transformation is local, so the fixed drawing of G from the beginning also induces a drawing of G^T in the plane. We shall say that an edge of G^T is *short* if it is an edge of one of complete graphs, and *long* otherwise (i.e., if it comes from an edge of G). The long edges of G^T which partly lie outside $\mathcal{P}$ will be also called *outside* edges, and *inside* otherwise. Given edge weights $x = (x_e)_{e \in E(G)}$ on G, we shall denote by x^T the edge weights on G^T obtained by assigning weight 1 to all long edges and weight $(x_e x_{e'})^{1/2}$ to the short edge corresponding to the two adjacent edges $e, e' \in E(G)$.

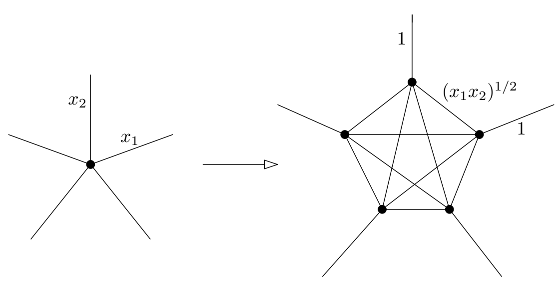


Figure 4.2: Local transformation at each vertex of G to obtain G^T.

We now turn to some specific orientations on G^T. First of all, let us assume from now on that the plane is oriented counterclockwise. As mentioned above, we always fix the drawing of G as well as the induced drawing of G^T in the plane. With this drawing of G^T, we are interested in two types of faces: a face f of G^T is called *inside* if it is homeomorphic

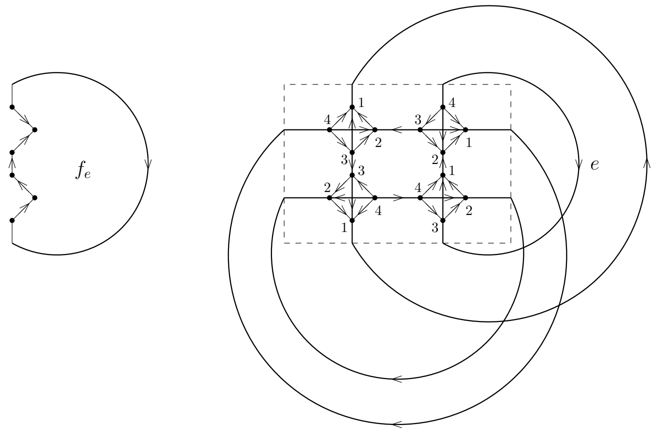


Figure 4.3: An outside face and a good orientation on the terminal graph of the 2×2 square lattice.

to a disc, and completely lies inside $\mathcal{P}$; the *outside* face f_e corresponding to an outside edge e, by ignoring all the other outside edges, is the unique face formed by e and some edges along the boundary of the subgraph obtained from G^T by removing all the other outside edges (see Figure 4.3). We also recall that each vertex $v \in V(G)$ of degree $d(v)$ gives rise to $d(v)$ vertices of the complete graph $K_{d(v)}$, and furthermore this complete graph can be drawn inside a small disc with the vertices on its boundary. Let us label these vertices by $1, \ldots, d(v)$ with respect to the clockwise orientation (the starting vertex is not important). Inspired by Kasteleyn [36] and Tesler [63] we define an orientation of interest as follows.

Definition 4.2.1. *An orientation K on the edges of the terminal graph G^T is called* good *if the three following conditions hold.*

(i) K is from big-labelled vertices to small-labelled vertices on short edges.

(ii) For every inside face f of G^T, the number $n^K(\partial f)$ of edges on its boundary ∂f where the orientation of ∂f is different from K is odd.

(iii) For every outside face f_e, $n^K(\partial f_e)$ is also odd.

Note that there always exists good orientations (see Figure 4.3 for example), however the proof of this fact is postponed until Proposition 4.3.5. Let us pick a good orientation and denote it by K_0. We now show how to derive other specific orientations from K_0. Recall that the $4g$-gon $\mathcal{P}$ has $4g$ sides symbolised by the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$, and this word induces the basis $\mathcal{B} := \{[a_1], \ldots, [a_g], [b_1], \ldots, [b_g]\}$ of $H_1(\Sigma; \mathbb{Z}_2)$. For each $\Delta \in H_1(\Sigma; \mathbb{Z}_2)$, let us write

$$\Delta = \sum_{i=1}^{g} \epsilon_i [a_i] + \sum_{i=1}^{g} \epsilon'_i [b_i],$$

with $\epsilon_i, \epsilon'_i \in \mathbb{Z}_2$. So by fixing the basis $\mathcal{B}$ (that we always do), one can identify Δ with $(\epsilon, \epsilon') \in \mathbb{Z}_2^{2g}$ where $\epsilon = (\epsilon_1, \ldots, \epsilon_g)$ and $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_g)$ are elements of $\mathbb{Z}_2^g$. Then for each

$\Delta \in H_1(\Sigma; \mathbb{Z}_2)$, or equivalently, for each $(\epsilon, \epsilon') \in \mathbb{Z}_2^{2g}$, let us denote by $K_{\epsilon,\epsilon'}$ the orientation obtained by inverting K_0 on every edge e each time e crosses a side a_i (resp. b_j) of $\mathcal{P}$ with $\epsilon_i = 1$ (resp. $\epsilon'_j = 1$). The collection $\{K_{\epsilon,\epsilon'} : (\epsilon, \epsilon') \in \mathbb{Z}_2^{2g}\}$ consists of orientations of interest.

We are now ready to state the first version of the Pfaffian formula.

4.2.2 Statement of the Pfaffian formula

Theorem 4.2.2. *Let (G, x) be a weighted graph embedded in the orientable surface Σ of genus g. Then the Ising partition function of G is given by*

$$Z_{\mathcal{I}}(G, x) = \frac{1}{2^g}\left| \sum_{(\epsilon,\epsilon')\in\mathbb{Z}_2^{2g}} (-1)^{\sum_{i=1}^{g} \epsilon_i\epsilon'_i} Pf(A^{K_{\epsilon,\epsilon'}}(G^T, x^T))\right|,$$

where $A^{K_{\epsilon,\epsilon'}}(G^T, x^T)$ is the adjacency matrix of the terminal graph (G^T, x^T) with respect to the orientation $K_{\epsilon,\epsilon'}$.

We now continue with the second version of the Pfaffian formula. Imagine that the $4g$-gon $\mathcal{P}$ is embedded in $\mathbb{R}^3$ together with the added strips. One can expand all the strips suitably so that we get back the original surface Σ with $\mathcal{P}$ as its subset. Then one can see that $H_1(\Sigma; \mathbb{Z}_2)$ is isomorphic to $H_1(\Sigma, \mathcal{P}; \mathbb{Z}_2)$, and thus a quadratic form $q \in \mathcal{Q}(\Sigma)$ can be considered as a $\mathbb{Z}_2$-valued function on $H_1(\Sigma, \mathcal{P}; \mathbb{Z}_2)$. Let q_0 be the quadratic form getting value 0 on the basis $\mathcal{B}$. For each quadratic form $q \in \mathcal{Q}(\Sigma)$, we define K_q as the orientation obtained by inverting K_0 on every outside edge e such that $q([e]) \neq q_0([e])$. (It is trivial that by this notation one has $K_0 = K_{q_0}$.) Then we can state the second version of the Pfaffian formula as follows.

Theorem 4.2.3. *Let (G, x) be a weighted graph embedded in an orientable surface Σ of genus g. Then the Ising partition function of G is given by*

$$Z_{\mathcal{I}}(G, x) = \frac{\epsilon_0}{2^g} \sum_{q\in\mathcal{Q}(\Sigma)} (-1)^{Arf(q)} Pf(A^{K_q}(G^T, x^T)),$$

where $\epsilon_0 = \pm 1$ is a constant. In this formula, $A^{K_q}(G^T, x^T)$ is the adjacency matrix of the terminal graph (G^T, x^T) with respect to the orientation K_q.

4.3 Proof of the Pfaffian formula

This section is spent to prove the two versions of the Pfaffian formula. However, before giving their proofs, let us again go through all the basis notions and terminology represented in the previous section to provide more details and needed results. In subsection 4.3.1 we provide necessary materials for our proofs. In subsection 4.3.2 we state our key result, Proposition 4.3.7, from which we can prove the two versions of the Pfaffian formula easily. Finally in the last subsection, we will show an elementary proof of Proposition 4.3.7.

4.3.1 Preliminaries

In this subsection, we recall some results of Loebl-Masbaum [48], Tesler [63] and Chelkak-Cimasoni-Kassel [7]; the easy proofs are included for completeness.

Let us first come back to the drawing of G and see how the intersection form $(\cdot)$ on $H_1(\Sigma;\mathbb{Z}_2)$ relates to the one in the plane. Recall that in the previous section, we identify $H_1(\Sigma;\mathbb{Z}_2)$ with $H_1(\Sigma,\mathcal{P};\mathbb{Z}_2)$.

Lemma 4.3.1. *Let e_1, e_2 be two outside edges of G which induce two relative homology classes $[e_1],[e_2]\in H_1(\Sigma,\mathcal{P};\mathbb{Z}_2)$. Then we have the following equality modulo 2*

$$[e_1]\cdot[e_2] = e_1\cdot e_2,$$

where the intersection in the left-hand side is in $H_1(\Sigma;\mathbb{Z}_2)$, while the right-hand side is the geometric intersection number in the plane.

Proof. For $i=1,2$ let e'_i be any loop obtained from e_i by connecting their two endpoints in an arbitrary way. By the fixed drawing of G, it is obvious that

$$[e_1]\cdot[e_2] + e_1\cdot e_2 = e'_1\cdot e'_2.$$

The last quantity is always 0 modulo 2 because any two loops in the plane, if cross each other transversely, will cross an even number of times. $\square$

Consequently, we get the following result, which is due to Loebl-Masbaum [48, Proposition 2.6].

Proposition 4.3.2. *For every $M\subset E(G)$, the number $t(M)$ of its self-intersections in the plane has the same parity as $q_0([M])$, where $[M]\in H_1(\Sigma,\mathcal{P};\mathbb{Z}_2)$ is the relative homology class of M.*

Proof. Assume that $M=\{e_k\}_{1\le k\le m}$, and for each k write $[e_k]=\sum_{i=1}^g \alpha_i^k[a_i]+\sum_{i=1}^g\beta_i^k[b_i]$ with $\alpha_i^k,\beta_i^k\in\mathbb{Z}_2$. By the fixed drawing of G and using the fact that $[a_i]\cdot[b_j]=\delta_{ij}$ and $[a_i]\cdot[a_j]=[b_i]\cdot[b_j]=0$, one sees that the number $t(e_k)$ of self-intersections of e_k is equal to $\sum_{i=1}^g\alpha_i^k\beta_i^k$. Using this fact and the definition of quadratic forms, as well as Lemma 4.3.1, we have the following equalities modulo 2

$$\begin{aligned}
q_0([M]) &= q_0(\sum_k [e_k]) = \sum_k q_0([e_k]) + \sum_{k<l}[e_k]\cdot[e_l]\\
&= \sum_k q_0(\sum_{i=1}^g \alpha_i^k[a_i] + \sum_{i=1}^g \beta_i^k[b_i]) + \sum_{k<l}[e_k]\cdot[e_l]\\
&= \sum_k \big(q_0(\sum_{i=1}^g \alpha_i^k[a_i]) + q_0(\sum_{i=1}^g \beta_i^k[b_i]) + (\sum_{i=1}^g \alpha_i^k[a_i])\cdot(\sum_{i=1}^g \beta_i^k[b_i])\big) + \sum_{k<l}[e_k]\cdot[e_l]\\
&= \sum_k (\sum_{i=1}^g \alpha_i^k[a_i])\cdot(\sum_{i=1}^g \beta_i^k[b_i]) + \sum_{k<l}[e_k]\cdot[e_l] = \sum_k\sum_{i=1}^g \alpha_i^k\beta_i^k + \sum_{k<l}[e_k]\cdot[e_l]
\end{aligned}$$

$$= \sum_k t(e_k) + \sum_{k<l} e_k \cdot e_l = t(M).$$

Note that the fifth equality comes from the fact that $q_0([a_i]) = q_0([b_i]) = 0$. The proof is concluded. □

Next let us move on by showing how to transform the partition function of the Ising model on G (twisted with some signs) to that of the dimer model on G^T. To do so, we need some notions and terminology. Recall that $\mathcal{D}(G^T)$ denotes the set of dimer configurations on G^T. We also write $\mathcal{D}_{2n}$ for $\mathcal{D}(K_{2n})$. For $D \in \mathcal{D}(G^T)$, we define $t_{\mathrm{in}}(D)$ (resp. $t_{\mathrm{out}}(D)$) its number of self-intersections lying inside (resp. outside) the polygon $\mathcal{P}$. Note that $t_{\mathrm{in}}(D)$ counts the number of crossings created by short edges of D, while $t_{\mathrm{out}}(D)$ counts the number of those created by long edges. It is trivial that the total number $t(D)$ of self-intersections of D satisfies $t(D) = t_{\mathrm{in}}(D) + t_{\mathrm{out}}(D)$. Representing $K_{d(v)}$ inside a closed disc with its vertices on the boundary of the disc, we see that, for every $D \in \mathcal{D}(G^T)$, the parity of $t_{\mathrm{in}}(D)$ does not depend on the way $K_{d(v)}$ is drawn. Furthermore, if $d(v)$ is even, we have the following result which is due to Chelkak-Cimasoni-Kassel [7, Lemma 2.1].

Lemma 4.3.3. *For any integer $n \geq 1$, $\sum_{D\in\mathcal{D}_{2n}} (-1)^{t_{in}(D)} = 1$.*

Proof. Fix two vertices of K_{2n} and consider the map $\sigma : \mathcal{D}_{2n} \to \mathcal{D}_{2n}$ given by exchanging them. The fixed point set $\mathrm{Fix}(\sigma)$ consists of all the dimer configurations of K_{2n} matching these two vertices. Since $t(D) = t(\sigma(D)) + 1$ for $D \notin \mathrm{Fix}(\Sigma)$ and σ is bijective, we get

$$\begin{aligned}
\sum_{D\in\mathcal{D}_{2n}} (-1)^{t(D)} &= \sum_{D\in\mathrm{Fix}(\sigma)} (-1)^{t_{\mathrm{in}}(D)} + \sum_{D\notin\mathrm{Fix}(\sigma)} (-1)^{t_{\mathrm{in}}(D)} \\
&= \sum_{D'\in\mathcal{D}_{2(n-1)}} (-1)^{t_{\mathrm{in}}(D')} + \frac{1}{2}\sum_{D\notin\mathrm{Fix}(\sigma)} \left((-1)^{t_{\mathrm{in}}(D)} + (-1)^{t_{\mathrm{in}}(\sigma(D))}\right) \\
&= \sum_{D'\in\mathcal{D}_{2(n-1)}} (-1)^{t_{\mathrm{in}}(D')}.
\end{aligned}$$

The lemma now follows by induction on $n \geq 1$. □

Now if we denote by $Z_1(G; \mathbb{Z}_2)$ the set of all 1-cycles modulo 2 in G, that is,

$$Z_1(G; \mathbb{Z}_2) = \{P = \sum_e e \in C_1(G; \mathbb{Z}_2) : \partial_1 P = 0 \in C_0(G; \mathbb{Z}_2)\}$$

where the sum is over some finite set of different edges of G, and set $x(P) := \prod_{e\in P} x_e$ for each $P \in Z_1(G; \mathbb{Z}_2)$, then we have the following result.

Lemma 4.3.4. *Setting $Z^q(G, x) := \sum\limits_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q(\alpha)} \sum\limits_{[P]=\alpha} x(P)$ for each quadratic form $q \in \mathcal{Q}(\Sigma)$, we have*

$$Z^q(G, x) = \sum_{D\in\mathcal{D}(G^T)} (-1)^{q([D\Delta D_0])+t_{in}(D)} x^T(D). \tag{4.1}$$

In this formula, D_0 is the standard dimer configuration of G^T, i.e., the one consisting of all the long edges.

Note that this equality was mentioned in [7, Subsection 4.2] but not with full details, so let us give its proof here for the sake of completeness.

Proof of Lemma 4.3.4. Recall that (G^T, x^T) is the associated terminal graph of the embedded graph (G, x) in Σ. Given a dimer configuration $D \in \mathcal{D}(G^T)$, let D_G denote the subgraph of G given by the edges of G corresponding to the long edges of D. We claim that $G \setminus D_G$ is an even subgraph of G. Indeed, consider a vertex v of G, and denote by $m(v)$ the degree of v in D_G. In and around the complete graph $K_{d(v)}$, the number of vertices that D matches is exactly $d(v) + m(v)$, which is even since D is a dimer configuration. Since the degree of v in $G \setminus D_G$ is equal to $d(v) - m(v)$, the claim follows. Therefore the assignment $D \mapsto G \setminus D_G$ defines a map $\varphi : \mathcal{D}(G^T) \to Z_1(G; \mathbb{Z}_2)$. Moreover, we have $x(P) = x(\varphi(D)) = x^T(D)$, and $\varphi^{-1}(P) = \prod_{v \in V} \mathcal{D}_{2n(v)}$, where V denotes the set of all vertices of G, and $2n(v)$ denotes the degree of v in P. From these facts and Lemma 4.3.3, we can write

$$\begin{aligned}
Z^q(G,x) &= \sum_{\alpha \in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q(\alpha)} \sum_{[P]=\alpha} x(P) \\
&= \sum_{\alpha \in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q(\alpha)} \sum_{[P]=\alpha} \Big(\prod_{v \in V} \sum_{D_v \in \mathcal{D}_{2n(v)}} (-1)^{t_{\mathrm{in}}(D_v)} \Big) x(P) \\
&= \sum_{\alpha \in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q(\alpha)} \sum_{[P]=\alpha} \sum_{D \in \varphi^{-1}(P)} (-1)^{t_{\mathrm{in}}(D)} x^T(D) \\
&= \sum_{D \in \mathcal{D}(G^T)} (-1)^{q([G \setminus D_G]) + t_{\mathrm{in}}(D)} x^T(D) \\
&= \sum_{D \in \mathcal{D}(G^T)} (-1)^{q([D \Delta D_0]) + t_{\mathrm{in}}(D)} x^T(D).
\end{aligned}$$

The last equality comes from the fact that $G \setminus D_G$ is homologous to $D \Delta D_0$. □

Let us now discuss good orientations. The proposition below shows that they do exist, and how to construct one of them.

Proposition 4.3.5. *Fixing a drawing of the graph G and its terminal graph G^T as before, there always exists good orientations on G^T.*

Proof. Firstly let us label vertices in each complete graph increasingly with respect to the clockwise orientation, and orient short edges by condition (i) of Definition 4.2.1. Secondly let us pick an arbitrary orientation K on the inside long edges of G^T. For each inside face f of G^T with $n^K(\partial f)$ even, we draw a path from the interior of f to the outside of $\mathcal{P}$ so that this path only crosses long edges transversely, and invert K on each inside long edge crossed by this path. Repeating this procedure for each face f with $n^K(\partial f)$ even, together with the orientation on short edges, we get an orientation satisfying both conditions (i) and

(ii) of Definition 4.2.1. Finally, the orientations on the outside long edges are determined uniquely by condition (iii) of Definition 4.2.1. □

Remark 4.3.6. The definition of good orientations can be extended to subgraphs of G^T in the following sense. Suppose that G' is a subgraph of G^T obtained by removing some interior short edges of complete graphs so that G' has no self-intersections inside $\mathcal{P}$. A good orientation K on G^T restricts to K' on G', which is still a good orientation by Definition 4.2.1. Indeed, the conditions (i) and (iii) hold for K' immediately. We only need to check that the condition (ii) also holds for newly-created inside faces of G'. Let f' be such a face. By the way we label vertices in each complete graph and by condition (i), $n^{K'}(\partial f')$ is always equal to 1, which is odd. We will use this fact later in our proof.

4.3.2 Proof of Theorems 4.2.2 and 4.2.3

The aim of this subsection is to prove the two theorems stated in subsection 4.2.2. In order to do that, we need the following result.

Proposition 4.3.7. *For every dimer configuration D of G^T, we have*

$$\epsilon^{K_0}(D) = \epsilon_0(-1)^{t(D)}$$

where $\epsilon^{K_0}(D)$ is defined by Equation (1.4)and $\epsilon_0 = \pm 1$ is a constant.

It should be mentioned that, by this proposition, the good orientation K_0 by our definition is a *crossing orientation* as defined by Tesler [63]. Moreover, using this property of crossing orientations and some properties of the Arf invariant, Loebl and Masbaum [48] gave a direct proof of the Pfaffian formula for the dimer partition function. It turns out that their proof can be adapted to our situation for the Ising partition function, as we will show now. The proof of Proposition 4.3.7 will be left until the end of this section.

Proof of Theorem 4.2.3. First of all we recall that the orientation K_q is obtained by inverting K_0 on every edge e such that $q([e]) \neq q_0([e])$. Hence by Equation (1.3) we get $\epsilon^{K_q}(D) = \epsilon^{K_0}(D)(-1)^{|\{e\in D: q([e])\neq q_0([e])\}|}$. By definition of quadratic forms, we can write

$$q([D]) - q_0([D]) = \sum_{e\in D} q([e]) - \sum_{e\in D} q_0([e]) = |\{e \in D : q([e]) \neq q_0([e])\}| \in \mathbb{Z}_2,$$

which implies that $\epsilon^{K_q}(D) = \epsilon^{K_0}(D)(-1)^{q([D])-q_0([D])}$. Now by Proposition 4.3.2 we have $q_0([D]) = t_{\text{out}}(D)$ modulo 2 for every $D \in \mathcal{D}(G^T)$. This fact and Proposition 4.3.7 lead to

$$\begin{aligned}\epsilon^{K_q}(D) &= \epsilon_0(-1)^{t(D)}(-1)^{q([D])-q_0([D])} = \epsilon_0(-1)^{t_{\text{in}}(D)+t_{\text{out}}(D)}(-1)^{q([D])-q_0([D])}\\ &= \epsilon_0(-1)^{t_{\text{in}}(D)+q_0([D])}(-1)^{q([D])-q_0([D])} = \epsilon_0(-1)^{t_{\text{in}}(D)+q([D])}.\end{aligned}$$

Secondly, for D_0 the standard dimer configuration of G^T, its relative homology class $[D_0] \in H_1(\Sigma, \mathcal{P}; \mathbb{Z}_2)$ induces a dual, denoted by $[D_0]^* \in Hom(H_1(\Sigma, \mathcal{P}; \mathbb{Z}_2); \mathbb{Z}_2) = H^1(\Sigma, \mathcal{P}; \mathbb{Z}_2)$, which can be simply given by $[D_0]^*([D]) := [D] \cdot [D_0]$ for every $[D] \in H_1(\Sigma, \mathcal{P}; \mathbb{Z}_2)$. Note that since $[D_0]^*$ is linear, $q + [D_0]^*$ is still a quadratic form. Using the equation above for $q + [D_0]^*$ together with Lemma 4.3.4, one can write

$$\begin{aligned}
Z^q(G,x) &= \sum_{D \in \mathcal{D}(G^T)} (-1)^{q([D \Delta D_0]) + t_{\text{in}}(D)} x^T(D) \\
&= (-1)^{q([D_0])} \sum_{D \in \mathcal{D}(G^T)} (-1)^{q([D]) + [D]\cdot[D_0] + t_{\text{in}}(D)} x^T(D) \\
&= (-1)^{q([D_0])} \sum_{D \in \mathcal{D}(G^T)} (-1)^{(q + D_0^*)([D]) + t_{\text{in}}(D)} x^T(D) \\
&= \epsilon_0 (-1)^{q([D_0])} \sum_{D \in \mathcal{D}(G^T)} \epsilon^{K_{q+[D_0]^*}}(D) x^T(D) \\
&\overset{(1.2)}{=} \epsilon_0 (-1)^{q([D_0])} \mathrm{Pf}(A^{K_{q+[D_0]^*}}(G^T, x^T)). \qquad (4.2)
\end{aligned}$$

Finally, let us recall that our purpose is to give a formula to calculate the Ising partition function on G

$$Z_{\mathcal{I}}(G,x) = \sum_{G' \subset G} \prod_{e \in E(G')} x_e,$$

where the sum is over all even subgraphs G' of G, or in other words, over the set $Z_1(G; \mathbb{Z}_2)$ of all 1-cycles modulo 2 in G (recall the proof of Lemma 4.3.4). Therefore the Ising partition function can be rewritten as follows

$$\begin{aligned}
Z_{\mathcal{I}}(G,x) &= \sum_{P \in Z_1(G;\mathbb{Z}_2)} x(P) = \sum_{\alpha \in H_1(\Sigma;\mathbb{Z}_2)} \sum_{[P]=\alpha} x(P) \\
&\overset{\text{Lem. } 1.2.3(i)}{=} \sum_{\alpha \in H_1(\Sigma;\mathbb{Z}_2)} \left(\frac{1}{2^g} \sum_{q \in \mathcal{Q}(\Sigma)} (-1)^{\mathrm{Arf}(q) + q(\alpha)} \right) \sum_{[P]=\alpha} x(P) \\
&= \frac{1}{2^g} \sum_{q \in \mathcal{Q}(\Sigma)} (-1)^{\mathrm{Arf}(q)} \sum_{\alpha \in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q(\alpha)} \sum_{[P]=\alpha} x(P) \\
&= \frac{1}{2^g} \sum_{q \in \mathcal{Q}(\Sigma)} (-1)^{\mathrm{Arf}(q)} Z^q(G,x) \\
&\overset{(4.2)}{=} \frac{\epsilon_0}{2^g} \sum_{q \in \mathcal{Q}(\Sigma)} (-1)^{\mathrm{Arf}(q) + q([D_0])} \mathrm{Pf}(A^{K_{q+[D_0]^*}}(G^T, x^T)) \\
&\overset{\text{Lem. } 1.2.3(ii)}{=} \frac{\epsilon_0}{2^g} \sum_{q \in \mathcal{Q}(\Sigma)} (-1)^{\mathrm{Arf}(q + [D_0]^*)} \mathrm{Pf}(A^{K_{q+[D_0]^*}}(G^T, x^T)) \\
&= \frac{\epsilon_0}{2^g} \sum_{q \in \mathcal{Q}(\Sigma)} (-1)^{\mathrm{Arf}(q)} \mathrm{Pf}(A^{K_q}(G^T, x^T)).
\end{aligned}$$

This concludes the proof. $\square$

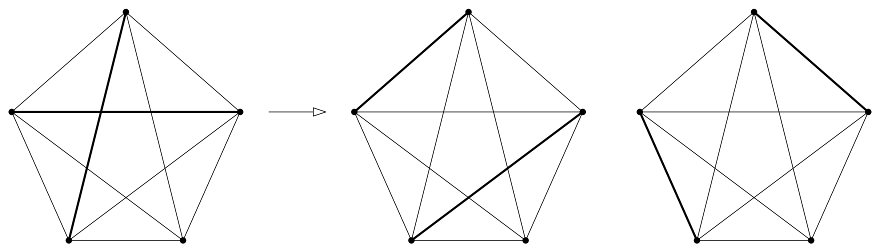


Figure 4.4: Reduce the number of self-intersections of a dimer configuration.

Let us continue by showing how Theorem 4.2.3 implies Theorem 4.2.2.

Proof of Theorem 4.2.2. From Theorem 4.2.3, one can write

$$Z_{\mathcal{I}}(G,x) = \frac{1}{2^g}\Bigg|\sum_{q\in\mathcal{Q}(\Sigma)} (-1)^{\mathrm{Arf}(q)+\mathrm{Arf}(q_0)}\mathrm{Pf}(A^{K_q}(G^T,x^T))\Bigg|.$$

By part (*ii*) of Lemma 1.2.3, for each quadratic form q, there exists an element $\Delta_q \in H_1(\Sigma;\mathbb{Z}_2)$ such that $\mathrm{Arf}(q)+\mathrm{Arf}(q_0) = q_0(\Delta_q)$. This element Δ_q is determined by its Poincaré dual $\Delta_q^* \in Hom(H_1(\Sigma;\mathbb{Z}_2);\mathbb{Z}_2)$ given by $\Delta_q^*(\alpha) = q(\alpha) + q_0(\alpha)$ for every $\alpha \in H_1(\Sigma;\mathbb{Z}_2)$. Moreover, the correspondence $q \mapsto \Delta_q$ is bijective, thus we can write

$$Z_{\mathcal{I}}(G,x) = \frac{1}{2^g}\Bigg|\sum_{\Delta\in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q_0(\Delta)}\mathrm{Pf}(A^{K_{q_0+\Delta^*}}(G^T,x^T))\Bigg|.$$

Now recall that, fixing the basis $\mathcal{B} = \{[a_1],\dots,[a_g],[b_1],\dots,[b_g]\}$ of $H_1(\Sigma;\mathbb{Z}_2)$, each element $\Delta \in H_1(\Sigma;\mathbb{Z}_2)$ can be written as $\Delta = \sum_{i=1}^g \epsilon_i[a_i] + \sum_{i=1}^g \epsilon'_i[b_i]$, and thus Δ can be identified with $(\epsilon,\epsilon') \in \mathbb{Z}_2^{2g}$. Since $q_0([a_i]) = q_0([b_i]) = 0$ and $[a_i]\cdot[b_j] = \delta_{ij}$ while $[a_i]\cdot[a_j] = [b_i]\cdot[b_j] = 0$ for every $1 \le i,j \le g$, by definition of quadratic forms we get

$$q_0(\Delta) = \sum_{i=1}^g \epsilon_i\epsilon'_i.$$

Therefore, the orientation $K_{q_0+\Delta^*}$, which is obtained by inverting $K_0 = K_{q_0}$ on every edge e such that $q_0([e]) + \Delta^*([e]) \neq q_0([e])$, or equivalently on every edge e such that $1 = \Delta^*([e]) = \Delta\cdot[e]$, can be understood as $K_{\epsilon,\epsilon'}$ defined before. This fact together with the two previous equations leads to the formula stated in Theorem 4.2.2. □

Now we are only left with the proof of Proposition 4.3.7.

4.3.3 Proof of Proposition 4.3.7

Let us recall that we need to prove the following equality

$$\epsilon^{K_0}(D) = \epsilon_0(-1)^{t(D)} \tag{4.3}$$

with $\epsilon_0 = \pm 1$ a constant. To do that, we first reduce this equation to the case where the dimer configuration D of G^T has no inside self-intersections. More precisely, if D has a self-intersection created by two short edges, let us replace these two by new ones (there are two ways to do so) as in Figure 4.4 to obtain a new dimer configuration D'. It is obvious that $t_{\text{in}}(D)$ and $t_{\text{in}}(D')$ have opposite parity, and $t_{\text{out}}(D) = t_{\text{out}}(D')$. Furthermore, by the choice of K_0 on short edges of G^T (recall Definition 4.2.1, condition (i)), one can verify easily that $\epsilon^{K_0}(D) = -\epsilon^{K_0}(D')$. By repeating this transformation (each time one decreases $t_{\text{in}}(D)$), one can finally obtain $D' \in \mathcal{D}(G^T)$ with $t_{\text{in}}(D') = 0$, $t_{\text{out}}(D') = t_{\text{out}}(D)$ and

$$\epsilon^{K_0}(D) = (-1)^{t_{\text{in}}(D)} \epsilon^{K_0}(D').$$

Hence Equation (4.3) now is equivalent to

$$\epsilon^{K_0}(D') = \epsilon_0 (-1)^{t_{\text{out}}(D')}.$$

Moreover, since $t_{\text{in}}(D') = 0$, we can remove some interior short edges inside complete graphs to obtain G' so that G' has no more crossings inside $\mathcal{P}$ and D' is still a dimer configuration of G'. Additionally, K_0 restricted on G' is still a good orientation by Remark 4.3.6. Therefore, Proposition 4.3.7 boils down to the following one.

Proposition 4.3.8. *Let G' be a subgraph of the terminal graph G^T with no inside crossings, and K a good orientation of G'. Then for every dimer configuration D of G' we have*

$$\epsilon^K(D) = \epsilon_0 (-1)^{t(D)},$$

where $\epsilon_0 = \pm 1$ is a constant.

Proof. First of all, until the end of this section, without stating explicitly, all equations and equalities will be understood in $\mathbb{Z}_2$. Note that D_0 is still the standard dimer configuration of G', and recall that the symmetric difference $D \Delta D_0$ is a vertex-disjoint union of cycles C_j's whose edges are long and short alternatively. Recall also that by Equation (1.4) one has

$$\epsilon^K(D_0)\epsilon^K(D) = (-1)^{\sum_j (n^K(C_j)+1)},$$

where $n^K(C_j)$ denotes the number of edges of C_j where K is different from a fixed orientation on C_j. We have the following result

$$\sum_j (n^K(C_j) + 1) = t(D) + t(D_0), \tag{4.4}$$

whose proof will be given later. Now let us finish the proof of Proposition 4.3.8 as well as Proposition 4.3.7 by using this equation. Indeed, with Equation (4.4) one gets

$$\epsilon^K(D) = \epsilon^K(D_0)(-1)^{\sum_j (n^K(C_j)+1)} = \epsilon^K(D_0)(-1)^{t(D)+t(D_0)} = \epsilon_0(-1)^{t(D)}$$

with $\epsilon_0 := \epsilon^K(D_0)(-1)^{t(D_0)}$. □

In the rest of this section, let us prove Equation (4.4), which is in fact a particular case of the main result shown in [63]. However, in our context, with the standard dimer configuration D_0 the proof in [63] can be simplified extremely so that we get a very elementary one, as we will show now.

Let us start by recalling that all the self-intersections (or simply crossings) of D now are only created by its outside edges, that is, the ones lying outside the polygon $\mathcal{P}$, and that D_0 consists of all the long edges, thus contains all the outside edges. Hence we can rewrite the right-hand side of Equation (4.4) as

$$t(D) + t(D_0) = (D_0 \setminus D) \cdot (D_0 \setminus D) + D \cdot (D_0 \setminus D).$$

Indeed, each self-intersection of D is also a self-intersection of D_0, therefore $t(D) + t(D_0)$ has the same parity as the number of crossings created by either 2 outside edges not in D, or by one outside edge in D and one not in D. The corresponding crossings contribute to $(D_0 \setminus D) \cdot (D_0 \setminus D)$ and $D \cdot (D_0 \setminus D)$ respectively. Now we are left with showing

$$\sum_j (n^K(C_j) + 1) = (D_0 \setminus D) \cdot (D_0 \setminus D) + D \cdot (D_0 \setminus D). \tag{4.5}$$

This equation will be proved in two steps: we start with the case where all C_j's are simple; if some of them are not, we can come back to the previous case by smoothing their self-intersections. Let us now treat the first case.

Lemma 4.3.9. *Equation* (4.5) *holds if all the cycles C_j's are simple.*

In order to prove this lemma, we need the following result.

Claim. If C is simple, then $n^K(C) + 1$ has the same parity as V_{int}, the number of vertices of G' inside C.

With this claim, one can prove Lemma 4.3.9 as follows.

Proof of Lemma 4.3.9. In fact we will show that the contributions of each C_j to both sides of (4.5) are equal, or equivalently, the number of crossings created by outside edges of C_j with D or $D_0 \setminus D$ has the same parity as the number of vertices inside C_j. Indeed, let v be a vertex inside C_j, since $G' \setminus C_j$ admits a dimer configuration, v is matched to another vertex v' by a dimer of this configuration. As v' is not on C_j, there are only two cases: either v' is inside C_j, or v' is outside. In the first case, whatever the dimer matching v and v' is a long or short edge, it must intersect C_j an even number of times (remember that C_j is drawn on the plane). Hence v (or the pair (v, v')) contributes $0 \in \mathbb{Z}_2$ to both V_{int} and the quantity $(D_0 \setminus D) \cdot (D_0 \setminus D) + D \cdot (D_0 \setminus D)$. In the latter case, the dimer matching v and v' must intersect C_j an odd number of times, and so it is a long edge. Depending on whether this long edge belongs to D or $D_0 \setminus D$, its crossings with outside edges of C_j (which are all long edges, and so belong to $D_0 \setminus D$ since C_j contains long and short edges alternatively) contribute to $D \cdot (D_0 \setminus D)$ or $(D_0 \setminus D) \cdot (D_0 \setminus D)$, but not both. Therefore, in this case v contributes $1 \in \mathbb{Z}_2$ to both V_{int} and $(D_0 \setminus D) \cdot (D_0 \setminus D) + D \cdot (D_0 \setminus D)$. □

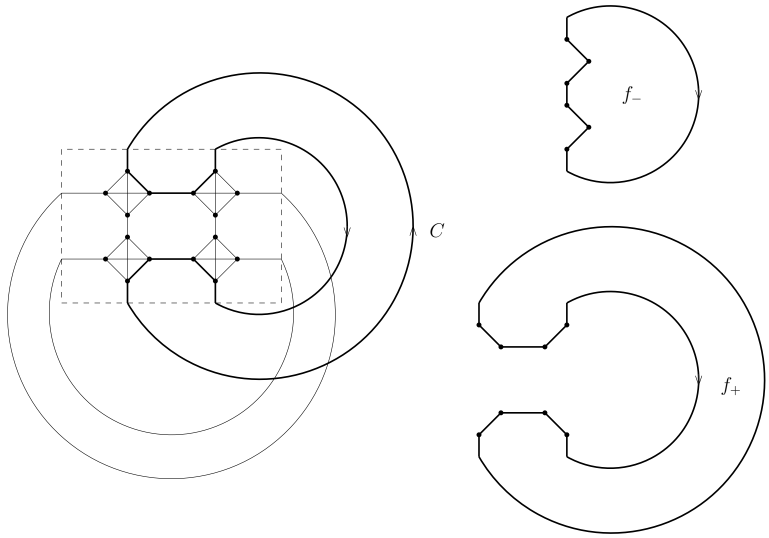


Figure 4.5: The case $m = 2$: a simple cycle C gives rise to an outside positive face f_+ and an outside negative face f_-.

Now let us complete the proof of Lemma 4.3.9 by proving the claim.

Proof of the claim. First of all, set m to be the number of outside edges belonging to C. Since C is a simple closed curve drawn on the plane, it must bound a disc. Let us consider the case where this disc is inside $\mathcal{P}$, that is, $m = 0$. In fact this case is the same as Kasteleyn's theorem displayed in Section 1.4: using the fact that K is good (recall Definition 4.2.1, condition (*ii*)), we get the statement in the claim.

Let us now continue with the case $m \geq 1$. Note that if $m = 1$, one can verify easily that the argument above works completely well with an additional fact: the orientation K satisfies condition (*iii*) of Definition 4.2.1 for the unique outside edge of C. This is also the case for general m if no two of the m outside edges are homologous. The situation only becomes more intricate when we have some outside edges of same homology classes. This is the content of what we will show next.

Recall that to each outside edge, there is a unique face, which is called the outside face, formed by this edge and some edges along the boundary of the subgraph obtained from G' by removing all the other outside edges (cf. Definition 4.2.1). Let us consider the m outside faces corresponding to m outside edges of C. Note that when we travel along C with the counterclockwise orientation, some outside faces have boundaries partly travelled with the same orientation, while the others have boundaries partly travelled with the opposite one. Let us call the former ones *positive* and the latter ones *negative* (see Figure 4.5 for example). For inside faces of the disc bounded by C, we also call them *positive*. With these terminology and the same notations as before, using conditions (*ii*) and (*iii*) of Definition 4.2.1 one gets

$$0 \;=\; \sum_{f \text{ positive}} (n^K(\partial f) + 1) + \sum_{f \text{ negative}} (n^K(\partial f) + 1)$$

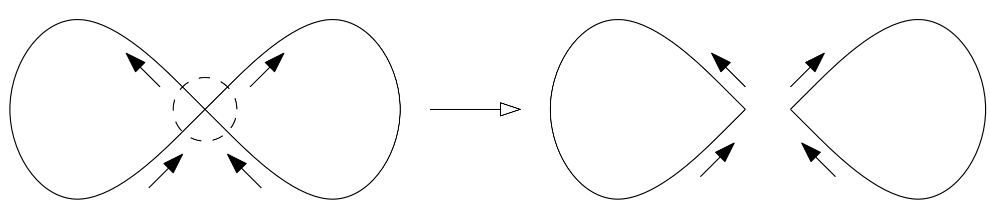
$$
\begin{aligned}
&= \sum_{f \text{ positive}} 1 + \sum_{f \text{ positive}} n^K(\partial f) + \sum_{f \text{ negative}} (n^K(-\partial f) + |\partial f| + 1) \\
&= F + \Big(\sum_{f \text{ positive}} n^K(\partial f) + \sum_{f \text{ negative}} n^K(-\partial f) \Big) + \sum_{f \text{ negative}} (|\partial f| + 1) \\
&= F + \Big(n^K(C) + E_{\text{int}} + \sum_{f \text{ negative}} (|\partial f| - 1) \Big) + \sum_{f \text{ negative}} (|\partial f| + 1) \\
&= F + n^K(C) + E_{\text{int}} \\
&= n^K(C) + V_{\text{int}} + 1,
\end{aligned}
$$

which implies the claim. In this equation, the third equality comes from the fact that each positive face contributes 1 to F, while negative ones do not contribute. The forth equality can be explained as follows: each edge belonging to ∂f (resp. $-\partial f$) for f positive (resp. negative) that is not a common edge of any two faces (among faces bounded by C) contributes to $n^K(C)$; moreover, common edges of two positive faces contribute to E_{int}, while common edges of a positive face and a negative face contribute to the length of that negative face minus 1. Finally, the last equality comes from the argument for $m = 0$ shown before. We have done with the proof of the claim, as well as Lemma 4.3.9. □

We now finish the proof of Equation (4.5) by considering the remaining case where some of the C_j's are not simple.

Lemma 4.3.10. *Equation* (4.5) *holds for every collection of cycles* $\{C_j\}$.

Proof. The idea of the proof is to transform the collection of cycles $\{C_j\}$ together with the good orientation K to have new ones so that we can apply Lemma 4.3.9. Let us start by supposing that there is only one cycle C of this collection which is not simple, and that C has only one self-intersection. Remember that since this self-intersection is created by two outside edges (which are also long edges and so belong to $D_0 \setminus D$), it contributes to the quantity $(D_0 \setminus D) \cdot (D_0 \setminus D)$. Smoothing C at its self-intersection as in Figure 4.6 we create two new outside edges splitting C into two simple closed curves C' and C''. Moreover, doing so introduces a new standard dimer configuration D_0', but still keeps D the same on the resulting graph. It is clear that C' and C'' still have alternative edges between D and $D_0' \setminus D$, and so have even lengths.



C C' C''

Figure 4.6: Smooth C at its self-intersection.

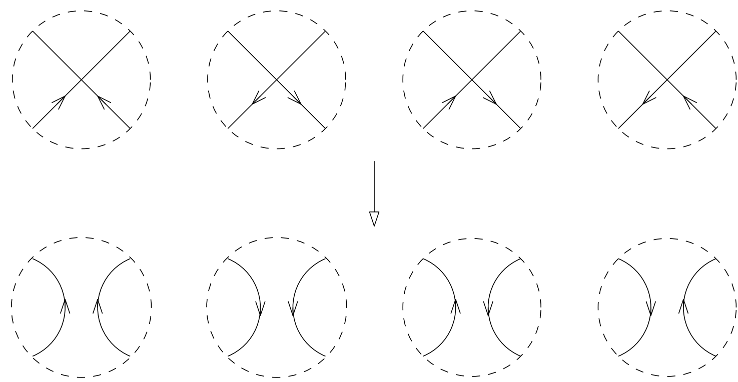


Figure 4.7: Transform K at the self-intersection of C.

Next we transform the orientation K to K' as in Figure 4.7, in which we have a certain flexibility. More precisely, one can switch two choices of K' in the first two cases, as well as those in the last two ones. With all these choices of K', one clearly has

$$n^K(C) = n^{K'}(C') + n^{K'}(C'').$$

This implies that replacing the cycle C by the pair (C', C'') (resulting to replace the collection $\{C_j\}$ by a new one of all simple cycles whose edges are still long and short alternatively) and the orientation K by K' changes the parity of both sides of Equation (4.5), and hence does not affect to the validity of this equation. To apply Lemma 4.3.9 for this new collection of cycles and the orientation K', one only needs to verify if K' is good on the new graph obtained by smoothing C. Recall that by smoothing C we introduce two new outside edges which give rise to two new outside faces, say f_1, f_2. We only need to check that K' satisfies condition (iii) of Definition 4.2.1 on these two new faces. Indeed, let us denote by f_3, f_4 the two outside faces corresponding to the two outside edges that create the self-intersection of C. By definition of K and K', it follows that

$$0 = n^K(\partial f_3) + n^K(\partial f_4) = n^{K'}(\partial f_1) + n^{K'}(\partial f_2) + 2n^K(l) = n^{K'}(\partial f_1) + n^{K'}(\partial f_2),$$

where l is the common path of ∂f_3 and ∂f_4. Now if both $n^{K'}(\partial f_1)$ and $n^{K'}(\partial f_2)$ are odd, we are done. If both of them are even, we only need to invert K' on the two newly created outside edges, using the flexibility mentioned above.

If C has many self-intersections, we smooth them one by one, and at each step, we transform K to K' as above. If $\{C_j\}$ has many cycles which are not simple, we treat them one by one as in the previous case. This concludes the proof of Equation (4.5), as well as Proposition 4.3.8 and Theorem 4.2.3. □

4.4 The non-orientable case

The method we have developed to calculate the Ising partition function for graphs embedded in orientable surfaces can be extended to the case of non-orientable surfaces with some slight modifications. This section is spent to show our Pfaffian formulas in this case. Similarly to the orientable case we obtain a practical version of the Pfaffian formula (see Theorem 4.4.1 below), which is an extension of Theorem 4.2.2 to non-orientable surfaces. Also, using quadratic enhancements and the Brown invariant, which are generalisations of quadratic forms and the Arf invariant to possibly non-orientable surfaces, we get a theoretical version of the Pfaffian formula (Theorem 4.4.4) which generalises Theorem 4.2.3. As a consequence, when we restrict our general Pfaffian formulas to the orientable case, we get back the ones stated before. Let us now go into detail with some new terminology and then state our general Pfaffian formulas. Their proofs will be left until the end of this section.

4.4.1 Statement of the general Pfaffian formulas

Throughout this section, let us assume that the graph G is embedded in a non-orientable surface Σ. We first need to fix a representation of Σ as described in subsection 1.3.1, that is, Σ is represented by the word $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}aba^{-1}b$ if Σ is homeomorphic to the connected sum of an orientable surface Σ_g of genus g with the Klein bottle $\mathscr{K}$, and by the word $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}cc$ if Σ is homeomorphic to the connected sum of Σ_g with the projective plane $\mathbb{R}P^2$. Obviously the side corresponding to the letter b (resp. c) in the first case (resp. in the second case) is a 1-side of the corresponding polygon representing Σ, hence the standard representation ω of the first Stiefel-Whitney class w_1 is simply given by $\omega(e) = e \cdot b$ (resp. $\omega(e) = e \cdot c$) modulo 2. Note that we still have the identification $H_1(\Sigma;\mathbb{Z}_2) \equiv H_1(\Sigma,\mathcal{P};\mathbb{Z}_2)$ as in the orientable context. The fixed drawing of G, as well as of its terminal graph G^T can be defined in the same way as in the orientable case.

Next we will define orientations of interest. Clearly one still can define good orientations by Definition 4.2.1 as before, and Proposition 4.3.5 holds in this new representation of Σ. Let us pick a good orientation K_0 and show how to derive other particular orientations from it. Recall that if $\Sigma = \Sigma_g \# \mathscr{K}$ the word representing Σ induces the basis $\mathcal{B}' := \{[a_1],\ldots,[a_g],[b_1]\ldots,[b_g],[a],[b]\}$ of $H_1(\Sigma;\mathbb{Z}_2)$. Then each element Δ of $H_1(\Sigma;\mathbb{Z}_2)$ can be written as

$$\Delta = \sum_{i=1}^{g} \epsilon_i[a_i] + \sum_{i=1}^{g} \epsilon_i'[b_i] + \alpha[a] + \beta[b],$$

and hence can be identified with $(\epsilon,\epsilon',\alpha,\beta) \in \mathbb{Z}_2^{b_1}$. For such $(\epsilon,\epsilon',\alpha,\beta) \in \mathbb{Z}_2^{b_1}$, let us denote by $K_{\epsilon,\epsilon',\alpha,\beta}$ the orientation obtained by inverting K_0 on every edge e each time e crosses a side a_i of $\mathcal{P}$ (resp. b_j) with $\epsilon_i = 1$ (resp. $\epsilon_j' = 1$) as well as the side a (resp. b) with $\alpha = 1$ (resp. $\beta = 1$). The set $\{K_{\epsilon,\epsilon',\alpha,\beta} : (\epsilon,\epsilon',\alpha,\beta) \in \mathbb{Z}_2^{2g+2}\}$ consists of orientations that we are interested in. Similarly if $\Sigma = \Sigma_g \# \mathbb{R}P^2$ we have the basis $\mathcal{B}'' := \{[a_1],\ldots,[a_g],[b_1]\ldots,[b_g],[c]\}$ of $H_1(\Sigma;\mathbb{Z}_2)$, and each $\Delta \in H_1(\Sigma;\mathbb{Z}_2)$ can be identified with $(\epsilon,\epsilon',\gamma) \in \mathbb{Z}_2^{2g+1}$. Then the orientations $K_{\epsilon,\epsilon',\gamma}$ of interest are obtained by inverting K_0 on every edge e each time e crosses a side a_i of $\mathcal{P}$ (resp. b_j) with $\epsilon_i = 1$ (resp. $\epsilon_j' = 1$) as well as the side c with $\gamma = 1$.

We are now ready to state the hands-on version of the general Pfaffian formula.

Theorem 4.4.1. *Let (G, x) be a weighted graph embedded in the non-orientable surface Σ. Then the Ising partition function of G is given by*

$$Z_{\mathcal{I}}(G,x) = \frac{1}{2^{g+1}} \Bigg| \sum_{(\epsilon,\epsilon',\alpha,\beta)\in\mathbb{Z}_2^{2g+2}} (-1)^{\sum_{i=1}^{g}\epsilon_i\epsilon_i'+\alpha\beta}(-i)^{\alpha} \mathit{Pf}(A^{K_{\epsilon,\epsilon',\alpha,\beta},\omega}(G^T,x^T)) \Bigg|$$

if $\Sigma = \Sigma_g \# \mathscr{K}$, and by

$$Z_{\mathcal{I}}(G,x) = \frac{1}{2^{g+1/2}} \Bigg| \sum_{(\epsilon,\epsilon',\gamma)\in\mathbb{Z}_2^{2g+1}} (-1)^{\sum_{i=1}^{g}\epsilon_i\epsilon_i'}\, i^{\gamma}\, \mathit{Pf}(A^{K_{\epsilon,\epsilon',\gamma},\omega}(G^T,x^T)) \Bigg|$$

if $\Sigma = \Sigma_g \# \mathbb{R}P^2$. In the above formulas, $A^{K_{\epsilon,\epsilon',\alpha,\beta},\omega}(G^T,x^T)$ and $A^{K_{\epsilon,\epsilon',\gamma},\omega}(G^T,x^T)$ are twisted adjacency matrices of the terminal graph (G^T,x^T) with respect to ω and the orientations $K_{\epsilon,\epsilon',\alpha,\beta}$, $K_{\epsilon,\epsilon',\gamma}$ respectively.

Note that if $\Sigma = \Sigma_g$ is an orientable surface of genus g, omitting the indices α, β (resp. the index γ) in the first formula (resp. in the second) and replacing $g+1$ (resp. $g+1/2$) by g (which is natural since they all indicate $\frac{1}{2}\dim H_1(\Sigma;\mathbb{Z}_2)$, see the proof below), we get back the formula stated in Theorem 4.2.2.

Remark 4.4.2. For further purpose let us quote the Pfaffian formula for graphs embedded in the Klein bottle. With K a good orientation, the Ising partition function of a weighted graph $(G,x) \subset \mathscr{K}$ can be given by

$$\begin{aligned} Z_{\mathcal{I}}(G,x) = \frac{1}{2}\big|&\mathrm{Pf}(A^{K_{0,0}}(G^T,x^T)) + \mathrm{Pf}(A^{K_{0,1}}(G^T,x^T)) \\ &- i\,\mathrm{Pf}(A^{K_{1,0}}(G^T,x^T)) + i\,\mathrm{Pf}(A^{K_{1,1}}(G^T,x^T))\big|, \end{aligned}$$

where the standard representation ω of the first Stiefel-Whitney class of $\mathscr{K}$ is dropped for simplicity. Note that in this formula K_{jk} is obtained by inverting $K \equiv K_{00}$ on every long edge e that crosses a (resp. b) with $j = 1$ (resp. $k = 1$). Moreover, it is clear by definition that $A^{K_{00}}(G^T,x^T)$ is conjugate to $A^{K_{01}}(G^T,x^T)$, while $A^{K_{10}}(G^T,x^T)$ is conjugate to $A^{K_{11}}(G^T,x^T)$. Therefore the above equation can be rewritten as

$$Z_{\mathcal{I}}(G,x) = \big|\mathrm{Re}\,\mathrm{Pf}(A^{K_{0,0}}(G^T,x^T)) + \mathrm{Im}\,\mathrm{Pf}(A^{K_{1,0}}(G^T,x^T))\big|, \tag{4.6}$$

which will be used later in our work.

Before giving another version of Theorem 4.4.1, let us give a very simple example to see how the Pfaffian formula works.

Example 4.4.3. Let G be the 1×1 square lattice embedded in the Klein bottle $\mathscr{K}$ with horizontal weight x and vertical weight y. Label the vertices of G^T as in Figure 4.8, and choose K a good orientation. By definition we have

$$A^K(G^T,x^T) = \begin{pmatrix} 0 & -\sqrt{xy} & 1-x & -\sqrt{xy} \\ \sqrt{xy} & 0 & -\sqrt{xy} & i-y \\ -1+x & \sqrt{xy} & 0 & -\sqrt{xy} \\ \sqrt{xy} & -i+y & \sqrt{xy} & 0 \end{pmatrix}$$

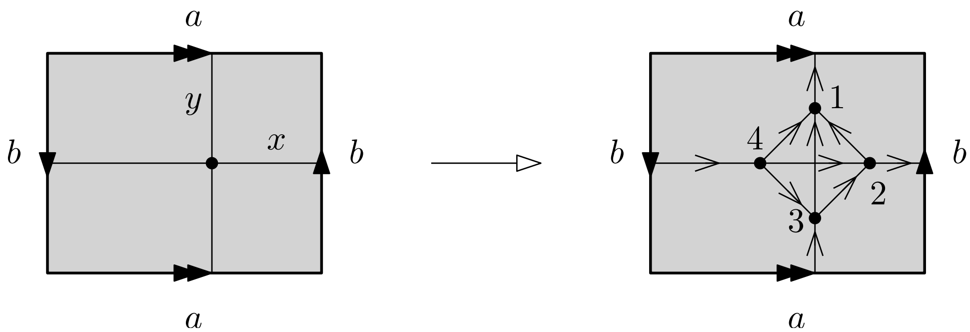


Figure 4.8: The graph $G \subset \mathscr{K}$ and its associated terminal graph G^T with a good orientation K.

whose Pfaffian is $\mathrm{Pf}(A^K(G^T, x^T)) = \mathrm{Pf}(A^{K_{0,0}}(G^T, x^T)) = xy - i + y + ix$. Similarly one can find $\mathrm{Pf}(A^{K_{1,0}}(G^T, x^T)) = xy + i - y + ix$. Then Equation (4.6) gives

$$Z_{\mathcal{I}}(G) = xy + y + 1 + x,$$

which is trivially correct.

Let us now continue with the alternative version of the general Pfaffian formula. To do so, we need a particular quadratic enhancement $\tilde{q}_0 \in \mathcal{Q}(\Sigma, \omega)$ which is defined as follows. If $\Sigma = \Sigma_g \# \mathscr{K}$, $\tilde{q}_0$ is given by its values on the basis $\mathcal{B}'$ of $H_1(\Sigma; \mathbb{Z}_2)$ by

$$\tilde{q}_0([a_i]) = \tilde{q}_0([b_i]) = \tilde{q}_0([b]) = 0 \quad \text{and} \quad \tilde{q}_0([a]) = 3.$$

If $\Sigma = \Sigma_g \# \mathbb{R}P^2$, we define $\tilde{q}_0$ by setting

$$\tilde{q}_0([a_i]) = \tilde{q}_0([b_i]) = 0 \quad \text{and} \quad \tilde{q}_0([c]) = 1$$

on the basis $\mathcal{B}''$. Then by fixing a good orientation K_0, for each quadratic enhancement $q \in \mathcal{Q}(\Sigma, \omega)$ we denote by K_q the orientation obtained by inverting K_0 on every edge e such that $q([e]) \neq \tilde{q}_0([e])$. We now can state the general version of the Pfaffian formula.

Theorem 4.4.4. *Let (G, x) be a weighted graph embedded in a non-orientable surface Σ. Then the Ising partition function on G is given by*

$$Z_{\mathcal{I}}(G, x) = \frac{\epsilon_0}{2^{b_1/2}} \sum_{q \in \mathcal{Q}(\Sigma, \omega)} \exp\left(\frac{i\pi}{4}\right)^{-Br(q)} Pf(A^{K_q,\omega}(G^T, x^T)),$$

where $\epsilon_0 = \pm 1$ is a constant and $b_1 := \dim H_1(\Sigma; \mathbb{Z}_2)$. In this formula, $A^{K_q,\omega}(G^T, x^T)$ is the twisted adjacency matrix of the terminal graph (G^T, x^T) with respect to the orientation K_q and the standard representation ω of w_1.

Recalling the relations between quadratic enhancements and quadratic forms on Σ, as well as the Brown and Arf invariants when Σ is orientable, it is easy to see that the formula in Theorem 4.4.4 restricted to the orientable surface Σ is the one stated in Theorem 4.2.3.

4.4.2 Proof of Theorems 4.4.1 and 4.4.4

The proofs of Theorems 4.4.1 and 4.4.4 are similar to those of Theorems 4.2.2 and 4.2.3, with some slight differences. However, our key result in the previous section, Proposition 4.3.7, is still valid in the non-orientable context. As in the previous section, let us go through new terminology and definitions once again to give more details and results that are needed, before giving our proofs.

First of all, Proposition 4.3.2 can be generalised to the following one.

Proposition 4.4.5. *For every $M \subset E(G)$, the number $t(M)$ of its self-intersections in the plane has the same parity as the number $\frac{1}{2}(\tilde{q}_0([M]) - \omega(M))$ where $[M] \in H_1(\Sigma, \mathcal{P}; \mathbb{Z}_2)$ is the relative homology class of M.*

Proof. Note that Lemma 4.3.1 is still valid when Σ is non-orientable. Using the fact that $[a_i] \cdot [b_j] = \delta_{ij}$ and $[a_i] \cdot [a_j] = [b_i] \cdot [b_j] = 0$ as before, together with $[a] \cdot [b] = 1$ and $[a] \cdot [a_i] = [a] \cdot [b_i] = [b] \cdot [a_i] = [b] \cdot [b_i] = 0$ if $\Sigma = \Sigma_g \# \mathcal{K}$, while $[c] \cdot [c] = 1$ and $[c] \cdot [a_i] = [c] \cdot [b_i] = 0$ if $\Sigma = \Sigma_g \# \mathbb{R}P^2$, the proof of Proposition 4.3.2 can be extended. □

We continue with the transformation of the twisted partition function of the Ising model on G to that of the dimer model on G^T. The following result is a generalisation of Lemma 4.3.4.

Lemma 4.4.6. *Setting $Z^q(G, x) := \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} i^{q(\alpha)} \sum_{[P]=\alpha} x(P)$ for each quadratic enhancement $q \in \mathcal{Q}(\Sigma, \omega)$, we have*

$$Z^q(G, x) = \sum_{D \in \mathcal{D}(G^T)} i^{q([D \Delta D_0]) + 2t_{in}(D)} x^T(D),$$

where D_0 is the standard dimer configuration of G^T.

Proof. Using the same argument and notations as in Lemma 4.3.4 one can write

$$\begin{aligned}
Z^q(G, x) &= \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} i^{q(\alpha)} \sum_{[P]=\alpha} x(P) \\
&= \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} i^{q(\alpha)} \sum_{[P]=\alpha} \Big(\prod_{v \in V} \sum_{D_v \in \mathcal{D}_{2n(v)}} (-1)^{t_{\text{in}}(D_v)}\Big) x(P) \\
&= \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} i^{q(\alpha)} \sum_{[P]=\alpha} \sum_{D \in \varphi^{-1}(P)} (-1)^{t_{\text{in}}(D)} x^T(D) \\
&= \sum_{D \in \mathcal{D}(G^T)} i^{q([G \setminus D_G]) + 2t_{\text{in}}(D)} x^T(D) \\
&= \sum_{D \in \mathcal{D}(G^T)} i^{q([D \Delta D_0]) + 2t_{\text{in}}(D)} x^T(D),
\end{aligned}$$

which concludes our proof. □

Finally before going to the proofs of Theorem 4.4.1 and 4.4.4, let us recall that the twisted adjacency matrix $A^{K,\omega}(G,x)$ defined by Equation (1.5) satisfies the following equation.

$$\mathrm{Pf}(A^{K,\omega}(G,x)) = \sum_{D\in\mathcal{D}(G)} \epsilon^K(D) i^{\omega(D)} x(D). \tag{4.7}$$

Proof of Theorem 4.4.4. First of all, as in the proof of Theorem 4.2.3, using Equation (1.3) one can write $\epsilon^{K_q}(D) = \epsilon^{K_0}(D)(-1)^{|\{e\in D: q([e])\neq \tilde{q}_0([e])\}}$. Note that $q([e])$ and $\tilde{q}_0([e])$ have the same parity (both have the parity of $\omega(e)$), thus the fact that $q([e]) \neq \tilde{q}_0([e])$ is equivalent to $q([e]) - \tilde{q}_0([e]) = 2 \in \mathbb{Z}_4$. Hence by definition of quadratic enhancements we can write

$$q([D]) - \tilde{q}_0([D]) = \sum_{e\in D} q([e]) - \sum_{e\in D} \tilde{q}_0([e]) = 2|\{e \in D : q([e]) \neq \tilde{q}_0([e])\}| \in \mathbb{Z}_4,$$

which implies that $\epsilon^{K_q}(D) = \epsilon^{K_0}(D) i^{q([D])-\tilde{q}_0([D])}$. By Proposition 4.4.5 we have $\tilde{q}_0([D]) - \omega(D) = 2t_{\mathrm{out}}(D)$ modulo 4 for every $D \in \mathcal{D}(G^T)$. Using this fact and Proposition 4.3.7 (which is still valid in the non-orientable case) we have

$$\begin{aligned}\epsilon^{K_q}(D) &= \epsilon_0(-1)^{t(D)} i^{q([D])-\tilde{q}_0([D])} = \epsilon_0(-1)^{t_{\mathrm{in}}(D)+t_{\mathrm{out}}(D)} i^{q([D])-\tilde{q}_0([D])} \\ &= \epsilon_0 i^{2t_{\mathrm{in}}(D)+\tilde{q}_0([D])-\omega(D)} i^{q([D])-\tilde{q}_0([D])} = \epsilon_0 i^{2t_{\mathrm{in}}(D)+q([D])-\omega(D)}.\end{aligned}$$

Secondly, recall that the dual $[D_0]^*$ of the relative homology class $[D_0] \in H_1(\Sigma, \mathcal{P}; \mathbb{Z}_2)$ is given by $[D_0]^*([D]) := [D]\cdot[D_0]$ for every $[D] \in H_1(\Sigma, \mathcal{P}; \mathbb{Z}_2)$. Since $[D_0]^*$ is linear, $q + 2[D_0]^*$ is still a quadratic enhancement. Using the previous equation for $q + 2[D_0]^*$ together with Lemma 4.4.6, one can write

$$\begin{aligned}Z^q(G,x) &= \sum_{D\in\mathcal{D}(G^T)} i^{q([D\Delta D_0])+2t_{\mathrm{in}}(D)} x^T(D) \\ &= i^{q([D_0])} \sum_{D\in\mathcal{D}(G^T)} i^{q([D])+2[D]\cdot[D_0]+2t_{\mathrm{in}}(D)} x^T(D) \\ &= i^{q([D_0])} \sum_{D\in\mathcal{D}(G^T)} i^{(q+2D_0^*)([D])+2t_{\mathrm{in}}(D)} x^T(D) \\ &= \epsilon_0 i^{q([D_0])} \sum_{D\in\mathcal{D}(G^T)} \epsilon^{K_{q+2[D_0]^*}}(D) i^{\omega(D)} x^T(D) \\ &\overset{(4.7)}{=} \epsilon_0 i^{q([D_0])} \mathrm{Pf}(A^{K_{q+2[D_0]^*},\omega}(G^T, x^T)).\end{aligned} \tag{4.8}$$

Finally, dropping the index ω, the Ising partition function can be written as follows

$$\begin{aligned}Z_{\mathcal{I}}(G,x) &= \sum_{P\in Z_1(G;\mathbb{Z}_2)} x(P) = \sum_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} \sum_{[P]=\alpha} x(P) \\ &\overset{\text{Lem. } 1.2.6(i)}{=} \sum_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} \left(\frac{1}{2^{b_1/2}} \sum_{q\in\mathcal{Q}(\Sigma,\omega)} \exp(i\pi/4)^{-\mathrm{Br}(q)} i^{q(\alpha)} \right) \sum_{[P]=\alpha} x(P)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{b_1/2}} \sum_{q\in\mathcal{Q}(\Sigma,\omega)} \exp(i\pi/4)^{-\mathrm{Br}(q)} \sum_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} i^{q(\alpha)} \sum_{[P]=\alpha} x(P) \\
&= \frac{1}{2^{b_1/2}} \sum_{q\in\mathcal{Q}(\Sigma,\omega)} \exp(i\pi/4)^{-\mathrm{Br}(q)} Z^q(G,x) \\
&\overset{(4.8)}{=} \frac{\epsilon_0}{2^{b_1/2}} \sum_{q\in\mathcal{Q}(\Sigma,\omega)} \exp(i\pi/4)^{-\mathrm{Br}(q)+2q([D_0])} \mathrm{Pf}(A^{K_{q+2[D_0]^*}}(G^T,x^T)) \\
&\overset{\text{Lem. }1.2.6(ii)}{=} \frac{\epsilon_0}{2^{b_1/2}} \sum_{q\in\mathcal{Q}(\Sigma,\omega)} \exp(i\pi/4)^{-\mathrm{Br}(q+2[D_0]^*)} \mathrm{Pf}(A^{K_{q+2[D_0]^*}}(G^T,x^T)) \\
&= \frac{\epsilon_0}{2^{b_1/2}} \sum_{q\in\mathcal{Q}(\Sigma,\omega)} \exp(i\pi/4)^{-\mathrm{Br}(q)} \mathrm{Pf}(A^{K_q}(G^T,x^T)).
\end{aligned}
$$

This completes the proof of Theorem 4.4.4. □

Now let us show the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. By the same manner of the proof of Theorem 4.2.2, using Theorem 4.4.4 and Lemma 1.2.6 part (ii) one can write

$$
\begin{aligned}
Z_{\mathcal{I}}(G,x) &= \frac{1}{2^{b_1/2}} \Bigg| \sum_{q\in\mathcal{Q}(\Sigma,\omega)} \exp(i\pi/4)^{\mathrm{Br}(\tilde{q}_0)-\mathrm{Br}(q)} \mathrm{Pf}(A^{K_q}(G^T,x^T)) \Bigg| \\
&= \frac{1}{2^{b_1/2}} \Bigg| \sum_{\Delta\in H_1(\Sigma;\mathbb{Z}_2)} i^{\tilde{q}_0(\Delta)} \mathrm{Pf}(A^{K_{\tilde{q}_0+2\Delta^*}}(G^T,x^T)) \Bigg|.
\end{aligned}
$$

Now recall that if $\Sigma = \Sigma_g \# \mathcal{K}$, the set $\mathcal{B}'$ given by $\{[a_1],\dots,[a_g],[b_1],\dots,[b_g],[a],[b]\}$ is a basis of $H_1(\Sigma;\mathbb{Z}_2)$. In this case $b_1 = 2g+2$ and each element $\Delta \in H_1(\Sigma;\mathbb{Z}_2)$ can be written as $\Delta = \sum_{i=1}^g \epsilon_i[a_i] + \sum_{i=1}^g \epsilon'_i[b_i] + \alpha[a] + \beta[b]$, and hence can be identified with $(\epsilon,\epsilon',\alpha,\beta) \in \mathbb{Z}_2^{2g+2}$. Using the fact that $[a]\cdot[b] = 1$ while $[a_i]\cdot[b_j] = \delta_{ij}$, $[a_i]\cdot[a_j] = [b_i]\cdot[b_j] = 0$ and $[a]\cdot[a_i] = [a]\cdot[b_j] = [b]\cdot[a_i] = [b]\cdot[b_j] = 0$ for every $i,j = 1,\dots,g$, by definition of $\tilde{q}_0$ and quadratic enhancements we get

$$
\begin{aligned}
\tilde{q}_0(\Delta) &= \tilde{q}_0\Big(\sum_{i=1}^g \epsilon_i[a_i] + \sum_{i=1}^g \epsilon'_i[b_i] + \alpha[a] + \beta[b]\Big) \\
&= \tilde{q}_0\Big(\sum_{i=1}^g \epsilon_i[a_i] + \sum_{i=1}^g \epsilon'_i[b_i]\Big) + \tilde{q}_0(\alpha[a] + \beta[b]) \\
&= 2\sum_{i=1}^g \epsilon_i\epsilon'_i + 2\alpha\beta + 3\alpha.
\end{aligned}
$$

Moreover the orientation $K_{\tilde{q}_0+2\Delta^*}$, which is obtained by inverting $K_0 = K_{\tilde{q}_0}$ on every edge e such that $\tilde{q}_0([e]) + 2\Delta^*([e]) \neq \tilde{q}_0([e])$, or equivalently on every edge e such that $1 = \Delta^*([e]) = \Delta\cdot[e]$, is exactly $K_{\epsilon,\epsilon',\alpha,\beta}$ defined before. This fact together with the equations above leads to the first formula stated in Theorem 4.4.1. The case $\Sigma = \Sigma_g \# \mathbb{R}P^2$ is treated similarly, concluding the proof. □

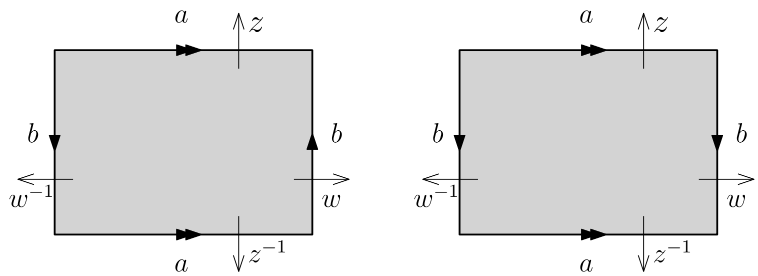


Figure 4.9: A fixed representation of $\mathscr{K}$ and a fixed representation of $\mathbb{T}$.

4.5 The Ising model on the Klein bottle

In this section, we shall study the behaviour of the Ising partition function of graphs embedded in the Klein bottle when the graph is enlarged periodically. Actually, what we will do is inspired by well-known results for the dimer model by Kenyon-Okounkov-Sheffield [42] for graphs embedded in the torus. In detail, we will define characteristic polynomials for graphs embedded in the Klein bottle. Relying on an unpublished work of Cimasoni-Kassel [12], we also establish a product formula relating the characteristic polynomial of enlarged graph and that of the original one. This formula, together with some symmetric properties of the characteristic polynomial, allows us to compute finite-size corrections.

4.5.1 Characteristic polynomials

Let (G, x) be a weighted graph embedded in the Klein bottle $\mathscr{K}$. Throughout this section, we will always represent $\mathscr{K}$ as a rectangle $\mathcal{R}$ with sides identified following the word $aba^{-1}b$, where a represents the two horizontal sides and b represents the two vertical ones (see Figure 4.9). Drawing G inside $\mathcal{R}$ so that its edges intersect a, b transversely, for e an oriented edge of G we define

$$e \cdot a = \begin{cases} +1 & \text{if } e \text{ crosses } a \text{ upward;} \\ -1 & \text{if } e \text{ crosses } a \text{ downward;} \\ 0 & \text{else,} \end{cases}$$

and

$$e \cdot b = \begin{cases} +1 & \text{if } e \text{ crossed } b \text{ from left to right;} \\ -1 & \text{if } e \text{ crossed } b \text{ from right to left;} \\ 0 & \text{else.} \end{cases}$$

Fix a function ω representing the first Stiefel-Whitney class of $\mathscr{K}$ and an orientation K on G^T. Recall that for every vertex u, v of G^T and every edge e between them,

$$\epsilon^K_{uv}(e) = \begin{cases} +1 & \text{if } K \text{ goes from } u \text{ to } v; \\ -1 & \text{if } K \text{ goes from } v \text{ to } u. \end{cases}$$

We have the following definition.

Definition 4.5.1. *Given two complex numbers $z, w \neq 0$, the* twisted adjacency matrix *$A(z,w) = (a_{uv})_{u,v\in V(G^T)}$ associated to the weighted graph (G^T, x^T), with respect to the function ω and the orientation K, is given by*

$$a_{uv} = \sum_{e=(u,v)} z^{e\cdot a} w^{e\cdot b} \epsilon^K_{uv}(e) i^{\omega(e)} x(e),$$

where the sum is taken over all oriented edges from u to v. Then the characteristic polynomial *of the graph G is defined by $P(z,w) := \det A(z,w)$.*

Remark 4.5.2. Using a similar representation for the torus $\mathbb{T}$ (recall Figure 4.9), Definition 4.5.1 still makes sense for any graph $G \subset \mathbb{T}$. In this case, following the argument in [7], the characteristic polynomial given in Definition 4.5.1 with respect to a good orientation K, i.e. a crossing orientation on G^T, and with respect to the standard representation $\omega \equiv 0$ is the same as the characteristic polynomial defined in [11, Definition 2.1] using Kac-Ward matrices.

We now give some properties of $P(z,w)$.

Proposition 4.5.3. *For a fixed orientation K on G^T and a fixed representation ω of the first Stiefel-Whitney class w_1 of the Klein bottle $\mathscr{K}$, the characteristic polynomial $P(z,w)$ with respect to K and ω satisfies the following conditions.*

(i) *$P(\pm 1, w)$ is of the form*

$$\sum a_{2k}(w^{2k} + \frac{1}{w^{2k}}) + i \sum a_{2k+1}(w^{2k+1} + \frac{1}{w^{2k+1}}),$$

where the sums are finite and a_k's are real numbers. As a consequence, we have $P(\pm 1, -w) = \overline{P(\pm 1, w)}$ for every $w \in S^1$.

(ii) *If K is a good orientation and ω is the standard representation of w_1, we have $P(\pm 1, \pm 1) \neq 0$.*

(iii) *With a good orientation K and the standard representation ω of w_1, the Ising partition function of $G \subset \mathscr{K}$ can be given by*

$$Z_{\mathcal{I}}(G) = \left|\operatorname{Re} P(1,1)^{1/2}\right| + \left|\operatorname{Im} P(-1,1)^{1/2}\right|. \tag{4.9}$$

Proof. Part (i) comes directly from the expansion of the determinant of $A(z,w)$. Indeed, let us denote by $\Gamma(G)$ the set of all collections $\gamma = \{\gamma_j\}$ consisting of oriented vertex-disjoint cycles γ_j that cover all the vertices of G^T. Then we get

$$\begin{aligned} P(z,w) &= \sum_{\gamma=\{\gamma_j\}\in\Gamma(G)} \prod_j z^{\gamma_j\cdot a} w^{\gamma_j\cdot b} i^{\omega(\gamma_j)} (-1)^{n^K(\gamma_j)+|\gamma_j|+1} x^T(\gamma_j) \\ &= \sum_{\gamma=\{\gamma_j\}\in\Gamma(G)} z^{\gamma\cdot a} w^{\gamma\cdot b} i^{\omega(\gamma)} (-1)^{(n^K(\gamma)+|\gamma|+\sum_j 1)} x^T(\gamma), \end{aligned}$$

where $|\gamma|$ denotes the length of γ. Firstly by definition $\omega(\gamma)$ has the same parity as $\gamma \cdot b$. Secondly, observe that γ belongs to $\Gamma(G)$ if and only if $\overline{\gamma}$ does, where the latter is the collection obtained by inverting the orientation of each cycle γ_j of γ. Finally since $|\gamma| = \sum_j |\gamma_j| = |V(G^T)| = 2|E(G)|$ is even, we have $n^K(\gamma) = n^K(\overline{\gamma})$. All these three facts lead to the stated form of $P(\pm 1, w)$.

We continue with part (ii) by first proving that $P(1,1)$ is non-zero with respect to a good orientation K. By Equation (4.8) we have

$$\det A^{K_q,\omega}(G^T, x^T) = \pm Z^{q+2[D_0]^*}(G,x)^2$$

for every quadratic enhancement $q \in \mathcal{Q}(\Sigma,\omega)$, and K_q is an orientation obtained by inverting the good orientation K on every edge e such that $q([e]) \neq \tilde{q}_0([e])$. Choosing $q := \tilde{q}_0$, writing $q' = \tilde{q}_0 + 2[D_0]^* \in \mathcal{Q}(\Sigma,\omega)$ and setting $Z_{jk} = \sum_{\substack{P \in Z_1(G;\mathbb{Z}_2) \\ [P]=j[a]+k[b]}} x(P)$ for $j,k \in \{0,1\}$, we have

$$P(1,1) = \det A^{K,\omega}(G^T, x^T) = \pm Z^{q'}(G,x)^2 = \pm\big(Z_{00} + i^{q'([a])} Z_{10} + i^{q'([b])} Z_{01} + i^{q'([a]+[b])} Z_{11}\big)^2.$$

Now we compute the coefficients of Z_{jk} in the right-hand side. By definition of quadratic enhancements, $q'([a])$ has the same parity to $\omega(a)$, which is equal to 1 and so is odd. Similarly we have $q'([b])$ even since $\omega(b) = 0$. Also we have $q'([a]+[b]) = q'([a]) + q'([b]) + 2$. Depending on whether $q'([b])$ is equal to 0 or 2, we will obtain

$$P(1,1) = \pm\big(Z_{00} + i^{q'([a])} Z_{10} + Z_{01} - i^{q'([a])} Z_{11}\big)^2,$$

or

$$P(1,1) = \pm\big(Z_{00} + i^{q'([a])} Z_{10} - Z_{01} + i^{q'([a])} Z_{11}\big)^2.$$

In both cases, $P(1,1)$ is different from 0 because Z_{jk}'s are positive and $q'([a])$ is odd.

To prove that $P(1,-1)$ does not vanish, note that $P(1,-1)$ with respect to K can be given by the determinant of $A^{K_{\tilde{q}_0+2[b]^*}}(G^T, x^T)$. Therefore the argument above can be applied for $q' = \tilde{q}_0 + 2[b]^* + 2[D_0]^*$. The cases of $P(-1,1)$ and $P(-1,-1)$ are treated similarly, concluding the proof of part (ii).

Let us now give the proof of part (iii). Note that the four values of P at $(\pm1,\pm1)$ correspond to the four determinant of the adjacency matrices $A^{K_{jk}}(G^T, x^T)$ described in Remark 4.4.2, so Equation (4.6) reads

$$Z_{\mathcal{I}}(G) = \big|\operatorname{Re} P(1,1)^{1/2} + \operatorname{Im} P(-1,1)^{1/2}\big|.$$

Following the proof of part (ii), both $P(1,1)^{1/2}$ and $P(-1,1)^{1/2}$ are the Ising partition function twisted by $\{\pm 1, \pm i\}$. Hence the equation above implies that $\operatorname{Re} P(1,1)^{1/2}$ and $\operatorname{Im} P(-1,1)^{1/2}$ have the same sign, and we obtain Equation (4.9). □

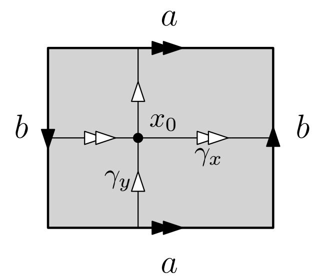


Figure 4.10: Two oriented loops generating the fundamental group $\pi_1(\mathscr{K})$.

4.5.2 Representation theory and product formula

In this subsection we will introduce a more general version of twisted adjacency matrices associated to graphs in the Klein bottle. Using an unpublished result of Cimasoni-Kassel [12], this version allows us to establish a product formula for twisted adjacency matrices in the case of the Klein bottle, which is similar to the one in the case of the torus [42, Theorem 3.3].

To begin, let us assume that (G, x) is a connected weighted graph. Let K be an orientation on G and ω a $\mathbb{Z}_2$-valued function on its edges. Assume that $\rho : \pi_1(G) \to GL(W)$ is a representation of the fundamental group of G (with respect to a certain base point), and W is a finite dimensional vector space over $\mathbb{C}$. Note that ρ can be represented by a collection $(\rho_e)_e$ with $\rho_e \in GL(W)$ for each oriented edge e of G as follows: choose a spanning tree T of G and set $\rho_e = id$ for every $e \in T$; for each oriented edge e not in T, there exists a unique oriented loop γ based at the base point of G, containing e as its unique edge not in T; we set $\rho_e := \rho([\gamma])$. With this representation of ρ we have the following definition.

Definition 4.5.4. *The* general twisted adjacency matrix *of G with respect to a representation $\rho : \pi_1(G) \to GL(W)$ is given by the linear map $A_G^\rho : W^{V(G)} \to W^{V(G)}$, where for each $f \in W^{V(G)}$ and $u \in V(G)$ we define*

$$A_G^\rho(f)(u) = \sum_{e=(u,v)} \epsilon_{uv}^K(e) i^{\omega(e)} x(e) \rho_e(f(v)).$$

In this equality the sum is over all the oriented edges e from u to v.

Remark 4.5.5. In fact one can prove that the determinant of A_G^ρ depends neither on the way we choose the representation $(\rho_e)_{e\in\mathbb{E}}$ of ρ, nor on the base point of G. Also one can check easily by definition that $A_G^{\rho_1\oplus\rho_2}$ is conjugate to $A_G^{\rho_1} \oplus A_G^{\rho_2}$, and hence we get

$$\det A_G^{\rho_1\oplus\rho_2} = \det A_G^{\rho_1} \det A_G^{\rho_2}.$$

Now consider the case where G is drawn inside the Klein bottle $\mathscr{K}$ (recall Figure 4.9) and ω is a function representing the first Stiefel-Whitney class of $\mathscr{K}$. Fix a base point x_0 on G and consider the two oriented loops γ_x, γ_y based at x_0 as in Figure 4.10. Then with

$\rho : \pi_1(G) \to \mathbb{C}^*$ satisfying $\rho([\gamma_x]) = w, \rho([\gamma_y]) = z$, the general twisted adjacency matrix A_G^ρ is exactly the twisted adjacency matrix $A_G(z,w)$ given in Definition 4.5.1. Similarly, if $G \subset \mathbb{T}$ is a bipartite graph embedded in the torus, $\omega \equiv 0$ and K is a Kasteleyn orientation, then the determinant of the corresponding general twisted adjacency matrix is exactly the characteristic polynomial in the case of the dimer model (see [42]).

We will use the following unpublished result of Cimasoni-Kassel [12].

Theorem 4.5.6. *Let (G,x) be a weighted graph with a fixed orientation K and a $\mathbb{Z}_2$-valued function ω on the edges of G. Assume that $\widetilde{G} \xrightarrow{p} G$ is a finite covering of G with an edge weight system $\widetilde{x}$, an orientation $\widetilde{K}$ and a function $\widetilde{\omega}$ lifted from those of G. Then for any representation $\rho : \pi_1(\widetilde{G}) \to GL(W)$ we have*

$$\det A_{\widetilde{G}}^\rho = \det A_G^{\rho^\#},$$

where $\rho^\#$ is the induced representation of ρ from $\pi_1(\widetilde{G}) \simeq p_*(\pi_1(\widetilde{G})) \leq \pi_1(G)$ to $\pi_1(G)$. $\square$

By Remark 4.5.5, we get the following consequence immediately.

Corollary 4.5.7. *With the setting as in Theorem 4.5.6 we have*

$$\det A_{\widetilde{G}}^\rho = \prod_{j=1}^{s} (\det A_G^{\rho_j})^{m_j},$$

where $\rho^\# \sim \rho_1^{\oplus m_1} \oplus \cdots \oplus \rho_s^{\oplus m_s}$ is a decomposition of $\rho^\#$ into irreducible representations. $\square$

By Corollary 4.5.7, to have a product formula for the determinant of $A_{\widetilde{G}}^\rho$ we need to find an irreducible decomposition of $\rho^\#$. Let us now clarify this induced representation $\rho^\#$ in the context of Theorem 4.5.6. First by definition, the degree of $\rho^\#$ is F times the degree of ρ, where F is the index of $p_*(\pi_1(\widetilde{G}))$ in $\pi_1(G)$, which is also equal to the number of sheets of the covering $\widetilde{G} \xrightarrow{p} G$. Fix a base point x_0 of G and a base point $\widetilde{x_0} \in p^{-1}(x_0)$ of $\widetilde{G}$. Recalling Definition 1.2.9, let us choose a set of loops $\{\gamma_1, \ldots, \gamma_F\}$ of G based at x_0 so that their homotopy classes form a complete set of representatives in $\pi_1(G)$ of the left cosets in $\pi_1(G)/p_*(\pi_1(\widetilde{G}))$. In fact by [25, Chapter 1], $\gamma_1, \ldots, \gamma_F$ are chosen such that their unique lifts to $\widetilde{G}$ starting at $\widetilde{x_0}$ have all ending points different. Then by definition, $\rho^\# : \pi_1(G) \to GL(\bigoplus_{j=1}^F [\gamma_j] W)$ is given by

$$\rho^\#([\gamma])(\sum_{i=1}^{F} [\gamma_i] w_i) = \sum_{i=1}^{F} [\gamma_{j(i)}] \rho(p_*^{-1}([\gamma_{j(i)}^{-1}][\gamma][\gamma_i])) w_i \tag{4.10}$$

for every $[\gamma] \in \pi_1(G)$, where $j(i)$ satisfies that $[\gamma][\gamma_i] \in [\gamma_{j(i)}] p_*(\pi_1(\widetilde{G}))$. We will use this equation to compute $\rho^\#$ and its irreducible decomposition in some cases of interest.

Recall that the Klein bottle $\mathscr{K}$ is always represented by a rectangle $\mathcal{R}$ with sides identified following the word $aba^{-1}b$. Gluing N copies of $\mathcal{R}$ along the sides b (with even-numbered copies flipped) with N odd, we obtain a Klein bottle $\mathscr{K}_{1N}$ which is an N-sheeted

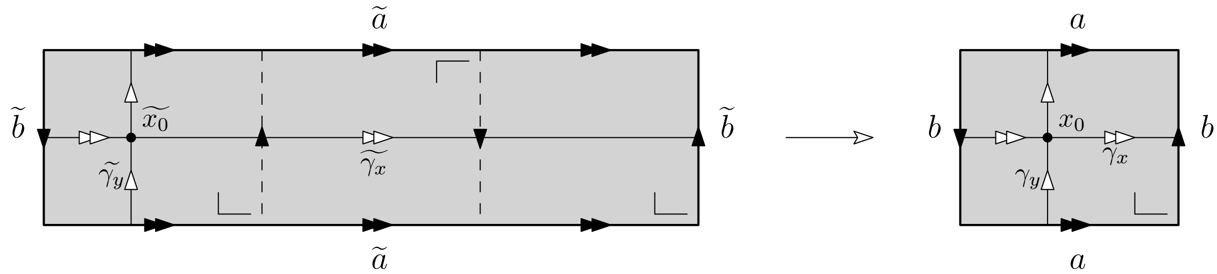


Figure 4.11: The "horizontal" covering $\mathscr{K}_{13} \xrightarrow{p} \mathscr{K}$.

covering of $\mathscr{K}$ (see Figure 4.11 for example). Also gluing M copies of $\mathscr{K}_{1N}$ along the sides a we get a Klein bottle $\mathscr{K}_{MN}$ which is an MN-sheeted covering of $\mathscr{K}$. Let us denote by G_{MN} and $(G^T)_{MN} = (G_{MN})^T =: G^T_{MN}$ the corresponding lift of G and G^T to $\mathscr{K}_{MN}$. We also denote by $P_{MN}(z,w)$ the associated characteristic polynomial of G_{MN} with respect to the lift K_{MN} of a fixed orientation K on G^T to G^T_{MN}, and with respect to the lift ω_{MN} of a fixed function ω representing the first Stiefel-Whitney class of $\mathscr{K}$ to $\mathscr{K}_{MN}$.

We have the following result, which can be considered as a product formula in the "horizontal" direction.

Lemma 4.5.8. *Let G be a weighted graph embedded in the Klein bottle $\mathscr{K}$. We have*

$$P_{1N}(\pm 1, w) = \prod_{u^N = w} P(\pm 1, u).$$

Proof. Fix two base points $x_0, \widetilde{x_0}$ of G^T, G^T_{1N} as well as oriented loops $\gamma_x, \gamma_y, \widetilde{\gamma_x}, \widetilde{\gamma_y}$ as in Figure 4.11. From now on, for ease of notation, let us write $\gamma_x, \gamma_y, \widetilde{\gamma_x}, \widetilde{\gamma_y}$ instead of their homotopy classes. Then the twisted adjacency matrix $A_{1N}(\pm 1, w)$ of G^T_{1N} can be given by the linear map $A^{\rho}_{G^T_{1N}}$ with $\rho : \pi_1(G^T_{1N}) \to \mathbb{C}^*$ given by $\rho(\widetilde{\gamma_x}) = w, \rho(\widetilde{\gamma_y}) = \pm 1$. We will look for the induced representation $\rho^{\#}$ of ρ, and an irreducible decomposition of $\rho^{\#}$.

It is obvious that the N-sheeted covering $G^T_{1N} \xrightarrow{p} G^T$ induces the map

$$p_* : \pi_1(\mathscr{K}_{1N}) = \langle \widetilde{\gamma_x}, \widetilde{\gamma_y} | \widetilde{\gamma_x}\widetilde{\gamma_y}\widetilde{\gamma_x}^{-1}\widetilde{\gamma_y} = 1 \rangle \to \pi_1(\mathscr{K}) = \langle \gamma_x, \gamma_y | \gamma_x \gamma_y \gamma_x^{-1} \gamma_y = 1 \rangle$$

given by $p_*(\widetilde{\gamma_x}) = \gamma_x^N$ and $p_*(\widetilde{\gamma_y}) = \gamma_y$, and therefore $p_*(\pi_1(\mathscr{K}_{1N})) = \langle \gamma_x^N, \gamma_y | \gamma_x\gamma_y\gamma_x^{-1}\gamma_y = 1 \rangle$. Choosing $\gamma_i = \gamma_x^i$ for $1 \le i \le N$, one can compute $\rho^{\#}$ as follows. First from the relation $\gamma_x\gamma_y\gamma_x^{-1}\gamma_y = 1$ one can show easily that $\gamma_y\gamma_x^i = \gamma_x^i\gamma_y^{\pm 1}$ for every i, and hence we have $\gamma_y\gamma_i = \gamma_y\gamma_x^i \in \gamma_i p_*(\pi_1(\mathscr{K}_{1N}))$. Then using Equation (4.10), we have $\gamma_{j(i)}^{-1}\gamma_y\gamma_i = \gamma_x^{-i}\gamma_y\gamma_x^i = \gamma_y^{\pm 1}$. Thus $\rho(p_*^{-1}(\gamma_{j(i)}^{-1}\gamma_y\gamma_i)) = \pm 1$ for every i and we get $\rho^{\#}(\gamma_y) = \pm\, id_{\mathbb{C}^N}$. Next we have $\gamma_x\gamma_i = \gamma_{i+1} \in \gamma_{i+1}p_*(\pi_1(\mathscr{K}_{1N}))$ for $1 \le i \le N-1$ and $\gamma_x\gamma_N = \gamma_x^{N+1} \in \gamma_1 p_*(\pi_1(\mathscr{K}_{1N}))$. Therefore using Equation (4.10) we have $\gamma_{j(i)}^{-1}\gamma_x\gamma_i = 1$ for every $1 \le i \le N-1$, and $\gamma_{j(N)}^{-1}\gamma_x\gamma_N = \gamma_x^N$. Then $p_*^{-1}(\gamma_{j(i)}^{-1}\gamma_x\gamma_i) = 1$ for $1 \le i \le N-1$ and $p_*^{-1}(\gamma_{j(N)}^{-1}\gamma_x\gamma_N) = \widetilde{\gamma_x}$, and hence by Equation (4.10) we obtain

$$\rho^{\#}(\gamma_x) : (w_1, \dots, w_N) \mapsto (w\, w_N, w_1, \dots, w_{N-1}).$$

Let us now look for a decomposition of $\rho^\#$ into irreducible representations. Observing that $\rho^\#(a)^N = w\, id_{\mathbb{C}^N}$, and using the fact that $\mathbb{Z}_N$ has precisely N degree one (so irreducible) representations given by $1 \mapsto e^{2k\pi i/N}$ for $1 \leq k \leq N$, it is not difficult to see that $\rho^\# \sim \bigoplus_{k=1}^N \rho_k$ with $\rho_k : \pi_1(G^T) \to \mathbb{C}^*$ given by

$$\rho_k(\gamma_x) = w^{1/N} e^{2k\pi i/N} \quad \text{and} \quad \rho_k(\gamma_y) = \pm 1,$$

where $w^{1/N}$ denotes any N^{th} root of w. Since the linear map $A^{\rho_k}_{G^T}$ gives us exactly the adjacency matrix $A(\pm 1, u)$ of G^T with $u^N = w$, the stated equality follows from Corollary 4.5.7. □

Remark 4.5.9. Lemma 4.5.8 restricts to the case N odd so that G_N is still inside the Klein bottle. Nevertheless, the formula stated there still holds when N is even (and then G_N is inside the torus $\mathbb{T}$). The proof is similar to that of Lemma 4.5.8 (and even easier since $\pi_1(\mathbb{T})$ is abelian) and hence is omitted here.

We now turn to the "vertical" version of Lemma 4.5.8.

Lemma 4.5.10. *Let G be a weighted graph embedded in the Klein bottle $\mathscr{K}$. We have*

$$P_{M1}(1,w) = \begin{cases} P(1,w)\prod_{k=1}^{\frac{M-1}{2}} \det A^{\rho^w_{2k}}_{G^T} & \text{if } M \text{ is odd;} \\ P(1,w)P(-1,-w)\prod_{k=1}^{\frac{M}{2}-1} \det A^{\rho^w_{2k}}_{G^T} & \text{if } M \text{ is even,} \end{cases}$$

where $\rho^w_k : \pi_1(\mathscr{K}) \simeq \langle a,b | aba^{-1}b = 1\rangle \to GL(\mathbb{C}^2)$ is given by

$$\rho^w_k(a) = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} \quad \textit{and} \quad \rho^w_k(b) = \begin{pmatrix} e^{k\pi i/M} & 0 \\ 0 & e^{-k\pi i/M} \end{pmatrix}.$$

Proof. As before, let us fix two base points of G^T, G^T_{M1} and oriented loops $\gamma_x, \gamma_y, \widetilde{\gamma}_x, \widetilde{\gamma}_y$ as in Figure 4.12. Then the twisted adjacency matrix $A_{M1}(1,w)$ of G^T_{M1} can be given by the linear map $A^\rho_{G^T_{M1}}$ with $\rho : \pi_1(G^T_{M1}) \to \mathbb{C}^*$ defined by $\rho(\widetilde{\gamma}_x) = w, \rho(\widetilde{\gamma}_y) = 1$. We now compute the induced representation $\rho^\#$, and its irreducible decomposition.

In this case the induced map $p_* : \pi_1(\mathscr{K}_{M1}) \to \pi_1(\mathscr{K})$ is given by $p_*(\widetilde{\gamma}_x) = \gamma_y^{M-1}\gamma_x$ and $p_*(\widetilde{\gamma}_y) = \gamma_y^M$, and the index of $p_*(\pi_1(\mathscr{K}_{M1}))$ in $\pi_1(\mathscr{K})$ is equal to M. Choosing $\gamma_i = \gamma_y^i$ for $1 \leq i \leq M$, let us compute $\rho^\#(\gamma_x)$ and $\rho^\#(\gamma_y)$ as follows. First note that by the relation $\gamma_x\gamma_y\gamma_x^{-1}\gamma_y = 1$ one can verify easily that $\gamma_x\gamma_y^i = \gamma_y^{-i}\gamma_x$ for every i, so we get

$$p_*(\pi_1(\mathscr{K}_{M1})) = \langle \gamma_x\gamma_y^{1-M}, \gamma_y^M\rangle = \langle \gamma_x\gamma_y, \gamma_y^M\rangle = \langle \gamma_y^{-1}\gamma_x, \gamma_y^M\rangle$$

and then $\gamma_x\gamma_i = \gamma_x\gamma_y^i = \gamma_y^{-i}\gamma_x \in \gamma_y^{1-i}p_*(\pi_1(\mathscr{K})) = \gamma_y^{M+1-i}p_*(\pi_1(\mathscr{K}_{M1}))$. Then using Equation (4.10) we obtain

$$p_*^{-1}(\gamma_{j(i)}^{-1}\gamma_x\gamma_i) = p_*^{-1}(\gamma_y^{i-M-1}\gamma_x\gamma_y^i) = p_*^{-1}(\gamma_x\gamma_y^{M+1}) = \widetilde{\gamma}_x\widetilde{\gamma}_y{}^2$$

and therefore

$$\rho^\#(\gamma_x)(w_1,\ldots,w_M) = w(w_M,\ldots,w_1).$$

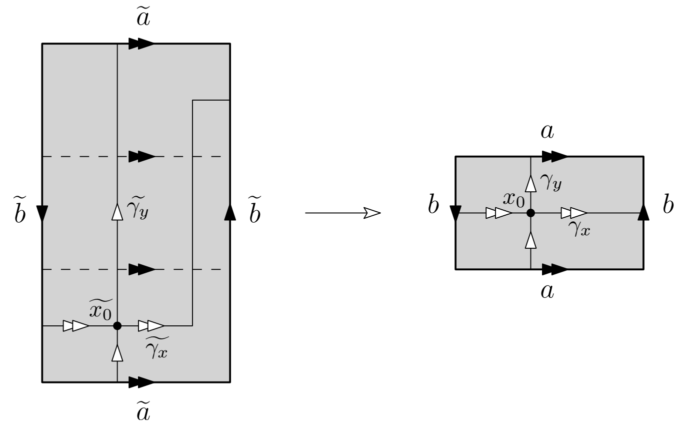


Figure 4.12: The "vertical" covering $\mathscr{K}_{31} \xrightarrow{p} \mathscr{K}$.

Next note that $\gamma_y\gamma_i = \gamma_y^{i+1} \in \gamma_{i+1}p_*(\pi_1(\mathscr{K}_{M1}))$ for every $1 \leq i \leq M-1$ and that $\gamma_y\gamma_M = \gamma_y^{M+1} \in \gamma_1 p_*(\pi_1(\mathscr{K}_{M1}))$. Then Equation (4.10) implies

$$p_*^{-1}(\gamma_{j(i)}^{-1}\gamma_y\gamma_i) = p_*^{-1}(\gamma_y^{-i-1}\gamma_y\gamma_y^i) = p_*^{-1}(1) = 1$$

for every $1 \leq i \leq M-1$, and

$$p_*^{-1}(\gamma_{j(M)}^{-1}\gamma_y\gamma_i) = p_*^{-1}(\gamma_y^{-1}\gamma_y\gamma_y^M) = p_*^{-1}(\gamma_y^M) = \widetilde{\gamma}_y.$$

Then Equation (4.10) gives

$$\rho^\#(\gamma_y)(w_1, \dots, w_M) = (w_M, w_1, \dots, w_{M-1}).$$

We now continue by looking for a decomposition of $\rho^\#$ into irreducible representations. For simplicity, let us first suppose that $w = 1$. Then in this case $\rho^\#(\gamma_x)^2 = \rho^\#(\gamma_y)^M = id_{\mathbb{C}^M}$, and thus $\rho^\#$ factors through the dihedral group $D_M = \langle r, s | r^M = s^2 = (sr)^2 = 1\rangle$ (recalling Example 1.2.10). If M is odd, since the degree of $\rho^\#$ is M and D_M has precisely 2 irreducible representations of degree one and $\frac{M-1}{2}$ ones of degree two, it follows that in the irreducible decomposition of $\rho^\#$ there are exactly $\frac{M-1}{2}$ degree two irreducible representations of D_M, and only 1 degree one representation of D_M. Observe that $\rho^\#(s) = \rho^\#(\gamma_x)$ can be given by a matrix A of size M with entries

$$A_{ij} = \begin{cases} 1 & \text{if } i+j = M+1; \\ 0 & \text{else,} \end{cases}$$

so $\chi_{\rho^\#}(s) = 1$ if M is odd. Since $\chi_{\oplus_{k=1}^{(M-1)/2}\rho_{2k}}(s) = 0$ (recalling the notations in Example 1.2.10), the remaining degree one constituent of $\rho^\#$ is the trivial one. The case M even requires computing the character of $\rho^\#$. Looking at the matrix A represented $\rho^\#(s)$, we see that if M is even then $\chi_{\rho^\#}(s) = 0$. Moreover, given the form of $\rho^\#$, one can represent $\rho^\#(r) = \rho^\#(\gamma_y)$ by a matrix B of size M with entries

$$B_{ij} = \begin{cases} 1 & \text{if } i = j+1 \text{ or } j-i = M-1; \\ 0 & \text{else.} \end{cases}$$

Then one can prove easily by induction on l that for every $1 \le l \le M$ we have

$$(B^l)_{ij} = \begin{cases} 1 & \text{if } i = j + l \text{ or } j - i = M - l; \\ 0 & \text{else.} \end{cases}$$

This implies that

$$\chi_{\rho^\#}(r^l) = Tr(B^l) = \begin{cases} 0 & \text{if } l \neq M; \\ M & \text{if } l = M. \end{cases}$$

Similarly we have

$$(AB^l)_{ij} = \begin{cases} 1 & \text{if } i + j = M + 1 - l \text{ or } i + j = 2M + 1 - l; \\ 0 & \text{else,} \end{cases}$$

and hence we obtain

$$\chi_{\rho^\#}(sr^l) = Tr(AB^l) = \begin{cases} 2 & \text{if } l \text{ is odd;} \\ 0 & \text{if } l \text{ is even.} \end{cases}$$

It is also straightforward to find $\chi_{\rho_k}(sr^l) = 0$ and $\chi_{\rho_k}(r^l) = 2\cos(kl\pi/M)$ for any l, and hence we get

$$\begin{aligned} \langle \chi_{\rho^\#}, \chi_{\rho_{2k}} \rangle &= \frac{1}{2M} \sum_{l=1}^{M} \left(\chi_{\rho^\#}(r^l)\overline{\chi_{\rho_{2k}}(r^l)} + \chi_{\rho^\#}(sr^l)\overline{\chi_{\rho_{2k}}(sr^l)} \right) \\ &= \frac{1}{2M} \sum_{l=1}^{M} \chi_{\rho^\#}(r^l)\overline{\chi_{\rho_{2k}}(r^l)} = \frac{1}{2M}\chi_{\rho^\#}(r^M)\overline{\chi_{\rho_{2k}}(r^M)} = 1. \end{aligned}$$

By Theorem 1.2.8, this means that $\rho^\#$ admits an irreducible decomposition consisting of all degree two irreducible representations of D_M, and so contains 2 degree one representations of D_M. Note also that (recalling Example 1.2.10)

$$\begin{aligned} \langle \chi_{\rho^\#}, \chi_{\tau_1} \rangle &= \frac{1}{2M} \sum_{l=1}^{M} \left(\chi_{\rho^\#}(r^l)\overline{\chi_{\tau_1}(r^l)} + \chi_{\rho^\#}(sr^l)\overline{\chi_{\tau_1}(sr^l)} \right) \\ &= \frac{1}{2M} \sum_{l=1}^{M} \left(\chi_{\rho^\#}(r^l) + \chi_{\rho^\#}(sr^l) \right) = \frac{1}{2M} \cdot M + \frac{1}{2M} \sum_{\substack{l=1 \\ l \text{ odd}}}^{M} 2 = 1, \end{aligned}$$

hence the trivial representation of D_M is a constituent of $\rho^\#$. Similarly one also gets τ_3 the remaining degree one constituent of $\rho^\#$.

To summarise, for $w = 1$ we have

$$\rho^\# \sim \begin{cases} \bigoplus_{k=1}^{(M-1)/2} \rho_{2k} \oplus \tau_1 & \text{if } M \text{ is odd;} \\ \bigoplus_{k=1}^{(M-2)/2} \rho_{2k} \oplus \tau_1 \oplus \tau_3 & \text{if } M \text{ is even.} \end{cases}$$

For general w, it is not difficult to see that

$$\rho^{\#} \sim \begin{cases} \bigoplus_{k=1}^{(M-1)/2} \rho_{2k}^w \oplus \tau_1^w & \text{if } M \text{ is odd;} \\ \bigoplus_{k=1}^{(M-2)/2} \rho_{2k}^w \oplus \tau_1^w \oplus \tau_3^w & \text{if } M \text{ is even,} \end{cases}$$

with $\tau_1^w(r) = 1, \tau_1^w(s) = w$ and $\tau_3^w(r) = -1, \tau_3^w(s) = -w$ while

$$\rho_k^w(r) = \begin{pmatrix} e^{k\pi i/M} & 0 \\ 0 & e^{-k\pi i/M} \end{pmatrix} \quad \text{and} \quad \rho_k^w(s) = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}.$$

Finally $A_{G^T}^{\tau_1^w}$ gives exactly the twisted adjacency matrix $A(1, w)$ of G^T, and $A_{G^T}^{\tau_3^w}$ gives $A(-1, -w)$. Therefore the stated equality follows from Corollary 4.5.7. □

Lemma 4.5.11. *With the same terminology and notations as in Lemma 4.5.10 we have*

$$P_{M1}(-1, w) = \begin{cases} P(-1, w) \prod_{k=0}^{\frac{M-3}{2}} \det A_{G^T}^{\rho_{2k+1}^w} & \textit{if } M \textit{ is odd;} \\ \prod_{k=0}^{\frac{M}{2}-1} \det A_{G^T}^{\rho_{2k+1}^w} & \textit{if } M \textit{ is even.} \end{cases}$$

Proof. As in the previous lemma, the twisted adjacency matrix $A_{M1}(-1, w)$ of G_{M1}^T corresponds to the linear map $A_{G_{M1}^T}^{\rho}$ with the representation $\rho : \pi_1(G_{M1}^T) \to \mathbb{C}^*$ given by $\rho(\widetilde{\gamma_x}) = w$ and $\rho(\widetilde{\gamma_y}) = -1$. The same computation as in the proof of Lemma 4.5.10 leads to

$$\rho^{\#}(\gamma_x)(w_1, \ldots, w_M) = w(w_M, \ldots, w_1),$$
$$\rho^{\#}(\gamma_y)(w_1, \ldots, w_M) = (-w_M, w_1, \ldots, w_{M-1}).$$

Again, to find an irreducible representation of $\rho^{\#}$, let us start with the case $w = 1$. Then we have $\rho^{\#}(\gamma_x)^2 = \rho^{\#}(\gamma_y)^{2M} = id_{\mathbb{C}^M}$, and hence $\rho^{\#}$ factors through the dihedral group $D_{2M} = \langle r, s | r^{2M} = s^2 = (sr)^2 = 1\rangle$. Note that D_{2M} admits 4 representations of degree 1, and $M - 1$ degree two irreducible representations ρ_k with $1 \leq k \leq M - 1$ (using the notations in Example 1.2.10). In the following we will compute the character of $\rho^{\#}$. Given its form, one can see easily that $\rho^{\#}(r) = \rho^{\#}(\gamma_y)$ can be represented by a matrix B of size M with entries

$$B_{ij} = \begin{cases} 1 & \text{if } i = j + 1; \\ -1 & \text{if } i = j - (M - 1); \\ 0 & \text{else.} \end{cases}$$

Then it is not difficult to prove by induction on l that for every $1 \leq l \leq M$ we have

$$(B^l)_{ij} = \begin{cases} 1 & \text{if } i = j + l; \\ -1 & \text{if } i = j - (M - l); \\ 0 & \text{else.} \end{cases}$$

It follows that

$$\chi_{\rho^{\#}}(r^l) = Tr(B^l) = \begin{cases} M & \text{if } l = 2M; \\ -M & \text{if } l = M; \\ 0 & \text{else.} \end{cases}$$

Now using $\chi_{\rho_k}(sr^l) = 0$ and $\chi_{\rho_k}(r^l) = 2\cos(kl\pi/M)$ for any l we can compute

$$\begin{aligned}\langle \chi_{\rho^\#}, \chi_{\rho_k}\rangle &= \frac{1}{4M}\sum_{l=1}^{2M}\left(\chi_{\rho^\#}(r^l)\overline{\chi_{\rho_k}(r^l)} + \chi_{\rho^\#}(sr^l)\overline{\chi_{\rho_k}(sr^l)}\right) = \frac{1}{4M}\sum_{l=1}^{2M}\chi_{\rho^\#}(r^l)\cdot 2\cos(\frac{kl\pi}{M})\\ &= \frac{1}{2M}(-M\cos(Mk\pi/M) + M\cos(2Mk\pi/M)) = \begin{cases} 0 & \text{if } k \text{ is even;}\\ 1 & \text{if } k \text{ is odd.}\end{cases}\end{aligned}$$

Therefore using Theorem 1.2.8 and paying attention to the degree of $\rho^\#$, we see that $\rho^\# \sim \bigoplus_{\substack{1\le k\le M-1\\ k \text{ odd}}}\rho_k$ if M is even, and that $\rho^\# \sim \bigoplus_{\substack{1\le k\le M-1\\ k \text{ odd}}}\rho_k \oplus \tau_i$ for some $1 \le i \le 4$ if M is odd. To determine i in the latter case, first see that $\chi_{\rho^\#}(s) = 1$ for M odd, while $\chi_{\bigoplus_{\substack{1\le k\le M-1\\ k \text{ odd}}}\rho_k}(s) = 0$. This implies that $\tau_i(s) = 1$. Similarly we have $\chi_{\rho^\#}(r) = 0$ while $\chi_{\bigoplus_{\substack{1\le k\le M-1\\ k \text{ odd}}}\rho_k}(r) = 2\sum_{\substack{1\le k\le M-1\\ k \text{ odd}}}\cos(k\pi/M) = 1$, which implies that $\tau_i(r) = -1$. Hence we get $i = 4$.

To summarise, for $w = 1$ we have

$$\rho^\# \sim \begin{cases} \bigoplus_{k=0}^{(M-3)/2}\rho_{2k+1}\oplus\tau_4 & \text{if } M \text{ is odd;}\\ \bigoplus_{k=0}^{M/2-1}\rho_{2k+1} & \text{if } M \text{ is even.}\end{cases}$$

For general w, it is not difficult to see that

$$\rho^\# \sim \begin{cases} \bigoplus_{k=0}^{(M-3)/2}\rho^w_{2k+1}\oplus\tau^w_4 & \text{if } M \text{ is odd;}\\ \bigoplus_{k=0}^{M/2-1}\rho^w_{2k+1} & \text{if } M \text{ is even.}\end{cases}$$

with $\tau_4^w(r) = -1, \tau_4^w(s) = w$ while

$$\rho^w_k(r) = \begin{pmatrix} e^{k\pi i/M} & 0\\ 0 & e^{-k\pi i/M}\end{pmatrix} \quad \text{and} \quad \rho^w_k(s) = \begin{pmatrix} 0 & w\\ w & 0\end{pmatrix}.$$

Finally since $A^{\tau_4^w}_{G^T}$ gives precisely the adjacency matrix $A(-1,w)$ of G^T, Lemma 4.5.11 follows. $\square$

Now let us study the general twisted adjacency matrix given by the linear map $A^{\rho^w_k}_{G^T}$ in the two previous lemma. For further purposes, we will consider a slightly more general version $A^{\rho^{z,w}}_{G^T}$ with $\rho^{z,w} : \pi_1(\mathscr{K}) \simeq \langle \gamma_x, \gamma_y | \gamma_x\gamma_y\gamma_x^{-1}\gamma_y = 1\rangle \to GL(\mathbb{C}^2)$ given by

$$\rho^{z,w}(\gamma_x) = \begin{pmatrix} 0 & w\\ w & 0\end{pmatrix} \quad \text{and} \quad \rho^{z,w}_k(\gamma_y) = \begin{pmatrix} z & 0\\ 0 & 1/z\end{pmatrix}.$$

By definition, this representation can be represented by a collection $(\rho^{z,w}_e)_{e\in\mathbb{E}}$ with

$$\rho^{z,w}_e = \begin{cases} \begin{pmatrix} z & 0\\ 0 & 1/z\end{pmatrix}^{\pm 1} & \text{if } e \text{ crosses } a \text{ upward (resp. downward);}\\ \begin{pmatrix} 0 & w\\ w & 0\end{pmatrix}^{\pm 1} & \text{if } e \text{ crosses } b \text{ from left to right (resp. from right to left);}\\ \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} & \text{else.}\end{cases}$$

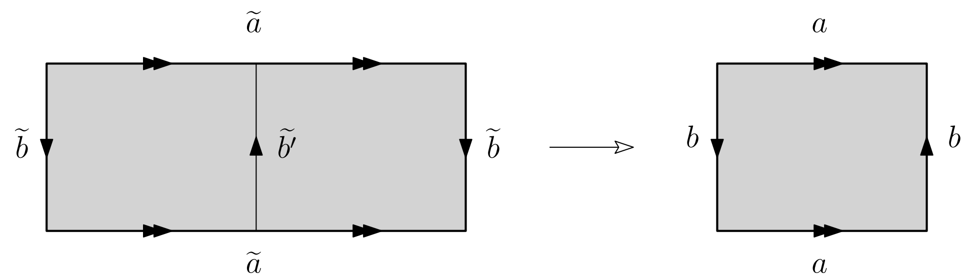


Figure 4.13: The orientation cover $\mathbb{T}$ of $\mathscr{K}$.

We will show that the determinant of $A_{G^T}^{\rho^{z,w}}$ is equal to the determinant of the twisted adjacency matrix of $\widetilde{G}^T$, the lift of G^T to the torus $\mathbb{T}$, the orientation cover of $\mathscr{K}$. To be more precise, recall that $A_{G^T}^{\rho^{z,w}}$ is with respect to some fixed orientation K on G^T and a fixed function ω representing the first Stiefel-Whitney class of $\mathscr{K}$. Let us choose ω the standard one. Recall also that the orientation cover of $\mathscr{K}$ (represented by the word $aba^{-1}b$) is the torus obtained by gluing two copies of $\mathscr{K}$ (with the second flipped) along the sides b. Representing this torus by the word $\widetilde{a}\widetilde{b}\widetilde{a}^{-1}\widetilde{b}^{-1}$ as in Figure 4.13, for e an oriented edge of $\widetilde{G}^T$, we also define $e\cdot\widetilde{a} = \pm 1$ if e crosses $\widetilde{a}$ upwards (resp. downwards), and define $e\cdot\widetilde{b} = \pm 1$ if e crosses $\widetilde{b}$ from left to right (resp. from right to left). Now denoting by $\widetilde{K}$ the orientation obtained by lifting K to $\widetilde{G}^T$, and then inverting the orientation of all the edges completely contained in the second copy of $\mathscr{K}$, we refer to $\widetilde{A}(z,w)$ the twisted adjacency matrix of $\widetilde{G}^T$ with respect to $\widetilde{K}$ and $\widetilde{\omega} \equiv 0$ (recall Definition 4.5.1). Setting $\widetilde{P}(z,w) := \det \widetilde{A}(z,w)$ we have the following proposition.

Proposition 4.5.12. *For every $z,w \in \mathbb{C}^*$,* $\det A_{G^T}^{\rho^{z,w}} = \widetilde{P}(z,w^2)$.

Proof. We will transform the matrix given by $A_G^{\rho^{z,w}}$ to the matrix $\widetilde{A}(z,w^2)$ without changing the determinant. To begin, let us first denote by ω' the lift of ω to $\widetilde{G}^T$, that is, $\omega'(e)$ is equal to the parity of the number of times e crosses $\widetilde{b}$ and $\widetilde{b}'$ (and hence ω' represents the first Stiefel-Whitney class of $\mathbb{T}$). We also denote by $M(z,w) = (m_{uv})_{u,v\in V(\widetilde{G}^T)}$ the "natural" twisted adjacency matrix of $\widetilde{G}^T$ with respect to the lift K' of K and the lift ω' of ω to $\widetilde{G}^T$, that is, its entries are given by

$$m_{uv} = \sum_{e=(u,v)} z^{e\cdot\widetilde{a}} w^{e\cdot(\widetilde{b}+\widetilde{b}')} \epsilon_{uv}^{K'}(e) i^{\omega'(e)} \widetilde{x}(e),$$

where the sum is taken over all oriented edges from u to v. Assume that $V(G^T) = \{v_1,\dots,v_n\}$ and $V(\widetilde{G}^T) = \{v_1^+,\dots,v_n^+, v_1^- \dots, v_n^-\}$ with n even, where $v_i^\pm$ are the lifts of v_i to the first and the second copy of $\mathscr{K}$. Considering the matrix $N(z,w)$ of $A_{G^T}^{\rho^{z,w}}$ with the vertex labelling $v_1,\dots,v_n$, and the matrix $M(z,w)$ with respect to the labelling $v_1^+, v_1^-, \dots, v_n^+, v_n^-$ so that these matrices are both in blocks of 2×2 submatrices, one can compare their corresponding blocks as follows. First note that the block of $N(z,w)$ corresponding to an edge $e = (v_i, v_j)$ corresponds to the block of $M(z,w)$ created by

$(v_i^{\pm}, v_j^{\pm})$. If the edge e does not cross either a or b, then $\rho_e^{z,w} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and so the corresponding block in $N(z,w)$ is equal to $\begin{pmatrix} \epsilon^K_{v_iv_j}(e)x(e) & 0 \\ 0 & \epsilon^K_{v_iv_j}(e)x(e) \end{pmatrix}$. This is precisely the corresponding block in $M(z,w)$ because the orientation from v_i to v_j and the orientation from $v_i^{\pm}$ to $v_j^{\pm}$ are the same, as well as the weights on corresponding edges. Similarly, if e crosses a upward (resp. downwards), then in $N(z,w)$ the corresponding block is twisted by $\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}^{\pm 1}$ and in $M(z,w)$ we twist (v_i^+, v_j^+) with $z^{\pm 1}$ and twist (v_i^-, v_j^-) with $z^{\mp 1}$, and hence the corresponding block in $M(z,w)$ is also twisted by $\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}^{\pm 1}$. Thus in this case $N(z,w)$ and $M(z,w)$ also have equal blocks. By the same argument, we also see that if e crosses b, the corresponding blocks of $N(z,w)$ and $M(z,w)$ are equal. Therefore we get $N(z,w) = M(z,w)$.

Let us continue with transforming $M(z,w)$. With respect to the vertex labelling $v_1^+, \dots, v_n^+, v_1^-, \dots, v_n^-$, this matrix has the form

$$\begin{pmatrix} M_1(z) & M_2(w) \\ M_2(w) & M_1(1/z) \end{pmatrix}.$$

Observe that the non-zero entries of $M_2(w)$ correspond exactly to edges of $\widetilde{G}^T$ crossing $\widetilde{b}$ or $\widetilde{b}'$, thus they are of the form $\pm i w^{\pm 1}$. Now let us multiply all the rows and columns of $M(z,w)$ corresponding to $v_1^-, \dots, v_n^-$ by $-i$ to obtain $M'(z,w)$. Since n is even, we have $\det M(z,w) = \det M'(z,w)$. Moreover $M'(z,w)$ is almost equal to $\widetilde{A}(z,w)$ except that in $M'(z,w)$ one still twists an edge e by $w^{\pm 1}$ each time e crosses $\widetilde{b}'$. To this end, note that the determinant of $M'(z,w)$, which is equal to that of $M(z,w)$, is a sum (with signs) of terms where each term corresponds to the weight of some oriented cycles of $\widetilde{G}^T$ (here z, w can be considered as new weights on $\widetilde{G}^T$). For C such an oriented cycle, one can see that, instead of twisting an edge e of C by $w^{\pm 1}$ each time e crosses $\widetilde{b}$ or $\widetilde{b}'$, twisting e by $w^{\pm 2}$ each time e crosses only $\widetilde{b}$ does not change the weight of C. Therefore we get $\det M'(z,w) = \det \widetilde{A}(z,w^2) = \widetilde{P}(z,w^2)$, concluding the proof. □

Remark 4.5.13. An immediate consequence of Proposition 4.5.12 is that the determinant $\widetilde{P}(z,w)$ of the matrix given by $A_G^{\rho^{z,w}}$ is a Laurent polynomial of z, w with real coefficients. Furthermore, since $A_{G^T}^{\rho^{z,w}}$ is conjugate to $A_{G^T}^{\rho^{1/z,w}}$ by definition, one has

$$\widetilde{P}(z,w^2) = \det A_{G^T}^{\rho^{z,w}} = A_{G^T}^{\rho^{1/z,w}} = \widetilde{P}(1/z,w^2)$$

for every z, w. Thus we get $\widetilde{P}(z,w) = \widetilde{P}(1/z,w)$ for every z, w.

Note also that, by the proof of Proposition 4.5.12 above, we have

$$\widetilde{P}(\pm 1, w^2) = \det M(\pm 1, w)$$

$$
\begin{aligned}
&= \det \begin{pmatrix} M_1(\pm 1) & M_2(w) \\ M_2(w) & M_1(\pm 1) \end{pmatrix} \\
&= \det \big(M_1(\pm 1) + M_2(w)\big) \det \big(M_1(\pm 1) - M_2(w)\big) \\
&= P(\pm 1, w) P(\pm 1, -w),
\end{aligned}
$$

recalling Definition 4.5.1.

Let us now combine Lemmas 4.5.8, 4.5.10 and 4.5.11 together with Proposition 4.5.12 to get a product formula for the Ising characteristic polynomial $P_{MN}(z,w)$ of the graph G_{MN}. Assume that N is odd so that G_{MN} is still embedded in the Klein bottle $\mathscr{K}_{MN}$. Setting $z_k := \exp(k\pi i/M)$ and $w_l := \exp(2l\pi i/N)$ we have the following formulas.

Theorem 4.5.14. *Given G a graph embedded in the Klein bottle $\mathscr{K}$ and G_{MN} its MN-sheeted covering. For N odd we have*

$$P_{MN}(1, \pm 1) = \prod_{l=1}^{N} P(1, \pm w_l) \prod_{l=1}^{N} \prod_{k=1}^{(M-1)/2} \widetilde{P}(z_{2k}, w_l^2),$$

$$P_{MN}(-1, \pm 1) = \prod_{l=1}^{N} P(-1, \pm w_l) \prod_{l=1}^{N} \prod_{k=0}^{(M-3)/2} \widetilde{P}(z_{2k+1}, w_l^2)$$

if M is odd, and

$$P_{MN}(1, \pm 1) = \prod_{l=1}^{N} P(1, \pm w_l) \prod_{l=1}^{N} P(-1, \mp w_l) \prod_{l=1}^{N} \prod_{k=1}^{(M-2)/2} \widetilde{P}(z_{2k}, w_l^2),$$

$$P_{MN}(-1, \pm 1) = \prod_{l=1}^{N} \prod_{k=0}^{(M-2)/2} \widetilde{P}(z_{2k+1}, w_l^2)$$

if M is even.

Proof. Let us prove the first equality. For M odd we have

$$
\begin{aligned}
P_{MN}(1, \pm 1) &\overset{Lem. 4.5.8}{=} \prod_{l=1}^{N} P_{M1}(1, \pm w_l) \\
&\overset{Lem. 4.5.10}{=} \prod_{l=1}^{N} \Big(P(1, \pm w_l) \prod_{k=1}^{(M-1)/2} \det A_{G^T}^{\rho_{2k}^{\pm w_l}}\Big) \\
&= \prod_{l=1}^{N} P(1, \pm w_l) \prod_{l=1}^{N} \prod_{k=1}^{(M-1)/2} \det A_{G}^{\rho^{z_{2k}, \pm w_l}} \\
&\overset{Prop. 4.5.12}{=} \prod_{l=1}^{N} P(1, \pm w_l) \prod_{l=1}^{N} \prod_{k=1}^{(M-1)/2} \widetilde{P}(z_{2k}, w_l^2).
\end{aligned}
$$

The other equalities can be proved similarly. $\square$

4.5.3 Finite-size corrections

In this last part of the thesis, we will study the free energy and the finite-size corrections. More precisely, we will show that the free energy can be given by an integral of $\log \widetilde{P}(z,w)$. Furthermore, setting $Q_{\pm}(w) := P(\pm 1, w)$, one can show that the finite-size corrections at criticality are determined by the leading coefficient and the roots of modulus bigger than 1 of this polynomial, together with certain functions coming from the correction term of the Ising partition function of $\widetilde{G} \subset \mathbb{T}$, the lift of G to the torus $\mathbb{T}$. These results are based on what we have developed so far together with a result of Cimasoni and Duminil-Copin [11], which itself relies on a famous result of Kenyon-Okounkov-Sheffield [42]. Throughout this subsection we will assume that N is odd.

Before going into detail, let us first show how to obtain a good orientation on G^T_{MN} from that on G^T. We first draw G inside the rectangle $\mathcal{R} = aba^{-1}b$ representing $\mathscr{K}$ and fix a good orientation K on G^T from now on. As mentioned before, the Klein bottle $\mathscr{K}_{MN}$ is obtained by gluing MN copies of $\mathcal{R}$ together, and thus can be represented by a rectangle with sides identified following the word $a_N b_M a_N^{-1} b_M$. For K a good orientation on G^T denote by K_{MN} its lift to G^T_{MN}. We also denote by $\widetilde{K}_{MN}$ the orientation obtained by inverting K_{MN} on every edge of G^T_{MN} which is contained completely inside the c^{th} "column" of $\mathscr{K}_{MN}$ with c even. For $j,k \in \{0,1\}$, we refer to $\widetilde{K}^{jk}_{MN}$ as the orientation obtained by inverting $\widetilde{K}_{MN}$ on every edge crossing a_N with $j=1$ and crossing b_M with $k=1$. The following lemma can be proved using the very definitions of $\widetilde{K}_{MN}$ and of good orientations.

Lemma 4.5.15. *Assume that K is a good orientation on G^T. Then we have*

(i) The orientation $\widetilde{K}_{MN}$ is good for $M=1$ and $N=2$.

(ii) For M odd, the orientation $\widetilde{K}_{MN}$ is good if $N \equiv 1$ modulo 4, and $\widetilde{K}^{01}_{MN}$ is good if $N \equiv 3$ modulo 4.

(ii) For M even, the orientation $\widetilde{K}^{11}_{MN}$ is good if $N \equiv 1$ modulo 4, and $\widetilde{K}^{10}_{MN}$ is good if $N \equiv 3$ modulo 4. □

For simplicity, let us from now on consider the case M odd and $N \equiv 1$ modulo 4 so that $\widetilde{K}_{MN}$ is a good orientation. The other cases will be discussed at the end of this subsection.

Let us denote by $P'_{MN}(z,w)$ the characteristic polynomial of G_{MN} with respect to the orientation $\widetilde{K}_{MN}$ and the standard representation $\widetilde{\omega}_{MN}$ of $\mathscr{K}_{MN}$ (that is, $\widetilde{\omega}_{MN}(e) = e \cdot b_M$ for any edge e of G^T_{MN}). Recall that $P_{MN}(z,w)$ is the characteristic polynomial of G_{MN} with respect to the orientation K_{MN} and the function ω_{MN} which also represents the first Stiefel-Whitney class of $\mathscr{K}_{MN}$. To be more precise, $\omega_{MN}(e)$ is equal to the parity of the intersection number of e with MN copies of the side b of $\mathcal{R}$ in $\mathscr{K}_{MN}$. We have the following result.

Lemma 4.5.16. *For every $z,w \in \mathbb{C}^*$, $P'_{MN}(z,w) = P_{MN}(z,w)$.*

Proof. Let $A_{MN}(z,w)$ and $A'_{MN}(z,w)$ be the corresponding twisted adjacency matrices of G^T whose determinants are $P_{MN}(z,w)$ and $P'_{MN}(z,w)$. To each vertex v of G^T_{MN} contained in the c^{th} column of $\mathscr{K}_{MN}$ with c even, we multiply the weight of each edge adjacent to v by $-\sqrt{-1}$. These multiplications simply transform $A_{MN}(z,w)$ to $A'_{MN}(z,w)$ and do not change the determinant since G^T has an even number of vertices. Hence we get $P_{MN}(z,w) = P'_{MN}(z,w)$. □

The following proposition, being a consequence of [11, Lemma 4.2] (which relies on [42]), tells us the zero locus of $\widetilde{P}(z,w)$ on the unit torus $S^1 \times S^1$.

Proposition 4.5.17. *The polynomial $\widetilde{P}(z,w)$ is real and non-negative on $S^1\times S^1$. Moreover if the model is not at criticality, $\widetilde{P}(z,w)$ never vanishes for $|z| = |w| = 1$. Otherwise, it vanishes at only one of two points $(\pm 1, -1)$.*

Proof. Recall by definition that $\widetilde{P}(z,w) = P'_{12}(z,w) = P_{12}(z,w)$, using Lemma 4.5.16. By Remark 4.5.2 and Lemma 4.5.15 part (i), $\widetilde{P}(z,w)$ is the same as the characteristic polynomial defined in [11, Definition 2.1]. Hence by [11, Lemma 4.2], $\widetilde{P}(z,w)$ is real and non-negative on $S^1 \times S^1$. Also by [11, Lemma 4.2], if the model is not at criticality, $\widetilde{P}(z,w)$ never vanishes for $|z| = |w| = 1$. Otherwise, it vanishes at only one of four points $(\pm 1, \pm 1)$. Moreover, by Remark 4.5.13 we have $\widetilde{P}(\pm 1, w^2) = P(\pm 1, w)P(\pm 1, -w)$, and since $P(\pm 1, \pm 1)$ never vanishes (recall Proposition 4.5.3 part (ii)), $\widetilde{P}$ can only vanish at $(\pm 1, -1)$. This concludes the proof. □

Denoting by $\widetilde{P}_{MN}(z,w)$ the characteristic polynomial of $\widetilde{G}_{MN}$, the MN-sheeted covering of $\widetilde{G}$ (note that $\widetilde{G}_{MN}$ is also embedded in the torus), we have the following lemma.

Lemma 4.5.18. *For every $M, N \geq 1$ and N odd we have*

$$|P_{MN}(1, \pm 1)| = \widetilde{P}_{MN}(1,1)^{1/2} \quad \textit{and} \quad |P_{MN}(-1, \pm 1)| = \widetilde{P}_{MN}(-1,1)^{1/2}.$$

Consequently $\widetilde{P}_{MN}(z,w)$ never vanishes at $(\pm 1, 1)$.

Proof. Let us consider the case M odd. By Theorem 4.5.14 and Proposition 4.5.3 part (i), as well as the properties of $\widetilde{P}(z,w)$ mentioned in Remark 4.5.13 we get

$$\begin{aligned}
|P_{MN}(1,1)|^2 &= \prod_{l=1}^{N} |P(1,w_l)|^2 \prod_{l=1}^{N} \prod_{k=1}^{(M-1)/2} \widetilde{P}(z_{2k}, w_l^2)^2 \\
&= \prod_{l=1}^{N} \big(|P(1,w_l)||P(1,-w_l)|\big) \prod_{l=1}^{N} \prod_{k=1}^{(M-1)/2} \big(\widetilde{P}(z_{2k}, w_l^2)\widetilde{P}(z_{2M-2k}, w_l^2)\big) \\
&= \prod_{l=1}^{N} \widetilde{P}(1, w_l^2) \prod_{l=1}^{N} \prod_{k=1}^{M-1} \widetilde{P}(z_{2k}, w_l^2) = \prod_{l=1}^{N} \prod_{k=1}^{M} \widetilde{P}(z_{2k}, w_l^2) = \widetilde{P}_{MN}(1,1),
\end{aligned}$$

where the last equality comes from the product formula for characteristic polynomials defined on the torus [11, Lemma 4.1] (recall that $\widetilde{P}(z,w)$ is the characteristic polynomial defined on the orientation cover of $\mathscr{K}$, which is the torus), and from the fact that N is odd. Other equalities and the case M even can be treated similarly. The fact that $\widetilde{P}_{MN}(\pm 1,1)$ is non-zero then follows from Lemma 4.5.3 part (ii) applied to $P'_{MN}(z,w)$, together with Lemma 4.5.16. □

Now let us quickly discuss the free energy. The *free energy per fundamental domain* of the model on the graph G is defined by

$$\mathrm{f}_0 := \lim_{M,N\to\infty} \frac{1}{MN} \log Z_{\mathcal{I}}(G_{MN}).$$

Using the product formula and the fact that $P_{MN}(\pm 1,\pm 1)$ is non-zero (recalling Proposition 4.5.3 part (ii)), one can show that the free energy per fundamental domain can be written as

$$\mathrm{f}_0 = \frac{1}{(4\pi i)^2} \int_{S^1\times S^1} \log \widetilde{P}(z,w) \frac{dz}{z}\frac{dw}{w}.$$

By [11, Lemma 4.3], the free energy $\widetilde{\mathrm{f}_0}$ per fundamental domain of the model on $\widetilde{G} \subset \mathbb{T}$ is given by

$$\widetilde{\mathrm{f}_0} = \frac{1}{2(2\pi i)^2} \int_{S^1\times S^1} \log \widetilde{P}(z,w) \frac{dz}{z}\frac{dw}{w}.$$

Hence we get $\widetilde{\mathrm{f}_0} = 2\,\mathrm{f}_0$, which is certainly correct: the free energy does not depend on boundary conditions, and the fundamental domain of $\widetilde{G}$ is twice as big as that of G.

We continue with studying finite-size corrections. By Lemma 4.5.16, Equation (4.9) reads

$$Z_{\mathcal{I}}(G_{MN}) = \left|\operatorname{Re} P_{MN}(1,1)^{1/2}\right| + \left|\operatorname{Im} P_{MN}(-1,1)^{1/2}\right|. \tag{4.11}$$

To have an asymptotic expansion for $Z_{\mathcal{I}}(G_{MN})$, let us investigate the behaviour of each term $P_{MN}(\pm 1,1)$ for M,N big. Recall that $Q_{\pm}(w) = P(\pm 1,w)$, by Theorem 4.5.14 and Lemma 4.5.18 we have for M odd

$$P_{MN}(1,1) = \exp(i \arg \prod_{1\le l\le N} Q_+(w_l))\widetilde{P}_{MN}(1,1)^{1/2}, \tag{4.12}$$

$$P_{MN}(-1,1) = \exp(i \arg \prod_{1\le l\le N} Q_-(w_l))\widetilde{P}_{MN}(-1,1)^{1/2}, \tag{4.13}$$

recalling that $w_l = \exp(2l\pi i/N)$.

The following result by Kenyon-Sun-Wilson [44, Theorem 1], restricted to our context, gives an asymptotic expansion of $\widetilde{P}_{MN}(\pm 1,1)$.

Theorem 4.5.19. *Suppose $\widetilde{P}(z,w)$ is an analytic non-negative function defined on the unit torus $S^1 \times S^1$, non-vanishing except at (z_0, w_0) with associated Hessian H. Then for M, N big we have*

$$\log \widetilde{P}_{MN}(z,w) = 2MN\,\widetilde{\mathrm{f}}_0 + 2\log \Xi\left(\frac{z}{z_0^M}, \frac{w}{w_0^N}|\tau\right) + o(1),$$

where τ is a parameter which can be computed from the Hessian of $\widetilde{P}$ at (z_0, w_0). In this expansion we have

$$\Xi(-e^{2\pi i\phi}, -e^{2\pi i\psi}|\tau) \equiv \left|\frac{\vartheta_{00}(\phi\tau - \psi|\tau)e^{\pi i\tau\phi^2}}{\eta(\tau)}\right|$$

in which ϑ_{00} is the Jacobi theta function *and η is the* Dedekind η function. □

Now we are left with estimating the argument of $\prod_{l=1}^N Q_\pm(\pm w_l)$. It turns out that this quantity only depends on the roots of $Q_\pm$ with modulus bigger than 1 and on the leading coefficient of $Q_\pm$ (which is of the form $i^{l_\pm}$ by Proposition 4.5.3 part (i)), as the following lemma states.

Lemma 4.5.20. *Let $i^{l_\pm}$ be the leading coefficient of $Q_\pm$, $m_\pm$ the number of its roots of modulus bigger than 1 and $n_\pm$ the number of such roots in $i\mathbb{R}^-$. Then for N odd and big enough we have*

$$\exp(i \arg \prod_{w^N=1} Q_\pm(w)) = i^{N(l_\pm + 2n_\pm - m_\pm)} + o(1).$$

Proof. First of all, for simplicity let us drop the subindex $\pm$ in the statement. Recall by Proposition 4.5.3 part (i) that $Q(w) = P(\pm 1, w)$ has the form

$$Q(w) = \sum_k a_{2k}(w^{2k} + \frac{1}{w^{2k}}) + i\sum_k a_{2k+1}(w^{2k+} + \frac{1}{w^{2k+1}})$$

with $a_k \in \mathbb{R}$. Let us write $Q(w) = \dfrac{i^l}{w^d}\prod_{k=1}^{2d}(w - \mu_k)$ where the μ_k's are roots of Q. Observe that for N odd we have $\prod_{w^N=1}(w - s) = 1 - s^N$ for every s. Using this equality for $s = 0$ and $s = \mu_k$ we find $\prod_{w^N=1} Q(w) = i^{Nl}\prod_{k=1}^{2d}(1 - \mu_k^N)$ and hence

$$\exp(i \arg \prod_{w^N=1} Q(w)) = i^{Nl}\prod_{k=1}^{2d}\exp(i\arg(1 - \mu_k^N)). \tag{4.14}$$

If Q has a root μ_k on S^1, by Proposition 4.5.17 and the fact that $\widetilde{P}(\pm 1, w^2) = Q_+(w)Q_-(w)$ (recall Remark 4.5.13), we have $\mu_k = \pm i$ (and the model is at criticality). Since N is odd and note that $\arg(1-i^N)+\arg(1-(-i)^N) = \arg((1+i^N)(1-i^N)) = \arg 2 = 0$, we can ignore the roots of Q with modulus equal to 1 (if they exist) in the product on the right hand side of Equation (4.14). For other roots μ_k of Q, writing $\mu_k = r_k\exp(i\varphi_k)$ (with $r_k \neq 1$) one gets

$$\exp(i\arg(1 - \mu_k^N)) = \frac{1 - \mu_k^N}{|1 - \mu_k^N|} = \frac{1 - r_k^N\cos(N\varphi_k) - r_k^N\sin(N\varphi_k)}{\sqrt{1 + r_k^{2N} - 2r_k^N\cos(N\varphi_k)}}$$

$$= \begin{cases} o(1) & \text{if } r_k < 1; \\ -\cos(N\varphi_k) - \sin(N\varphi_k) + o(1) & \text{if } r_k > 1 \end{cases}$$

for N big. As N is always odd, the equation above and Equation (4.14) lead to

$$\exp(i \arg \prod_{w^N=1} Q(w)) = i^{Nl} \prod_{r_k>1} \exp(iN(\varphi_k + \pi) + o(1)).$$

We now show that $\sum_{r_k>1} \varphi_k$ is a multiple of $\pi/2$. Using Proposition 4.5.3 part (i), the Laurent polynomial $Q_1(w) := Q(iw)$ is of real coefficients, and hence Q_1 has real roots, or complex roots which are conjugate in pairs. It follows that the roots of Q are pure imaginary, or fall into pairs $(iz, i\overline{z})$. Denote by p the number of roots of Q in $i\mathbb{R}^+$ of modulus bigger than 1, and note that $\arg(iz) + \arg(i\overline{z}) = \pi$, we can write modulo 2π

$$\sum_{r_k>1} \varphi_k = (m-p-n)\frac{\pi}{2} + p\frac{\pi}{2} - n\frac{\pi}{2} = m\frac{\pi}{2} + n\pi.$$

The two equations displayed above conclude the proof. $\square$

We now use Lemma 4.5.20 and Theorem 4.5.19 to express $Z_{\mathcal{I}}(G_{MN})$ asymptotically. First let us recall Equation (4.11)

$$Z_{\mathcal{I}}(G_{MN}) = \left|\operatorname{Re} P_{MN}(1,1)^{1/2}\right| + \left|\operatorname{Im} P_{MN}(-1,1)^{1/2}\right|.$$

By Proposition 4.5.17, at criticality $\widetilde{P}(z,w)$ vanishes at only one of two points $(\pm 1, -1)$. Let us assume that it vanishes at $(-1,-1)$. Then setting

$$\iota_1 := \left|\operatorname{Re}\left(i^{N(l_+ + 2n_+ - m_+)/2}\right)\right| \quad \text{and} \quad \iota_2 := \left|\operatorname{Im}\left(i^{N(l_- + 2n_- - m_-)/2}\right)\right|,$$

by Theorem 4.5.19, Lemma 4.5.20 and by Equations (4.12), (4.13) we have at criticality

$$\left|\operatorname{Re} P_{MN}(1,1)^{\frac{1}{2}}\right| = \left|\operatorname{Re}\left(\exp\left(\frac{1}{2}\arg \prod_{w^N=1} Q_+(w)\right)\right)\widetilde{P}_{MN}(1,1)^{\frac{1}{4}}\right| = \iota_1\, \Xi \exp\left(\frac{MN}{2}\widetilde{\mathrm{f}_0} + o(1)\right),$$

$$\left|\operatorname{Im} P_{MN}(-1,1)^{\frac{1}{2}}\right| = \left|\operatorname{Im}\left(\exp\left(\frac{1}{2}\arg \prod_{w^N=1} Q_-(w)\right)\right)\widetilde{P}_{MN}(-1,1)^{\frac{1}{4}}\right| = \iota_2\, \Xi' \exp\left(\frac{MN}{2}\widetilde{\mathrm{f}_0} + o(1)\right)$$

with

$$\Xi := \left|\frac{\vartheta_{00}(0|\tau)}{\eta(\tau)}\right|^{1/2}, \quad \Xi' := \left|\frac{\vartheta_{10}(0|\tau)}{\eta(\tau)}\right|^{1/2}.$$

Note that to have Ξ' displayed as above, we use the following transformation between theta functions

$$\vartheta_{00}(\nu + \frac{1}{2}\tau|\tau) = e^{-i\pi(\nu+\tau/4)}\vartheta_{10}(\nu|\tau).$$

Then using the fact that $\widetilde{\mathrm{f}_0} = 2\,\mathrm{f}_0$ mentioned before, we obtain

$$Z_{\mathcal{I}}(G_{MN}) = \left(\iota_1\, \Xi + \iota_2\, \Xi'\right) \exp\left(MN\, \mathrm{f}_0 + o(1)\right).$$

Note that for M odd and $N \equiv 3$ modulo 4, we also obtain the same result as above: in this case one can consider the characteristic polynomial $P''_{MN}(z,w)$ of G_{MN} with respect to $\widetilde{K}^{01}_{MN}$, and similarly to Lemma 4.5.16 one gets $P''_{MN}(z,w) = P_{MN}(z,-w)$; the calculation presented above works verbatim for $P_{MN}(z,-w)$. By the same argument, for M even we also get

$$Z_{\mathcal{I}}(G_{MN}) = \big(\iota\,\Xi + \Xi'\big)\exp\big(MN\,\mathrm{f}_0 + o(1)\big)$$

with

$$\iota := \big|\mathrm{Im}\big(i^{N(l_+ - l_- + 2n_+ - 2n_- + m_- - m_+)/2}\big)\big|.$$

We summarise our result in the following theorem.

Theorem 4.5.21. *The finite-size correction terms to the free energy of the Ising model on a weighted graph $G \subset \mathscr{K}$ at criticality is given by $\log\big(\iota\,\Xi + \Xi'\big)$ for M even, and by $\log\big(\iota_1\,\Xi + \iota_2\,\Xi'\big)$ for M odd. The numbers ι, ι_1, ι_2 belong to $\{0, 1, \sqrt{2}/2\}$ and depend only on N modulo 8, while the functions Ξ, Ξ' are given by*

$$\Xi = \left|\frac{\vartheta_{00}(0|\tau)}{\eta(\tau)}\right|^{1/2}, \quad \Xi' = \left|\frac{\vartheta_{10}(0|\tau)}{\eta(\tau)}\right|^{1/2}$$

with ϑ_{rs} Jacobi theta functions, η the Dedekind η function, and τ being computable from the Hessian of $\widetilde{P}(z,w)$ at $(-1,-1)$.

Proof. The proof comes straight forward from the computation above. □

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