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Khovanov homology of torus links: Structure and Computations

THÈSE

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par
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«Khovanov Homology of Torus Links: Structure and Computations»

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Résumé de la thèse en français

L'objet de cette thèse est l'étude de la topologie en basse dimension, et plus précisément de la théorie des noeuds. Elle se décompose globalement en 2 parties. La première consiste en l'étude de l'homologie réduite de Khovanov et ses propriétés combinatoires déterminantes pour la calculer. La seconde applique ces outils à la famille des entrelacs toriques avec deux points de vue. D'une part nous calculerons l'homologie d'entrelacs toriques standards, et d'autre part nous considèrerons l'homologie d'entrelacs toriques "infinis". Ces étapes sont décrites brièvement en les deux paragraphes suivants.

Un entrelacs est une collection de cercles plongée dans un espace de dimension 3. Un tel entrelacs peut être représenté en 2 dimensions par un diagramme, qui est donc une immersion de cercles dans un plan, avec des données de croisements à chaque point double. Ceux-ci sont analysés via des quantités algébriques, appelées invariants de diagrammes. Cette thèse se concentre sur l'étude d'un invariant de noeuds et d'entrelacs de type homologique, appelé homologie de Khovanov. Dans une première partie, après avoir rappelé les définitions de bases de théorie des noeuds, nous donnerons la définition de l'homologie de Khovanov et de sa version réduite, que nous plaçerons dans un contexte qui lui est propre. Nous étudierons aussi une version dégénérée de cette théorie, l'homologie de Bar-Natan. Nous continuerons avec l'étude d'outils calculatoires, qui seront utilisés dans la deuxième partie. En particulier, nous nous concentrerons sur l'opération de somme connexe de noeuds, ainsi que sur la suite exacte longue et la suite spectrale en homologie de Khovanov. Enfin, nous développerons un dernier outil sous forme d'une opération cohomologique, qui sera utilisée en conjonction avec les outils sus-mentionnés.

La deuxième partie de cette thèse se concentre sur des calculs de l'homologie de Khovanov. Nous nous concentrerons en particulier sur la famille des entrelacs dit toriques. Ces entrelacs vivent naturellement sur un tore et dépendent uniquement de deux paramètres, le nombre de brins, et le nombre de tours. Pour un nombre de tours fini, nous calculerons entièrement l'homologie de Khovanov pour les familles à 3 brins. Pour un nombre de tours infinis, nous montrerons que l'homologie correspondante admet la structure supplémentaire d'une algèbre. Nous déterminerons précisement cette structure d'algèbre pour la familles à 2 brins.

Remerciements

Tout d'abord, je tiens à remercier mes directeurs de thèse, David Cimasoni et Paul Turner, pour leur aide, leurs conseils, leur patience et leur enthousiasme. Ce fut pour moi un honneur et un grand plaisir que d'effectuer ma thèse sous votre direction. J'aimerais également remercier le Professeur Rinat Kashaev et le Dr. Lukas Lewark d'avoir accepté de faire partie du jury.

À qui lirait ces quelques mots, excusez-moi si vous ne trouvez pas votre nom ici. Une liste exhaustive de vous tous serait certainement plus longue que cette thèse.

Merci à tous les membres de la Section, collaborateurs et personnel administratif pour leur présence. En particulier, j'aimerais remercier mes nombreux "co-bureaux" pour toutes les discussions et bons moments passés. Merci Fabien, Minh, Aitor, Mucyo, Cyril, Lida et, last but not least, merci Fathi, j'ai énormément appris de toi. Merci Anthony, Jérémy, Maxime, Grégoire, Seb, Xavier, Anders, pour les nombreux moments de sérieux, et pour les autres qui ne l'étaient pas trop. Sandie, Élise, vous c'est surtout pour les apéros.

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Contents

Introduction

Background

Given a topological space X, its homology groups $H_i(X)$ are topological invariants which contain information about the Euler characteristic $\chi(X)$. More precisely we have

$$\sum_{i\in\mathbb{Z}}(-1)^i \text{ rank } (H_i(X)) = \chi(X).$$

Singular homology is also functorial: given two topological spaces, X, Y and a continuous map $f : X \longrightarrow Y$, there is an induced map

$$f_* : H_*(X) \longrightarrow H_*(Y).$$

Thus, in a sense, homology upgrades the Euler characteristic to the level of categories and we say that it *categorifies* the Euler characteristic.

In his seminal paper [**Kho00**], Khovanov categorifies the Jones polynomial of knots and links: he introduces a bigraded homology theory $Kh^{*,*}(L)$, defined from a diagram D that represents a link L, that is an isotopy invariant satisfying

$$\sum_{i,j\in\mathbb{Z}}(-1)^i q^j \text{ rank } (Kh^{i,j}(L)) = V_L(q^2),$$

where $V_L(q^2)$ is the normalized Jones polynomial in variable q^2.

Rather than using Jones' original construction of the epochal polynomial, Khovanov uses Kauffman's state-sum formula as a starting point. Given a diagram D, he constructs a bigraded chain complex $(C(D), d_{Kh})$ inspired by Kauffman's approach whose homology $Kh^{*,*}(D)$ is Khovanov homology.

First and foremost, Khovanov homology is a link invariant. Indeed, Khovanov shows that if two diagrams D and D' are related by Reidemeister moves, then the associated homologies are isomorphic

$$Kh^{*,*}(D) \cong Kh^{*,*}(D').$$

The resulting invariant is stronger than the Jones polynomial: there are examples of knots with identical Jones polynomial that are distinguished by Khovanov homology. But the real interest in Khovanov homology is that it gives more than just an invariant, it actually gives a functor.

There is a category whose objects are links and morphisms are *link cobordisms*, i.e. surfaces with boundary the disjoint union of two links L and L', up to an equivalence relation. To any such cobordism Σ from L to L', Khovanov homology associates a graded map

$$\phi_\Sigma : Kh^{*,*}(L) \longrightarrow Kh^{*,*+m}(L').$$

These maps are defined by using *movies*, which are a diagrammatic way to encode cobordisms. Over the integers, this theory is not fully functorial: two movies that represent the same link cobordism produce maps that are only equal up to sign (Jacobson [**Jac04**]). One way to avoid this issue of sign is to change the coefficients to $\mathbb{Z}_2$, and that is the option we take in this thesis. There are two other ways of making the theory functorial and both rely on considering enriched link diagrams and their movies. The first approach is due to Clark, Morrison and Walker [**CMW09**], and requires $\mathbb{Z}[i]$ coefficients and uses links and link cobordisms augmented with extra data (seams). The second approach, due to Blanchet [**Bla10**], uses so-called webs and foams and categorifies the Murakami-Ohtsuki-Yamada [**MOY98**] bracket rather than the Kauffman bracket.

There is also a reduced version of Khovanov homology, introduced by Khovanov [**Kho03**] which requires the additional data of a chosen point on a diagram.

So apart from functoriality, why is Khovanov homology so important? First, through the work of Kronheimer-Mrowka [**KM11**], it detects the unknot. Secondly, there is a filtered version of Khovanov homology for $\mathbb{Q}$ coefficients, introduced by Lee [**Lee05**], which has been used by Rasmussen [**Ras10**] to extract topological information about knots. By using the functoriality of Lee's theory, he defined a lower bound for the slice genus. Finally, there is the seminal work of Bar-Natan [**BN05**] who categorifies the Temperley-Lieb algebra. He works in a formal category of circles and associates to tangles a formal bracket in the form of a chain complex in that category. He shows invariance at that geometric level, and also studies functoriality. His version is universal: all other versions, and their invariance can be recovered by applying a $(1+1)$-TQFT to his setting. With a particular choice of TQFT, he defined a new filtered theory over $\mathbb{Z}_2$ known as Bar-Natan homology.

There are many other Jones-like polynomials, in particular one for each Lie algebra $\mathfrak{sl}_N$. Khovanov and Rozansky [**KR08a, KR08b**] categorified these polynomials too. The HOMFLY-PT polynomial has also been categorified into a triply-graded theory [**Kho07**].

Overview of results

Khovanov homology and how to compute it. In Chapter 1, we recall basic notions of knot theory: the definition of knots, links and their diagrams, as well as some variants, namely braids and tangles. Following Khovanov [**Kho03**], we also introduce the notion of pointed links and their diagrams, which are simply usual links and diagrams with an additional choice of a basepoint and the key objects used to define the reduced theory. Pointed links with multiple basepoints have been used by Baldwin-Levine-Sarkar [**BLS17**]. We then present Turner's construction of the Khovanov chain complex over $\mathbb{Z}_2$ [**Tur14**] and mention selected properties. With pointed diagrams, we give an construction the reduced Khovanov chain complex $\widetilde{C}(D,p)$ of a pointed diagram (D,p), different than Khovanov's, and a new proof of the independence from the choice of basepoint. Similarly, we present a reduced version of Bar-Natan theory, mentionned first in [**Tur06**], and explore some of its properties.

From Chapter 2 onwards, we will only use the reduced mod 2 Khovanov homology and reduced Bar-Natan homology. We introduce the main tools that will be used for computations throughout this thesis. Though they are probably known to the experts, they are not found in the litterature in the context of the reduced mod 2 theory and we will give all the details.

We begin with studying the behaviour of Khovanov homology with respect to connected sums of pointed links, which, contrary to the non pointed case, is a well-defined operation.

Then, given a pointed diagram (D,p) and a crossing c in D, we produce a long exact sequence in homology. Denote by D_0 (resp. D_1) the diagram obtained from D by doing surgery near the chosen crossing as indicated here.

Such a triple (D_1, D, D_0) will be called *exact triple*. Given an exact triple, there is a short exact sequence of (ungraded) chain complexes

$$0 \to \widetilde{C}(D_1) \overset{i}{\to} \widetilde{C}(D) \overset{\pi}{\to} \widetilde{C}(D_0) \to 0.$$

By carefully considering orientations one can re-introduce gradings. For example if c is negative, there is a long exact sequence of the form:

$$\cdots \overset{\partial^*}{\to} \widetilde{Kh}^{i,j+1}(D_1) \overset{i^*}{\to} \widetilde{Kh}^{i,j}(D) \overset{\pi^*}{\to} \widetilde{Kh}^{i-w_-,j-3w_--1}(D_0) \overset{\partial^*}{\to} \widetilde{Kh}^{i+1,j+1}(D_1) \overset{i^*}{\to} \cdots$$

This sequence was implicit in [**Kho00**], and made explicit by Viro [**Vir02**] for integer coefficients version. Finally, if one chooses more than one crossing this long exact sequence generalizes to a spectral sequence (Turner [**Tur08**]).

For all three computational tools -connected sum, the long exact sequence and Turner's spectral sequence-, we will rely heavily on the naturality with respect to maps induced by 1-handles. Finally in Chapter 2, we discuss how computers come into play in this thesis to provide initial data crucial to our calculations.

In Chapter 3, we construct a new cohomology operation on Khovanov homology. It arises by comparison of Bar-Natan and Khovanov homologies. These two theories are obtained via chain complexes $(\widetilde{C}(D,p), d_{\widetilde{BN}})$ and $(\widetilde{C}(D,p), d_{\widetilde{Kh}})$ and following Turner [**Tur06**], one can compare these two differentials by setting

$$d_{\widetilde{BN}} = \widetilde{\beta} + d_{\widetilde{Kh}}.$$

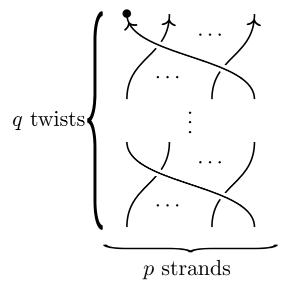
This produces a chain map $\widetilde{\beta}$ on the Khovanov complex, with bidegree $(1,2)$, which induces a map in homology

$$\widetilde{\beta}^* : \widetilde{Kh}^{i,j}(D) \longrightarrow \widetilde{Kh}^{i+1,j+2}(D).$$

The first new result of this thesis is that this map is natural: that is, it is a cohomology operation.

THEOREM (3.8). *The map* $\widetilde{\beta}^* : \widetilde{Kh}^{i,j}(D) \longrightarrow \widetilde{Kh}^{i+1,j+2}(D)$ *is a cohomology operation.*

Torus links: finite and infinite. The bulk of our work concerns the *torus links* $T_{p,q}$. Each of them is uniquely determined by a pair of integers (p,q). They have a standard diagram $D_{p,q}$, given by the closure of the braid depicted below.



$D_{p,q}$

Given any diagram D, there is an isomorphism relating its homology with that of its mirror, therefore we will only consider negative torus links. For torus knots, i.e. those for which $gcd(p,q) = 1$, Jones computed their Jones polynomial [**Jon87**]. This formula was generalized to any torus link by Isodro-Labastida-Ramallo [**ILR93**]. The formula, where $d = gcd(p,q)$ is given by

$$V_{T_{p,q}}(t) = (-1)^{d+1}\frac{t^{(p-1)(q-1)/2}}{1-t^2}\sum_{i=0}^{d}\binom{d}{i}t^{\frac{p}{d}(1+\frac{q}{d})(d-i)}\left(t^{\frac{q}{d}(d-i)} - t^{1+\frac{q}{d}i}\right).$$

It is therefore natural to ask oneself the question: what is the Khovanov homology of torus links? In general such computations are very hard. Two reasons can be cited to explain this. First computing the Jones polynomial $V_D(t)$ exactly is $\sharp$P-hard [**JVW90**] for almost all t so any theory containing it is at least as hard. Second is the nature of Khovanov's theory itself, the size of the underlying chain complex grows exponentially with respect to the number of crossings.

As it stands, only the cases $p = 2$ (Khovanov [**Kho00**]) and $p = 3$ (Turner [**Tur08**], Stošić [**Sto09**], and Gillam [**Gil12**]) have been treated, with some indeterminacy for the homology over $\mathbb{Z}_2$ for the family $T_{3,q}$. In Chapter 4, we remove this indeterminacy and compute the mod 2 homology of all 3-stranded torus links. In this case, we use the δ-graded Khovanov homology $\widetilde{Kh}^i_\delta(D)$, a normalization of $\widetilde{Kh}^{i,j}(D)$ obtained by setting $\delta = j - 2i$. This version focuses on diagonals and the structure of the homology appears very neatly. For a pointed diagram (D,p) and the associated δ-graded homology, we define the δ-graded Poincaré polynomial of D as follows:

$$P_\delta(t,q)(D) := \sum_{i,\delta\in\mathbb{Z}} t^i q^\delta \dim(\widetilde{Kh}^i_\delta(D)).$$

Our result is as follows.

THEOREM (4.13). *(i) For any $N \geq 1$, the δ-graded Poincaré polynomial of $T_{3,3N-1}$ is the following*

$$\begin{aligned} P_\delta(t,q)(T_{3,3N-1}) &= q^{-6N+4}(1+t^{-2}+t^{-3}+t^{-5})\left(\sum_{k=0}^{N-2} q^{2k}t^{-4k}\right) \\ &\quad + q^{-4N+2}t^{-4N+4}(1+t^{-2}+t^{-3}). \end{aligned}$$

(ii) For any $N \geq 1$, the δ-graded Poincaré polynomial of $T_{3,3N}$ is the following

$$\begin{aligned} P_\delta(t,q)(T_{3,3N}) &= q^{-6N+2}(1+t^{-2}+t^{-3}+t^{-5})\left(\sum_{k=0}^{N-2} q^{2k}t^{-4k}\right) \\ &\quad + q^{-4N}t^{-4N+4}(1+t^{-2}+t^{-3}+2t^{-4}) + q^{-4N+2}t^{-4N}. \end{aligned}$$

(iii) For any $N \geq 0$, the δ-graded Poincaré polynomial $T_{3,3N+1}$ is the following

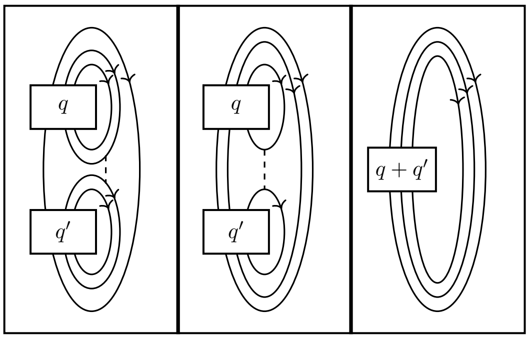
$$P_\delta(t,q)(T_{3,3N+1}) = q^{-6N}(1+t^{-2}+t^{-3}+t^{-5})\left(\sum_{k=0}^{N-1} q^{2k}t^{-4k}\right) + q^{-4N}t^{-4N}.$$

Alternatively, these spaces are given by the grids below.

Paradoxally, one way to circumvent the difficulty in computations is to consider torus links with an infinite number of twists. These infinite torus links were developped independently by Rozansky [**Roz14**] and Cooper-Krushkal [**CK12**]. Their point of views are similar but they work at different levels: the latter within Bar-Natan's categorification of the Temperley-Lieb algebra [**BN05**], and the former after application of a properly normalized Khovanov bracket.

The homology associated to these infinite torus links is the object of a conjecture of Gorsky, Oblomkov and Rasmussen [**GOR13**], that gives an explicit description of the vector spaces, but also suggests that it admits the additional structure of an algebra.

In Chapter 5, we follow in Rozansky's footsteps in the algebraic world. For any $p \geq 2$, we produce torus links with an infinite number of crossings as a limit, whose associated family of homologies is known to converge by a result of Stošić [**Sto07**]. Let us denote this limit by $\widetilde{Kh}^{*,*}(T_{p,\infty})$. It is obtained through a directed system, therefore we begin the chapter by recalling some known facts about these objects. We then give the definition of $\widetilde{Kh}^{*,*}(T_{p,\infty})$ and compute it explicitly for $p = 2$. This infinite dimensional vector space exhibits extra structure compared to its finite counterparts, namely that of an algebra induced by a well-chosen family of "fusion" movies, that induce a product in the limit:



And we obtain the following.

THEOREM (5.7). *For any $p \geq 2$, the vector space $\widetilde{Kh}^{*,*}(T_{p,\infty})$ can be endowed with the structure of a bigraded commutative algebra with unit.*

We describe the algebra for $p = 2$, for the δ-graded version of Khovanov homology:

THEOREM (5.10). *There is a bi-graded algebra isomorphism:*

$$\widetilde{Kh}^*_*(T_{2,\infty}) \cong \mathbb{Z}_2[x,y]/(x^3 = y^2).$$

The degrees of the generators are given by $|x| = (-2,0)$, $|y| = (-3,0)$.

This theorem relies heavily on the fact that we know the homologies of $T_{2,q}$ in advance, and that we can understand the maps induced by our fusion movie. Moreover, this algebra does not coincide with the one predicted by the Gorsky-Oblomkov-Rasmussen, and we propose an explanation to this discrepancy.

$-4N$				$-4N+4$										-2	-1	0	$i \backslash \delta$	
$\mathbb{Z}_2$																	$-4N+2$	
$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$														
			$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$										$N-2$ diagonals with the same pattern
						$\ddots$	$\ddots$	$\ddots$										
							$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$						
											$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$-6N+2$	

	$-4N$				$-4N+4$										-2	-1	0	$i \backslash \delta$	
	$\mathbb{Z}_2$																	$-4N$	
$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$														
				$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$										$N-2$ diagonals with the same pattern
							$\ddots$	$\ddots$	$\ddots$										
								$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$						
												$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$-6N$	

			$-4N$				$-4N+4$										-2	-1	0	$i \backslash \delta$	
$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$																	$-4N+2$	
		$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$														
						$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$										$N-2$ diagonals with the same pattern
									$\ddots$	$\ddots$	$\ddots$										
										$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$						
														$\mathbb{Z}_2$		$\mathbb{Z}_2$	$\mathbb{Z}_2$		$\mathbb{Z}_2$	$-6N-2$	

FIGURE 1. From top to bottom, the Khovanov homology of $T_{3,3N}, T_{3,3N+1}, T_{3,3N-1}$.

Before we move on to describing the rest of the contents of the final chapter of this thesis, let us mention that, unbeknownst to us, this algebra has been studied by Hogancamp [**Hog14**]. Hogancamp's work is centered around the Cooper-Krushkal point of view, i.e the construction of an unbounded chain complex P_n that categorifies the Jones-Wenzl projectors. Therefore he works at the level of the categorification of the Temperley-Lieb algebra. He provides a bounded chain complex Q_n - that roughly corresponds to a full twist over n strands, constructed from P_{n-1}. The idea relating Hogancamp's and our version is that an algebra with a unit is isomorphic to its endomorphism ring. Under this correspondance the product becomes a composition. So when we work with movies and cobordisms, that realize the product, he works immediately with composition of tangles. He then proceeds to relate the bounded Q_n's and the unbounded P_n's. The process is the following: he "glues" $\mathbb{N}$ copies of Q_n appropriately shifted, and shows that the result is P_n. Additionnally, this exhibits P_n as a polynomial algebra over P_{n-1}. Our methods are very different as we work directly with the homology spaces. The polynomial character of the algebra also appears, by studying the fusion movies precisely. Note that in the case we study, we actually compute the algebra structure explicitly. We also believe that our methods will eventually extend to the general case of $\widetilde{Kh}^{*,*}(T_{p,\infty})$.

Finally, in Chapter 7, we discuss a possible proof to compute the algebra structure of $\widetilde{Kh}^{*,*}(T_{3,\infty})$ and make some comments about the general case of $\widetilde{Kh}^{*,*}(T_{p,\infty})$. We examine each step leading to the conjectured algebra structure of $\widetilde{Kh}^{*,*}(T_{3,\infty})$ and discuss which of these might or might not work in the general case. We propose a strategy to overcome the aspects that fail in the general case.

CHAPTER 1

Khovanov homology: basics

In this first chapter, we introduce the basic concepts upon which we have built our work. We start with knot theory, and describe knots and links and some variants. A fundamental variant that we develop is that of *pointed links* and their diagrams. In a second section, we describe Turner's construction of the Khovanov chain complex [**Tur14**]. We adapt this construction to pointed link diagrams, and obtain a chain complex whose homology is the main object of this thesis: the mod 2 reduced Khovanov homology, first introduced by Khovanov [**Kho03**]. A version of Khovanov homology for multiple basepoints was developped by Baldwin, Levine and Sarkar [**BLS17**]. For both constructions, the process is to endow the Khovanov chain complex with extra-structure where as we give an explicit chain complex for the reduced version. Finally, we present a variant of Khovanov homology, the so-called Bar-Natan homology, introduced by Bar-Natan [**BN05**] and explore its reduced version.

1. Knots and Links

In this opening section, we discuss the fundamentals of knot theory. We introduce the notions of knots, links and their diagrams, as well as some variants: braids and tangles. We discuss the natural concept of morphisms between two links, *link cobordisms* and the equivalent notion of *movies* for diagrams. Next we give to links additional data in the form of a choice of basepoint and examine how this affects the diagrams.

Classical links. We begin with the definition of a link, one of the main objects of interest in this thesis.

DEFINITION 1.1. A link L with μ components is the image of a smooth embedding of μ circles

$$L : \mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1 \longrightarrow \mathbb{R}^3.$$

If each component is equipped with an orientation, we say the link is *oriented*. For any oriented link L, we define $-L$ as the link obtained from L by reversing the orientation of every component.

Two oriented links L and L' are isotopic if there is a smooth map

$$H : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$$

such that

(1) $h_t := H(\cdot, t)$ is a diffeomorphism for all $t \in [0,1]$
(2) h_0 is the identity.
(3) $h_1(L) = L'$.

For links, being isotopic is an equivalence relation and captures the non-rigid nature of embeddings in $\mathbb{R}^3$. Knot theory is interested in distinguishing isotopy classes of links, through the use of *link invariants*.

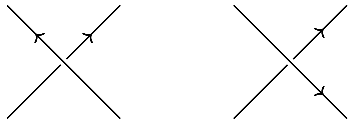


FIGURE 3. On the left a negative crossing. On the right, a positive crossing.

DEFINITION 1.2. A link invariant with values in a set S is a map

$$I : \{links\} \longrightarrow S$$

such that $I(L) = I(L')$ whenever L and L' are isotopic.

There are a lot of invariants available to a knot theorist, and the set S can be as simple as $\mathbb{N}$ or something more complicated. In this thesis we will focus on a bigraded vector space valued invariant: Khovanov homology, introduced by Khovanov [**Kho00**]. In this case, S is the set of bigraded vector spaces.

Given as smooth embeddings, links can be difficult to work with, however there exists a more combinatorial approach using *link diagrams*.

DEFINITION 1.3. A *link diagram* D is the image of a smooth immersion

$$D : \mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1 \longrightarrow \mathbb{R}^2$$

with finitely many tranversal double points, called *crossings*, such that at each crossing, one of the two arcs is distinguished and called *overpassing*. The other is called *underpassing*. If the circles are oriented, we say the diagram is *oriented*.

For oriented diagrams, we can assign a sign to crossings, with the rules given in Figure 3.

Links and link diagrams are related through the following steps. First, a link diagram D defines a link L_D, well-defined up to isotopy. A diagram D of a link L is a link diagram D such that L is isotopic to L_D and any link has a link diagram. The notion of isotopy can be realized in this combinatorial world, through the use of the so-called *Reidemeister moves* in Figure 5 and isotopies of $\mathbb{R}^2$. This is the famous Reidemeister Theorem.

THEOREM 1.1 (Reidemeister, [**Rei74**]). *Two (oriented) links are isotopic if and only if their (oriented) diagrams are related by a finite sequence of (oriented)Reidemeister moves and isotopies of* $\mathbb{R}^2$.

REMARK 1.2. *Any isotopy can be realized by using the three unoriented Reidemeister moves of Figure 5. For oriented links, Polyak* [**Pol10**] *showed that only two type I, one type II and one type III moves are sufficient to describe diagramatically any isotopy.*

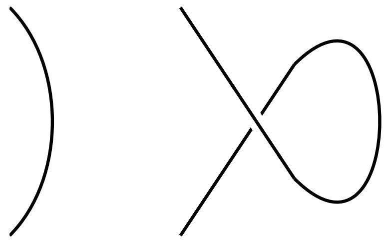
Isotopies have a very interesting property: given an isotopy H from an oriented L to an oriented L', one can construct a surface in $\mathbb{R}^3 \times [0,1]$ parametrized by

$$F : \left(\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1\right) \times [0,1] \longrightarrow \mathbb{R}^3 \times [0,1],$$

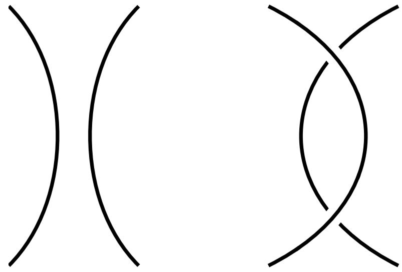
given by

$$F(x,t) = (h_t(L(x)), t).$$

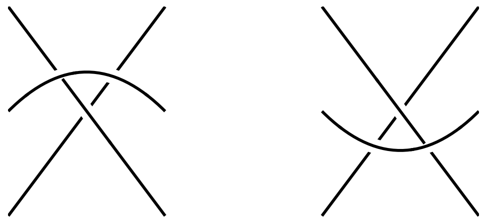
This surface has boundary $L \sqcup -L'$. This naturally leads to the notion of *link cobordism*.



(A) An instance of type I Reidemeister move.



(B) An instance of type II Reidemeister move.



(C) An instance of type III Reidemeister move.

FIGURE 5. The three types of Reidemeister moves.

DEFINITION 1.4. Let L and L' be two oriented links in $\mathbb{R}^3$. A *link cobordism* from L to L' is an compact orientable embedded smooth surface Σ in $\mathbb{R}^3 \times [0,1]$, such that

(1) The composition
$$f : \Sigma \hookrightarrow \mathbb{R}^3 \times [0,1] \longrightarrow [0,1]$$
is a Morse function with finitely many critical points.

(2) $\partial\Sigma = \Sigma_0 \sqcup \Sigma_1$, with $\Sigma_i \subset \mathbb{R}^3 \times \{i\}$, for $i = 0,1$.

(3) $\Sigma_0 = L$, $\Sigma_1 = -L'$ as smooth oriented manifolds.

Two cobordisms Σ, Σ' from L to L' are said to be *equivalent* if there exists an isotopy
$$H : (\mathbb{R}^3 \times [0,1]) \times [0,1] \longrightarrow \mathbb{R}^3 \times [0,1]$$
such that

(1) $H(x,t) = x$ for any $t \in [0,1], x \in \mathbb{R}^3 \times \{0,1\}$.

(2) $h_t := H(\cdot,t)$ is diffeomorphism for all $t \in [0,1]$.

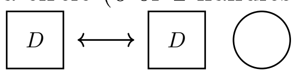
(3) h_0 is the identity.

(4) $H(\Sigma,1) = \Sigma'$.

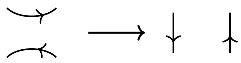
Combinatorially, this notion of cobordism corresponds to that of *movies*, which are sequences of diagrams. This notion was explored by Carter and Saito [**CS93**], following work of Roseman [**Ros98**] on surfaces in four-dimensional spaces.

DEFINITION 1.5. Let D and D' be two oriented link diagrams. A *movie* M from D to D' is a finite sequence of oriented link diagrams, called *frames*. Two consecutive frames must differ locally at most by one of the following possibilites:

(i) An oriented Reidemeister move.
(ii) The birth or death of a circle (0 or 2 handles moves).



(iii) An oriented 1 handle move.



Every cobordism from L to L' can be represented by a movie starting at a diagram D, representing L, to a diagram D', representing L'. Conversely every such movie represents a link cobordism. Moreover, Carter-Saito identified whenever two movies represent equivalent cobordisms, via transformations of movies. These *Carter-Saito movie moves* express diagramatically some isotopies of surfaces in $\mathbb{R}^3 \times [0,1]$.

THEOREM 1.3 ([**CS93**]). *Two movies represent equivalent cobordisms if and only they can be related by a finite sequence of Carter-Saito movie moves and exchanging distant critical points.*

Together, these notions fit into two catogories: the category *Links*, whose objects are oriented links and morphisms from L to L' are link cobordisms up to equivalence, and the category *Diag*, whose objects are oriented link diagrams and morphisms from D to D' are movies, up to Carter-Saito movie moves and exchange of distant critical points. In the former, the composition is given by gluing cobordisms along their common boundary. In the latter, composition is just the concatenation of movies.

Variants. There are other knot-like objects which we will consider, namely braids and tangles. These two concepts are closely related to links and are, for the latter especially, very handy for working with "pieces" of knots. We will only discuss their diagrams, not their topological equivalent since Khovanov homology relies exclusively on diagrams.

DEFINITION 1.6. Let $k, l \geq 0$ be integers with $k + l$ even. A *(k,l)-tangle* τ is the image of an immersion of $(k+l)/2$ copies of the interval $I = [0,1]$ and μ copies of a circle S^1 into $\mathbb{R} \times [0,1]$:

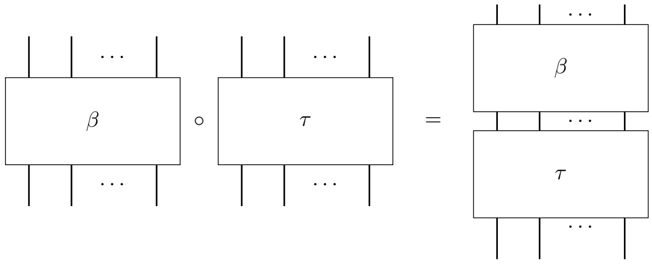
$$\tau : \underbrace{I \sqcup \cdots \sqcup I}_{(k+l)/2 \text{ times}} \sqcup \underbrace{S^1 \sqcup \cdots \sqcup S^1}_{\mu \text{ times}} \longrightarrow \mathbb{R} \times [0,1]$$

with finitely many double points, additional information of a crossing at each of them, and $k + l$ boundary points separated into two families

- l boundary points *on top*: $\tau \cap (\mathbb{R} \times \{1\}) = \{(i,1) | 1 \leq i \leq l\}$.
- k boundary points *at the bottom*: $\tau \cap (\mathbb{R} \times \{1\}) = \{(i,0) | 1 \leq i \leq k\}$.

If all the intervals and circles are oriented, we say the tangle is *oriented*.

If τ is a (k,l)-tangle and β is a (l,m)-tangle, we define their composition $\beta \circ \tau$ to be the (k,m)-tangle defined by stacking β on top of τ and rescaling the interval. If the tangles are oriented, we require the orientation at $\tau \cap (\mathbb{R} \times \{1\})$ and $\beta \cap (\mathbb{R} \times \{0\})$ to match for the composition to be defined.



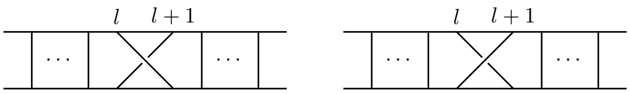
As for link diagrams, tangles are subject to the Reidemeister moves and planar isotopies but with the additional requirement that the boundary points are fixed. There is a particular family of (n,n)-tangles that is fundamental in knot theory: braids.

DEFINITION 1.7. A *braid* β over n strands is a (n,n)-tangle with no closed components and the additional property that for each interval I, the map:

$$I \longrightarrow \mathbb{R} \times [0,1] \longrightarrow [0,1]$$

is monotone increasing.

Braids over n strands are (n,n)-tangles so two braids can be composed by using the composition for tangles. Additionally any braid can be decomposed as a composition of *elementary crossings*, two for each $l \in \{1,\dots,n-1\}$: σ_l and σ_l^{-1} pictured below.



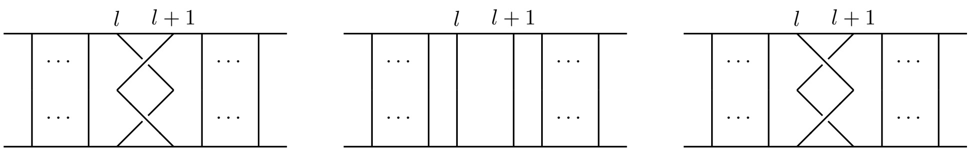
σ_l σ_l^{-1}

REMARK 1.4. *When the braid is oriented, the crossing on the left is negative while the one on the right is positive. We will mostly use the negative crossing, therefore our notation for these crossings is opposite to the usual one.*

Over a fixed number of strands n, braids form a group: the so-called *Artin braid group*, introduced by Artin [**Art25**]. This group B_n is given by the presentation below

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \rangle,$$

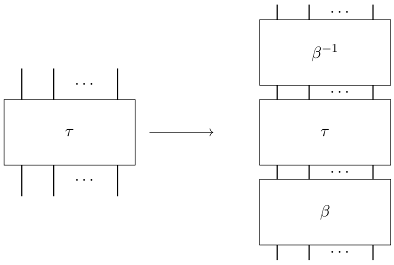
where in the first group of relations $1 \le i \le n-2$, and in the second $|i-j| \ge 2$. For a given braid, given as a composition of elementary crossings, its inverse β^{-1} is then composition in reverse order, and exchanging σ_l and σ_l^{-1}. For example, in B_n, the three braids depicted below are equal.



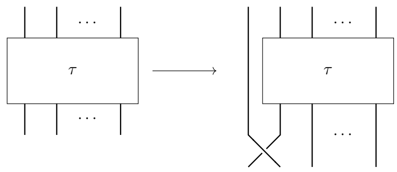
$\sigma_l^{-1} \circ \sigma_l$ $\sigma_l \circ \sigma_l^{-1}$

REMARK 1.5. *The first group of relations corresponds to type III Reidemeister moves, where as the second corresponds to planar isotopies. The inversibility of the generators σ_l corresponds to type II Reidemeister moves, as pictured above.*

Given a (n,n)-tangle or a braid over n strands, one can reconstruct a link diagram by using the *closure* operation illustrated below.

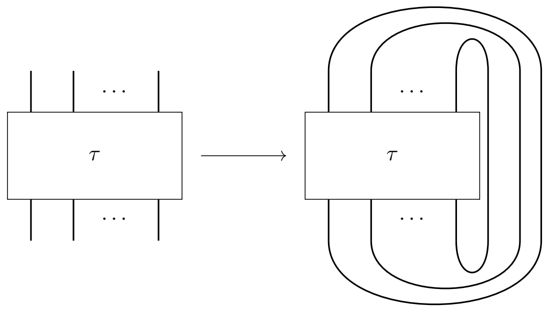


(A) Conjugation of braids.



(B) Stabilization.

FIGURE 7. Markov moves for braids.



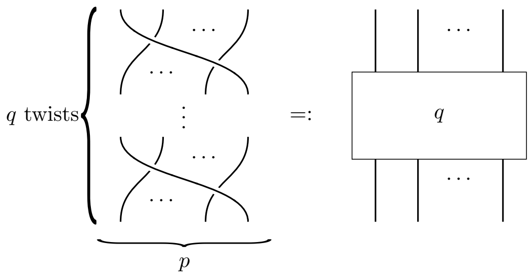
If the tangle is oriented, we require the orientations at $\tau \cap (\mathbb{R} \times \{0\})$ and $\tau \cap (\mathbb{R} \times \{1\})$ to match for the closure to be defined. Closing a braid gives a link diagram, it is then natural to ask oneself whether any link diagram can be obtained in such a way. The famous Alexander theorem answers this question: it states that any link diagram is isotopic to the closure of some braid.

THEOREM 1.6 (Alexander [**Ale23**]). *Let D be a link diagram. Then there exists a braid β such that its closure $\widehat{\beta}$ is equivalent to D.*

Such a *braid representative* is not unique, and two representatives might not even have the same number of strands. Markov's theorem, first proved by Birman [**Bir75**], provides us with a process to understand whether two given braids produce two equivalent link diagrams. These *Markov moves* given in Figure 7 are well-defined on equivalence classes of braids, i.e. they induce homomorphisms of the corresponding braid groups.

THEOREM 1.7 (Markov's Theorem). *Two braids close to equivalent link diagrams if and only they are related by a finite sequence of the two Markov moves, up to braid-equivalence.*

The most important family of links we will be confronted to is the family of torus links $T_{p,q}$, for $p \geq 1, q \geq 0$ both integers. As embeddings, these are collections of non-intersecting closed curves on the boundary of a torus. We define them via a braid, pictured below. This standard braid $D_{p,q}$ yields $T_{p,q}$ after closure.



$D_{p,q}$

Pointed links and diagrams. In this last part of the section, we introduce another kind of variant of links: *pointed links* which have the extra data of a basepoint. We also present the corresponding diagrams. The reason we are interested in these objects will become clear in Section 2.

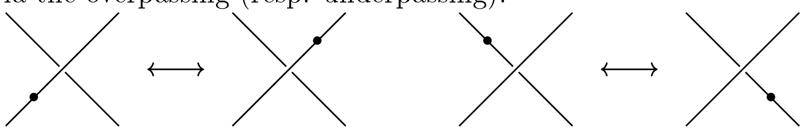
DEFINITION 1.8. A *pointed link* is a pair (L, p) where L is a non-empty oriented link in $\mathbb{R}^3$ and p is a point in L, called the *basepoint*.

An *isotopy of pointed links* from (L, p) to (L', p') is an ambient isotopy $F : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3$ from L to L' such that $F(p,1) = p'$. Note that at each time $t \in [0,1]$, a pointed link is specified by $(F(L,t), F(p,t))$. Two pointed links (L,p) and (L',p') are said to be equivalent if and only if there exists an isotopy of pointed links from (L,p) to (L',p').

We now describe the combinatorial description of pointed links, namely *pointed diagrams.*

DEFINITION 1.9. A *pointed link diagram* is a pair (D, p) where D is a non-empty oriented link diagram and a point $p \in D$, the *basepoint*, such that p is not a double point. The *Reidemeister moves* for pointed link diagrams are the following local operations on diagrams:

(1) Planar isotopies not allowed to move the basepoint through crossings.
(2) The usual oriented Reidemeister move of type I,II,III acting in a small disc not containing the basepoint.
(3) An overcrossing (resp. undercrossing) move that slides the basepoint through a crossing via the overpassing (resp. underpassing).



Two pointed link diagrams (D,p) and (D',p') are said to be *equivalent* if and only if one can be obtained from the other through a finite sequence of Reidemeister moves.

For both pointed links and pointed link diagrams, being equivalent is an equivalence relation. Moreover, any pointed link admits a pointed link diagram and any pointed link diagram defines a pointed link, well defined up to isotopy. Of course, such a description of links and their diagrams wouldn't be complete without a Reidemeister-type theorem.

PROPOSITION 1.8. *Two pointed links are equivalent if and only their pointed link diagrams are equivalent.*

Proof. Let (L, p) and (L', p') be two isotopic pointed links. In particular, we have that the links L and L' are isotopic. The classical Reidemeister theorem provides us with a sequence of Reidemeister moves from D, a diagram for L, to D', a diagram for L'. From this sequence, we construct a sequence of pointed Reidemeister moves that begins with (D, p) and ends with (D', p') as follows. First we set the basepoint to be p in the first diagram D and we consider the first Reidemeister move in the sequence. If it occurs away from the basepoint, then it is a valid pointed Reidemeister move. If it does not, we first move the basepoint with over/undercrossing moves until it is away from the local transformation, which is then a pointed Reidemeister move. We repeat this process until all Reidemeister moves in the original sequence become a sequence of pointed Reidemeister moves. At the end of the sequence, we have a pointed diagram (D, p''). If $p'' = p'$, then we have a sequence of pointed Reidemeister moves as claimed. If $p'' \neq p'$, then we move the basepoint with over/undercrossing moves until we reach p'.

For the other implication, it is clear that there is an isotopy that realizes the overcrossing and undercrossing moves. This concludes the proof.

□

2. Mod 2 Khovanov homology

The main object of study in our work is the so-called Khovanov homology, introduced by Khovanov [**Kho00**]. Khovanov homology is a vector space-valued invariant of knots and links whose construction relies on choosing a diagram to represent a given link and then building a chain complex based on the diagram. Up to quasi-isomorphism the choice of diagram turns out to be unimportant: one may show invariance with respect to Reidemeister moves. Moreover, we don't have just an invariant, but a functor. In this section we describe a construction of the Khovanov chain complex over $\mathbb{Z}_2$, following Turner's Hitchihiker's guide [**Tur14**], and mention selected properties. Then we give the construction of reduced Khovanov chain complex whose definition relies on pointed diagrams.

The non-reduced Khovanov homology. Let D be a link diagram and denote by χ_D the set of crossings of D. Any crossing $c \in \chi_D$ can be smoothed in two different ways, described in figure 8.

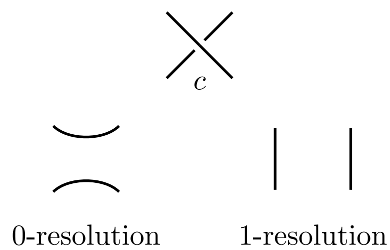


FIGURE 8. The two possible resolutions of a crossing c.

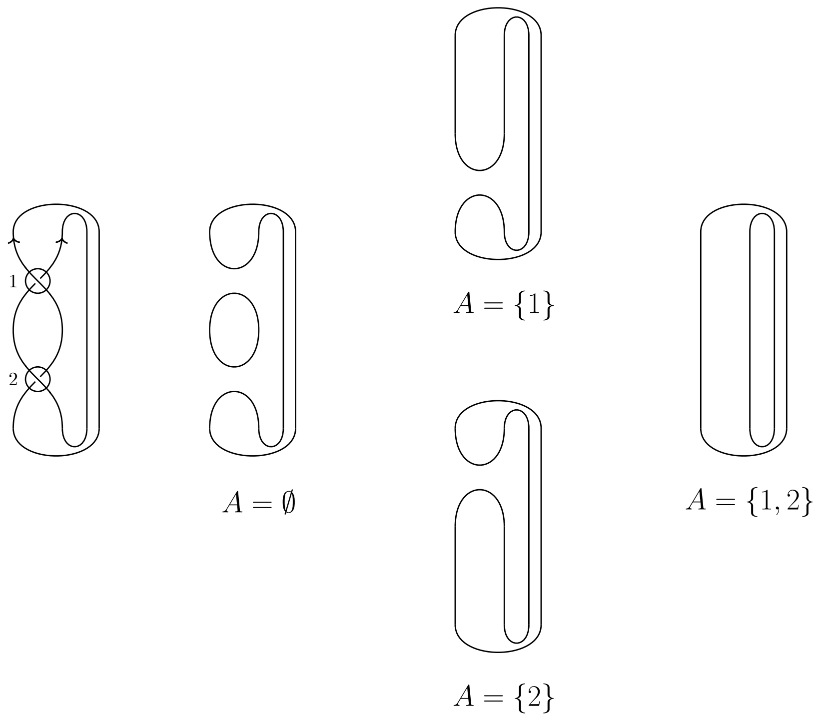
DEFINITION 1.10. A *smoothing* s of a diagram D is a map $s : \chi_D \to \{0, 1\}$. By extension, the diagram obtained from D by smoothing each crossing c with the $s(c)$ resolution, which is just a collection of circles, will also be called a *smoothing*.

For a given smoothing s of D, we denote by $|s|$ the number of circles in the smoothing.

To any subset $A \subset \chi_D$, one can associate a smoothing s_A defined by:

$$s_A: \chi_D \longrightarrow \{0,1\}$$
$$c \longmapsto \begin{cases} 0 & \text{if } c \notin A \\ 1 & \text{if } c \in A. \end{cases}$$

Let us start off with the various smoothings of the Hopf link, represented by the diagram below with circled crossings. It will be our running example for the whole section. It has 2 crossings and therefore 4 different smoothings, pictured below.



To any subset $A \subset \chi_D$, one can associate a vector space

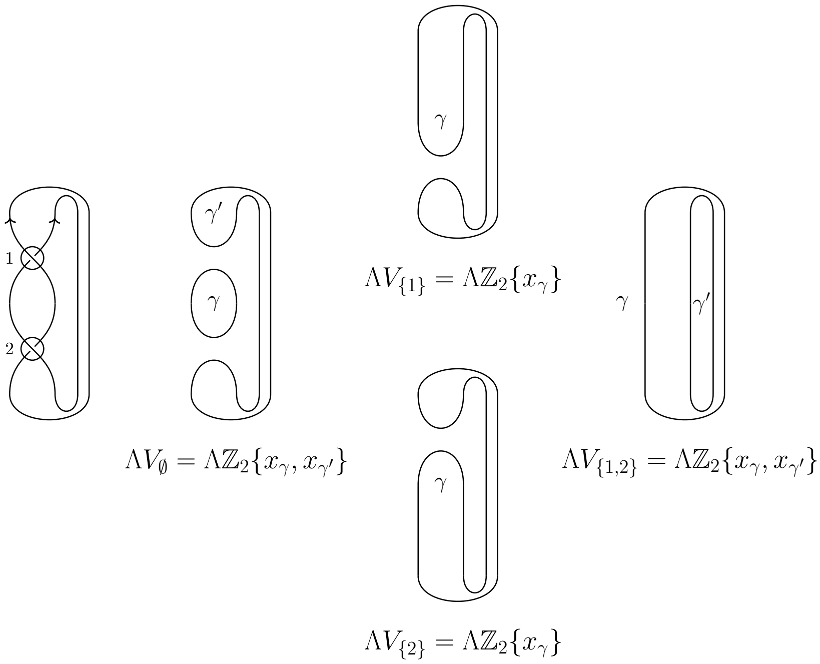
$$V_A := \mathbb{Z}_2\{x_\gamma \mid \gamma \text{ circle in } s_A\}$$

and the exterior algebra ΛV_A over V_A. Note that we have

$$\Lambda V_A = \frac{\mathbb{Z}_2[x_1, \cdots, x_{|s_A|}]}{(x_i^2 = 0\ \forall i)}$$

and that $\dim(\Lambda V_A) = 2^{|s_A|}$.

In our example for the negative Hopf link we then have the 4 exterior algebras:



Before we construct the Khovanov chain complex of a diagram, let us clarify our conventions for graded objects.

DEFINITION 1.11. Let W be a $\mathbb{Z}^l$-graded vector space, i.e W splits as a sum

$$W = \bigoplus_{i_1,\dots,i_l} W^{i_1,\dots,i_l}.$$

For $(k_1,\dots,k_l) \in \mathbb{Z}^l$, the *shifted vector space* $W[k_1,\dots,k_l]$ is the $\mathbb{Z}^l$-graded vector space defined by

$$W[k_1,\dots,k_l] = \bigoplus_{i_1,\dots,i_l} W^{i_1,\dots,i_l}[k_1,\dots,k_l], \text{ where } W^{i_1,\dots,i_l}[k_1,\dots,k_l] := W^{i_1-k_1,\dots,i_l-k_l}.$$

We will encounter various types of degrees and degree-preserving maps. Therefore, we fix some notations and explore the behavior of graded maps. We will use spaces that are $\mathbb{Z}^2$-graded or *bigraded* for short.

DEFINITION 1.12. Let V, W be two bigraded vector spaces. We say that a map $f : V \longrightarrow W$ has bidegree (a,b) if it satisfies

$$f(V^{i,j}) \subset W^{i+a,j+b}.$$

Any map f with bidegree (a,b) can be forcefully made into a map with bidegree $(0,0)$, by shifting either the domain or co-domain.

LEMMA 1.9. *Let V, W be two bigraded vector space and $f : V \longrightarrow W$ be a bidegree (a,b) map. Then the maps*

$$g : V[a,b] \longrightarrow W, \quad h : V \longrightarrow W[-a,-b]$$

defined by $g(x) = h(x) = f(x)$ both have bidegree $(0,0)$.

We can now turn to the main object of this section, the Khovanov chain complex of a diagram. Given an oriented diagram D, let $n_-(D)$ (resp. $n_+(D)$) be the number of negative (resp. positive) crossings of D.

Definition 1.13. The algebra ΛV_A can be $\mathbb{Z}$-graded: the *quantum grading*

$$q : \Lambda V_A \longrightarrow \mathbb{Z}$$

is defined on monomials by

$$q(x_{\gamma_1} \cdots x_{\gamma_k}) = |s_A| + |A| + n_+(D) - 2n_-(D) - 2k.$$

Here all the quantities except the last are fixed by the diagram D and the crossing set A. A monomial $v \in \Lambda V_A$ is said to have *quantum degree* j if $q(v) = j$. We will denote by $\Lambda^q V_A \subset \Lambda V_A$ the subspace of monomials with quantum degree q, so that

$$\Lambda V_A = \bigoplus_q \Lambda^q V_A.$$

For the negative Hopf link, we have collected the quantum degrees in the tables below.

v	$q(v)$
1	-2
$x_\gamma, x_{\gamma'}$	-4
$x_\gamma x_{\gamma'}$	-6

$A = \emptyset$

v	$q(v)$
1	-2
x_γ	-4

$A = \{1\}$ or $A = \{2\}$

v	$q(v)$
1	0
$x_\gamma, x_{\gamma'}$	-2
$x_\gamma x_{\gamma'}$	-4

$A = \{1, 2\}$

Remark 1.10. *This quantum grading is not the usual grading on the exterior algebra but a shift of -2 times the usual one. Moreover, for $A \subset \chi_D$, we have the inequalities*

$$-|s_A| + |A| + n_+(D) - 2n_-(D) \leq q(v) \leq |s_A| + |A| + n_+(D) - 2n_-(D).$$

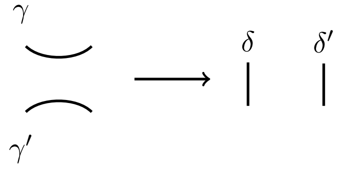
For $i \in \mathbb{Z}$, we define the vector space

$$C^i(D) = \bigoplus_{\substack{A \subset \chi_D \\ |A| = i + n_-(D)}} \Lambda V_A.$$

Any element $v \in C^i(D)$ is said to have *homological degree* i. Moreover each $C^i(D)$ inherits a quantum grading from the exterior algebra summands. Therefore these spaces are bigraded and we will denote by $C^{i,j}(D)$ the subspace of monomials with homological degree i and quantum degree j, so that

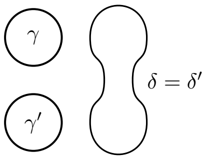
$$C^{*,*}(D) = \bigoplus_{i,j \in \mathbb{Z}} C^{i,j}(D).$$

The family $\{C^{i,j}(D)\}_{i = -n_-(D), \dots, n_+(D)}$ can be endowed with the structure of a chain complex. Suppose we have $A \subset B \subset \chi_D$, with $|B| = |A| + 1$. Then the two smoothings s_A and s_B are identical except in a small disk centered around the crossing $c \in B \backslash A$. Diagrammatically, we have:



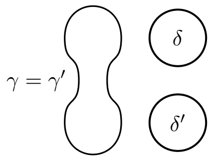
The labels $\gamma, \gamma', \delta, \delta'$ all indicate a particular component of the smoothing but are not necessarily distinct. For example, if the two strands shown on the left are in the same component then $\gamma = \gamma'$, and $\delta \neq \delta'$. There are exactly two possible configuration of circles, which allow us to define two maps as follows.

(i) If $\gamma \neq \gamma'$, then $\delta = \delta'$. We define



$$\begin{array}{rrcl} m_{A,B}: & \Lambda V_A & \longrightarrow & \Lambda V_B \\ & x_\gamma,\ x_{\gamma'} & \longmapsto & x_\delta, \\ & x_i & \longmapsto & x_i \ \textit{otherwise}. \end{array}$$

(ii) If $\gamma = \gamma'$, then $\delta \neq \delta'$. For a monomial v not featuring x_γ, we define



$$\begin{array}{rrcl} \Delta_{A,B}: & \Lambda V_A & \longrightarrow & \Lambda V_B \\ & v & \longmapsto & (x_\delta + x_{\delta'})v \\ & x_\gamma v & \longmapsto & x_\delta x_{\delta'} v. \end{array}$$

We extend the maps linearly to ΛV_A. Finally for $v \in \Lambda V_A \subset C^i(D)$, let

$$d^i(v) = \sum_{\substack{A \subset B \subset \chi_D \\ |B| = 1+i+n_-(D)}} d_{A,B}(v),$$

where $d_{A,B}$ is either $m_{A,B}$ if $|s_A| > |s_B|$ or $\Delta_{A,B}$ if $|s_A| < |s_B|$, extended linearly.

The construction depends a priori on the choice of diagram, yet Khovanov's fundamental result states that up to isomorphism, the resulting homology is in fact independent of such a choice.

THEOREM 1.11. [**Kho00**] *The differential d preserves the quantum grading and endows $C^{*,*}(D)$ with the structure of a bigraded chain complex. The isomorphism class of the bigraded homology $Kh^{*,*}(D) = H_*(C^{*,*}(D), d)$ is an invariant of oriented link.*

For the Hopf link, the chain complex is then the following

$$\begin{array}{ccccc} & & \Lambda V_{\{1\}} = \Lambda\mathbb{Z}_2\{x_\gamma\} & & \\ & \overset{m}{\nearrow} & & \overset{\Delta}{\searrow} & \\ \Lambda V_\emptyset = \Lambda\mathbb{Z}_2\{x_\gamma, x_{\gamma'}\} & & \bigoplus & & \Lambda V_{\{1,2\}} = \Lambda\mathbb{Z}_2\{x_\gamma, x_{\gamma'}\} \\ & \underset{m}{\searrow} & & \underset{\Delta}{\nearrow} & \\ & & \Lambda V_{\{2\}} = \Lambda\mathbb{Z}_2\{x_\gamma\} & & \end{array}$$

$$C^{-2,*}(T_{2,2}) \xrightarrow{\quad d^{-2} \quad} C^{-1,*}(T_{2,2}) \xrightarrow{\quad d^{-1} \quad} C^{0,*}(T_{2,2})$$

For d^{-2}, we compute

$$m(1) = (1,1),\ \ m(x_\gamma) = (x_\gamma, x_\gamma) = m(x_{\gamma'}),\ \ m(x_\gamma x_{\gamma'}) = 0.$$

The kernel of this map is generated by $\{x_\gamma + x_{\gamma'}, x_\gamma x_{\gamma'}\}$ and its image by $\{(1,1),(x_\gamma, x_\gamma)\}$. For d^{-1}, we have:

$$\Delta(1,0) = x_\gamma + x_{\gamma'} = \Delta(0,1),\ \Delta(x_\gamma, 0) = x_\gamma x_{\gamma'} = \Delta(0, x_\gamma).$$

Hence the kernel is generated by $\{(1,1),(x_\gamma, x_\gamma)\}$ and the image by $\{x_\gamma + x_{\gamma'}\}$. Consequently, we compute the Khovanov homology of the Hopf link:

$$\begin{array}{lcl} Kh^{0,0}(T_{2,2}) & = & \mathbb{Z}_2\{[1]\}, \\ Kh^{0,-2}(T_{2,2}) & = & \mathbb{Z}_2\{[x_\gamma]\}, \\ Kh^{-2,-4}(T_{2,2}) & = & \mathbb{Z}_2\{[x_\gamma + x_{\gamma'}]\}, \\ Kh^{-2,-6}(T_{2,2}) & = & \mathbb{Z}_2\{[x_\gamma x_{\gamma'}]\}. \end{array}$$

All others are zero.

Let us mention selected properties of this invariant:

(1) The graded Euler characteristic $\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim(Kh^{i,j}(D))$ is equal to $(q+q^{-1})$ times the Jones polynomial $V_D(q^2)$ of the diagram D.

(2) For the unknot U, represented by a trivial diagram D_u,

$$Kh^{i,j}(U) = \begin{cases} \mathbb{Z}_2 & \text{if } (i,j) \in \{(0,1),(0,-1)\}. \\ 0 & \text{otherwise.} \end{cases}$$

The vector space supported in bidegree $(0,1)$ is generated by 1, and the one in bidegree $(0,-1)$ is generated by x_u, the variable associated to the unknot itself.

(3) To each Reidemeister move R, one can associate a *quasi-isomorphism* ϕ_R, i.e a chain map with bidegree $(0,0)$

$$\phi_R : C^{*,*}(D) \longrightarrow C^{*,*}(D')$$

that induces an isomorphism in homology.

(4) For any two oriented diagrams D, D', and any $i, j \in \mathbb{Z}$, there is an isomorphism

$$Kh^{i,j}(D \sqcup D') \cong \bigoplus_{\substack{i_1+i_2=i \\ j_1+j_2=j}} Kh^{i_1,j_1}(D) \otimes Kh^{i_2,j_2}(D').$$

(5) If $D^!$ is the *mirror of* D, i.e it is obtained from D by exchanging the over and under passing of every crossing, then there is an isomorphism

$$Kh^{i,j}(D^!) \cong Kh^{-i,-j}(D).$$

Here property (5) is the main reason why we consider only negative torus links. Positive torus links are their mirror, and therefore their homology is completely determined by their negative counterparts.

Before turning to the reduced version of this homology theory, let us say a few words about functoriality. This fundamental property of homology theories provides us with *maps* connecting the homology of D and D', whenever they are related by a movie. These maps actually contain lots of information and functoriality will be one of the most important features of Khovanov homology we use. We will be especially interested in the maps induced by 1-handles, these are the cornerstone of our work.

As described in Section 1, two frames of a movie are related by a collection of local moves. To each of these moves, Khovanov [**Kho00**] associates a map, which we describe now.

(1) For Reidemeister moves, let $\phi_R^* : Kh^{i,j}(D) \longrightarrow Kh^{i,j}(D')$ be the isomorphism induced by ϕ_R the corresponding quasi-isomorphism.
(2) For 0-handle moves, let

$$\phi : Kh^{i,j}(D) \to Kh^{i,j+1}(D \sqcup D_u) \cong Kh^{i,j}(D) \otimes Kh^{0,1}(D_u) \bigoplus Kh^{i,j+2}(D) \otimes Kh^{0,-1}(D_u)$$

be defined by $\phi(v) = v \otimes 1$.
(3) For 2-handle moves, let

$$\phi : Kh^{i,j}(D \sqcup D_u) \cong Kh^{i,j-1}(D) \otimes Kh^{0,1}(D_u) \bigoplus Kh^{i,j+1}(D) \otimes Kh^{0,-1}(D_u) \to Kh^{i,j+1}(D)$$

be defined by $\phi(v \otimes 1) = 0, \phi(v \otimes x_u) = v$.
(4) The map for 1-handle moves is trickier. The move doesn't involve any crossing, so $\chi_D = \chi_{D'}$. Thus any $A \subset \chi_D$ gives rise to two smoothings: s_A for D and s'_A for D', as well as two corresponding vector spaces $\Lambda V_A \subset C^i(D)$ and $\Lambda V'_A \subset C^i(D')$. Define

$$\phi_A : \Lambda V_A \to \Lambda V'_A$$

by the same formula as $m_{A,B}$ or $\Delta_{A,B}$, depending on whether $|s_A| > |s'_A|$ or $|s_A| < |s'_A|$. Let

$$\phi^i = \bigoplus_{\substack{A \subset \chi_D \\ |A| = i + n_-(D)}} \phi_A.$$

The map ϕ^i is checked to have bidegree $(0, -1)$ and the family $\{\phi^i\}$ is a chain map. Finally, set

$$\phi : Kh^{i,j}(D) \to Kh^{i,j-1}(D')$$

to be the map induced on homology.

For a movie M with more than two frames, define the map ϕ_M as the composition of the maps associated to each pair of consecutive frames. This final map has bidegree $(0, m)$, where m is the number of 0 and 2-handle moves minus the number of 1-handle moves. In order to have a real functorial theory, we still need to know that these maps are well-defined, with respect to equivalence of movies. This is true, as shown by Jacobsson.

THEOREM 1.12. [**Jac04**] *Let M_1 and M_2 be two movies related by a sequence of Carter-Saito movie moves or exchanging distant critical points. Then $\phi_{M_1} = \phi_{M_2}$.*

Jacobsson worked with the integral version of Khovanov homology and showed that it was functorial up to sign, a defect already predicted by Khovanov in his original work. Since we work with $\mathbb{Z}_2$ coefficients, we have no problem of signs and the theory is fully functorial.

The reduced Khovanov homology. We define a chain complex from which arises a reduced version of Khovanov homology and explore some of its properties. In order to reduce, we need the extra data of basepoints, so from now on we will work with pointed diagrams.

Let (D, p) be a pointed diagram. In this context, any smoothing of D is a collection of circles containing exactly one pointed circle. For every $A \subset \chi_D$, let $x_\bullet$ be the variable of the exterior algebra ΛV_A over

$$V_A = \mathbb{Z}_2\{x_\gamma \mid \gamma \text{ circle in } s_A\}$$

associated to the pointed circle. Let $x_\bullet : C^{*,*}(D) \to C^{*,*-2}(D)$ be the map defined by $x_\bullet(v) = x_\bullet v$.

LEMMA 1.13. *The map* $x_\bullet$ *is a chain map such that* $\ker(x_\bullet) = Im(x_\bullet)$. *In particular, the differential* $d : C^{*,*}(D) \to C^{*+1,*}$ *restricts to a map* $d_{\widetilde{Kh}} : \ker(x_\bullet)^{*,*} \to \ker(x_\bullet)^{*+1,*}$. *Moreover, the induced map* $x_\bullet^*$ *commutes with the maps associated to* $0, 1$ *and* 2*-handles moves.*

Proof. To show that $x_\bullet$ is a chain map, it is sufficient to check that it commutes with the maps $m_{A,B}$ and $\Delta_{A,B}$ used to define the differential d. Using the notation γ, γ', δ in the definition of $m_{A,B}$, it can be assumed that $\bullet = \gamma$. The statement then follows from the equalities:

$$x_\bullet \circ m_{A,B}(x_\bullet^{\varepsilon_\gamma} x_{\gamma'}^{\varepsilon_{\gamma'}} v) = x_\bullet(x_\bullet^{\varepsilon_\gamma + \varepsilon_{\gamma'}} v) = x_\bullet^{1+\varepsilon_\gamma+\varepsilon_{\gamma'}} v,$$
$$m_{A,B} \circ x_\bullet(x_\bullet^{\varepsilon_\gamma} x_{\gamma'}^{\varepsilon_{\gamma'}} v) = m_{A,B}(x_\bullet^{1+\varepsilon_\gamma} x_{\gamma'}^{\varepsilon_{\gamma'}} v) = x_\bullet^{1+\varepsilon_\gamma+\varepsilon_{\gamma'}} v,$$

for $\varepsilon_\gamma, \varepsilon_{\gamma'} \in \{0, 1\}$. Using the notation γ, δ, δ' in the definition of $\Delta_{A,B}$, there are two cases to consider: $\bullet \neq \gamma$ and $\bullet = \gamma$. In the former, if $\bullet \neq \gamma$, then $\bullet \neq \delta, \delta'$ and $x_\bullet$ is the identity, so the statement follows immediately. In the latter, we have $\bullet = \gamma$. We can assume $\bullet = \delta$, since the formula that defines $\Delta_{A,B}$ is symmetric in δ and δ'. Let $v \in \Lambda V_A$ be a monomial that is not a factor of $x_\bullet$. The statement then follows from the equations below:

$$\begin{array}{lcccl} x_\bullet \circ \Delta_{A,B}(v) & = & x_\bullet((x_\bullet + x_{\delta'})v) & = & x_\bullet x_{\delta'} v, \\ \Delta_{A,B} \circ x_\bullet(v) & = & \Delta_{A,B}(x_\bullet v) & = & x_\bullet x_{\delta'} v. \\ & & & & \\ x_\bullet \circ \Delta_{A,B}(x_\bullet v) & = & x_\bullet(x_\bullet x_{\delta'} v) & = & 0, \\ \Delta_{A,B} \circ x_\bullet(x_\bullet x_{\delta'} v) & = & 0. & & \end{array}$$

To show the equality $\ker(x_\bullet) = \mathrm{Im}(x_\bullet)$, first note that one inclusion follows from the equality $x_\bullet^2 = 0$. To show the other inclusion $\ker(x_\bullet) \subset \mathrm{Im}(x_\bullet)$, first note that since

$$x_\bullet = \bigoplus_A x_{\bullet}{}_{|_{\Lambda V_A}},$$

we only need to show that

$$\ker\left(x_{\bullet}{}_{|_{\Lambda V_A}}\right) \subset \mathrm{Im}\left(x_{\bullet}{}_{|_{\Lambda V_A}}\right).$$

The statement now follows from the fact, if $v \in \Lambda V_A$ is not a factor of $x_\bullet$, then $x_\bullet v$ will not vanish.

For the naturality statements, note that maps induced by 1-handle moves are defined algebraically by using the formulas for $m_{A,B}$ and $\Delta_{A,B}$. Since $x_\bullet$ commutes with both, it also commutes with the chain map associated to 1-handles and the statement in homology follows.

For 0-handle moves, i.e. the birth of an unmarked circle, we need to show that the diagram below commutes:

$$\begin{array}{ccccccccc} Kh^{i,j}(D) & \xrightarrow{\phi} & Kh^{i,j}(D) & \otimes & Kh^{0,1}(D_u) & \oplus & Kh^{i,j+2}(D) & \otimes & Kh^{0,-1}(D_u) \\ x_\bullet^* \downarrow & & & \downarrow & x_\bullet^* \otimes id & & & \downarrow & x_\bullet^* \otimes id \\ Kh^{i,j+2}(D) & \xrightarrow{\phi} & Kh^{i,j+2}(D) & \otimes & Kh^{0,1}(D_u) & \oplus & Kh^{i,j+4}(D) & \otimes & Kh^{0,-1}(D_u). \end{array}$$

Note that the rightmost vertical arrow is irrelevant, since by definition

$$\phi(v) = v \otimes 1 \in Kh^{i,j}(D) \otimes Kh^{0,1}(D_u).$$

The commutativity then follows from the equation:

$$((x_\bullet^* \otimes 1) \circ \phi)(v) = x_\bullet^*(v \otimes 1) = x_\bullet^*(v) \otimes 1 = (\phi \circ x_\bullet^*)(v),$$

where the second equality is due to the fact that the new circle is unmarked.

For 2-handle moves, i.e. the death of an unmarked circle, we show that the diagram below is commutative:

$$\begin{array}{ccccccccc} Kh^{i,j-1}(D) & \otimes & Kh^{0,1}(D_u) & \oplus & Kh^{i,j+1}(D) & \otimes & Kh^{0,-1}(D_u) & \xrightarrow{\phi} & Kh^{i,j}(D) \\ & \downarrow & x^*_\bullet \otimes id & & & \downarrow & x^*_\bullet \otimes id & & x^*_\bullet \downarrow \\ Kh^{i,j+1}(D) & \otimes & Kh^{0,1}(D_u) & \oplus & Kh^{i,j+3}(D) & \otimes & Kh^{0,-1}(D_u) & \xrightarrow{\phi} & Kh^{i,j+2}(D), \end{array}$$

where the map ϕ is given by $\phi(v \otimes 1) = 0, \phi(v \otimes x_u) = v$. The commutative follows from the equality

$$(x^*_\bullet \circ \phi)(v \otimes 1) = 0 = \phi(x^*_\bullet(v) \otimes 1) = (\phi \circ (x^*_\bullet \otimes id))(v \otimes 1),$$

and the equality below, where x_u is the variable associated to the additional circle:

$$(x^*_\bullet \circ \phi)(v \otimes x_u) = x^*_\bullet(v) = \phi(x^*_\bullet(v) \otimes x_u) = (\phi \circ (x^*_\bullet \otimes id))(v \otimes x_u).$$

This concludes the proof. □

Definition 1.14. For a pointed diagram (D,p), the *reduced Khovanov chain complex* is defined as the shifted graded chain complex

$$\widetilde{C}^{*,*}(D,p) := \ker(x_\bullet)^{*,*}[0,1] = (x_\bullet C)^{*,*}[0,1].$$

The *reduced Khovanov homology* of the pointed diagram (D,p) is the bigraded vector space defined by $\widetilde{Kh}^{*,*}(D,p) := H_*(\widetilde{C}^{*,*}(D,p), d_{\widetilde{Kh}})$.

For example, if (D,p) is the trivial diagram, then $\widetilde{Kh}^{*,*}(D,p) = \mathbb{Z}_2$ in bidegree $(0,0)$, generated by $x_u = x_\bullet$. Moreover, if (D,p) is a pointed diagram and D' is an oriented diagram, then $(D \sqcup D', p)$ is a pointed diagram and for any $i,j \in \mathbb{Z}$ there is an isomorphism

$$\widetilde{Kh}^{i,j}(D \sqcup D', p) \cong \bigoplus_{\substack{i_1+i_2=i \\ j_1+j_2=j}} \widetilde{Kh}^{i_1,j_1}(D,p) \otimes Kh^{i_2,j_2}(D).$$

Remark 1.14. *The point p might be omitted from the notation if the basepoint is clear from the context. Moreover, the shift we apply to the chain complex has only one purpose: that reduced homology of the unknot is supported in bidegree* $(0,0)$ *rather that* $(0,-1)$.

Over $\mathbb{Z}_2$, there is a clear relationship between the unreduced and reduced Khovanov homologies. Shumakovitch [**Shu14**] defines a map $\nu : C^{*,*}(D) \to C^{*,*+2}(D)$ as follows. For a monomial $x_{\gamma_1} \cdots x_{\gamma_k} \in \Lambda V_A$, define

$$\nu(v) = \sum_{i=1}^{k} x_{\gamma_1} \cdots \hat{x}_{\gamma_i} \cdots x_{\gamma_k},$$

and extend this map linearly to the whole complex. In particular, any element $v = x_\bullet w$ is sent to

$$\nu(v) = \nu(x_\bullet w) = w + x_\bullet \nu(w). \tag{2.1}$$

Theorem 1.15. [**Shu14**] *The map* ν *has the following properties:*

(i) ν *is a chain map such that* $\nu \circ \nu = 0$.
(ii) $x_\bullet \circ \nu + \nu \circ x_\bullet = id$.
(iii) The following short exact sequence splits, where $i_\bullet$ *is the inclusion of subcomplexes:*

$$0 \longrightarrow \widetilde{C} \xrightarrow{i_\bullet} C \underset{\nu}{\overset{x_\bullet}{\rightleftarrows}} \widetilde{C} \longrightarrow 0$$

(iv) The long exact sequence in homology induced by the previous short exact sequence of chain complexes splits into split short exact sequences of the form:

$$0 \longrightarrow \widetilde{Kh}^{m,*+1}(D,p) \xrightarrow{i_\bullet^*} Kh^{m,*}(D) \xrightarrow{x_\bullet^*} \widetilde{Kh}^{m,*-1}(D,p) \longrightarrow 0, \quad \nu^* : \widetilde{Kh}^{m,*-1}(D,p) \to Kh^{m,*}(D)$$

This theorem has two main consequences:

(1) For all $m \in \mathbb{Z}$, the map $i_\bullet^*$ is injective so we have

$$\widetilde{Kh}^{*,*}(D,p)[0,-1] \cong i_\bullet^*(\widetilde{Kh}^{*,*}(D,p)[0,-1]) = \text{ Im } (i_\bullet^*) = \ker(x_\bullet^*),$$

where the last equality follows from the exactness of the sequence.

(2) For all $m \in \mathbb{Z}$, there is an isomorphism

$$Kh^{m,*}(D) \cong \widetilde{Kh}^{m,*+1}(D,p) \oplus \widetilde{Kh}^{m,*-1}(D,p),$$

thusreduced Khovanov homology does not depend on the choice of basepoint.

With (1), we can compute the reduced Khovanov homology of the Hopf link. We set $x_\bullet = x_\gamma$ for each smoothing. Recall we had a description for the Khovanov homology as

$$\begin{array}{lcl} Kh^{0,0}(T_{2,2}) & = & \mathbb{Z}_2\{[1]\}, \\ Kh^{0,-2}(T_{2,2}) & = & \mathbb{Z}_2\{[x_\gamma]\}, \\ Kh^{-2,-4}(T_{2,2}) & = & \mathbb{Z}_2\{[x_\gamma + x_{\gamma'}]\}, \\ Kh^{-2,-6}(T_{2,2}) & = & \mathbb{Z}_2\{[x_\gamma x_{\gamma'}]\}. \end{array}$$

The map $x_\bullet^*$ has kernel generated by $\{[x_\bullet],[x_\bullet x_{\gamma'}]\}$. Therefore after a shift of $[0,1]$, we obtain the reduced Khovanov homology of the Hopf link:

$$\begin{array}{lcl} \widetilde{Kh}^{0,-1}(T_{2,2}) & = & \mathbb{Z}_2\{[x_\bullet]\}, \\ \widetilde{Kh}^{-2,-5}(T_{2,2}) & = & \mathbb{Z}_2\{[x_\bullet x_{\gamma'}]\}. \end{array}$$

We provide an alternate proof of (2), that also features naturality properties with respect to the maps 0, 1 and 2-handle moves on the reduced Khovanov homology.

PROPOSITION 1.16. *Let D be an oriented diagram and $p, p' \in D$ be two points which are not double points. There is an isomorphism of chain complexes*

$$f : \widetilde{C}^{*,*}(D,p) \longrightarrow \widetilde{C}^{*,*}(D,p').$$

Moreover, the induced isomorphism commutes with the maps induced by 0, 1 and 2-handles.

Proof. Let D be an oriented diagram and $p, p' \in D$ be two points which are not double points. Let $x_\bullet$ (resp. $x'_\bullet$) be the variable associated to the circles containing p (resp. p'). Consider the chain maps

$$\begin{array}{l} f := x'_\bullet \circ \nu : x_\bullet C^{*,*}(D,p) \longrightarrow C^{*,*+2}(D) \longrightarrow x'_\bullet C^{*,*}(D,p'), \\ g := x_\bullet \circ \nu : x'_\bullet C^{*,*}(D,p') \longrightarrow C^{*,*+2}(D) \longrightarrow x_\bullet C^{*,*}(D,p). \end{array}$$

The proof is just a computation of the equalities $f \circ g = id$ and $g \circ f = id$. We write v as $v = x_\bullet w$, as $\ker(x_\bullet) = \mathrm{Im}(x_\bullet)$. Under this decomposition, we obtain from equation 2.1

$$\nu(v) = \nu(x_\bullet w) = w + x_\bullet \nu(w).$$

Thus we have

$$\begin{aligned}(f \circ g)(v) &= f \circ (x'_\bullet w + x'_\bullet x_\bullet \nu(w)) \\ &= x_\bullet(w + x'_\bullet \nu(w) + x_\bullet \nu(w) + x'_\bullet \nu(w) + x'_\bullet x_\bullet \nu^2(w)) \\ &= x_\bullet w = v,\end{aligned}$$

since we work over $\mathbb{Z}_2$ and $\nu^2 = 0$. The other equality is obtained in the same way, by exchanging the roles of $x_\bullet$ and $x'_\bullet$.

For the naturality statement, let D and D' be two oriented diagrams related by a 0,1 or 2-handle move. Let p and p' be two points in D, which are not double points, outside the range of the local operation. They are fixed by the move and consequently $p, p' \in D'$ and are not double points either.

For 1-handles, recall that the associated map arises as chain map, defined algebraically by the same formula as $m_{A,B}$ and $\Delta_{A,B}$, then restricted to $x_\bullet C(D)$. Since f is a chain map, it commutes with the chain map associated to a 1-handle and the result in homology follows.

For the remaining maps, we first need to understand the isomorphism f^* for oriented diagrams of the form $(D \sqcup D_u, p)$, where $p \in D$. Using the isomorphism

$$\widetilde{Kh}^{i,j}(D \sqcup D', p) \cong \bigoplus_{\substack{i_1+i_2=i \\ j_1+j_2=j}} \widetilde{Kh}^{i_1,j_1}(D,p) \otimes Kh^{i_2,j_2}(D),$$

we obtain, by setting $D' = D_u$, an isomorphism

$$\widetilde{Kh}^{i,j}(D \sqcup D_u, p) \cong Kh^{i,j-1}(D) \otimes Kh^{0,1}(D_u) \oplus Kh^{i,j+1}(D) \otimes Kh^{0,-1}(D_u).$$

Under this isomorphism, the map f^* can be made explicit. Let $v \in \widetilde{Kh}^{i,j-1}(D)$ and $w \in \widetilde{Kh}^{i,j+1}(D)$ then:

$$f^*(v \otimes 1 + w \otimes x_u) = (f^*(v) + x'^*_\bullet(w)) \otimes 1 + f^*(w) \otimes x_u.$$

For maps induced by 0-handles, we show that the diagram below commutes:

$$\begin{array}{ccccc}\widetilde{Kh}^{i,j}(D) & \xrightarrow{\phi} & \widetilde{Kh}^{i,j}(D) \otimes Kh^{0,1}(D_u) & \oplus & \widetilde{Kh}^{i,j+2}(D) \otimes Kh^{0,-1}(D_u) \\ f^* \big\downarrow & & & \big\downarrow f^* & \\ \widetilde{Kh}^{i,j}(D) & \xrightarrow{\phi} & \widetilde{Kh}^{i,j}(D) \otimes Kh^{0,1}(D_u) & \oplus & \widetilde{Kh}^{i,j+2}(D) \otimes Kh^{0,-1}(D_u).\end{array}$$

This is achieved by a direct computation, where $v \in \widetilde{Kh}^{i,j}(D)$:

$$(f^* \circ \phi)(v) = f^*(v \otimes 1) = f^*(v) \otimes 1 = (\phi \circ f^*)(v).$$

For 2-handles, we consider the diagram below

$$\begin{array}{ccccc}\widetilde{Kh}^{i,j-1}(D) \otimes Kh^{0,1}(D_u) & \oplus & \widetilde{Kh}^{i,j+1}(D) \otimes Kh^{0,-1}(D_u) & \xrightarrow{\phi} & \widetilde{Kh}^{i,j}(D) \\ & f^* \big\downarrow & & & \big\downarrow f^* \\ \widetilde{Kh}^{i,j-1}(D) \otimes Kh^{0,1}(D_u) & \oplus & \widetilde{Kh}^{i,j+1}(D) \otimes Kh^{0,-1}(D_u) & \xrightarrow{\phi} & \widetilde{Kh}^{i,j}(D).\end{array}$$

Let $v \in \widetilde{Kh}^{i,j-1}(D)$ and $w \in \widetilde{Kh}^{i,j+1}(D)$, then on the one hand we have

$$(\phi \circ f^*)(v \otimes 1 + w \otimes x_u) = \phi((f^*(v) + x'^*_\bullet(w)) \otimes 1 + f^*(w) \otimes x_u) = f^*(w),$$

while on the other hand we have

$$(f^* \circ \phi)(v \otimes 1 + w \otimes x_u) = f^*(w).$$

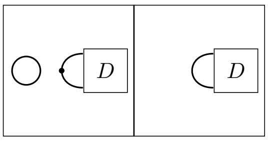
Therefore, the diagram above commutes. This concludes the proof. □

Finally, we can relate movies that differ by an exchange of distant 1-handle moves. The following statement follows immediately from the the fact that the maps induced in reduced homology are just restrictions, together with Jacobsson's theorem.

LEMMA 1.17. *Let M_1 and M_2 be two movies that differ by an exchange of distant 1-handles. Then for any choice of basepoint, the maps induced in reduced Khovanov homology are equal.*

Invariance of the reduced theory. In this last part of the section, we prove invariance of the reduced theory. This is achieved by considering a more geometrical model for the reduced theory, following Khovanov's originial definition [**Kho03**] and comparing it with our explicit model.

The homology of an unknot acts on the homology of any pointed diagram (D, p), via the movie below.



In the diagram above, the basepoint is an indication of where the 1-handle move acts, rather than additional structure that gets carried to the chain complex. By functoriality, this movie produces a map:

$$\Phi_p^* : Kh^{*,*}(U) \otimes Kh^{*,*}(D) \longrightarrow Kh^{*,*-1}(D).$$

This map is very simple: for any smoothing s_A, the chain map Φ_p associated to that particular 1-handle is, by definition, given by the formula for $m_{A,B}$ as it always merges two circles. Recall that $Kh^{0,-1}(U) = \mathbb{Z}_2\{x_u\}$, where x_u is the variable associated to the unknot. When restricted to the subspace $Kh^{0,-1}(U) \otimes Kh^{*,*+1}(D)$, the map Φ_p^* is explicitly given by

$$\Phi_p^*(x_u \otimes [v]) = [x_\bullet v] = x_\bullet^*([v]),$$

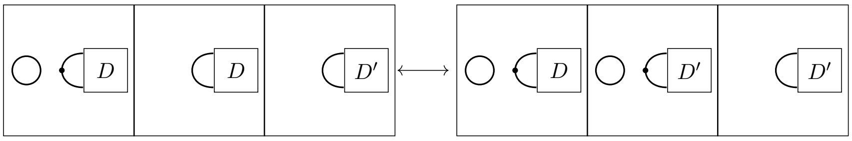
for any cycle $v \in C^{*,*+1}(D)$. In particular, the kernel of the restriction of Φ_p^* coincides with that of $x_\bullet^*$, i.e. it is the reduced Khovanov homology.

We now have a more geometrical flavor for reduced Khovanov homology, and we use that version, together with Khovanov's fundamental theorem, to prove invariance of the reduced theory.

THEOREM 1.18. *The isomorphism type of $\widetilde{Kh}^{*,*}(D, p)$ is an invariant of pointed diagrams.*

Proof. To show invariance, we must check that if two pointed diagrams (D, p) and (D', p') are related by either an under/over-crossing move or a Reidemeister move away from the basepoint, then the associated homologies are isomorphic. First note that for the over/under-crossing moves, we already have invariance by Proposition 1.16. Let us treat the remaining moves.

Let (D, p) and (D', p) be two diagrams related by a Reidemeister move. We consider the two movies M, M' below.



These movies only differ by the order of critical points, so by Theorem 1.12 they induce the same map in homology, i.e. we have a commutative diagram

$$\begin{array}{ccc} Kh^{*,*}(U)\otimes Kh^{*,*}(D) & \xrightarrow{\Phi_\bullet} & Kh^{*,*-1}(D) \\ \big\downarrow 1\otimes\Phi_R & & \big\downarrow \Phi_R \\ Kh^{*,*}(U)\otimes Kh^{*,*}(D') & \xrightarrow{\Phi_\bullet} & Kh^{*,*-1}(D') \end{array}$$

where Φ_R is the isomorphism in non reduced Khovanov homology corresponding to the Reidemeister move in question. We extend to the left by restricting Φ_p and by inclusion of the kerner, which is the reduced Khovanov homology. This yields another commutative diagram:

$$\begin{array}{ccccc} \widetilde{Kh}^{*,*+1}(D,p) & \hookrightarrow & Kh^{0,-1}(U)\otimes Kh^{*,*+1}(D) & \xrightarrow{\Phi_\bullet} & Kh^{*,*-1}(D) \\ \big\downarrow \widetilde{\Phi}_R & & \big\downarrow 1\otimes\Phi_R & & \big\downarrow \Phi_R \\ \widetilde{Kh}^{*,*+1}(D',p) & \hookrightarrow & Kh^{0,-1}(U)\otimes Kh^{*,*+1}(D') & \xrightarrow{\Phi_\bullet} & Kh^{*,*-1}(D') \end{array}$$

Since $1\otimes\Phi_R$ is an isomorphism, it restricts to an isomorphisms of the kernels:

$$\widetilde{\Phi}_R : \widetilde{Kh}^{*,*+1}(D,p) \longrightarrow \widetilde{Kh}^{*,*+1}(D',p).$$

This concludes the proof. □

3. Reduced Bar-Natan homology

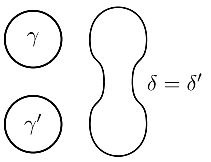
Bar-Natan homology is a variant of Khovanov homology defined in [**BN05**]. This new theory is also functorial, and very easy to understand. It was first studied by Turner [**Tur06**], where existence of a reduced version was mentionned. To the best of our knowledge, the reduced version of Bar-Natan homology we present has not been studied before.

The construction follows that of Khovanov, but rather than the exterior algebra, where variables square to 0, we consider the quotient polynomial algebra

$$V'_A = \mathbb{Z}_2[x_\gamma \mid \ \gamma \text{ circle in } s_A]/(x_\gamma^2 = 1).$$

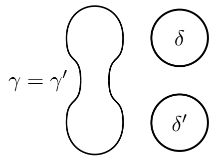
Note that as vector spaces, we have $V'_A = \Lambda V_A$, and V'_A inherits the quantum grading. Moreover, it still makes sense to talk about $C(D)$ in this context. Recall that at its most basic state, the differential of the Khovanov complex $(C^{*,*},d)$ was defined by two maps, $m_{A,B}$ and $\Delta_{A,B}$ where $A,B\in\chi_D, |B| = |A|+1$. Let us consider two other maps, which will play the same role.

(i) If $\gamma\neq\gamma'$, then $\delta=\delta'$. We define



$$\begin{array}{rccl} \widetilde{m}_{A,B}: & V'_A & \longrightarrow & V'_B \\ & x_\gamma,\ x_{\gamma'} & \longmapsto & x_\delta, \\ & x_i & \longmapsto & x_i \textit{ otherwise.} \end{array}$$

(ii) If $\gamma=\gamma'$, then $\delta\neq\delta'$. For a monomial v not featuring x_γ, we define



$$\begin{array}{rccc} \widetilde{\Delta}_{A,B}: & V'_A & \longrightarrow & V'_B \\ & v & \longmapsto & (1 + x_\delta + x_{\delta'})v \\ & x_\gamma v & \longmapsto & x_\delta x_{\delta'} v. \end{array}$$

Then for $v \in V'_A \subset C^i(D)$, we define

$$d^i_{BN}(v) = \sum_{\substack{A \subset B \subset \chi_D \\ |B| = 1 + i + n_-(D)}} d_{BN,A,B}(v),$$

where $d_{BN,A,B}$ is either $\widetilde{m}_{A,B}$ or $\widetilde{\Delta}_{A,B}$. This gives rise to a map

$$d_{BN} : C^*(D) \longrightarrow C^{*+1}(D),$$

that endows $C^*(D)$ with a structure of chain complex, different from that of Khovanov. The homology $BN^*(D) := H_*(C(D), d_{BN})$ is the *Bar-Natan homology* of D. Note that this differential d_{BN} does not respect the quantum grading, so the theory has only one grading, the homological grading. However it never decreases it either: the theory is filtered.

For a pointed diagram (D, p), a quick glance at the formulas for $\widetilde{m}_{A,B}$ and $\widetilde{\Delta}_{A,B}$ shows that

$$d_{BN}(x_\bullet C(D)) \subset x_\bullet C(D),$$

or in others terms: the new differential on $C(D)$ descends to a differential $d_{\widetilde{BN}}$ on $\widetilde{C}(D)$. Hence the following definition.

Definition 1.15. The *reduced Bar-Natan homology* of (D, p), denoted by, $\widetilde{BN}(D, p)$, is defined as the homology $H_*(\widetilde{C}(D), d_{\widetilde{BN}})$.

We delay the proof of invariance for later, though it can be obtained from Corollary 1.22. Apart from invariance, Bar-Natan homology provides a lower bound on Khovanov homology. More precisely, for each $i \in \mathbb{Z}$ we have a lower bound on the dimension of $\widetilde{Kh}^{i,*}(D)$:

$$\dim(\widetilde{BN}^i(D)) \leq \dim(\widetilde{Kh}^{i,*}(D)).$$

Both the invariance and the lower bound are proved in Corollary 3.5. Let us first show, in the spirit of Shumakovitch [**Shu14**], that Bar-Natan homology splits into two copies of its reduced version. In order to do that, we follow [**Lee05**] or rather Turner [**Tur06**] since we work mod 2 and change the basis of each V'_A as follows.

Lemma 1.19. *Suppose that a smoothing s_A is a collection of n circles $\gamma_1, \dots, \gamma_n$. As a vector space, V'_A has basis*

$$\left\{ \prod_{i \in I} a_{\gamma_i} \prod_{j \notin I} x_{\gamma_j} \mid I \subset \{1, \dots, n\} \right\},$$

where $a_\gamma = 1 + x_\gamma$.

Proof. Suppose that the smoothing s_A is a collection of n circles. Consider the vector space V''_A generated by unordered monomials of the form

$$v = a_{\gamma_{i_1}} \cdots a_{\gamma_{i_k}} x_{\gamma_{i_{k+1}}} \cdots x_{\gamma_{i_n}}.$$

Clearly V''_A has dimension 2^n: a generator is uniquely described by a choice, for each γ, of either a_γ or x_γ. Moreover, V'_A also has dimension 2^n. Consider the map

$$\begin{array}{cccc} f_A: & V''_A & \longrightarrow & V'_A \\ & a_{\gamma_{i_1}} \cdots a_{\gamma_{i_k}} x_{\gamma_{i_{k+1}}} \cdots x_{\gamma_{i_n}} & \longmapsto & (1+x_{\gamma_{i_1}})\cdots(1+x_{\gamma_{i_k}}) x_{\gamma_{i_{k+1}}} \cdots x_{\gamma_{i_n}} \end{array}$$

This map is injective, any two different combinations of a_γ's and x_γ's are mapped to sums of different monomials in V'_A. The two vector spaces have the same dimension so the map must be an isomorphim. □

With this new basis, the differential d_{BN} can be expressed in a simpler way. Using the notations in the definition of $\widetilde{m}_{A,B}$, we set $\gamma_1 = \gamma, \gamma_2 = \gamma'$. The map $\widetilde{m}_{A,B}$ then becomes:

$$\widetilde{m}_{A,B}(\prod_{i\in I} a_{\gamma_i} \prod_{j\notin I} x_{\gamma_j}) = \begin{cases} a_\delta \prod_{i\in I} a_{\gamma_i} \prod_{j\notin I} x_{\gamma_j} & \text{if } I\cap\{1,2\} = \{1,2\} \\ 0 & \text{if } I\cap\{1,2\} \text{ is either equal to } \{1\} \text{ or } \{2\}. \\ x_\delta \prod_{i\in I} a_{\gamma_i} \prod_{j\notin I} x_{\gamma_j} & \text{if } I\cap\{1,2\} = \varnothing. \end{cases}$$

Similarly, using the notations in the definition of $\widetilde{\Delta}_{A,B}$, we set $\gamma_1 = \gamma$ and obtain

$$\widetilde{\Delta}_{A,B}(\prod_{i\in I} a_{\gamma_i} \prod_{j\notin I} x_{\gamma_j}) = \begin{cases} a_\delta a_{\delta'} & \text{if } \{1\}\subset I, \\ x_\delta x_{\delta'} & \text{otherwise.} \end{cases}$$

Defining reduced Khovanov homoloy, we unified the variables associated to the marked circle under the name $x_\bullet$, we shall also define $a_\bullet = 1 + x_\bullet$. Let us associate to $a_\bullet$ a chain map. Here we assume v to contain neither $a_\bullet$ nor $x_\bullet$:

$$\begin{array}{cccc} a_\bullet: & C^*(D) & \longrightarrow & C^*(D) \\ & a_\bullet v & \longmapsto & a_\bullet v \\ & x_\bullet v & \longmapsto & 0 \end{array}$$

Let $a_\bullet C(D)$ denote the image of that map. One checks easily that $d_{BN}(a_\bullet C(D)) \subset a_\bullet C(D)$, so there is a well defined homology $H_*(a_\bullet C(D), d_{BN})$, which turns out to be isomorphic to the reduced Bar-Natan theory, as we show in the proof of Proposition 1.20. This fact is key to proving the splitting of Bar-Natan homology below.

Proposition 1.20. *Bar-Natan homology splits into a direct sum*

$$BN^*(D) = \widetilde{BN}^*(D,p) \oplus \widetilde{BN}^*(D,p).$$

Consequently, the reduced Bar-Natan homology is independent from the choice of basepoint.

Proof. We show that this homology splits by constructing a short split exact sequence of chain complexes. Using the definition, it is clear that $\ker(a_\bullet) = \widetilde{C}(D) = x_\bullet C(D)$ so the following sequence, where i is the inclusion of the subcomplex, is exact

$$0 \longrightarrow x_\bullet C(D) \xrightarrow{i} C(D) \xrightarrow{a_\bullet} a_\bullet C(D) \longrightarrow 0$$

This short exact sequence of chain complexes splits on the right with the inclusion. Therefore the long exact sequence in homology splits into split short exact sequences of the form

$$0 \longrightarrow \widetilde{BN}^*(D,p) \longrightarrow BN(D) \xrightarrow{a_\bullet^*} H_*(a_\bullet C(D)) \longrightarrow 0.$$

Let $r : C^*(D) \longrightarrow C^*(D)$ be the map defined by

$$r(a_{\gamma_{i_1}} \cdots a_{\gamma_{i_k}} x_{\gamma_{i_{k+1}}} \cdots x_{\gamma_{i_n}}) = x_{\gamma_{i_1}} \cdots x_{\gamma_{i_k}} a_{\gamma_{i_{k+1}}} \cdots a_{\gamma_{i_n}}.$$

This map clearly verifies $r^2(v) = v$ as it only exchanges the a_γ's and x_γ's. It is a chain map, as one checks easily on $\widetilde{m}$ and $\widetilde{\Delta}$. Moreover, by setting $a_{\gamma_1} = a_\bullet$ one obtains $r(a_\bullet C(D)) \subset x_\bullet C(D)$ while setting $x_{\gamma_{k+1}} = x_\bullet$ yields $r(x_\bullet C(D)) \subset a_\bullet C(D)$. Hence we have a sequence of inclusions

$$a_\bullet C(D) = r^2(a_\bullet C(D)) \subset r(x_\bullet C(D)) \subset a_\bullet C(D).$$

Consequently, $r(\widetilde{C}(D)) = a_\bullet C(D)$. It follows that r is an isomorphism of chain complexes and thus induces an isomorphism in homology. Thence we have an isomorphism

$$BN^*(D) \cong \widetilde{BN}^*(D,p) \oplus \widetilde{BN}^*(D,p),$$

and the reduced theory is independent from the choice of basepoint. □

A great feature of that theory is that it can be described explicitly. This will be extremely useful when computing homologies later on.

Theorem 1.21. [**Tur06**] *The dimension of $BN^*(D)$ is 2^μ where μ is the number of components in D. Moreover if $L_1, \dots, L_\mu$ are the components then*

$$\dim(BN^i(D)) = \text{ Card } \{E \subset \{1,2,\dots,\mu\} |\ 2 \sum_{l \in E, m \notin E} lk(L_l, L_m) = i\}$$

where $lk(L_l, L_m)$ is the linking number between the components L_l and L_m.

This theorem follows from the fact that the generators of $BN^*(D)$ are in 1-1 correspondance with the orientations of the (unoriented) diagram D. This correspondance can be described as follows: given an orientation θ of D, there exists a unique smoothing of D which inherits an orientation from D. The circles of that smoothing are partitionned into two groups A and B. Assigning the value a_γ to all circles in group A and x_γ to all those that belong to group B produces a generator of Bar-Natan homology.

In our framework of reduced Bar-Natan homology, the conventions made force the pointed circle to belong to the group B. These make up half of all possible orientations. We thus get an explicit description of $\widetilde{BN}^*(D)$.

Corollary 1.22. *The dimension of $\widetilde{BN}^*(D)$ is $2^{\mu-1}$ where μ is the number of components in D. Moreover if $L_1, \dots, L_\mu$ are the components, with L_1 the pointed one, then*

$$\dim(\widetilde{BN}^i(D,p)) = \text{ Card } \{E \subset \{2,\dots,\mu\} |\ 2 \sum_{l \in E \cup \{1\}, m \notin E} lk(L_l, L_m) = i\}$$

where $lk(L_l, L_m)$ is the linking number between the components L_l and L_m.

Remark 1.23. *The map r which induced the isomorphism $\widetilde{BN}^*(D) \cong H_*(a_\bullet C(D))$ is in fact just the map which corresponds, in non reduced Bar-Natan homology, to reversing the orientation of every component of D.*

Example 1.24. For any pointed diagram (D,p) the Bar-Natan homology in homological degree 0 is non trivial. Indeed it suffices to take $E = \{2,\dots,\mu\}$ in the statement of Corollary 1.22. In particular, if D is a knot diagram, i.e $\mu = 1$, then we have $\dim \widetilde{BN}^*(D,p) = 1$, so

$$\widetilde{BN}^i(D,p) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

As another example, let us treat the case of the torus links.

EXAMPLE 1.25. Let $T_{2,q}$ be the 2-stranded torus link with q negative half twists. Assume q is even so that $T_{2,q}$ is a 2 component link. The linking number of these two components L_1, L_2 is easily computed as $2lk(L_1, L_2) = -q$. Hence we can describe its reduced Bar-Natan homology as

$$\widetilde{BN}^i(D,p) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0 \text{ or } i = -q \\ 0 & \text{otherwise.} \end{cases}$$

CHAPTER 2

Computational Tools

In this chapter, we present the main tools we use to compute Khovanov homology. In algebraic topology, if one wishes to compute homology, taking the direct approach of describing entirely the differentials quickly becomes unmanageable. There are two algebraic tools one can use to avoid such involved computations: the long exact sequence arising from a short exact sequence and its generalization, the spectral sequence arising from a filtration of a chain complex. Before we describe the form of these two in our context of mod 2 reduced Khovanov homology, we will first focus our attention on connected sums of pointed diagrams. Then we will present the short exact sequence of surgery and the associated long exact sequence and spectral sequence. This sequence was implicit in Khovanov's original work [**Kho00**] and made explicit by Viro [**Vir02**] for the non reduced version. The spectral sequence we use is a reduced version Turner's skein spectral sequence [**Tur08**]. In each case, we will study how these objects behave with respect to maps induced by 1-handles, as these will be heavily featured in our work.

1. Connected sums

Before we discuss the various computational tools we will use in this thesis, let us mention the fact that Khovanov homology can be regraded with a focus on diagonals: that is the purpose of the so called *δ-grading.* For a given pair $i, j \in \mathbb{Z}$, this grading is defined as $\delta = j - 2i$. This grading is relative, as it depends on i, so one should be extremely careful when using it for computations. In order to avoid confusion between the quantum and δ graded versions of Khovanov homology, the latter will be denoted $\widetilde{Kh}_*^*(D)$.

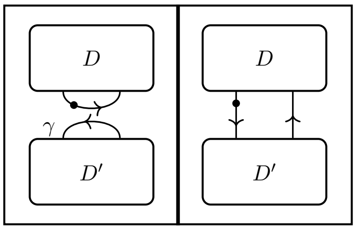
DEFINITION 2.1. Let (D, p) be a pointed diagram. It is said to be *$\widetilde{Kh}$-thin*, if $\widetilde{Kh}_*^*(D, p)$ is supported in exactly one δ-grading, i.e if there exists $\delta \in \mathbb{Z}$ such that $\widetilde{Kh}_{\delta'}^*(D, p) = 0$ if $\delta' \neq \delta$.

REMARK 2.1. *The family of $\widetilde{Kh}$-thin pointed diagrams contains infinitely many links: Lee* [**Lee05**] *proved that non split alternating links are all $\widetilde{Kh}$-thin. In particular, any 2-stranded torus link is $\widetilde{Kh}$-thin.*

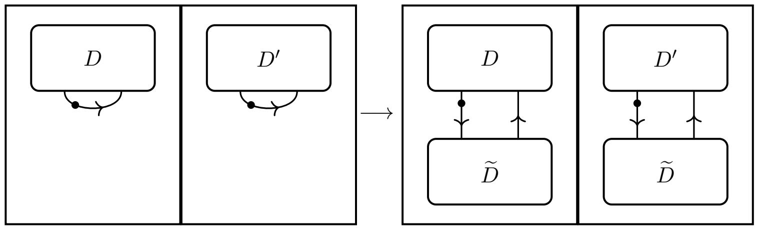
Apart from the (natural) independence of basepoint, the reduced Khovanov homology has a very interesting property with respect to connected sums of pointed links, which we explore now. Of course, the notion of connected sum of links is usually not well defined, however, with the additional data of basepoints, there is a canonical choice of which components to connect. The original result, due to Khovanov [**Kho03**], is stated for the non-reduced version as an isomorphism of A-modules, where $A = \mathbb{Z}[x_\bullet]/(x_\bullet^2)$:

$$C(D \sharp D') \cong C(D) \otimes_A C(D').$$

In our context of reduced homology, the result is slightly different.



(A) A 1-handle realizing a connected sum



(B) On the left, a movie M with 2 frames. On the right, the induced movie $\widetilde{M}$ obtained by connected sum with another diagram $\widetilde{D}$.

FIGURE 11. Connected sums of pointed links and naturality of that operation.

Consider the picture in Figure 11B. On the left, we have a movie M starting at a pointed diagram (D,p) and ending at a pointed diagram (D',p). Given such a movie, and an additional oriented diagram $\widetilde{D}$, we can construct a new movie $\widetilde{M}$ from M as follows: at each frame, we replace the diagram D_i by its connected sum $D_i \sharp D$ with D as in Figure 11A. In the following lemma, we give an explicit description of the homology of a connected sum and how maps induced by movies behave with respect to that operation.

LEMMA 2.2. *For every $i,j \in \mathbb{Z}$, there is an isomorphism of $\mathbb{Z}_2$-vector spaces*

$$S^* : \bigoplus_{\substack{i_1+i_2=i \\ j_1+j_2=j}} \widetilde{Kh}^{i_1,j_1}(D) \otimes_{\mathbb{Z}_2} \widetilde{Kh}^{i_2,j_2}(D') \longrightarrow \widetilde{Kh}^{i,j}(D \sharp D').$$

Moreover, if Φ_M and $\Phi_{\widetilde{M}}$ are induced by the movies in Figure 11b, then we have the following commutative diagram.

$$\begin{array}{ccc}
\widetilde{Kh}(\widetilde{D}) \otimes \widetilde{Kh}(D) & \xrightarrow{\;1 \otimes \Phi_M\;} & \widetilde{Kh}(\widetilde{D}) \otimes \widetilde{Kh}(D') \\
\downarrow S^* & & \downarrow S^* \\
\widetilde{Kh}(\widetilde{D} \sharp D) & \xrightarrow{\;\Phi_{\widetilde{M}}\;} & \widetilde{Kh}(\widetilde{D} \sharp D')
\end{array}$$

.

REMARK 2.3. *In terms of δ-grading, this isomorphism takes the form*

$$S^* : \bigoplus_{\substack{i_1+i_2=i \\ \delta_1+\delta_2=\delta}} \widetilde{Kh}^{i_1}_{\delta_1}(D) \otimes_{\mathbb{Z}_2} \widetilde{Kh}^{i_2}_{\delta_2}(D') \longrightarrow \widetilde{Kh}^{i}_{\delta}(D \sharp D').$$

Proof. We start with an explicit description of the chain complex for $D \sharp D'$. First notice that $\chi_{D \sharp D'} = \chi_D \cup \chi_{D'}$. Let $A_\sharp \subset \chi_{D \sharp D'}$. Such a subset is in 1-1 correspondence with a pair $(A, A') \subset \chi_D \times \chi_{D'}$. By definition, we associate to $A_\sharp$ the algebra

$$\Lambda V_{A_\sharp} = \mathbb{Z}_2[x_\delta \mid \delta \text{ circle in } s_{A_\sharp}]/(x_\delta^2 = 0 \forall \delta).$$

Except for the marked circle, any circle in $s_{A_\sharp}$ corresponds to a circle in either s_A or $s_{A'}$. The marked circle itself corresponds to both the marked circle in s_A and the circle labelled γ in $s_{A'}$, where γ is given in Figure 11a. Therefore, any $y \in \Lambda V_{A_\sharp}$ can be written as

$$y = v x_\bullet w,$$

for some $v \in \Lambda V_A, w \in \Lambda V_{A'}$. The differential $d_\sharp$ can also be made explicit. Denote by d_D (resp. $d_{D'}$) the differential of the complex $\widetilde{C}(D)$ (resp. $\widetilde{C}(D')$). If we add a crossing c to $A_\sharp$, we add a crossing to either A or A'. The differential $d_\sharp$ splits into two pieces: in the first we sum over all crossing that can be added to A, in the second over all those that can be added to A'. Therefore:

$$d_\sharp(y) = d_D(vx_\bullet)w + vd_{D'}(x_\gamma w) = v'x_\bullet w + vx_\bullet w',$$

where $d_D(vx_\bullet) = x_\bullet v'$ and $d_{D'}(x_\gamma w) = x_\gamma w'$. We define a map

$$S : \bigoplus_{\substack{i_1+i_2=i \\ j_1+j_2=j}} x_\bullet C^{i_1,j_1}(D) \otimes x_\gamma C^{i_2,j_2}(D') \longrightarrow x_\bullet C^{i,j+1}(D \,\sharp\, D'),$$

by the formula $S(vx_\bullet \otimes x_\gamma w) = vx_\bullet w$. The map S is a chain map. Indeed, we have an equality:

$$(d_\sharp \circ S)(vx_\bullet \otimes x_\gamma w) = d_\sharp(vx_\bullet w) = v'x_\bullet w + vx_\bullet w'.$$

For the tensor product $\widetilde{C}(D) \otimes \widetilde{C}(D')$, the relation $d_\otimes = d_D \otimes 1 + 1 \otimes d_{D'}$ holds by definition, hence:

$$(S \circ d_\otimes)(vx_\bullet \otimes x_\gamma w) = S(v'x_\bullet \otimes x_\gamma w + vx_\bullet \otimes x_\gamma w') = v'x_\bullet w + vx_\bullet w'.$$

We show that S is an isomorphism of chain complexes by presenting an explicit inverse, the map $h : \widetilde{C}(D \,\sharp\, D') \to \widetilde{C}(D) \otimes \widetilde{C}(D')$ defined by $h(vx_\bullet w) = vx_\bullet \otimes x_\gamma w$. It remains to show that h also is a chain map. A direct computation shows

$$h \circ d_\sharp(y) = h(v'x_\bullet w + vx_\bullet w') = v'x_\bullet \otimes x_\gamma w + vx_\bullet \otimes x_\gamma w',$$

and

$$d_{D\otimes D'} \circ h(y) = d_{D\otimes D'}(vx_\bullet \otimes x_\gamma w) = (d_D \otimes 1 + 1 \otimes d_{D'})(vx_\bullet \otimes x_\gamma w) = v'x_\bullet \otimes x_\gamma w + vx_\bullet \otimes x_\gamma w'.$$

Hence h is a chain map and S an isomorphism of chain complexes. There is one remaining detail to check: the compatibility of S with the shifts we added in the definition of the reduced chain complex. Recall that we had $\widetilde{C}^{*,*}(D,p) = x_\bullet C^{*,*}(D)[0,1]$. We shift each of $x_\bullet C^{i_1,j_1}(D)$ and $x_\gamma C^{i_2,j_2}(D')$ by $[0,1]$, i.e. a total shift of $[0,2]$ which we apply to $x_\bullet C^{i,j+1}(D \,\sharp\, D')$. By definition of the shifts, we have

$$x_\bullet C^{i,j+1}(D \,\sharp\, D')[0,2] = x_\bullet C^{i,j}(D \,\sharp\, D')[0,1] = \widetilde{C}^{i,j}(D \,\sharp\, D', p).$$

After these shifts, we obtain a shifted version of S, with bidegree $(0,0)$, which induces the isomorphism in the statement.

For the naturality statement, recall that a map of reduced chain complexes Φ_M send an element $v = x_\bullet w$ to $x_\bullet \Phi_M(w)$ so the image of v is completely determined by $\Phi_M(w)$. We show that the diagram commutes at the level of chain complexes. Indeed we have

$$(S \circ (1 \otimes \Phi_M))(vx_\gamma \otimes x_\bullet w) = S(vx_\gamma \otimes x_\bullet \Phi_M(w)) = vx_\bullet \Phi_M(w).$$
$$(\Phi_{\widetilde{M}} \circ S)(vx_\gamma \otimes x_\bullet w) = \Phi_{\widetilde{M}}(vx_\bullet w) = vx_\bullet \Phi_M(w),$$

where the last equality is due to the fact that the variables in v arise from circles which are fixed by the movie M. □

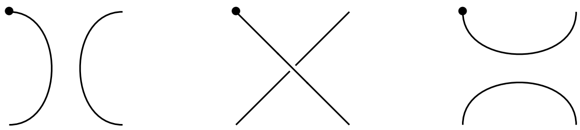
We conclude this section on connected sums by showing that the family of $\widetilde{Kh}$-thin diagrams is stable under connected sums. Indeed, the definition of $\widetilde{Kh}$-thin, together with the δ-graded version of the isomorphism in Proposition 2.2 give the corollary below.

Corollary 2.4. *Let D, D' be two $\widetilde{Kh}$-thin diagrams supported in δ-gradings δ_D and $\delta_{D'}$ respectively. Then any connected sum $D \,\sharp\, D'$ is $\widetilde{Kh}$-thin, supported in δ grading $\delta = \delta_D + \delta_{D'}$.*

2. The skein long exact sequence

In this section we describe a short exact sequence of chain complexes naturally associated to a pointed diagram (D,p) with an additional choice of crossing and explore its behaviour with respect to the connected sum of diagrams, and with maps induced by 1-handles. There is a long exact sequence in Khovanov homology that mimics the Kauffman bracket, whose properties we explore. Note that most results will be stated for both the quantum and δ graded versions.

DEFINITION 2.2. Let (D,p) be a pointed diagram and $c \in \chi_D$. Denote by D_k the diagram obtained from D by k-smoothing the crossing c, for $k = 0,1$. Both D_0 and D_1 are pointed diagrams, with basepoint p, as pictured below.



D_1 D D_0

Such a triple (D_1, D, D_0, p) will be called *exact triple*.

Given an exact triple, we construct a short exact sequence of chain complexes that relates the complexes of $(D_1,p),(D,p)$ and (D_0,p). For now we consider the chain complexesas vector spaces. We will re-introduce gradings later.

Let c be a crossing in D and $A \subset \chi_D$. Recall from the definition of $C(D)$ the equality

$$C(D) = \bigoplus_{A\subset\chi_D} \Lambda V_A.$$

In reduced setting, this decomposition is then

$$\widetilde{C}(D,p) = \bigoplus_{A\subset\chi_D} x_\bullet \Lambda V_A.$$

This sum can be split further by separating the subsets of χ_D that contain c and the others:

$$\widetilde{C}(D,p) = \left(\bigoplus_{A\subset\chi_D\setminus\{c\}} x_\bullet \Lambda V_A\right) \oplus \left(\bigoplus_{c\in A\subset\chi_D} x_\bullet \Lambda V_A\right).$$

If $c \in A$, then $A = A' \cup \{c\}$ for some $A' \subset \chi_{D_1}$ and $s_A = s_{A'}$. Therefore we have an equality

$$\Lambda V_A = \Lambda V_{A'},$$

and it follows that, as vector spaces we have an equality:

$$\bigoplus_{c\in A\subset\chi_D} x_\bullet \Lambda V_A = \widetilde{C}(D_1,p).$$

Similarly if $c \notin A$, then $A \subset \chi_{D_0}$ and we obtain an equality of vector spaces

$$\bigoplus_{A\subset\chi_D\setminus\{c\}} x_\bullet \Lambda V_A = \widetilde{C}(D_0,p).$$

Thus we have a decomposition of the vector space $\widetilde{C}(D,p)$ as

$$\widetilde{C}(D,p) = \widetilde{C}(D_0,p) \oplus \widetilde{C}(D_1,p)$$

Under this decomposition, the differential $d_{\widetilde{Kh}}$ becomes a matrix

$$d_{\widetilde{Kh}} = \begin{pmatrix} d_0 & 0 \\ \partial & d_1 \end{pmatrix}.$$

This matrix is lower triangular for a simple reason. If $A \subset \chi_D, c \in A$, we have

$$d_{\widetilde{Kh}} = \sum_{\substack{A \subset B \subset \chi_D \\ |A|+1=|B|}} d_{A,B},$$

and all such B's must also contain c. Hence the map $d_{\widetilde{Kh}}$ restricted to $\widetilde{C}(D_1, p)$ is just its counterpart for $\widetilde{C}(D_1, p)$, denoted by d_1.

Let us now assume that $c \notin A$. Since $x_\bullet \Lambda V_A \subset \widetilde{C}(D_0, p)$, we have

$$d_{\widetilde{Kh}} = \sum_{\substack{A \subset B \subset \chi_D \\ |A|+1=|B|}} d_{A,B} = d_{A,A\cup\{c\}} + \sum_{\substack{A \subset B \subset \chi_D \setminus \{c\} \\ |A|+1=|B|}} d_{A,B} = \partial + d_0.$$

The splitting of the sum is due to the fact that there is a unique $B \subset \chi_D, |B| = |A| + 1$, which contains both A and c, that is $B = A \cup \{c\}$. The map d_0 is identified exactly as d_1.

In particular $\widetilde{C}(D_1, p)$ is a subcomplex of $\widetilde{C}(D_0, p)$ and $\widetilde{C}(D_0, p)$ is the quotient complex, i.e. we have a short exact sequence of complexes:

$$0 \longrightarrow \widetilde{C}(D_1, p) \xrightarrow{i} \widetilde{C}(D, p) \xrightarrow{\pi} \widetilde{C}(D_0, p) \longrightarrow 0,$$

where i is the inclusion of the subcomplex $\widetilde{C}(D_1, p)$ and π is the quotient map.

REMARK 2.5. *The map ∂ connects smoothings where c has been replaces by it's 0-smoothing and it's 1-smoothing. Moreover, it is defined as $m_{A,A\cup\{c\}}$ or $\Delta_{A,A\cup\{c\}}$ depending on which of s_A or $s_{A\cup\{c\}}$ has the most crossing. If we forget about gradings, then this map ∂ coincides with the 1-handle that changes D_0 into D_1.*

One can reintroduce gradings as follows. If c is a negative crossing, D_1 inherits an orientation from D. Choose any orientation for D_0 and let $w_- = n_-(D_0) - n_-(D)$. Then for each $j \in \mathbb{Z}$, we obtain a short exact sequence of graded chain complexes:

$$0 \to \widetilde{C}^{*,j}(D_1)[0,-1] \xrightarrow{i} \widetilde{C}^{*,j}(D) \xrightarrow{\pi} \widetilde{C}^{*,j}(D_0)[w_-, 3w_- + 1] \to 0$$

If c is positive, D_0 inherits an orientation from D. Choose any orientation for D_1 and let $w_+ = n_-(D_1) - n_-(D)$. The corresponding short exact sequence is then

$$0 \to \widetilde{C}^{*,j}(D_1)[w_+, 3w_+ - 1] \xrightarrow{i} \widetilde{C}^{*,j}(D) \xrightarrow{\pi} \widetilde{C}^{*,j}(D_0)[0,1] \to 0$$

The various grading shifts can be computed by hand. Note that with these shifts, in both the case c positive and c negative the maps i and π have bidegree $(0,0)$.

Before we give an explicit description of the associated long exact sequence, let us first study how the short exact sequence reacts with respect to changes of basepoint.

LEMMA 2.6. *Let D be an oriented diagram and c be a crossing in D. For any two choices of basepoint p, p', the short exact sequences associated to the triples (D_1, D, D_0, p) and (D_1, D, D_0, p') are isomorphic.*

Proof. Let D be an oriented diagram and c be a crossing in D. Let p, p' be two points in D that are not crossings and consider the two exact triples (D_1, D, D_0, p) and (D_1, D, D_0, p'). Recall from Proposition 1.16 that the chain map $f = x'_\bullet \circ \nu$ gives an isomorphism for the change of basepoint. Given a smoothing s_A, both $x'_\bullet$ and ν are endomorphisms of the associated exterior algebra ΛV_A. Thus, under the isomorphism

$$\widetilde{C}(D,p) \cong \widetilde{C}(D_0,p) \oplus \widetilde{C}(D_1,p),$$

they both give diagonal matrices. Consequently, so does f. Hence, for each $j \in \mathbb{Z}$, the chain map f provides a map of short exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow & \widetilde{C}^{*,j}(D_1,p)[w_+, 3w_+ - 1] & \xrightarrow{i} & \widetilde{C}^{*,j}(D,p) & \xrightarrow{\pi} & \widetilde{C}^{*,j}(D_0,p)[w_-, 3w_- + 1] & \longrightarrow 0 \\ & \downarrow f & & \downarrow f & & \downarrow f & \\ 0 \longrightarrow & \widetilde{C}^{*,j}(D_1,p)[w_+, 3w_+ - 1] & \xrightarrow{i} & \widetilde{C}^{*,j}(D,p) & \xrightarrow{\pi} & \widetilde{C}^{*,j}(D_0,p)[w_-, 3w_- + 1] & \longrightarrow 0, \end{array}$$

an each vertical arrow is an isomorphism. This concludes the proof. □

REMARK 2.7. *As a consequence of Lemma 2.6, we will drop the basepoint from the exact triple notation, as well as in the long exact sequence of Proposition 2.8.*

We then obtain long exact sequences in homology, described below.

PROPOSITION 2.8. *For each $j \in \mathbb{Z}$, there is a long exact sequence in reduced Khovanov homology relating the homologies of D_0, D_1 and D.*

If c is negative, we have

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^{i,j+1}(D_1) \xrightarrow{i^*} \widetilde{Kh}^{i,j}(D) \xrightarrow{\pi^*} \widetilde{Kh}^{i-w_-,j-3w_--1}(D_0) \xrightarrow{\partial^*} \widetilde{Kh}^{i+1,j+1}(D_1) \xrightarrow{i^*} \cdots$$

If c is positive, we have

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^{i-w_+-1,j-3w_+-2}(D_1) \xrightarrow{i^*} \widetilde{Kh}^{i,j}(D) \xrightarrow{\pi^*} \widetilde{Kh}^{i,j-1}(D_0) \xrightarrow{\partial^*} \widetilde{Kh}^{i-w_+,j-3w_+-2}(D_1) \xrightarrow{i^*} \cdots$$

In terms of the δ-grading, for each $\delta \in \mathbb{Z}$ and $i \in \mathbb{Z}$,we have a long exact sequence, for c negative:

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^{i}_{\delta+1}(D_1) \to \widetilde{Kh}^{i}_{\delta}(D) \to \widetilde{Kh}^{i-w_-}_{\delta-w_--1}(D_0) \xrightarrow{\partial^*} \widetilde{Kh}^{*+1}_{\delta-1}(D_1) \to \cdots$$

Similarly, if c is positive, we have

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^{i-w_+-1}_{\delta-w_+}(D_1) \to \widetilde{Kh}^{i}_{\delta}(D) \to \widetilde{Kh}^{i}_{\delta-1}(D_0) \xrightarrow{\partial^*} \widetilde{Kh}^{i-w_+}_{\delta-w_+-2}(D_1) \to \cdots$$

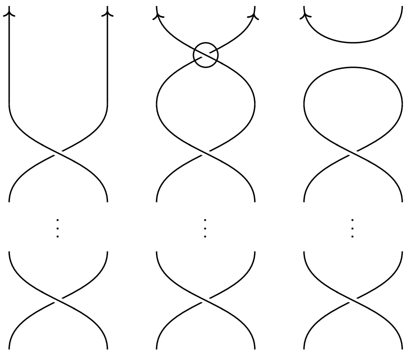
EXAMPLE 2.9. As a first use of the long exact sequence, we can compute the reduced Khovanov homology of any negative 2-stranded torus link $T_{2,q}$, already computed in [**Kho00**]. For $q = 1$, the link is an unknot so we already know it's homology. The $q = 2$ case, i.e when $T_{2,q}$ is a negative Hopf link has also been computed earlier in Chapter 1 Section 2. We aim to show by induction that the following description holds:

$$\widetilde{Kh}^{i,j}(T_{2,q}) = \begin{cases} \mathbb{Z}_2 & \text{if } (i,j) = (k, -q+1+2k) \text{ for } k \in \{0,-2,-3,\dots,-q\}. \\ 0 & \text{otherwise.} \end{cases}$$

In terms of δ-grading, using $\delta = j - 2i$, this can be re-stated as follows:

$$\widetilde{Kh}^{i}_{\delta}(T_{2,q}) = \begin{cases} \mathbb{Z}_2 & \text{if } \delta = -q+1 \text{ and } i \in \{0,-2,-3,\dots,-q\} \\ 0 & \text{otherwise.} \end{cases}$$

In order to use any long exact sequence, we must first choose a crossing and form an exact triple. We consider the triple of diagrams pictured below.



D_1 D D_0

The 1-smoothing yields a diagram D_1 for $T_{2,q-1}$ and the 0-smoothing is a diagram for the unknot U, with only positive crossings for any choice of orientation. Since $n_-(D) = q$, we can compute the shift w_- with the definition

$$w_- = n_-(D_0) - n_-(D) = 0 - q = -q.$$

Our crossing is negative, so for fixed $j \in \mathbb{Z}$, we have the long exact sequence:

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^{i,j+1}(T_{2,q-1}) \xrightarrow{i^*} \widetilde{Kh}^{i,j}(T_{2,q}) \xrightarrow{\pi^*} \widetilde{Kh}^{i+q,j+3q-1}(U) \xrightarrow{\partial^*} \widetilde{Kh}^{i+1,j+1}(T_{2,q-1}) \xrightarrow{i^*} \cdots$$

The homology of the unknot is one dimensional, supported in bidegree $(0,0)$, so we begin with $j = -3q+1$. The sequence at $i = -q$ is:

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^{-q,-3q+2}(T_{2,q-1}) \xrightarrow{i^*} \widetilde{Kh}^{-q,-3q+1}(T_{2,q}) \xrightarrow{\pi^*} \mathbb{Z}_2 = \widetilde{Kh}^{0,0}(U) \xrightarrow{\partial^*} \widetilde{Kh}^{-q+1,-3q+2}(T_{2,q-1}) \xrightarrow{i^*} \cdots$$

From our induction hypothesis, we know that

$$\widetilde{Kh}^{-q,-3q+2}(T_{2,q-1}) \cong 0 \cong \widetilde{Kh}^{-q+1,-3q+2}(T_{2,q-1}),$$

where the left hand side isomorphism is due to the homological degree, and the right hand side to the quantum degree. Hence the sequence becomes:

$$0 \to \widetilde{Kh}^{-q,-3q+1}(T_{2,q}) \to \mathbb{Z}_2 \xrightarrow{\partial^*} 0.$$

By exactness, we have an isomorphism

$$\widetilde{Kh}^{-q,-3q+1}(T_{2,q}) \cong \mathbb{Z}_2.$$

In particular we have $\widetilde{Kh}^{i,j}(T_{2,q}) \cong \mathbb{Z}_2$ if $(i,j) = (k, -q+1+2k)$ for $k = -q$. Let us now assume that $j \neq -3q+1$, so that $\widetilde{Kh}^{i,j}(U) = 0$ for any $i \in \mathbb{Z}$. This translates into the long exact sequence as

$$0 \xrightarrow{\partial^*} \widetilde{Kh}^{i,j+1}(T_{2,q-1}) \to \widetilde{Kh}^{i,j}(T_{2,q}) \to 0$$

By exactness, and by induction we have isomorphisms:

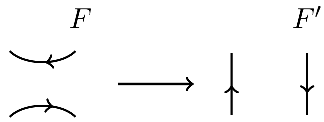
$$\widetilde{Kh}^{i,j}(T_{2,q}) \cong \widetilde{Kh}^{i,j+1}(T_{2,q-1}) = \begin{cases} \mathbb{Z}_2 & \text{if } (i,j) = (k, -q+1+2k) \text{ for } k \in \{0,-2,-3,\dots,-q-1\}. \\ 0 & \text{otherwise.} \end{cases}$$

We combine this last isomorphism with the one at $(i,j) = (-q, -3q+1)$ to obtain:

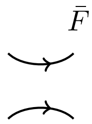
$$\widetilde{Kh}^{i,j}(T_{2,q}) = \begin{cases} \mathbb{Z}_2 & \text{if } (i,j) = (k, -q+1+2k) \text{ for } k \in \{0, -2, -3, \dots, -q\}. \\ 0 & \text{otherwise.} \end{cases}$$

According to remark 2.5, the boundary map ∂^* in the long exact sequence coincides up to some grading shift with the map induced by a 1-handle move between D_0 and D_1. Thus it is natural to ask oneself if up to some shift in degrees, one could compute the map induced by the oriented 1-handle move by using the long exact sequence. With that purpose in mind, we introduce a new technique that does exactly that. More precisely, we show how to associate an exact triple to any oriented 1-handle move. This procedure, which we describe now, will be called *completing the triple*.

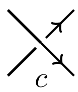
Consider an oriented 1-handle move between two diagrams F and F'.



If the two strands shown in F belong to different components, then reverse the orientation of one of them to obtain a diagram $\bar{F}$.

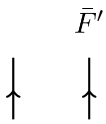


Let D be the diagram identical to $\bar{F}$ except in the region shown where it is:

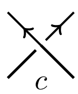


Let c be the additional crossing that is in D. This crossing is positive and $D_0 = \bar{F}$. Moreover the two strands appearing in D_1 must be in the same component and we may choose the orientation to be that of F'.

If the two strands shown in F belong to the same component, then the two strands shown in F' must belong to different components. Thus one can reverse the orientation of any of these, yielding a diagram $\bar{F}'$.



Let D be the diagram identical to $\bar{F}'$ except in the region shown, where it is



D has one more crossing than $\bar{F}'$. This crossing is negative and $D_1 = \bar{F}'$. The orientation of D_0 can be chosen to be the orientation of F.

In both case we obtain an exact triple (D_1, D, D_0). We have summarized this procedure in Figure 12: on the left is the negative case, on the right the positive one.

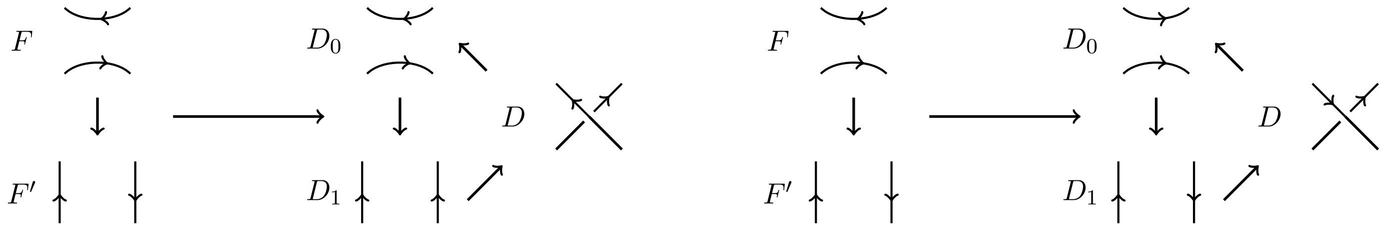


FIGURE 12. The two ways of completing a triple. For each pair of local diagrams related by a horizontal arrow, on the left is the oriented 1-handle, and on the right the local result of the procedure "completing the triple".

The main ingredient of the procedure is the reversal of the orientation of a component, therefore we must know how the reduced Khovanov homology behaves with respect to this operation.

PROPOSITION 2.10. *Let (D,p) be an pointed diagram. If (D^r,p) is obtained from L by reversing the orientation of the r^{th} component, then there is an isomorphism*

$$\widetilde{Kh}^{i,j}(D^r) \cong \widetilde{Kh}^{i+2l_r,j+6l_r}(D),$$

where $l_r = \sum_{i\neq r} lk(L_r,L_i)$ and L_i is the i^{th} component of D. In terms of δ-grading, this isomorphism is

$$\widetilde{Kh}^{i}_{\delta}(D^r) \cong \widetilde{Kh}^{i+2l_r}_{\delta+2l_r}(D)$$

The isomorphism in question is in fact just the identity. However, reversing an orientation changes the crossings and thus the degrees. This means the identity has (a priori) a non trivial bi-degree. The δ-graded version follows since $\delta = j-2i$.

Proof. We can assume without loss of generality that $r=1$. Let D be a pointed diagram with components $L_1\cup\cdots\cup L_\mu$, so that $D^1 = -L_1\cup\cdots\cup L_\mu$. Denote by $n_+(L_1,D\backslash L_1)$ (resp. $n_-(L_1,D\backslash L_1)$) the number of positive (resp. negative) crossings with one strand in L_1 and one strand in $D\backslash L_1$. One derives easily the equalities

$$\begin{array}{lcl} 2l_1 & = & n_+(L_1,D\backslash L_1) - n_-(L_1,D\backslash L_1),\\ n_-(D) & = & n_-(L_1,D\backslash L_1) + n_-(L_1) + n_-(D\backslash L_1),\\ n_-(D^1) & = & n_-(-L_1,D^1\backslash -L_1) + n_-(-L_1) + n_-(D^1\backslash -L_1),\\ n_+(D) & = & n_+(L_1,D\backslash L_1) + n_+(L_1) + n_+(D\backslash L_1),\\ n_+(D^1) & = & n_+(-L_1,D^1\backslash -L_1) + n_+(-L_1) + n_+(D^1\backslash -L_1). \end{array}$$

Moreover, as D^1 is obtained from D by reversing the orientation of L_1, one gets

$$\begin{array}{l} n_+(-L_1,D^1\backslash -L_1) = n_-(L_1,D\backslash L_1),\\ n_-(-L_1,D^1\backslash -L_1) = n_+(L_1,D\backslash L_1). \end{array}$$

Finally we can compute

$$\begin{array}{lcl} n_-(D) - n_-(D^1) & = & n_-(L_1,D\backslash L_1) + n_-(L_1) + n_-(D\backslash L_1)\\ & = & -n_-(-L_1,D^1\backslash -L_1) - n_-(-L_1) - n_-(D^1\backslash -L_1)\\ & = & n_-(L_1,D\backslash L_1) - n_+(L_1,D\backslash L_1)\\ & = & -2l_1 \end{array}$$

And similarly

$$n_+(D) - n_+(D^1) = 2l_1.$$

From $A \subset \chi_D = \chi_{D^1}$, we construct two copies of ΛV_A that only differ by some quantum degree shift. The two chain complexes are isomorphic, up to some shift in homological and quantum degrees. Once we determine these shifts, the identity map will be the desired isomorphism. Consider a monomial $x_{\gamma_1} \cdots x_{\gamma_k} \in \Lambda V_A \subset C^{i_D}(D)$. It has two degrees:

$$\begin{aligned} i_D &= |A| - n_-(D), \\ q_D &= |s_A| - 2k + |A| + n_+(D) - 2n_-(D). \end{aligned}$$

The same monomial seen as an element of $C^{i_{D^1}}(D^1)$ has degrees:

$$\begin{aligned} i_{D^1} &= |A| - n_-(D^1), \\ q_{D^1} &= |s_A| - 2k + |A| + n_+(D^1) - 2n_-(D^1). \end{aligned}$$

We can now compute the shift in homological degree.

$$\begin{aligned} i_D - i_{D^1} &= |A| - n_-(D) - |A| + n_-(D^1) \\ &= -n_-(D) + n_-(D^1) = 2l_1, \end{aligned}$$

as well as the shift in quantum degree

$$\begin{aligned} q_D - q_{D^1} &= |s_A| - 2k + |A| + n_+(D) - 2n_-(D) \\ &\quad -|s_A| + 2k - |A| - n_+(D^1) + 2n_-(D^1) \\ &= n_+(D) - n_+(D^1) + 2(n_-(D^1) - n_-(D)) \\ &= 2l_1 + 4l_1 = 6l_1. \end{aligned}$$

Thus the identity map $id : C(D^1)^{i,j} \to C(D)^{i+2l_1, j+6l_1}$ is an isomorphism of graded chain complexes and as such induces an isomorphism of degree $(2l_1, 6l_1)$ on homology. □

Now we have all the necessary elements to compute the map induced by an oriented 1-handle move. More precisely, we have the following statement.

PROPOSITION 2.11. *Let F and F' be related by an oriented 1-handle move. Denote by Φ^* the map induced in both quantum and δ-graded homology. Let D be obtained from F and F' by completing the triple as above. For each $j \in \mathbb{Z}$, we have an long exact sequences:*

If $c \in \chi_D \backslash \chi_F$ is negative

$$\cdots \xrightarrow{\Phi^*} \widetilde{Kh}^{i,j+1}(F') \to \widetilde{Kh}^{i-2l,j-6l}(D) \to \widetilde{Kh}^{i+1,j+2}(F) \xrightarrow{\Phi^*} \widetilde{Kh}^{i+1,j+1}(F') \to \cdots$$

If $c \in \chi_D \backslash \chi_{F'}$ is positive

$$\cdots \xrightarrow{\Phi^*} \widetilde{Kh}^{i-1,j-2}(F') \to \widetilde{Kh}^{i-2l,j-6l}(D) \to \widetilde{Kh}^{i,j-1}(F) \xrightarrow{\Phi^*} \widetilde{Kh}^{i,j-2}(F') \to \cdots$$

where in both cases, l is the linking number of the reversed component with its complement. In terms of δ-grading, for each $\delta, i \in \mathbb{Z}$ if $c \in \chi_D \backslash \chi_F$ is negative, we have

$$\cdots \xrightarrow{\Phi^*} \widetilde{Kh}^{i}_{\delta+1}(F') \to \widetilde{Kh}^{i-2l}_{\delta-2l}(D) \to \widetilde{Kh}^{i+1}_{\delta}(F) \xrightarrow{\Phi^*} \widetilde{Kh}^{i+1}_{\delta-1}(F') \to \cdots$$

If $c \in \chi_D \backslash \chi_{F'}$ is positive, we have

$$\cdots \xrightarrow{\Phi^*} \widetilde{Kh}^{i-1}_{\delta}(F') \to \widetilde{Kh}^{i-2l}_{\delta-2l}(D) \to \widetilde{Kh}^{i}_{\delta-1}(F) \xrightarrow{\Phi^*} \widetilde{Kh}^{i}_{\delta-2}(F') \to \cdots$$

Proof. The proof relies on the long exact sequence of the completed triple of Proposition 2.8, and the isomorphism of Proposition 2.10. We assume that the triple is completed with a negative crossing, i.e we are in the situation of Figure 12 left hand side. More precisely, we have an exact triple

$$(\bar{F}', D, F)$$

where $\bar{F}'$ is F' with the orientation of one component reversed. The diagrams F and F' differ by an oriented 1-handle move, so we have the equality

$$n_-(F) = n_-(F').$$

Moreover, from the proof of Proposition 2.10, we also have the equality

$$-2l = n_-(\bar{F}') - n_-(F'),$$

where l is the linking number number of the reversed component with its complement. Additionally, $\bar{F}'$ and D differ by a negative crossing, therefore

$$n_-(D) = n_-(\bar{F}') + 1.$$

We can combine these equalities into

$$\begin{array}{rcl} w_- & = & n_-(F) - n_-(D) \\ & = & n_-(F') - n_-(\bar{F}') - 1 \\ & = & -2l - 1. \end{array}$$

We can plug the value of w_- in the long exact sequence, and use the isomorphism of Proposition 2.10 at $D_1 = \bar{F}'$. This process yields

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^{i+2l,j+6l+1}(F') \xrightarrow{i^*} \widetilde{Kh}^{i,j}(D) \xrightarrow{\pi^*} \widetilde{Kh}^{i+2l+1,j+6l+2}(F) \xrightarrow{\partial^*} \widetilde{Kh}^{i+1+2l,j+1+6l}(F') \xrightarrow{i^*} \cdots$$

Finally we shift the whole exact sequence by $[2l, 6l]$, thus obtaining

$$\cdots \xrightarrow{\Phi^*} \widetilde{Kh}^{i,j+1}(F') \xrightarrow{i^*} \widetilde{Kh}^{i-2l,j-6l}(D) \xrightarrow{\pi^*} \widetilde{Kh}^{i+1,j+2}(F) \xrightarrow{\Phi^*} \widetilde{Kh}^{i+1,j+1}(F') \xrightarrow{i^*} \cdots$$

Here we have replaced ∂^* by Φ^* since they are algebraically the same and differ only by a gradind shift. The other case is treated similarly, and the δ-graded version follows by fixing $\delta, i \in \mathbb{Z}$. This concludes the proof. □

It will be very convenient for our future endeavours to treat all quantum gradings simultaneously. The motivation will become clear with Proposition 2.12 below. With this objective in mind, we use exact triangles and simply sum them up over $j \in \mathbb{Z}$, in the obvious way. We obtain an exact triangle, where the boundary map has bidegree $(1, 0)$:

$$\begin{array}{ccc} \widetilde{Kh}^{*,*}(D_1)[0,-1] & \xrightarrow{\quad i^* \quad} & \widetilde{Kh}^{*,*}(D) \\ & {\scriptstyle \partial^*} \nwarrow \qquad \swarrow {\scriptstyle \pi^*} & \\ & \widetilde{Kh}^{*,*}(D_0)[w_-, 3w_- + 1] & \end{array}$$

Since this triangle features all degrees at once, it can be regraded for the δ-graded version, where the boundary map has bidegree $(1, 2)$:

$$\begin{array}{ccc} \widetilde{Kh}^*_*(D_1)[0,-1] & \xrightarrow{\quad i^* \quad} & \widetilde{Kh}^*_*(D) \\ & {\scriptstyle \partial^*} \nwarrow \quad \swarrow {\scriptstyle \pi^*} & \\ & \widetilde{Kh}^*_*(D_0)[w_-, 3w_- + 1] & \end{array}$$

Definition 2.3. Let (D, p) be a oriented diagram and c be a crossing in D. The exact triangle with terms $\widetilde{Kh}^{*,*}(D_1), \widetilde{Kh}^{*,*}(D), \widetilde{Kh}^{*,*}(D_0)$ above will be called the *total exact triangle.*

In the following statement, that explores maps induced by 1-handles, the dimension $\dim(\widetilde{Kh}^{*,*}(F))$ we consider is the total dimension of $\widetilde{Kh}^{*,*}(F)$. The proof will use the total exact triangle.

Proposition 2.12. *Let F and F' be two diagrams related by a 1-handle move, and let D be obtained from F and F' by completing the triple. If*

$$\dim(\widetilde{Kh}^{*,*}(F)) = \dim(\widetilde{Kh}^{*,*}(F')) + \dim(\widetilde{Kh}^{*,*}(D)),$$

then the map induced by the 1-handle

$$\Phi^* : \widetilde{Kh}^{*,*}(F) \longrightarrow \widetilde{Kh}^{*,*-1}(F')$$

is surjective. Similarly, if

$$\dim(\widetilde{Kh}^{*,*}(F')) = \dim(\widetilde{Kh}^{*,*}(F)) + \dim(\widetilde{Kh}^{*,*}(D)),$$

then the map induced by the 1-handle

$$\Phi^* : \widetilde{Kh}^{*,*}(F) \longrightarrow \widetilde{Kh}^{*,*-1}(F')$$

is injective.

Remark 2.13. *The statement of Proposition 2.12 is also true for the δ-graded version.*

This proposition essentially tell us that - under certain conditions- if we have an exact triple (D_1, D, D_0) and a cyclic permutation of that triple, say (D, D_0, D_1), then the maps are also subject to a cyclic permutation.

This proposition will mostly be used as follows: let D be a diagram and c be a crossing of D. Consider the associated exact triple (D_1, D, D_0). If we assume that the boundary maps in every exact sequences (for each quantum degree) are zero, then we will have an isomorphism (up to some grading shift):

$$\widetilde{Kh}^{*,*}(D) \cong \widetilde{Kh}^{*,*}(D_1) \oplus \widetilde{Kh}^{*,*}(D_0),$$

and in particular the total dimensions are related by the formula

$$\dim(\widetilde{Kh}^{*,*}(D)) = \dim(\widetilde{Kh}^{*,*}(D_1)) + \dim(\widetilde{Kh}^{*,*}(D_0)).$$

If we have two diagrams F, F' related by a 1-handle, such that the completed triple is equivalent to (D_1, D_0, D) or (D_0, D_1, D) up to isotopies and reversal of orientations, then the induced map will be surjective.

Proof. The proof of this proposition will rely essentially on two things: the rank theorem, as well as the exactness of the total exact triangle for a completed triple. We will omit all indications of degrees in the notation.

$$\begin{array}{ccc} \widetilde{Kh}(F') & \xrightarrow{\quad i^* \quad} & \widetilde{Kh}(D) \\ & {\scriptstyle \Phi^*} \nwarrow \quad \swarrow {\scriptstyle \pi^*} & \\ & \widetilde{Kh}(F) & \end{array}$$

We start with the rank theorem at F':

$$\begin{aligned} \dim(\widetilde{Kh}(F')) = &\ \dim(\ker(i^*)) + \dim(\mathrm{Im}(i^*)) \\ = &\ \dim(\mathrm{Im}(\Phi^*)) + \dim(\ker(\pi^*)), \text{ by exactness at } F \text{ and } D, \\ = &\ \dim(\mathrm{Im}(\Phi^*)) + \dim(\widetilde{Kh}(D)) - \dim(\mathrm{Im}(\pi^*)), \text{ by the rank theorem at } D, \\ = &\ \dim(\mathrm{Im}(\Phi^*)) + \dim(\widetilde{Kh}(D)) - \dim(\ker(\Phi^*)), \text{ by exactness at } F, \\ = &\ \dim(\mathrm{Im}(\Phi^*)) + \dim(\widetilde{Kh}(D)) - (\dim(\widetilde{Kh}(F)) - \dim(\mathrm{Im}(\Phi^*)), \text{ by the rank theorem at } F, \\ = &\ \dim(\mathrm{Im}(\Phi^*)) - \dim(\widetilde{Kh}(F')) + \dim(\mathrm{Im}(\Phi^*)), \end{aligned}$$

where the last equality is a consequence of our hypothesis on dimensions. Hence it follows that

$$\dim(\mathrm{Im}(\Phi^*)) = \dim(\widetilde{Kh}(F')),$$

or equivalently, that the map is surjective, as claimed.

For the other case, the computation is similar. The first five equalities still hold, so we have

$$\begin{aligned} \dim(\widetilde{Kh}(F')) = &\ 2\dim(\mathrm{Im}(\Phi^*)) + \dim(\widetilde{Kh}(D)) - \dim(\widetilde{Kh}(F)) \\ = &\ \dim(\widetilde{Kh}(F)) - 2\dim(\ker(\Phi^*)) + \dim(\widetilde{Kh}(D)), \text{ by the rank theorem at } F, \\ = &\ \dim(\widetilde{Kh}(F')) - 2\dim(\ker(\Phi^*)) \end{aligned}$$

where the last equality follows from the hypothesis. Therefore we have the equality

$$\dim(\ker(\Phi^*)) = 0,$$

or equivalently, Φ^* is injective. This concludes the proof. □

There is one kind of oriented 1-handle that *always* induce a surjective map, namely those that realize a connected sum (see Figure 11a).

Corollary 2.14. *Let (D,p) be a pointed diagram and D' be an oriented diagram. The map induced by realizing a connected sum*

$$\Phi : \widetilde{Kh}^{*,*}(D \sqcup D') \longrightarrow \widetilde{Kh}^{*,*-1}(D \,\sharp\, D')$$

is surjective.

Proof. This is a straight-forward application of Proposition 2.12. We have the following isomorphisms

$$\begin{aligned} \widetilde{Kh}^{*,*}(D \sqcup D') &\cong \widetilde{Kh}^{*,*}(D) \otimes Kh^{*,*}(D') \\ \widetilde{Kh}^{*,*}(D \,\sharp\, D') &\cong \widetilde{Kh}^{*,*}(D) \otimes \widetilde{Kh}^{*,*}(D'). \end{aligned}$$

So we have an equality

$$\dim(\widetilde{Kh}^{*,*}(D \sqcup D')) = 2\dim(\widetilde{Kh}^{*,*}(D \,\sharp\, D')).$$

We complete the triple by reversing the orientation of the strand L_i in D'. Then the completed diagram is isotopic to $D \sharp \bar{D}'$, since reversing the orientation only changes degrees, we have

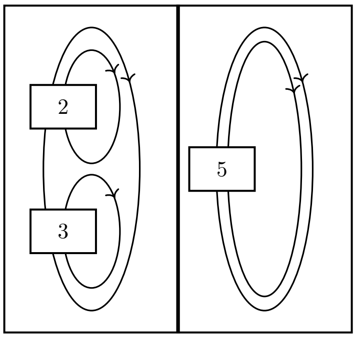
$$\dim(\widetilde{Kh}^{*,*}(D \sharp D')) = \dim(\widetilde{Kh}^{*,*}(D \sharp \bar{D}')).$$

It follows immediately that

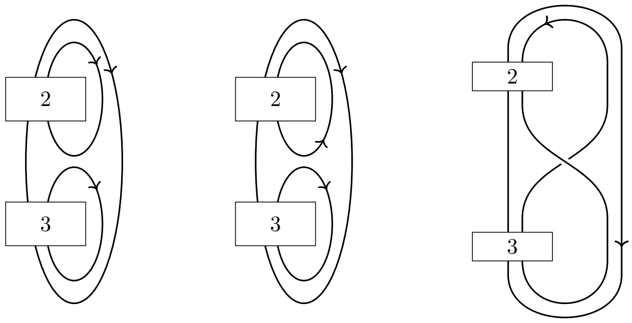
$$\dim(\widetilde{Kh}^{*,*}(D \sqcup D')) = \dim(\widetilde{Kh}^{*,*}(D \sharp D')) + \dim(\widetilde{Kh}^{*,*}(D \sharp \bar{D}')),$$

thus, by Proposition 2.12, the induced map is surjective as claimed. □

EXAMPLE 2.15. As an example, let us show that the map induced by the 1-handle pictured below is surjective.



The movie starts at $F = T_{2,2} \sharp T_{2,3}$ and ends at $F' = T_{2,5}$. We begin by completing the triple. We follow our recipe, whose steps are pictured below.



Since the two strands below to different components, we reverse the orientation of the upper strand in F to obtain $\bar{F}$. We then replace these two strands by a positive crossing, as prescribed by our method. This yields the completed diagram D, which is just a diagram for the unknot.

In order to apply Proposition 2.12, we still need information about the dimensions of the spaces involved. We know that the total dimension of $\widetilde{Kh}^{*,*}(T_{2,q})$ is q from Example 2.9. Moreover, Lemma 2.2 tell us that

$$\widetilde{Kh}^{*,*}(F) \cong \widetilde{Kh}^{*,*}(T_{2,2}) \otimes \widetilde{Kh}^{*,*}(T_{2,3}).$$

Therefore, we have an equality for the dimensions:

$$\dim(\widetilde{Kh}^{*,*}(F)) = \dim(\widetilde{Kh}^{*,*}(T_{2,2})) \cdot \dim(\widetilde{Kh}^{*,*}(T_{2,3})) = 6.$$

It follows that we have

$$\dim(\widetilde{Kh}^{*,*}(F)) = 6 = 5 + 1 = \dim(\widetilde{Kh}^{*,*}(T_{2,5})) + \dim(\widetilde{Kh}^{*,*}(U)).$$

By Proposition 2.12, the induced map is surjective.

Before we move on to the naturality properties of the long exact sequence, let us specialize it further to exact triples of $\widetilde{Kh}$-thin links. We will use the δ-graded version.

LEMMA 2.16. *Let D be a diagram which is $\widetilde{Kh}$-thin, supported in δ-grading δ. Let $c \in \chi_D$ and suppose that D_0 and D_1 are also $\widetilde{Kh}$-thin supported in grading δ_0 and δ_1 respectively.*
If c is negative and $\delta_1 = \delta_0 + w_-$, then

- *If $\dim(\widetilde{Kh}^*_{\delta_0}(D_0)) > \dim(\widetilde{Kh}^*_{\delta_1}(D_1))$, we have $\delta = \delta_1 + 1$ and*
$$\partial^* : \widetilde{Kh}^{*-w_- -1}_{\delta_0}(D_0) \to \widetilde{Kh}^*_{\delta_1}(D_1)$$
is surjective.
- *If $\dim(\widetilde{Kh}^*_{\delta_0}(D_0)) < \dim(\widetilde{Kh}^*_{\delta_1}(D_1))$, we have $\delta = \delta_1 - 1$ and*
$$\partial^* : \widetilde{Kh}^{*-w_- -1}_{\delta_0}(D_0) \to \widetilde{Kh}^*_{\delta_1}(D_1)$$
is injective.

If c is positive and $\delta_0 - 1 = \delta_1 + w_+$, then

- *If $\dim(\widetilde{Kh}^*_{\delta_0}(D_0)) > \dim(\widetilde{Kh}^*_{\delta_1}(D_1))$, we have $\delta = \delta_0 + 1$ and*
$$\partial^* : \widetilde{Kh}^*_{\delta_0}(D_0) \to \widetilde{Kh}^{*-w_+}_{\delta_1}(D_1)$$
is surjective.
- *If $\dim(\widetilde{Kh}^*_{\delta_0}(D_0)) < \dim(\widetilde{Kh}^*_{\delta_1}(D_1))$, we have $\delta = \delta_0 - 1$ and*
$$\partial^* : \widetilde{Kh}^*_{\delta_0}(D_0) \to \widetilde{Kh}^{*-w_+}_{\delta_1}(D_1)$$
is injective.

Proof. Let us treat the case where c is negative. The long exact sequence for the δ-grading $\delta_1 - 1$ is zero except at:
$$0 \to \widetilde{Kh}^{*-1}_{\delta_1+1}(D) \to \widetilde{Kh}^{*-w_- -1}_{\delta_0=\delta_1-w_-}(D_0) \xrightarrow{\partial^*} \widetilde{Kh}^*_{\delta_1}(D_1) \to \widetilde{Kh}^*_{\delta_1-1}(D) \to 0$$
If $\dim(\widetilde{Kh}^*_{\delta_0}(D_0)) > \dim(\widetilde{Kh}^*_{\delta_1}(D_1))$, then $\ker(\partial^*) \neq \{0\}$. The sequence is exact, so we must have
$$\widetilde{Kh}^*_{\delta_1+1}(D) \neq \{0\}, \text{ i.e } \delta = \delta_1 + 1.$$
Moreover L is $\widetilde{Kh}$-thin, so
$$\widetilde{Kh}^*_{\delta_1-1}(D) = \{0\}.$$
Therefore, by exactness again, ∂^* must be surjective. Similarly, if $\dim(\widetilde{Kh}^*_{\delta_0}(D_0)) < \dim(\widetilde{Kh}^*_{\delta_1}(D_1))$, then $\mathrm{coker}(\partial^*) \neq \{0\}$. It follows by exactness that
$$\widetilde{Kh}^*_{\delta}(D) \neq \{0\}, \text{ so } \delta = \delta_1 - 1 \text{ and } \widetilde{Kh}^*_{\delta_1+1}(D) = \{0\}$$
as L is $\widetilde{Kh}$-thin. The exactness of the sequence implies ∂^* is injective.

The positive case is treated similarly, with the long exact sequence at δ-grading $\delta_0 - 1$. □

REMARK 2.17. *Note that if all three D, D_1, D_0 are $\widetilde{Kh}$-thin, we must have:*

$$\dim(\widetilde{Kh}^*_{\delta_0}(D_0)) \neq \dim(\widetilde{Kh}^*_{\delta_1}(D_1)).$$

Indeed, suppose that $\dim(\widetilde{Kh}^*_{\delta_0}(D_0)) = \dim(\widetilde{Kh}^*_{\delta_1}(D_1))$ *and consider the long exact sequences of the proof. If the map ∂^* is either injective or surjective, then it must be an isomorphism and it follows that $\widetilde{Kh}^*_*(D) = 0$, which is impossible (the Jones polynomial is never constant equal to 0). Therefore the map is neither injective nor surjective, i.e it has non trivial kernel and cokernel. But this implies that D is not $\widetilde{Kh}$-thin by exactness of the sequence. It follows that the two dimensions cannot coincide.*

This lemma can be tailored to match the "completing the triple" procedure as follows.

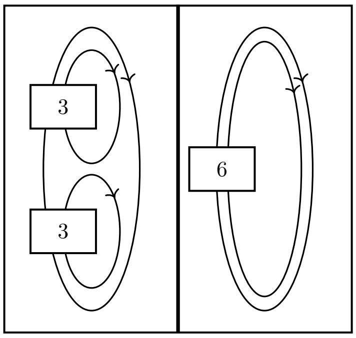
COROLLARY 2.18. *Let F and F' be two pointed diagrams related by a 1-handle move. Let D be obtained from F and F' by completing the triple. Assume that F, F' and D are $\widetilde{Kh}$-thin supported in δ-gradings $\delta_F, \delta_{F'}$ and δ respectively with $\delta_{F'} = \delta_F - 1$. Then the following holds.*

(i) If $\dim(\widetilde{Kh}^*_{\delta_F}(F)) > \dim(\widetilde{Kh}^*_{\delta_{F'}}(F'))$, *then* $\Phi^* : \widetilde{Kh}^*_{\delta_F}(F) \to \widetilde{Kh}^*_{\delta_{F'}}(F')$ *is surjective.*

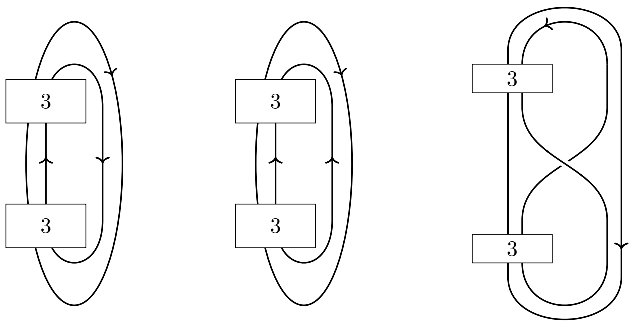
(ii) If $\dim(\widetilde{Kh}^*_{\delta_F}(F)) < \dim(\widetilde{Kh}^*_{\delta_{F'}}(F'))$, *then* $\Phi^* : \widetilde{Kh}^*_{\delta_F}(F) \to \widetilde{Kh}^*_{\delta_{F'}}(F')$ *is injective.*

We conclude this subsection with an example of application of Lemma 2.18, which will come in handy later on.

EXAMPLE 2.19. Consider the map induced by the 1-handle pictured below.



The movie starts at $F = T_{2,3} \sharp T_{2,3}$ and ends at $F' = T_{2,6}$. Both are alternating diagrams so they are $\widetilde{Kh}$-thin. The procedure is depicted below.



Completing the triple yields a diagram D equivalent to an alternating diagram, so it is $\widetilde{Kh}$-thin. The monoidality with respect to the connected sum, combined with the explicit value of $\widetilde{Kh}^*_*(T_{2,q})$ of Example 2.9 gives the following δ-gradings:

$$\delta_F = -2 + (-2) = -4, \quad \delta_{F'} = -5 = \delta_F - 1.$$

Additionally, we have

$$\dim(\widetilde{Kh}^*_*(F)) = \dim(\widetilde{Kh}^{*,*}(T_{2,3}))\cdot\dim(\widetilde{Kh}^{*,*}(T_{2,3})) = 9 > 6\dim(\widetilde{Kh}^{*,*}(T_{2,6})).$$

Hence by Lemma 2.18 (i), the map induced by the 1-handle is surjective.

2.1. Naturality properties. Since maps induced by oriented 1-handles are key to our work, we need to study naturality properties of the long exact sequence with respect to this local move.

Having to separate systematically the cases of c positive and negative is not practical, therefore we present a unified version of the short exact sequence. We defined before w_- when c is negative and w_+ if c is positive. In the former case, we set $w_+ = 0$, and $w_- = 0$ for the latter. For each $j \in \mathbb{Z}$ the short exact sequence becomes, regardless of the sign of the crossing:

$$0 \longrightarrow \widetilde{C}^{*,j}(D_1)[w_+, 3w_+ - 1] \xrightarrow{i} \widetilde{C}^{*,j}(D) \xrightarrow{q} \widetilde{C}^{*,j}(D_0)[w_-, 3w_- + 1] \longrightarrow 0$$

When one has a short exact sequence of chain complexes, there is a natural long exact sequence in homology. Since we work with graded chain complexes that are shifted in the short exact sequence, we need to check that maps connecting various instances ofshort exact sequences respect these grading shifts. We focus on maps induced by 1-handles.

First, we remark that the two maps induced on $\widetilde{C}^{*,j}(D_1)$ and $\widetilde{C}^{*,j}(D_0)$ are obtained from the movie starting at D ending at D' by replacing the crossing c by its 1 and 0 smoothing respectively. Note that one of these might not be an oriented 1-handle move, as it depends on a choice of orientation.

LEMMA 2.20. *Let D, D' be two diagrams related by a 1-handle move. For any choice of crossing c, the map induced by the move also induces a map of the associated short exact sequences.*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \widetilde{C}^{*,j}(D_1)[w_+, 3w_+ - 1] & \xrightarrow{i} & \widetilde{C}^{*,j}(D) & \xrightarrow{q} & \widetilde{C}^{*,j}(D_0)[w_-, 3w_- + 1] & \longrightarrow 0 \\ & \downarrow \Phi^1 & & \downarrow \Phi & & \downarrow \Phi^0 & \\ 0 \longrightarrow & \widetilde{C}^{*,j-1}(D'_1)[w'_+, 3w'_+ - 1] & \xrightarrow{i} & \widetilde{C}^{*,j-1}(D') & \xrightarrow{q} & \widetilde{C}^{*,j-1}(D'_0)[w'_-, 3w'_- + 1] & \longrightarrow 0. \end{array}$$

Proof. We only need to check that Φ^1 and Φ^0 have the proper degrees. No matter the sign of the crossing, if both induced movie, i.e. where c is replaced by its smoothing, can be made into an oriented 1-handle move by some choice of orientations then $n_-(D) = n_-(D')$, $n_-(D_1) = n_-(D'_1)$, $n_-(D_0) = n_-(D'_0)$ and it follows that $w_+ = w'_+$ and $w_- = w'_-$. Thus the induced map at the chain complex level has the form, for $m = 0, 1$:

$$\Phi^m : \widetilde{C}^{*,j}(D_m) \longrightarrow \widetilde{C}^{*,j-1}(D'_m).$$

They both appear in the short exact sequence as a shifted version, with domain and co-domain are shifted consistently.

Let us assume that c is negative and that the induced movie for the 0-smoothing cannot be oriented. First remark that D_1 and D'_1 inherit the orientations so the induced movie is always oriented and treated as above. The two strands of D_0 in the the 1-smoothing of c must point in the same direction. We consider an intermediate $\bar{D}$, identical to D_0 except we replace the 0-smoothing of c by a positive crossing $\bar{c}$. This produces an exact triple $(\bar{D}_1 = D'_0, \bar{D}, D_0)$ associated to c'. Let $\bar{w}_+$ the corresponding shift. We then have the following sequence of equalities:

$$\begin{aligned}\bar{w}_+ &= n_-(\bar{D}_1) - n_-(\bar{D}) + 1\\ &= n_-(D_0') - n_-(D_0) + 1\\ &= n_-(D_0') - n_-(D') + n_-(D) - n_-(D_0) + 1\\ &= w_-' - w_- + 1.\end{aligned}$$

The first equality is just the definition of $\bar{w}_+$, the second follows from $\bar{D}_1 = D_0'$ and $n_-(\bar{D}) = n_-(D_0)$ - since D_0 and $\bar{D}$ differ only by a positive crossing $\bar{c}$. The third uses $n_-(D) = n_-(D')$ and the last one follows from the definition of w_-' and w_-. We now have a graded map

$$\Phi^0 : \widetilde{C}^{*,j}(D_0)[0,1] \longrightarrow \widetilde{C}^{*+1,j}(D_0')[\bar{w}_+, 3\bar{w}_+ - 1].$$

The previous computation of shifts $\bar{w}_+$ yields

$$[\bar{w}_+, 3\bar{w}_+ - 1] = [w_-' - w_- + 1, 3(w_-' - w_- + 1) - 1] = [w_-' - w_- + 1, 3w_-' - 3w_- + 2].$$

There are two steps left. First, we shift both chain complexes by $[w_-, 3w_-]$:

$$\Phi^0 : \widetilde{C}^{*,j}(D_0)[w_-, 3w_- + 1] \longrightarrow \widetilde{C}^{*+1,j}(D_0')[w_-' + 1, 3w_-' + 2].$$

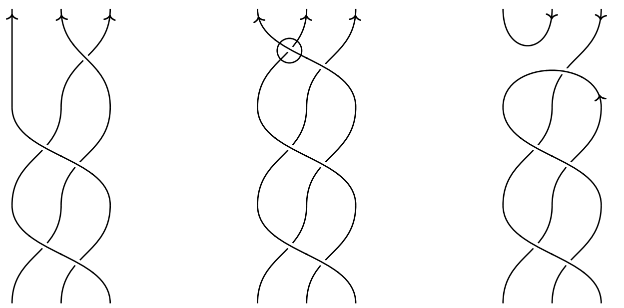
Finally, we apply the definition of the shift to change $\widetilde{C}^{*+1,j}(D_0')$ into $\widetilde{C}^{*,j-1}(D_0')$. This yields:

$$\Phi^0 : \widetilde{C}^{*,j}(D_0)[w_-, 3w_- + 1] \longrightarrow \widetilde{C}^{*,j-1}(D_0')[w_-', 3w_-' + 1].$$

The map Φ_0 has the proper degree, and we get a map relating the two graded short exact sequences as claimed. The positive case can be treated similarly.

□

EXAMPLE 2.21. We study the 3-stranded torus link $T_{3,3}$, with diagram given by the closure of the middle braid below (with the circled crossing). One checks easily that $n_-(D) = 6$. Let c be the circled crossing. We obtain an exact triple of diagrams.



One checks easily that, up to isotopy, we have the exact triple:

$$(D_1, D, D_0) = (T_{2,4}, T_{3,3}, U \coprod U).$$

With the orientations prescribed above, $n_-(D_0) = 2$ thus we have $w_- = 2 - 6 = -4$ and $w_+ = 0$. We consider the δ-graded skein exact sequence below:

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^i_{\delta+1}(T_{2,4}) \to \widetilde{Kh}^i_{\delta}(T_{3,3}) \to \widetilde{Kh}^{i+4}_{\delta+3}(U \sqcup U) \xrightarrow{\partial^*} \widetilde{Kh}^{i+1}_{\delta-1}(T_{2,4}) \to \cdots$$

Recall that

$$\widetilde{Kh}^i_{\delta}(U \coprod U) = Kh^i_{\delta}(U) = \begin{cases}\mathbb{Z}_2 & \text{if } i = 0 \text{ and } \delta = \pm 1.\\ 0 & \text{otherwise.}\end{cases}$$

We computed the δ-graded homology of $T_{2,4}$ in Example 2.9. It is given by:

$$\widetilde{Kh}^i_\delta(T_{2,4}) = \begin{cases} \mathbb{Z}_2 & \text{if } \delta = -3 \text{ and } i \in \{0,-2,-3,-4\}, \\ 0 & \text{otherwise.} \end{cases}$$

For $i \in \{0,-2\}$, $\delta = -3$, the sequence becomes

$$0 = \widetilde{Kh}^{i-1}_3(U \sqcup U) \xrightarrow{\partial^*} \widetilde{Kh}^i_{-3}(T_{2,4}) \to \widetilde{Kh}^i_{-3}(T_{3,3}) \to \widetilde{Kh}^{i+4}_{-1}(U \sqcup U) = 0$$

The leftmost term is zero since it has δ-grading 3, and the rightmost is zero since $i+4 \neq 0$ for $i \in \{0,-2\}$. Therefore, we have an equality

$$\widetilde{Kh}^i_\delta(T_{3,3}) = \mathbb{Z}_2 \text{ for } (i,\delta) \in \{(0,-4),(-2,-4)\}.$$

For $i = -4$ and $\delta = -4$, the sequence becomes:

$$\cdots \to \widetilde{Kh}^{-5}_3(U \sqcup U) = 0 \xrightarrow{\partial^*} \widetilde{Kh}^{-4}_{-3}(T_{2,4}) \to \widetilde{Kh}^{-4}_{-4}(T_{3,3}) \to \widetilde{Kh}^0_{-1}(U \sqcup U) \xrightarrow{\partial^*} \widetilde{Kh}^{-3}_{-5}(T_{2,4}) = 0 \to \cdots$$

The two terms can be checked to be zero with the descriptions given above. Thus we have a short exact sequence

$$0 \xrightarrow{\partial^*} \widetilde{Kh}^{-4}_{-3}(T_{2,4}) \to \widetilde{Kh}^{-4}_{-4}(T_{3,3}) \to \widetilde{Kh}^0_{-1}(U \sqcup U) \xrightarrow{\partial^*} 0$$

that splits, since we work with finite dimensional vector spaces. This yields an isomorphism:

$$\widetilde{Kh}^{-4}_{-4}(T_{3,3}) \cong \widetilde{Kh}^{-4}_{-2}(T_{2,4}) \oplus \widetilde{Kh}^0_{-1}(U \coprod U) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Finally, fix $i = -4$ and $\delta = -2$. The sequence then becomes:

$$\cdots \xrightarrow{\partial^*} \widetilde{Kh}^{-4}_{-1}(T_{2,4}) = 0 \to \widetilde{Kh}^{-4}_{-2}(T_{3,3}) \to \widetilde{Kh}^0_1(U \sqcup U) \xrightarrow{\partial^*} \widetilde{Kh}^{-3}_{-3}(T_{2,4}) \to \widetilde{Kh}^{-3}_{-4}(T_{3,3}) \to 0$$

We use the Bar-Natan homology of $T_{3,3}$ to study the boundary map. The link $T_{3,3}$ has 3 components. Say L_1 is the marked component. For any $k \neq l \in \{1,2,3\}$ the linking number is

$$lk(L_k, L_l) = -1.$$

If we reverse the orientation of exactly one component, then, by Corollary 1.22, we get one generator for Bar-Natan homology in homological degree

$$\begin{aligned} i &= 2lk(L_1,L_2) + 2lk(L_3,L_2) = -4, \\ i &= 2lk(L_1,L_3) + 2lk(L_2,L_3) = -4, \end{aligned}$$

If we reverse the orientation of both L_2 and L_3, then the contribution goes to the homological degree

$$i = 2lk(L_1,L_2) + 2lk(L_1,L_3) = -4.$$

Therefore $\dim(\widetilde{BN}^{-4}(T_{3,3})) = 3$. If the boundary map in the grid is non zero then the Khovanov homology of $T_{3,3}$ is given by the grid below, with columns the homological grading, and rows the δ-grading. Each entry of the grid contains the dimension of the space $\widetilde{Kh}^i_\delta(T_{3,3})$ and empty entries are 0 dimensional vector spaces.

−4	−3	−2	−1	0	i \ δ
2		1		1	−4

In particular, we have $\dim(\widetilde{Kh}_*^{-4}(T_{3,3})) = 2$. This contradicts the lower bound from Bar-Natan homology of Corollary 3.5. Therefore the boundary map is zero and the description of $\widetilde{Kh}_*^*(T_{3,3})$ is that of the grid below. Note that in particular, $T_{3,3}$ is not $\widetilde{Kh}$-thin.

-4	-3	-2	-1	0	i \ δ
1					-2
2	1	1		1	-4

3. The skein spectral sequence

In this section we present general facts about spectral sequences then focus our attention on the skein spectral sequence in Khovanov homology, first constructed in [**Tur08**], which generalises the skein long exact sequence.

Let us start with the generalities for which we follow [**Wei95**, Chapter 5]. All details and proofs will be omitted but can be found there.

Definition 2.4. Let $\mathcal{A}$ be the category of vector spaces over $\mathbb{Z}_2$. A *cohomology spectral sequence (starting with E_a)* in $\mathcal{A}$ is a family $\{E_a^{p,q}\}$ of vector spaces for $r \geq a$, together with maps

$$d_r^{p,q} : E_a^{p,q} \longrightarrow E_a^{p+r,q+1-r},$$

such that $d_r \circ d_r = 0$, and isomorphims between E_{r+1} and the homology of E_r. Note that d_r increases the total degree $p+q$ of $E_r^{p,q}$ by one.

Each $E_{r+1}^{p,q}$ is a quotient of a subspace of the previous term $E_r^{p,q}$. Thus there is a nested family of subobjects of $E_a^{p,q}$:

$$0 = B_a^{p,q} \subseteq \cdots \subseteq B_r^{p,q} \subseteq B_{r+1}^{p,q} \subseteq \cdots \subseteq Z_{r+1}^{p,q} \subseteq Z_r^{p,q} \subseteq \cdots \subseteq Z_a^{p,q} = E_a^{p,q},$$

such that $E_r^{p,q} \cong Z_r^{p,q}/B_r^{p,q}$. Define the intermediate objects

$$B_\infty^{p,q} = \bigcup_{r=a}^{\infty} B_r^{p,q} \text{ and } Z_\infty^{p,q} = \bigcap_{r=a}^{\infty} Z_r^{p,q}$$

and let $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$.

Definition 2.5. If we are given a family H^* of objects in $\mathcal{A}$, we say that a spectral sequence *converges* to H^* if the conditions hold:

(i) For each p and q, the outgoing differentials $d_r^{p,q}$ are zero for all large r.
(ii) Each H^n has a filtration

$$\cdots \subseteq F^{p+1}H^n \subseteq F^pH^n \subseteq \cdots \subseteq H^n$$

together with isomorphisms $\alpha^{p,q} : E_\infty^{p,q} \to F^pH^{p+q}/F^{p+1}H^{p+q}$ for all p and q.
(iii) For each n, $H^n = \varprojlim H^n/F^pH^n$.

Since we work with vector spaces, there is no extension problem, i.e all short exact sequences split. This means that we can reconstruct H^n from the spectral sequence without ambiguity. The filtration on H^n provides us with short exact sequences

$$0 \longrightarrow F^{p+1}H^n \longrightarrow F^pH^n \longrightarrow \frac{F^pH^n}{F^{p+1}H^n} \longrightarrow 0.$$

Using a splitting, we have an isomorphism

$$F^pH^n \cong F^{p+1}H^n \oplus \frac{F^pH^n}{F^{p+1}H^n} \cong F^{p+1}H^n \oplus E_\infty^{p,n-p}$$

If we iterate this argument, and start at p such that $F^pH^n = H^n$, we obtain the description

$$H^n \cong \oplus_{p+q=n} E_\infty^{p,q}.$$

We can add more structure to spectral sequences: there is a category of cohomology spectral sequences. A morphism $f : E \to E'$ is a family of maps $f_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ in $\mathcal{A}$ (for r suitably large) with $d_r \circ f_r = f_r \circ d_r$ such that each $f_{r+1}^{p,q}$ is the map induced on homology by $f_r^{p,q}$.

Lemma 2.22. *Let $f : E \to E'$ be a morphism of spectral sequences such that for some fixed r, $f^r : E_r^{p,q} \to E_r'^{p,q}$ is an isomorphism for all p and q. Then $f^s : E_s^{p,q} \to E_s'^{p,q}$ is also an isomorphim for all $s \geq r$ (including $s = \infty$).*

If $\{E_a^{p,q}\}$ and $\{E_a'^{p,q}\}$ converge to H^* and H'^* respectively, we say that a map $h : H^* \to H'^*$ is *compatible* with a morphism $f : E \to E'$ if h maps F^pH^n to $F^pH'^n$ and the next diagram commutes.

$$\begin{array}{ccc} E_\infty^{p,n-p} & \xrightarrow{f^\infty} & E_\infty'^{p,n-p} \\ \downarrow \alpha & & \downarrow \alpha' \\ F^pH^n/F^{p+1}H^n & \xrightarrow{h} & F^pH'^n/F^{p+1}H'^n \end{array}$$

Theorem 2.23. *Let $\{E_a^{p,q}\}$ and $\{E_a'^{p,q}\}$ be two spectral sequences that converge to H^* and H'^* respectively. Suppose given a map $h : H^* \to H'^*$ compatible with a morphism $f : E \to E'$. If $f^r : E_r^{p,q} \to E_r'^{p,q}$ is an isomorphism for all p and q for some r, then h is an isomorphism*

We continue our generalities section by introducing a special type of spectral sequences, those that arise from a filtration of a chain complex.

Definition 2.6. A *filtration* F on a chain complex C is an ordered family of chain subcomplexes

$$\cdots \subseteq F^{p+1}C^n \subseteq F^pC^n \subseteq \cdots$$

of C. Such a filtration is said to be *exhaustive* if $C = \bigcup_p F^pC$. It is said to be *bounded below* if, for each n, there exists $s \in \mathbb{Z}$ such that $F^sC^n = 0$ and *bounded above* if, for each n, there exists $t \in \mathbb{Z}$ such that $F^tC^n = C^n$. A filtration which is bounded below and above will be referred to as *bounded.*

Theorem 2.24. *The following properties hold:*

(1) A filtration F on a chain complex naturally determines a spectral sequence starting with $E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}$ and $E_1^{p,q} = H^{p+q}E_0^{p,*}$.

(2) If $f : C \longrightarrow C'$ is a map of filtered chain complexes, then there is an associated map of spectral sequences.

(3) Suppose that the filtration on C is bounded below and exhaustive. Then the spectral sequence is bounded below and converges to $H^*(C)$. Moreover this convergence is natural in the sense that if $f : C \longrightarrow C'$ is a map of filtered chain complexes, then the induced map $f^* : H^*(C) \longrightarrow H^*(C')$ is compatible with the corresponding map of spectral sequences.

Now equipped with some general knowledge about spectral sequences, we can move on the precise case of the skein spectral sequence.

Let (D,p) be a pointed diagram. Choose a crossing set of size m and for $k = 1,\dots,m$ denote by $\widetilde{D}_{(k)}$ the diagram obtained from D by replacing crossings 1 to $k-1$ by their 1-resolution and crossing k by its 0-resolution. Similarly, let $D_{(k)}$ the diagram obtained from D by replacing crossings 1 to k by their 1-resolution. For convenience we also define $D_0 = \widetilde{D}_0 = D$.

Define the following quantities, for $k = 0,\dots,m$:

$$\begin{array}{rcl} n_+(D_{(k)}) & = & \text{number of positive crossings in } D_{(k)} \\ n_-(D_{(k)}) & = & \text{number of positive crossings in } D_{(k)} \\ n_+(\widetilde{D}_{(k)}) & = & \text{number of positive crossings in } \widetilde{D}_{(k)} \\ n_-(\widetilde{D}_{(k)}) & = & \text{number of positive crossings in } \widetilde{D}_{(k)}. \end{array}$$

If the kth crossing is positive, set

$$a_k = n_-(D_{(k)}) - n_-(D_{(k-1)}) + 1,\ \ \widetilde{a}_k = 0.$$

If the kth crossing is negative, set

$$a_k = 0,\ \ \widetilde{a}_k = n_-(\widetilde{D}_{(k)}) - n_-(D_{(k-1)}).$$

For $k \geq 1$, and any type of crossing, define

$$b_k = 3a_k - 1,\ \ \widetilde{b}_k = 3\widetilde{a}_k + 1,$$

and

$$A_k = \sum_{i=1}^{k} a_k,\ \ B_k = \sum_{i=1}^{k} b_k.$$

Finally, set $A_0 = B_0 = 0$.

For each $j \in \mathbb{Z}$ and $k \geq 1$, there is a short exact sequence of chain complexes - the same as before-:

$$0 \longrightarrow \widetilde{C}^{*,j}(D_{(k)})[a_k, b_k] \longrightarrow \widetilde{C}^{*,j}(D_{k-1}) \longrightarrow \widetilde{C}^{*,j}(\tilde{D}_{(k)})[\widetilde{a}_k, \widetilde{b}_k] \longrightarrow 0.$$

These sequences give rise to a filtration on $\widetilde{C}^{*,j}(D,p)$, defined by

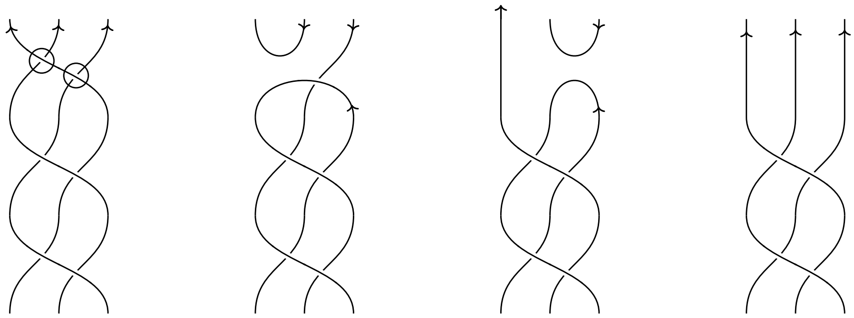
$$\begin{array}{rcll} F^k\widetilde{C}^{*,j}(D,p) & = & \widetilde{C}^{*,j}(D_{(k)},p)[A_k, B_k] & \text{for } k = 0,\dots,m. \\ F^k\widetilde{C}^{*,j}(D,p) & = & 0 & \text{for } k > m. \end{array}$$

This filtration is bounded and thus there is an associated spectral sequence.

THEOREM 2.25. [**Tur08**] *For each $j \in \mathbb{Z}$, there is a spectral sequence ${}^jE(D)$ which converges to $\widetilde{Kh}^{*,j}(D)$, whose E_1-page is given by*

$${}^jE_1^{s,t} = \begin{cases} \widetilde{Kh}^{s+t-A_s-\widetilde{a}_{s+1},j-B_s-\widetilde{b}_{s+1}}(\tilde{D}_{(s+1)}) & \textit{if } s = 0,\dots,m-1. \\ \widetilde{Kh}^{m+t-A_m,j-B_m}(D_{(m)}) & \textit{if } s = m. \\ 0 & \textit{otherwise.} \end{cases}$$

EXAMPLE 2.26. Let us give an example of use of that spectral sequence. More precisely we will use it to compute $\widetilde{Kh}^{*,*}(T_{3,3})$ again, but with respect to the two crossings circled in the leftmost diagram below. Note that we consider the closure of the depicted braid diagrams.



From left to right the diagrams are $D = T_{3,3}, \widetilde{D}_{(1)}, \widetilde{D}_{(2)}$ and $D_{(2)} = T_{3,2}$. Note that we assume the basepoint to lie on a strand of the closure.

First we note that all the crossings we use are negative so all $D_{(i)}$ will be equipped with the induced orientation. By definition, we have

$$a_1 = a_2 = A_1 = A_2 = 0, \;\; B_1 = -1, \;\; B_2 = -2.$$

For the other diagrams, we have indicated orientations with arrows. Without much effort, one finds that $\widetilde{D}_{(1)}$ is equivalent to a diagram for a disjoint union of unknots $U \coprod U$, with $n_-(\widetilde{D}_{(1)}) = 2$. The diagram $\widetilde{D}_{(2)}$ is a diagram for an unknot, and $n_-(\widetilde{D}_{(2)}) = 1$. So the shifts are

$$\begin{aligned} \widetilde{a}_1 &= 2-6 = -4, & \widetilde{b}_1 &= -11, \\ \widetilde{a}_2 &= 1-5 = -4, & \widetilde{b}_2 &= -11. \end{aligned}$$

We can now describe the E_1-page of our spectral sequence for a generic j.

$$\begin{aligned} {}^jE_1^{0,t} &= \widetilde{Kh}^{t+4,j+11}(U \textstyle\coprod U), \\ {}^jE_1^{1,t} &= \widetilde{Kh}^{t+5,j+12}(U), \\ {}^jE_1^{2,t} &= \widetilde{Kh}^{t+2,j+2}(T_{2,3}). \end{aligned}$$

For $j \notin \{-4,-8,-10,-12\}$, the E_1-page is zero. For $j \in \{-4,-8\}$, only the second column contributes, so ${}^jE_1 = {}^jE_\infty$. It remains to treat the cases $j = -10$ and $j = -12$. We present the E_1-pages for both cases below by grids with columns parametrized by s and rows by t. The entries represent the dimensions of the space $E_1^{s,t}$, with empty entries being 0-dimensional.

	0	1	2
−4	1	0	0
−5	0	0	1

(A) $j = -10$.

	0	1	2
−4	1	0	0
−5	0	1	0

(B) $j = -12$.

For $j = -12$ all differentials have to be zero, so ${}^{-12}E_1 = {}^{-12}E_\infty$. Both entries contribute to the same homological degree $i = s + t = -4$ in $\widetilde{Kh}^{*,*}(T_{3,3})$. For $j = -10$, there might be a non trivial differential

$$d_2^{0,-4} : E_2^{0,-4} = \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 = E_2^{2,-5}.$$

If this differential is non zero, then $\dim(\widetilde{Kh}^{-4,-10}(T_{3,3})) = 0$ and thus

$$\dim(\widetilde{Kh}^{-4,*}(T_{3,3})) = \dim(\widetilde{Kh}^{-4,-12}(T_{3,3})) = 2.$$

We obtain the same contradiction with the lower bound from reduced Bar-Natan homology. Thus the d_2 above must be the zero map and ${}^{-10}E_1 = {}^{-10}E_\infty$. The resulting homology is described in the table below, with columns the homological degree and rows the quantum degree. Each entry (i,j) contains the dimension of $\widetilde{Kh}^{i,j}(T_{3,3})$, and empty entries have dimension 0.

-4	-3	-2	-1	0	$i \backslash j$
				1	-4
					-6
		1			-8
1	1				-10
2					-12

This concludes the example.

We will only ever use this spectral sequence with m negative crossings. For simplicity, we give a description of that particular form of the spectral sequence. In this case we have

$$n_-(D_{(k)}) = n_-(D) - (k-1),$$

and

$$a_k = 0,\ A_k = 0,\ b_k = -1,\ B_k = -k,$$

We define intermediate quantities, for $1 \le k \le m$

$$\begin{array}{rcl} \alpha_k & = & n_-(\widetilde{D}_{(k)}) - n_-(D), \\ \beta_k & = & 3\alpha_k + 1. \end{array}$$

With these notations, we have

$$\begin{array}{rclcl} \widetilde{a}_k & = & n_-(\widetilde{D}_{(k)}) - n_-(D_{(k-1)}) & = & n_-(\widetilde{D}_{(k)}) - (n_-(D) - (k-1)) = \alpha_k + k - 1. \\ \widetilde{b}_k & = & 3\widetilde{a}_k + 1 & = & \beta_k + 3(k-1). \end{array}$$

It is enough to plug these values α_k and β_k in the description of the spectral sequence to obtain the following statement.

Corollary 2.27. *For each $j \in \mathbb{Z}$, the skein spectral sequence ${}^jE(D)$ with respect to m negative crossings has E_1-page given by*

$${}^jE_1^{s,t} = \begin{cases} \widetilde{Kh}^{t-\alpha_{s+1}, j-\beta_{s+1}-2s}(\tilde{D}_{(s+1)}) & \text{if } s = 0, \dots, m-1. \\ \widetilde{Kh}^{m+t, j+m}(D_{(m)}) & \text{if } s = m. \\ 0 & \text{otherwise.} \end{cases}$$

With a good grasp on the behaviour of that spectral sequence, we can now relate it to the various structures introduced previously.

Proposition 2.28. *The skein spectral sequence does not depend on the choice of basepoint. More precisely, the map f of Proposition 1.16 is a map of filtered chain complexes and it induces an isomorphism of spectral sequences.*

Proof. Both maps $x_\bullet$ and ν commute with the inclusions of subcomplexes induced by 1-smoothing a crossing. Moreover, for each k, their composition f is an endomorphism of $\widetilde{C}(D_{(k)})$. Thus f is a map of filtered chain complexes and as such induces a map $\{f^r\}$ of spectral sequences. The map $f_0^{p,q}$ on the E_0-page is an isomorphism for all p,q so f^r is an isomorphism for all $r \geq 0$. Consequently, the spectral sequence itself does not depend on the choice of basepoint. □

LEMMA 2.29. *Let (D,p) and (D',p) be two pointed diagrams related by a 1-handle. The induced map*

$$\Phi : \widetilde{C}^{*,*}(D) \longrightarrow \widetilde{C}^{*,*-1}(D')$$

preserves the skein filtration. Consequently, for each $j \in \mathbb{Z}$, there is map of spectral sequences

$${}^j\Phi^* : {}^jE(D) \longrightarrow {}^{j-1}E(D').$$

Moreover, Φ induces maps

$$\Phi_s : \widetilde{C}^{*,*}(\widetilde{D}_{(s+1)}) \longrightarrow \widetilde{C}^{*,*-1}(\widetilde{D'}_{(s+1)})$$

and

$$\Phi_m : \widetilde{C}^{*,*}(D_{(m)}) \longrightarrow \widetilde{C}^{*,*-1}(D'_{(m)})$$

such that the induced maps in homology agree with ${}^j\Phi^*$ on the E_1-page.

This lemma is a direct consequence of Lemma 2.20, since the skein filtration arises from such short exact sequences. Therefore the map Φ_m is obtained as the 1-handle $D_{(m)}$ and $D'_{(m)}$ by their 1-smoothing. The maps Φ_s are obtained as 1-handles relating $\widetilde{D}_{(s+1)}$ and $\widetilde{D'}_{(s+1)}$.

We spent some time working out what happens with connected sums earlier. Let us see how the skein spectral sequence behaves with respect to that operation.

Let (D,p) a pointed diagram and D' be an oriented diagram, and choose m crossings in D. We assume that all m crossings are negative, so that all intermediate diagrams inherits the orientation of D. Just as the total exact triangle could be simplified, the spectral sequence also takes on a very simple form. We fix $j \in \mathbb{Z}$, and define the chain complex:

$$C_{\otimes}^{*,j} = \bigoplus_{j=j_1+j_2} \widetilde{C}^{*,j_1}(D) \otimes \widetilde{C}^{*,j_2}(D')$$

By additivity of the homology functor and the Künneth formula, this complex has homology

$$H_*(C_{\otimes}^{*,j}) = \bigoplus_{j=j_1+j_2} \widetilde{Kh}^{*,j_1}(D) \otimes \widetilde{Kh}^{*,j_2}(D').$$

Moreover, this complex is equipped with a filtration which we describe now. We consider the short exact sequences that gave us the filtration for the skein spectral sequence for D:

$$0 \longrightarrow \widetilde{C}^{*,j_1}(D_{(k)})[a_k,b_k] \xrightarrow{i} \widetilde{C}^{*,j_1}(D_{(k-1)}) \xrightarrow{\pi} \widetilde{C}^{*,j_1}(\tilde{D}_{(k)})[\widetilde{a}_k,\widetilde{b}_k] \longrightarrow 0$$

The tensor product is exact over $\mathbb{Z}_2$, so for every $j_2 \in \mathbb{Z}$ we have a short exact sequence of complexes:

$$0 \longrightarrow \widetilde{C}^{*,j_1}(D_{(k)})[a_k,b_k]\otimes\widetilde{C}^{*,j_2}(D') \xrightarrow{i\otimes 1} \widetilde{C}^{*,j_1}(D_{(k-1)})\otimes\widetilde{C}^{*,j_2}(D') \xrightarrow{\pi\otimes 1} \widetilde{C}^{*,j_1}(\tilde{D}_{(k)})[\widetilde{a}_k,\widetilde{b}_k]\otimes\widetilde{C}^{*,j_2}(D') \longrightarrow 0$$

We sum these short exact sequences over all $j_1 + j_2 = j$ and obtain a filtration on $C_{\otimes}^{*,j}$ given by

$$\begin{aligned} F^k C_{\otimes}^{*,j} &= \bigoplus_{j=j_1+j_2} F^k \widetilde{C}^{*,j_1}(D) \otimes \widetilde{C}^{*,j_2}(D') && \text{for } k = 0, \dots, m. \\ F^k C_{\otimes}^{*,j} &= 0 && \text{for } k > m. \end{aligned}$$

Thus we have a spectral sequence ${}^jE(D, D')$ converging to

$$\bigoplus_{j=j_1+j_2} \widetilde{Kh}^{*,j_1}(D) \otimes \widetilde{Kh}^{*,j_2}(D')$$

whose E_1-page is given by

$${}^jE_1^{s,t}(D, D') = \begin{cases} \bigoplus_{j=j_1+j_2} \widetilde{Kh}^{s+t-A_s-\widetilde{a}_{s+1}, j_1-B_s-\widetilde{b}_{s+1}}(\widetilde{D}_{(s+1)}) \otimes \widetilde{Kh}^{*,j_2}(D') & \text{if } s = 0, \dots, m-1. \\ \bigoplus_{j=j_1+j_2} \widetilde{Kh}^{m-A_m, j_1-B_m}(D_{(m)}) \otimes \widetilde{Kh}^{*,j_2}(D') & \text{if } s = m. \\ 0 & \text{otherwise.} \end{cases}$$

This spectral sequence can now be compared to the skein spectral sequence for $D \sharp D'$ with respect to the same crossings.

PROPOSITION 2.30. *For all $j \in \mathbb{Z}$, the spectral sequence ${}^jE(D, D')$ is isomorphic to the spectral sequence ${}^jE(D \sharp D')$.*

Proof. The map S of proposition 2.2 is a map of filtered chain complexes, and it induces an isomorphism of E_1-pages. Thus the spectral sequences are isomorphic. □

CHAPTER 3

A cohomology operation on reduced Khovanov homology

In this chapter we define a new cohomology operation that arises naturally when comparing Khovanov anad Bar-Natan homology. In algebraic topology in general, cohomology operations are extremely natural to consider. Though we haven't discussed the functoriality of the reduced theory, we still have maps induced by Reidemeister moves away from basepoints and by 0,1 and 2-handle moves. There are already some known cohomology operations in Khovanov homology, for example Lipshitz-Sarkar [**LS14**] constructed Steenrod squares via their theory of homotopy types of Khovanov homology. Another example, due to Putyra and Shumakovitch [**PS16**], is the Bockstein cohomology operation. Our construction yields a new operation. Our main use for that map is computational shortcuts when using the skein long exact sequence. In a first section, we define the cohomology operation. In a second section, we explore some of its properties and use it to show invariance of Bar-Natan homology. Finally we close the chapter with some remarks of a more general nature.

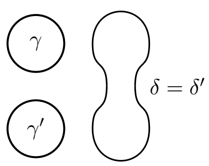
1. Constructing a cohomology operation.

In this section, we define a cohomology operation on the reduced Khovanov homology. More precisely, we define a map β at the level of the Khovanov chain complex. We then explore a spectral sequence that features this map as a differential. We will start at the non-reduced level and work our way to the reduced case. The map we consider was defined for non-reduced homology by Turner [**Tur06**]. It is defined in a similar fashion to d and d_{BN}, i.e. we only need to give maps "$m_{A,B}$"and "$\Delta_{A,B}$". We consider again the quotiented polynomial algebra

$$V'_A = \mathbb{Z}_2[x_\gamma \mid \ \gamma \text{ circle in } s_A]/(x_\gamma^2 = 1).$$

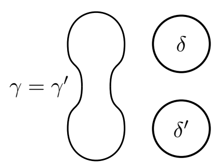
In this setting, we define the maps as follows.

(i) If $\gamma \neq \gamma'$, then $\delta = \delta'$. For a monomial v not featuring x_γ or $x_{\gamma'}$, we define



$$\begin{array}{rrcl} m^\beta_{A,B}: & V'_A & \longrightarrow & V'_B \\ & v & \longmapsto & v, \\ & x_\gamma v,\ x_{\gamma'} v & \longmapsto & 0, \\ & x_\gamma x_{\gamma'} v & \longmapsto & x_\delta. \end{array}$$

(ii) If $\gamma = \gamma'$, then $\delta \neq \delta'$. For a monomial v not featuring $x_\gamma = x_{\gamma'}$, we define



$$\begin{array}{rrcl} \Delta^\beta_{A,B}: & V'_A & \longrightarrow & V'_B \\ & v & \longmapsto & v \\ & x_\gamma v & \longmapsto & 0. \end{array}$$

This map is extended linearly to define

$$\beta : C^{i,j}(D) \longrightarrow C^{i+1,j+2}(D),$$

and verifies $d = \beta + d_{BN}$. Moreover, as proved by Turner in [**Tur06**], one easily checks that $\beta \circ \beta = 0$ and that $d \circ \beta = \beta \circ d$. With these tools in hand we can define the reduced version of this map β.

LEMMA 3.1. *The map β descends to a map $\widetilde{\beta} : \widetilde{C}^{i,j}(D) \longrightarrow \widetilde{C}^{i+1,j+2}(D)$, such that*
(i) $\widetilde{\beta} \circ \widetilde{\beta} = 0$.
(ii) $\widetilde{\beta}$ *commutes with the differential* $d_{\widetilde{Kh}}$ *so it induces a map*

$$\widetilde{\beta} : \widetilde{Kh}^{i,j}(D) \longrightarrow \widetilde{Kh}^{i+1,j+2}(D).$$

Note that, in homology, we can also consider this map as acting on the δ-graded version, that is

$$\widetilde{\beta}^* : \widetilde{Kh}^{i}_{\delta}(D) \longrightarrow \widetilde{Kh}^{i+1}_{\delta}(D).$$

Proof. As in the previous cases (for d and d_{BN}), one sees that the maps descends directly from the definition of $m^{\beta}_{A,B}$ and $\Delta^{\beta}_{A,B}$. The result of (i) follows immediately by restriction of Turner's result mentioned above. For (ii), the property $d_{BN} = \beta + d$ descends to the reduced versions, therefore we have:

$$0 = d^2_{\widetilde{BN}} = (\widetilde{\beta} + d_{\widetilde{Kh}})^2 = \widetilde{\beta}^2 + d^2_{\widetilde{Kh}} + d_{\widetilde{Kh}} \circ \widetilde{\beta} + \widetilde{\beta} \circ d_{\widetilde{Kh}} = d_{\widetilde{Kh}} \circ \widetilde{\beta} + \widetilde{\beta} \circ d_{\widetilde{Kh}},$$

where the last equality follows from (i) and $d^2_{\widetilde{Kh}} = 0$. Therefore $\widetilde{\beta}$ commutes with the Khovanov differential and induces a map

$$\widetilde{\beta}^* : \widetilde{Kh}^{i,j}(D) \longrightarrow \widetilde{Kh}^{i+1,j+2}(D),$$

as claimed.

□

There is a spectral sequence, the so-called *Lee-Rasmussen spectral sequence*, first constructed by Rasmussen [**Ras10**] for the rational homology that connects Khovanov and Lee homologies. This particular occurence leads to Rasmussen's *s-invariant*, which he used to prove the Milnor conjecture (now called Kronheinmer-Mrowka theorem [**KM93**], since they proved it first). The mod 2 equivalent of that spectral sequence was constructed by Turner [**Tur06**] for non-reduced homology and relates the Khovanov and Bar-Natan homologies. Given a diagram D, this spectral sequence has E_1-page given by $Kh^{*,*}(D)$ and converges to $BN^*(D)$, moreover the first differential is the map β^*. In fact the whole spectral sequence is a link invariant. We work with reduced Khovanov homology and therefore we need to set it up in our context.

This sequence arises from a filtration on $\widetilde{C}(D)$, which we shall describe now. Let μ denote the number of components of D modulo 2 and set

$$F^k\widetilde{C}^i(D) = \left\{ v = \sum_{i \in I} v_i \;\mid\; q(v_i) \geq 2k + \mu + 1, \text{ for all } i \in I \right\}.$$

The differential $d_{\widetilde{BN}}$ never decreases the quantum degree so $d_{\widetilde{BN}}(F^k) \subset F^k$. This filtration is bounded since the complex itself is bounded. Thence there is a spectral sequence, the *Lee-Rasmussen spectral sequence* for reduced Bar-Natan homology converging to $\widetilde{BN}^*(D)$. The

E_0-page can be made explicit

$$E_0^{k,l}(D) = \frac{F^k}{F^{k+1}} = \widetilde{C}^{k+l,2k+\mu+1}(D).$$

The d_0 differential has degree $(0,1)$ in the spectral sequence and so it maps

$$d_0 : \widetilde{C}^{k+l,2k+\mu+1}(D) = E_0^{k,l} \longrightarrow E_0^{k,l+1}(D) = \widetilde{C}^{k+l+1,2k+\mu+1}(D).$$

Hence d_0 is the quantum degree preserving part of $d_{\widetilde{BN}} = d + \widetilde{\beta}$, namely d. It follows that the E_1-page can be described as

$$E_1^{k,l}(D) = \widetilde{Kh}^{k+l,2k+\mu+1}(D).$$

The general theory of spectral sequences obtained through a filtration tells us that the d_1 differential is given by the boundary map in the long exact sequence induced by the short exact sequence

$$0 \longrightarrow \frac{F^{k+1}}{F^{k+2}} \longrightarrow \frac{F^k}{F^{k+2}} \longrightarrow \frac{F^k}{F^{k+1}} \longrightarrow 0$$

In our context, this statement can be made more precise:

$$d_1 : \widetilde{Kh}^{k+l,2k+\mu+1}(D) \longrightarrow \widetilde{Kh}^{k+l+1,2k+\mu+3}(D).$$

The d_1 differential is then identified with the part of $d_{\widetilde{BN}}$ that increases the quantum degree by 2, namely $\widetilde{\beta}^*$.

Let us once again use the family of 2-stranded torus links as an example. In the following, we compute the map $\widetilde{\beta}^*$ completely for any 2-stranded torus link.

EXAMPLE 3.2. For $q = 1$, i.e when $T_{2,q}$ is an unknot, the reduced Khovanov homology is one dimensional, generated by a_0 in degree $(0,0)$. Hence $\widetilde{\beta}^*(a_0) = 0$.

For the cases $q \geq 2$, we will use the spectral sequence to compute the map $\widetilde{\beta}^*$. The reduced Khovanov homology of 2-stranded torus links is well-known has been computed in Example 2.9. We had

$$\widetilde{Kh}^{i,j}(T_{2,q}) = \begin{cases} \mathbb{Z}_2 & \text{if } (i,j) = (i, -q+1+2i) \text{ for } i = 0, -2, -3, \ldots, -q. \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by a_i the generator of $\widetilde{Kh}^{-i,*}(T_{2,q})$. The first page of the spectral sequence is given by

$$E_1^{k,l} = \widetilde{Kh}^{k+l,2k+\mu+1}(T_{2,q}),$$

where μ is the number of components of $T_{2,q}$ modulo 2. Recall that the differential d_1 coincides with $\widetilde{\beta}^*$. The first step towards our goal is to describe explicitly the E_1-page. From the description of $\widetilde{Kh}^{i,j}(T_{2,q})$, we see $E_1^{k,l} = 0$ unless

$$2k + \mu + 1 = j = -q + 1 + 2(k+l), \text{ so } l = \frac{\mu + q}{2}.$$

Hence there is exactly one row in the first page which is non-zero. This implies that the spectral sequence collapses at the E_1-page. In this context, we recover $\widetilde{BN}^i(D)$ by the formula

$$\widetilde{BN}^i(D) = \bigoplus_{k+l=i} E_2^{k,l} = E_2^{i-\frac{\mu+q}{2}, \frac{\mu+q}{2}}.$$

The E_1-page with columns k and row l, is given by:

	$-\frac{\mu+q}{2}-q$	$-\frac{\mu+q}{2}-q+1$	$-\frac{\mu+q}{2}-q+2$	$\cdots$	$-\frac{\mu+q}{2}-3$	$-\frac{\mu+q}{2}-2$	$-\frac{\mu+q}{2}-1$	$-\frac{\mu+q}{2}$
$\frac{\mu+q}{2}$	$\mathbb{Z}_2\{a_q\}$	$\mathbb{Z}_2\{a_{q-1}\}$	$\mathbb{Z}_2\{a_{q-2}\}$	$\cdots$	$\mathbb{Z}_2\{a_3\}$	$\mathbb{Z}_2\{a_2\}$	0	$\mathbb{Z}_2\{a_0\}$

First we notice that for all q we must have $\widetilde{\beta}^*(a_0)=0$, because the table is zero outside the region shown. If q is even, then the leftmost element of the page must survive, since $\widetilde{BN}^{-q}(T_{2,q})$ has a generator there, therefore:

$$\widetilde{\beta}^*(a_q)=0.$$

As $\widetilde{BN}^i(D)=0$ for $i\neq 0,-q$, all others generator must be paired by $d_1=\widetilde{\beta}^*$. This fact, together with $(\widetilde{\beta}^*)^2=0$, determines a unique pairing of the generators of $\widetilde{Kh}^{*,*}(T_{2,q})$:

$$\widetilde{\beta}^*(a_i)=\begin{cases} a_{i-1} & \text{if } i \text{ is odd.} \\ 0 & \text{otherwise.}\end{cases}$$

If q is odd, then $\widetilde{BN}^i(D)\neq 0$ for $i=0$ only, and the same argument applies, yielding the same result.

2. Properties of β^* and invariance of Bar-Natan homology.

In this section, we explore some properties of the map $\widetilde{\beta}^*$. We begin by proving invariance of Bar-Natan homology through the use of $\widetilde{\beta}^*$. Then we show that $\widetilde{\beta}^*$ is indeed a cohomology operation. Finally, we consider its behaviour with respect to connected sums and the skein long exact sequence.

Let us now return to the proof of the invariance of Bar-Natan homology. Note that we already have invariance of that theory since it is explicitly given in terms of linking numbers and we have shown that it is independent from the choice of basepoint. The strategy for this new proof is to show that the whole Lee-Rasmussen spectral sequence is an invariant of pointed links.

PROPOSITION 3.3. *The quasi-isomorphisms associated to the Reidemeister moves and the isomorphism for changing the basepoint f of Proposition 1.16 induce maps of Lee-Rasmussen spectral sequences.*

Proof. Let (D,p) and (D',p) be two pointed diagrams that differ by a Reidemeister move away from the basepoint. Denote by $E(D,p)$ and $E(D',p)$ their respective Lee-Rasmussen spectral sequences. To show that the quasi-isomorphism ϕ_R associated to the Reidemeister move induces a map of Lee-Rasmussen spectral sequences, it is enough by Theorem 2.24 to show that ϕ_R preserves the filtration used to define the spectral sequence. The quasi-isomorphism associated to that move has bidegree $(0,0)$:

$$\phi:\widetilde{C}^{i,j}(D)\longrightarrow\widetilde{C}^{i,j}(D').$$

Consider an element $v\in F^k\widetilde{C}^i(D)$. We show that $\phi_R(v)\in F^k\widetilde{C}^i(D')$. By definition we have

$$v=\sum_{i\in I}v_i,\ \text{such that}\ q(v_i)\geq 2k+\mu+1,\ \text{for all}\ i\in I.$$

So, as ϕ_R is linear with bidegree $(0,0)$, it follows that

$$\phi_R(v)=\sum_{i\in I}\phi_R(v_i),\ \text{such that}\ q(\phi_R(v_i))\geq 2k+\mu+1,\ \text{for all}\ i\in I.$$

In other words, $\phi(v) \in F^k\widetilde{C}^i(D')$: ϕ is a map of filtered chain complexes. Thence by Theorem 2.24, ϕ_R induces a map of Lee-Rasmussen spectral sequences

$$\phi : E(D,p) \longrightarrow E(D',p).$$

For the change of basepoint, let (D,p) and (D,p') be two pointed diagrams with the same underlying oriented diagram. Denote by $E(D,p)$ and $E(D,p')$ their respective Lee-Rasmussen spectral sequences. In this case, the same argument applies: indeed the chain map f of Proposition 1.16 also has bidegree $(0,0)$. Therefore it preserves the filtration and induces a map of spectral sequences

$$f : E(D,p) \longrightarrow E(D,p').$$

This concludes the proof. □

Corollary 3.4. *The map $\widetilde{\beta}^*$ commutes with the isomorphisms associated to Reidemeister moves and the isomorphism f^* induced by the chain map f of Proposition 1.16.*

Proof. Let (D,p) and (D',p) be two pointed diagrams that differ by a Reidemeister move away from the basepoint. Denote by $E(D,p)$ and $E(D',p)$ their respective Lee-Rasmussen spectral sequences. To show that $\widetilde{\beta}$ commutes with the isomorphisms associated to the Reidemeister moves, we use the fact that the d_1 differential of the Lee-Rasmussen spectral sequence is $\widetilde{\beta}^*$. By Proposition 3.3, the quasi-isomorphism ϕ_R associated to a Reimeister induces a map of spectral sequences. In particular, at the E_1-page, we have a commutative diagram, for every $k,l \in \mathbb{Z}$:

$$\begin{array}{ccc} E_1^{k,l}(D,p) & \xrightarrow{\phi_1} & E_1'^{k,l}(D',p) \\ \downarrow d_1 & & \downarrow d_1' \\ E_1^{k+1,l}(D,p) & \xrightarrow{\phi_1} & E_1'^{k+1,l}(D',p). \end{array}$$

Recall that for any $k,l \in \mathbb{Z}$ we have an equality: $E_1^{k,l}(D,p) = \widetilde{Kh}^{k+l,2k+\mu+1}(D,p)$. Moreover, the d_1 differential coincides with $\widetilde{\beta}^*$. Finally, by definition, the map ϕ_1 is the map induced by ϕ at the E_1-page, i.e. it is the isomorphism ϕ_R^* associated to the Reidemeister move in homology. The commutative diagram above then becomes:

$$\begin{array}{ccc} \widetilde{Kh}^{k+l,2k+\mu+1}(D,p) & \xrightarrow{\phi_R^*} & \widetilde{Kh}^{k+l,2k+\mu+1}(D',p) \\ \downarrow \widetilde{\beta}^* & & \downarrow \widetilde{\beta}^* \\ \widetilde{Kh}^{k+l+1,2+2k+\mu+1}(D,p) & \xrightarrow{\phi_R^*} & \widetilde{Kh}^{k+l+1,2+2k+\mu+1}(D',p). \end{array}$$

This diagrams gives the first part of the statement.

For the change of basepoint, let (D,p) and (D,p') be two pointed diagrams with the same underlying oriented diagram. Denote by $E(D,p)$ and $E(D,p')$ their respective Lee-Rasmussen spectral sequences. Since the map f of Proposition 1.16 also induces a map of spectral sequences by Proposition 3.3, and the induced map at the E_1-page is f^*, the isomorphism induced in homology, we then have the commutative diagram:

$$\begin{array}{ccc} \widetilde{Kh}^{k+l,2k+\mu+1}(D,p) & \xrightarrow{f^*} & \widetilde{Kh}^{k+l,2k+\mu+1}(D,p') \\ \downarrow \widetilde{\beta}^* & & \downarrow \widetilde{\beta}^* \\ \widetilde{Kh}^{k+l+1,2+2k+\mu+1}(D,p) & \xrightarrow{f^*} & \widetilde{Kh}^{k+l+1,2+2k+\mu+1}(D,p'). \end{array}$$

This diagram gives the second part of the statement and concludes the proof. □

Corollary 3.5. *The Lee-Rasmussen spectral sequence for reduced Bar-Natan homology is an invariant of pointed links. Consequently* $\widetilde{BN}^*(D)$ *is an invariant of pointed links. Moreover, for each* $i \in \mathbb{Z}$*, we have a lower bound on the dimension of* $\widetilde{Kh}^{i,*}(D)$*:*

$$\dim(\widetilde{BN}^i(D)) \leq \dim(\widetilde{Kh}^{i,*}(D)).$$

Proof. Let (D,p) and (D',p) be two pointed diagrams that differ by a Reidemeister move away from the basepoint. Denote by $E(D,p)$ and $E(D',p)$ their respective Lee-Rasmussen spectral sequences and let ϕ_R be the quasi-isomorphism associated to that move. From Proposition 3.3, we know that ϕ_R induces a map of spectral sequences and that ϕ_1 is just the map induced in homology, which is an isomorphism. Hence, by Theorem 2.23, for all $r \geq 1$, and for all k,l we have an isomorphism

$$\phi_r : E_r^{k,l}(D,p) \longrightarrow E_r'^{k,l}(D',p),$$

and the induced map in Bar-Natan homology

$$\phi^* : \widetilde{BN}^*(D,p) \longrightarrow \widetilde{BN}^*(D',p)$$

is also an isomorphism.

For the change of basepoint, let (D,p) and (D,p') be two pointed diagrams with the same underlying oriented diagram. Denote by $E(D,p)$ and $E(D,p')$ their respective Lee-Rasmussen spectral sequences. By Proposition 3.3, the map f of Proposition 1.16 induces a map of spectral sequences that is an isomorphism on the E_1-page. By Theorem 2.23, for all $r \geq 1$, and for all k,l we have an isomorphism

$$f_r : E_r^{k,l}(D,p) \longrightarrow E_r'^{k,l}(D,p'),$$

and the induced map in Bar-Natan homology

$$f^* : \widetilde{BN}^*(D,p) \longrightarrow \widetilde{BN}^*(D,p'),$$

is also an isomorphism. Consequently, both the Lee-Rasmussen spectral sequence for $r \geq 1$ and Bar-Natan homology are invariants of pointed links.

For the lower bound, recall that reduced Bar-Natan homology is obtained from the E_∞-page of the Lee-Rasmussen spectral sequence. Indeed, for each $i \in \mathbb{Z}$, we have

$$\widetilde{BN}^i(D) = \bigoplus_{k+l=i} E_\infty^{k,l}.$$

By definition, $E_\infty^{k,l}$ is a quotient of a subspace of $E_1^{k,l} = \widetilde{Kh}^{k+l,2k+\mu+1}(D)$, where μ is the number of components in D modulo 2. In particular, we have an inequality

$$\dim(E_\infty^{k,l}) \leq \dim(\widetilde{Kh}^{k+l,2k+\mu+1}(D)).$$

It follows that we have sequence of inequalities below

$$\begin{aligned} \dim(\widetilde{BN}^i(D)) &= \dim(\bigoplus_{k+l=i} E_\infty^{k,l}) \\ &= \sum_{k+l=i} \dim(\widetilde{Kh}^{i,2k+\mu+1}(D)) \\ &\leq \sum_{j\in\mathbb{Z}} \dim(\widetilde{Kh}^{i,j}(D)). \end{aligned}$$

This concludes the proof. □

Now that we have invariance of Bar-Natan homology, we can focus on other properties of the map $\widetilde{\beta}$. The first we explore is of course the fact that $\widetilde{\beta}^*$ is a cohomology operation.

To do this, we first turn to the relationship between the map $\widetilde{\beta}^*$ and the skein spectral sequence. This will provide us with the naturality of $\widetilde{\beta}^*$ with respect to maps induced by 1-handles. Recall that given an exact triple (D_1, D, D_0), there is an short exact sequence of ungraded chain complexes

$$0 \longrightarrow \widetilde{C}(D_1) \xrightarrow{i} \widetilde{C}(D) \xrightarrow{\pi} \widetilde{C}(D_0) \longrightarrow 0$$

As a sequence of vector spaces over $\mathbb{Z}_2$, it splits: there is an isomorphism

$$\widetilde{C}(D) \cong \widetilde{C}(D_0) \bigoplus \widetilde{C}(D_1).$$

Both complexes $\widetilde{C}(D_0)$ and $\widetilde{C}(D_1)$ are equipped with their own $\widetilde{\beta}$, denoted by $\widetilde{\beta}_0$ and $\widetilde{\beta}_1$ respectively. With respect to the splitting, the map $\widetilde{\beta}$ becomes a matrix

$$\widetilde{\beta} = \begin{pmatrix} \widetilde{\beta}_0 & 0 \\ \delta_\beta & \widetilde{\beta}_1 \end{pmatrix}.$$

This matrix is lower triangular for a simple reason. Let $c \in \chi_D$ be the crossing used to obtain the exact triple. If $A \subset \chi_D, c \in A$ then for $v \in x_\bullet \Lambda V_A \subset \widetilde{C}(D_1)$, we have

$$\widetilde{\beta}(v) = \sum_{\substack{A \subset B \subset \chi_D \\ |A|+1=|B|}} \widetilde{\beta}_{A,B}(v),$$

and all such B's must also contain c, i.e $x_\bullet \Lambda V_B \subset \widetilde{C}(D_1)$. Hence the map $\widetilde{\beta}$ restricted to $\widetilde{C}(D_1)$ is just its counterpart for $\widetilde{C}(D_1)$, namely $\widetilde{\beta}_1$. Let us now assume that $c \notin A$. For $v \in x_\bullet \Lambda V_A \subset \widetilde{C}(D_0)$, we have

$$\widetilde{\beta}(v) = \sum_{\substack{A \subset B \subset \chi_D \\ |A|+1=|B|}} \widetilde{\beta}_{A,B}(v) = \widetilde{\beta}_{A,A\cup\{c\}}(v) + \sum_{\substack{A \subset B \subset \chi_D \setminus \{c\} \\ |A|+1=|B|}} \widetilde{\beta}_{A,B}(v) = \delta_\beta + \widetilde{\beta}_0.$$

The splitting of the sum is due to the fact that there is a unique $B \subset \chi_D, |B| = |A| + 1$, which contains both A and c, that is $B = A \cup \{c\}$. The map $\widetilde{\beta}_0$ is identified exactly as $\widetilde{\beta}_1$ above.

PROPOSITION 3.6. *The map $\widetilde{\beta}$ preserves the skein filtration. Hence it induces a map of skein spectral sequences*

$$\widetilde{\beta}^* : {}^jE_r^{s,t} \longrightarrow {}^{j+2}E_r^{s,t}.$$

Proof. This is mostly a consequence of the discussion above. We want to show that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 \longrightarrow & \widetilde{C}(D_1) & \xrightarrow{i} & \widetilde{C}(D) & \xrightarrow{\pi} & \widetilde{C}(D_0) & \longrightarrow 0 \\ & \big\downarrow \widetilde{\beta}_1 & & \big\downarrow \widetilde{\beta} & & \big\downarrow \widetilde{\beta}_0 & \\ 0 \longrightarrow & \widetilde{C}(D_1) & \xrightarrow{i} & \widetilde{C}(D) & \xrightarrow{\pi} & \widetilde{C}(D_0) & \longrightarrow 0. \end{array}$$

First we show that the square on the left commutes. With respect to the isomorphism $\widetilde{C}(D) \cong \widetilde{C}(D_0) \oplus \widetilde{C}(D_1)$, the inclusion of the subcomplex $\widetilde{C}(D_1)$ into $\widetilde{C}(D)$ can be written as $i(v) =$

$(0, v)$ and it follows that

$$\widetilde{\beta}(i(v)) = \begin{pmatrix} \widetilde{\beta}_0 & 0 \\ \delta_\beta & \widetilde{\beta}_1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\beta}_1(v) \end{pmatrix} = i(\widetilde{\beta}_1(v)),$$

therefore the square commutes. We also need to show that the square on the right commutes. The map π can be explicited as $\pi(v, w) = v$, and it follows that

$$\pi(\widetilde{\beta}(v)) = \pi\left(\begin{pmatrix} \widetilde{\beta}_0 & 0 \\ \delta_\beta & \widetilde{\beta}_1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}\right) = \pi \begin{pmatrix} \widetilde{\beta}_0(v) \\ \delta_\beta(w) \end{pmatrix} = \widetilde{\beta}_0(v) = \widetilde{\beta}_0\left(\pi\begin{pmatrix} v \\ w \end{pmatrix}\right).$$

Therefore the second square commutes and $\widetilde{\beta}$ does indeed induce a map of short exact sequences, as well as a map of the corresponding long exact sequences in homology. □

Corollary 3.7. *The map $\widetilde{\beta}^*$ commutes with the maps induced by 1-handle moves.*

Proof. Let F and F' be two pointed diagrams related by an oriented 1-handle move. By completing the triple, one obtains an exact triple (D_1, D, D_0) such that, up to some shift, the boundary map in the long exact sequence coincides with the one induced by the 1-handle. The map $\widetilde{\beta}^*$ is a morphism of long exact sequences by Proposition 3.6. In particular it commutes with the boundary map. The isomorphism relating the homologies of F and D_0, and F' and D_1 is just the identity, with a grading. So $\widetilde{\beta}^*$ also commutes with that map. Hence it commutes with the map induced by the 1-handle move. □

The map $\widetilde{\beta}^*$ will prove to be a very useful tool to understand the boundary map in skein exact sequences. We shall delay that discussion until a bit later. For now we focus on proving the main theorem of this section.

Theorem 3.8. *The map $\widetilde{\beta}^*$ is a cohomology operation.*

Proof. To prove this statement we need to show that the map $\widetilde{\beta}^*$ commutes with the maps induced by movies. Thus it enough to show that it commutes with the maps induced by 2 consecutive frames of a movie. There are 4 types of such maps (see Chapter 1 Section 1).

The first type is maps induced by Reidemeister moves. This has already been proved in Corollary 3.3.

The second type is maps induced by oriented 1-handle moves. This is the content of Corollary 3.7.

The two remaining types are maps induced by 0 and 2-handle moves. Let us first describe the map $\widetilde{\beta}^*$ for a diagram $D \sqcup D_u$, where D_u is the trivial diagram for the unknot. Note that D contains the pointed component. The adjunction of a trivial component doesn't create any crossing, therefore the map $\widetilde{\beta}^*$ acts on this new component as the identity. At the level of the chain complexes, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{C}(D \sqcup D_u) & \xrightarrow{\widetilde{\beta}} & \widetilde{C}(D \sqcup D_u) \\ \downarrow & & \downarrow \\ \widetilde{C}(D) \otimes C(D_u) & \xrightarrow{\widetilde{\beta} \otimes 1} & \widetilde{C}(D) \otimes C(D_u) \end{array}$$

Here the vertical maps are is the identity. For the 2-handle move we are interested in the following diagram,

$$\begin{array}{ccc}
\widetilde{Kh}^{i,j}(D \sqcup D_u) & \xrightarrow{\widetilde{\beta}^*} & \widetilde{Kh}^{i+1,j+2}(D \sqcup D_u) \\
\downarrow & & \downarrow \\
\widetilde{Kh}^{i,j-j_u}(D) \otimes \widetilde{Kh}^{0,j_u}(D_u) & \xrightarrow{\widetilde{\beta}^* \otimes 1} & \widetilde{Kh}^{i+1,j-j_u+2}(D) \otimes \widetilde{Kh}^{0,j_u}(D_u) \\
\downarrow \phi & & \downarrow \phi \\
\widetilde{Kh}^{i,j+1}(D) & \xrightarrow{\widetilde{\beta}^*} & \widetilde{Kh}^{i+1,j+3}(D)
\end{array}$$

The top square commutes because it is the homology version of the diagram above. The bottom square commutes thanks to the definition of ϕ. Indeed the image of ϕ depends entirely on j_u, which is preserved when going right. Thus the big square commutes. For the 0-handle move, we are interested in similar diagram:

$$\begin{array}{ccc}
\widetilde{Kh}^{i,j}(D) & \xrightarrow{\widetilde{\beta}^*} & \widetilde{Kh}^{i+1,j+2}(D) \\
\downarrow \phi & & \downarrow \phi \\
\widetilde{Kh}^{i,j}(D) \otimes \widetilde{Kh}^{0,1}(D_u) & \xrightarrow{\widetilde{\beta}^* \otimes 1} & \widetilde{Kh}^{i+1,j+2}(D) \otimes \widetilde{Kh}^{0,1}(D_u) \\
\downarrow & & \downarrow \\
\widetilde{Kh}^{i,j+1}(D \sqcup D_u) & \xrightarrow{\widetilde{\beta}^*} & \widetilde{Kh}^{i+1,j+3}(D \sqcup D_u)
\end{array}$$

The argument is also similar: the bottom square commutes as the homology version of the first diagram in the proof. The top square commutes because $\phi(v) = v \otimes 1$. Therefore the big square also commutes. This concludes the proof. □

Now that we know that $\widetilde{\beta}^*$ is a cohomology operation, we return to its main use: computing pieces of the boundary map with limited information. The lemma that follows is essentially a corollary of Proposition 3.6. To make the statement intelligible we need a few definitions.

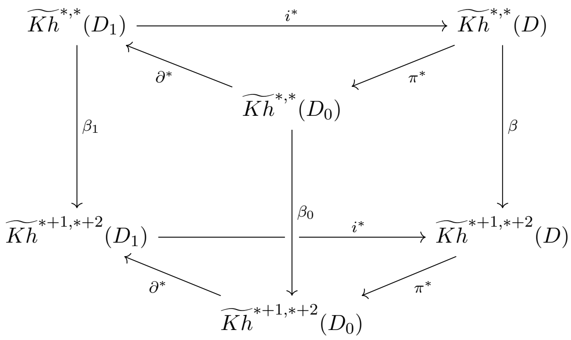
Definition 3.1. The map $\widetilde{\beta}^*$ allows us to group the elements of $\widetilde{Kh}^{*,*}(D)$ into three families.

- The *β-cycles* are elements $v \in \widetilde{Kh}^{*,*}(D)$ such that $\widetilde{\beta}^*(v) = 0$.
- The *β-boundaries* are elements $v \in \widetilde{Kh}^{*,*}(D)$ such that $v \in \mathrm{Im}(\widetilde{\beta}^*)$.

Lemma 3.9. *Let (D_1, D, D_0) be an exact triple and i^*, π^*, ∂^* denote the maps in the long exact sequence. Then the following properties hold.*

(i) If $v \in \widetilde{Kh}^{*,*}(D_0)$ is a β-cycle, then so is $\partial^*(v) \in \widetilde{Kh}^{*,*}(D_1)$.
(ii) If $v \in \widetilde{Kh}^{*,*}(D_1)$ is a β-cycle, then so is $i^*(v) \in \widetilde{Kh}^{*,*}(D)$.

Proof. The main tool in the proof is the fact that $\widetilde{\beta}^*$ induces a map of exact triangles, i.e. $\widetilde{\beta}^*$ commutes with the maps i^*, q^* and ∂^* in the long exact sequence. More precisely we have, shifts omitted, a commutative diagram for the total triangle:



For (i), let $v \in \widetilde{Kh}^{*,*}(D_0)$ be a β-cycle, i.e. $\widetilde{\beta}_0^*(v) = 0$. Therefore we have

$$\widetilde{\beta}^*(\partial^*(v)) = \partial^*(\widetilde{\beta}_0^*(v)) = 0,$$

so $\partial^*(v)$ is a β-cycle.

For (ii) let $v \in \widetilde{Kh}^{*,*}(D_1)$ be a β-cycle, i.e. $\widetilde{\beta}_1^*(v) = 0$. Therefore we have

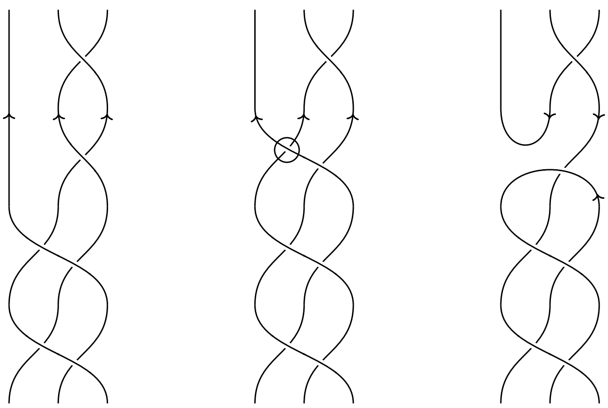
$$\widetilde{\beta}^*(i^*(v)) = i^*(\widetilde{\beta}_1^*(v)) = 0,$$

so $i^*(v)$ is a β-cycle.

□

As an example, we compute the homology of a diagram closely related to the standard braid that closes into $T_{3,3}$.

EXAMPLE 3.10. Let D be the closure of the braid with the circled crossing below. We assume the basepoint to be on any closing strand.



Consider the associated exact triple (D_1, D, D_0). All the crossings in D are negative so $n_-(D) = 7$. The diagram D_1 is a diagram for $T_{2,5}$ and D_0 is a diagram for the unknot U. For the latter, we choose to orient the leftmost strand up. With this orientation $n_-(D_0) = 3$, hence

$$w_- = n_-(D_0) - n_-(D) = -4.$$

Consider the long exact sequence for $\delta = -3$. It is non trivial only at $i = -4$, so that

$$\widetilde{Kh}^{i-w_-}_{\delta-w_--1}(D_0) = \widetilde{Kh}^0_0(U) = \mathbb{Z}_2\{x_u\}.$$

The sequence is thus

$$0 \longrightarrow \widetilde{Kh}^{-4}_{-3}(D) \longrightarrow \mathbb{Z}_2\{x_u\} \xrightarrow{\partial^*} \widetilde{Kh}^{-3}_{-5}(T_{2,5}) = \mathbb{Z}_2\{a_3\} \longrightarrow \widetilde{Kh}^{-3}_{-4}(D) \longrightarrow 0.$$

The only question to answer is: do we have $\partial^*(x_u) = a_3$? In Example 3.2, we showed that generators of the homology of a 2-stranded torus link were β-cycles for even homological degrees and not β-cycles for odd homological degrees. Since both the unknot and $T_{2,5}$ belong to this family, we have that x_u is a β-cycle and a_3 is not a β-cycle. By lemma 3.9 (i), we must have $\partial^*(x_u) \neq a_3$, so $\partial^*(x_u) = 0$.

$$\widetilde{Kh}^i_\delta(D) = \begin{cases} \mathbb{Z}_2 & \text{if } (i,\delta) = (-4,-3). \\ \mathbb{Z}_2 & \text{if } \delta = -5 \text{ and } i = 0,-2,-3,-4,-5. \\ 0 & \text{otherwise.} \end{cases}$$

Since $\partial^* = 0$, we may also choose to express this as:

$$\widetilde{Kh}^*_*(D) = \widetilde{Kh}^*_*(U)[-4,-3] \bigoplus \widetilde{Kh}^*_*(T_{2,5})[0,-1].$$

This concludes the example.

We conclude this section by discussing connected sums and how the cohomology operation behaves with respect to this operation

Proposition 3.11. *Let D be a pointed diagram and D' be an oriented link. Then the following diagram is commutative.*

$$\begin{array}{ccc} \widetilde{Kh}(D) \otimes \widetilde{Kh}(D') & \xrightarrow{\widetilde{\beta}^*_D \otimes 1 + 1 \otimes \widetilde{\beta}^*_{D'}} & \widetilde{Kh}(D) \otimes \widetilde{Kh}(D') \\ \downarrow S^* & & \downarrow S^* \\ \widetilde{Kh}(D\sharp D') & \xrightarrow{\widetilde{\beta}^*_{D\sharp D'}} & \widetilde{Kh}(D\sharp D') \end{array}$$

Proof. ◻

3. Further remarks

In this slightly informal section, we discuss the impact of the existence of such a cohomology operation. We begin by showing that our cohomology operation is different from Lipschitz-Sarkar's Steenrod square and Putyra-Shumakovitch's Bocksetin homomorphism. We then describe how it enriches reduced Khovanov homology. Finally we discuss Lee theory for rational homology.

The already existing cohomology operations can be described as follows. For a link L, we have:

(1) The Steenrod squares $Sq^k : Kh^{i,j}(L) \longrightarrow Kh^{i+k,j}(L)$.
(2) The two Bockstein homomorphisms: $\beta_e, \beta_o : Kh^{i,j}(L) \longrightarrow Kh^{i+1,j}(L)$.

Because our map $\widetilde{\beta}^*$ has bidegree $(1,2)$, it is necessarily different than these others.

This cohomology operation enriches the reduced Khovanov homology. Set R to be the polynomial ring

$$R := \frac{\mathbb{Z}_2[\beta]}{\beta^2 = 0}$$

Then for any diagram D, R acts on the reduced homology by $\widetilde{\beta}^*$, in the following sense:

$$\begin{array}{rccc} \Phi : R \otimes \widetilde{Kh}^{*,*}(D) & \longrightarrow & \widetilde{Kh}^{*,*}(D) \\ \beta \otimes v & \longmapsto & \widetilde{\beta}^*(v). \end{array}$$

The reduced Khovanov homology becomes a module over R. Having additional structure usually leads to a stronger invariant, however we have found no such example where the module structure distinguishes links with the same reduced homology.

Note that there is a secondary homology $\widetilde{Kh}_\beta^{*,*}(D)$, introduced by Turner [**Tur06**] and defined by

$$\widetilde{Kh}_\beta^{*,*}(D) := H_*(\widetilde{Kh}^{*,*}(D), \beta).$$

This homology is just the E_2-page of the Lee-Rasmussen spectral sequence.

This cohomology operation also has a defect. Because its degree is not restrictive enough, it is not always possible to compute easily the Lee-Rasmussen spectral sequence. The issue is the following. Let us assume that we have an exact triple (D_1, D, D_0). We consider the long exact sequence in Bar-Natan homology. If we assume the boundary map to be non-zero, i.e

$$\partial_{\widetilde{BN}} : \widetilde{BN}^i(D_0) \longrightarrow \widetilde{BN}^{i+1}(D_1)$$

is non-zero on a quantum degree monogeneous element v, then, since $d_{\widetilde{BN}} = d + \widetilde{\beta}$, we have

$$\partial^*(v) + \partial_{\widetilde{\beta}}(v) \neq 0.$$

Both maps have different degrees, therefore either $\partial^*(v)$ is non zero or $\partial_{\widetilde{\beta}}(v)$ is non-zero. Both cases are equally possible and we don't know how to choose. A possible way to adress this issue would be to obtain an explicit description of the secondary homology $\widetilde{Kh}_\beta^{*,*}(D)$.

CHAPTER 4

The homology of 3-stranded torus links

This chapter is dedicated to the computation of the Khovanov homology $\widetilde{Kh}^{*,*}(T_{3,q})$ for any negative torus link $T_{3,q}$. Quite a lot is known about their non reduced homologies, for other coefficients than $\mathbb{Z}_2$. Turner [**Tur08**] and Stošić [**Sto09**] independently computed the rational homology. Simultaneously, Turner also showed that the integer homology does not contain p-torsion, for p prime, $p \neq 2$ [**Tur08**]. Later, Gillam [**Gil12**] identified some of the 2-torsion in the integer version. His work splits the torsion into two groups: bigradings where there is 2-torsion, and bigradings where it *might or might not* appear. Therefore his work does not provide a complete description of the mod 2 homology. Though they use different techniques or versions of Khovanov homology, the overall strategy of their proofs is the same: an induction over q with three steps, from $q = 3N - 1$ to $q = 3N$ from $q = 3N$ to $q = 3N + 1$, and from $q = 3N + 1$ to $q = 3N + 2$. The fact that this cyclic proof works suggest that these steps are determined by the cases $N = 1$. We present a new approach which we believe will generalize to p-stranded torus links.

This chapter is divided into three sections. First, we specialize our study of Turner's skein spectral sequence to particular types of spectral sequences and use $T_{3,4}$ as an example. In a second part, we study two families of spectral sequences for $T_{3,3N}$ and $T_{3,3N+1}$ respectively. Our approach is to use a particular family of movies to show that the differentials in each family are entirely determined by the ones for $N = 1$. Finally, we combine the results of the second part to achieve a complete computation of the mod 2 homology of $T_{3,q}$.

1. Technical preliminaries

In this section, we introduce a special type of spectral sequence, that appears naturally in the context of the computation of $\widetilde{Kh}^{*,*}(T_{3,q})$. Along the way, as a warm-up for Section 2, we study a skein spectral sequence for $T_{3,4}$.

Recall that, by definition, the skein spectral sequence has a finite number of non-zero columns, whose indices range from 0 to m, where m is the number of crossings we choose in our original diagram D. More precisely we will say that a spectral sequence has $m+1$ columns if it satisfies

$$E_1^{m,*} \neq 0, \text{ and } E_1^{i,*} = 0 \text{ if } i < 0 \text{ and } i > m.$$

We open this section with some terminology related to spectral sequences, and the definition of a special type of spectral sequence.

DEFINITION 4.1. Let E be a convergent spectral sequence with $m + 1$ columns. We say that E *converges* at E_{r_0} if all differentials d_r are zero for $r \geq r_0$. If the E_{r_0}-page admits either a single non-zero column or a single non-zero row, we say that the spectral sequence *collapses* at E_{r_0}. We say that E is *nice* if it satisfies the following two conditions:

(1) for any $r \geq 1$, the differential $d_r : E_r^{m-r,*} \longrightarrow E_r^{m,*-1+r}$ is zero, and

(2) any non-zero differential $d_r^{s,t}$ is an isomorphism.

REMARK 4.1. *In the context of torus links in Khovanov homology, Stosic's Theorem* [**Sto09**] *tells us that if we choose enough crossings, then the associated skein spectral sequence verifies at least condition* (1).

The notion of nice spectral sequences is very restrictive, as we describe below. For example, the last column of such spectral sequences always survives to the E_∞-page.

LEMMA 4.2. *Let E be a nice spectral sequence with $m+1$ columns. For any page E_r, $r \geq 1$, any column $s \in \{0, \dots, m-1\}$, and any row $t \in \mathbb{Z}$, we consider the incoming and outgoing differentials at $E_r^{s,t}$:*

$$E_r^{s-r,t-r+1} \xrightarrow{d_r^{s-r,t-r+1}} E_r^{s,t} \xrightarrow{d_r^{s,t}} E_r^{s+r,t-1+r}$$

For every r, s, t, exactly one the following occurs:

(i) $d_r^{s-r,t-r+1} = d_r^{s,t} = 0$ *and* $E_{r+1}^{s,t} = E_r^{s,t}$.
(ii) $d_r^{s,t} = 0$, $d_r^{s-r,t-r+1}$ *is isomorphism, and* $E_{r+1}^{s,t} = E_{r+1}^{s-r,t-r+1} = 0$.
(iii) $d_r^{s-r,t-r+1} = 0$, $d_r^{s,t}$ *is an isomorphism and* $E_{r+1}^{s,t} = E_{r+1}^{s+r,t-1+r} = 0$.

Moreover $E_\infty^{m,*} = E_1^{m,*}$.

Proof. At every page r of a spectral sequence, we have $d_r \circ d_r = 0$. If it is nice, two consecutive arrows cannot both be non-zero, because of the surjectivity implied by condition (2). This yields the three possible configurations: both incoming and outgoing differentials zero or exactly one non-zero, which is then an isomorphism by (2). The characterizations of $E_{r+1}^{s,t}$ follow from the definition of a spectral sequence: $E_{r+1}^{s,t}$ is the homology at $E_r^{s,t}$. For the last column E^{m*}, we are in the configuration (i) for every page E_r by condition (1) in the definition of nice spectral sequence, so the claim holds. ☐

Before we study some examples, let us remind the reader how to reconstruct $\widetilde{Kh}^{i,j}(D)$ from the E_∞-page of a given skein spectral sequence ${}^jE(D)$ with respect to m crossings for a fixed quantum grading j. Since we work with vector spaces, there is no extension problem so we have an isomorphism

$$\widetilde{Kh}^{i,j}(D) \cong \bigoplus_{s+t=i} {}^jE_\infty^{s,t}.$$

Keeping this fact in the back of our mind, let us now present an example of a nice spectral sequence (along with many converging and collapsing at E_1). However, before we do this example, let us mention that we actually needed a computer for data that is used as a starting point in Example 4.3. Indeed, when one tries to compute the homology of $T_{3,4}$ the defect of $\widetilde{\beta}^*$ mentionned in Chapter 3 Section 3 appears. The computer program "JavaKh" we used is part of the Mathematica package "KnotTheory" available on Bar-Natan's Knot Atlas [**kno**]. It was originally written by Jeremy Green.

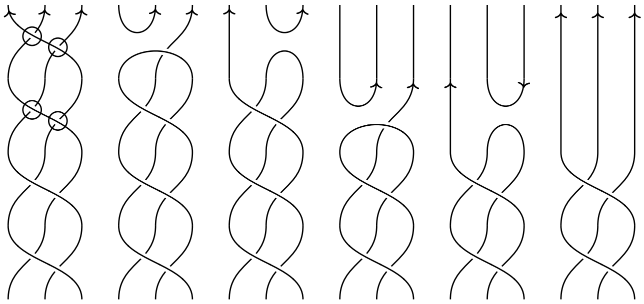


FIGURE 15. The choice of crossings and the intermediate diagrams for the skein spectral sequence in Example 4.3. From left to right: $D, \widetilde{D}_{(0)}, \widetilde{D}_{(1)}, \widetilde{D}_{(2)}, \widetilde{D}_{(3)}$ and $D_{(4)}$.

EXAMPLE 4.3. In this example, we study a skein spectral sequence for $T_{3,4}$. Using the "JavaKh" program, we found that the homology $\widetilde{Kh}^{*,*}(T_{3,4})$ is given by the following grid:

−5	−4	−3	−2	−1	0	i / j
					1	-6
						-8
			1			-10
	1	1				-12
						-14
1						-16

Equivalently, we have the quantum graded Poincaré polynomial:

$$\begin{aligned} P(q,t)(T_{3,4}) &= q^{-6} + q^{-10}t^{-2} + q^{-12}t^{-3} + q^{-12}t^{-4} + q^{-16}t^{-5} \\ &= q^{-6}(1 + q^{-4}t^{-2} + q^{-6}t^{-3} + q^{-10}t^{-5}) + q^{-12}t^{-4}. \end{aligned}$$

The δ-graded Poincaré polynomial is then the following:

$$P_\delta(q,t)(T_{3,4}) = q^{-6}(1 + t^{-2} + t^{-3} + t^{-5}) + q^{-4}t^{-4}$$

So we know the answer. We will use this to examine the possible behavior of the differentials. We choose to run our skein spectral sequence with respect to the 4-crossings indicated in Figure 15. Therefore, the sequence will have 5 columns. Let us begin with identifiying the isotopy class of the intermediate diagrams $\widetilde{D}_{(k)}$ for $k = 0,1,2,3$ and $D_{(4)}$, as well as the number of negative crossings $\widetilde{n}_-(\widetilde{D}_{(k)})$ of $\widetilde{D}_{(k)}$ with respect to the prescribed orientations. We have the following:

$$\begin{array}{lll} \widetilde{D}_{(0)} \sim U, & \text{and} & \widetilde{n}_-(\widetilde{D}_{(0)}) = 3, \\ \widetilde{D}_{(1)} \sim U \sqcup U, & \text{and} & \widetilde{n}_-(\widetilde{D}_{(1)}) = 2, \\ \widetilde{D}_{(2)} \sim U \sqcup U, & \text{and} & \widetilde{n}_-(\widetilde{D}_{(2)}) = 2, \\ \widetilde{D}_{(3)} \sim U, & \text{and} & \widetilde{n}_-(\widetilde{D}_{(3)}) = 1, \\ D_{(4)} \sim T_{3,2}. & & \end{array}$$

Since we rely only on negative crossings, we use the spectral sequence of Corollary 2.27. We compute the shifts $\alpha_k = n_-(\widetilde{D}_{(k)}) - n_-(T_{3,4})$ and $\beta_k = 3\alpha_k + 1$ associated to that sequence below:

$$\begin{aligned} \alpha_0 &= 3-8=-5, & \beta_0 &= -15+1=-14,\\ \alpha_1 &= 2-8=-6, & \beta_1 &= -18+1=-17,\\ \alpha_2 &= 2-8=-6, & \beta_2 &= -18+1=-17,\\ \alpha_4 &= 1-8=-7, & \beta_3 &= -21+1=-20. \end{aligned}$$

For fixed $j \in \mathbb{Z}$ and $k \in \{0,1,2,3\}$, we have

$$^jE_1^{k,*} = \widetilde{Kh}^{*+\alpha_{k+1},j+\beta_{k+1}-2k}(\widetilde{D}_{(k)}),$$

Note that $^jE_1^{k,*}$ forms the k'th column of the E_1-page. The last column ($k=4$) has the form

$$^jE_1^{4,*} = \widetilde{Kh}^{*+4,j+4}(D_{(4)}).$$

In our case, for a generic $j \in \mathbb{Z}$, the E_1-page is given by:

$$\begin{aligned} {}^jE_1^{0,t} &= \widetilde{Kh}^{t+5,j+14}(U),\\ {}^jE_1^{1,t} &= \widetilde{Kh}^{t+6,j+15}(U\sqcup U),\\ {}^jE_1^{2,t} &= \widetilde{Kh}^{t+6,j+13}(U\sqcup U),\\ {}^jE_1^{3,t} &= \widetilde{Kh}^{t+7,j+14}(U),\\ {}^jE_1^{4,t} &= \widetilde{Kh}^{t+4,j+4}(T_{3,2}). \end{aligned}$$

All links involved here are simple and we know their homologies, so we already have that if $j \notin \{-6,-10,-12,-14,-16\}$, then the E_1-page is zero. If $j \in \{-6,-10\}$, only the fifth column contributes, so the spectral sequence collapses at E_1 and we have an isomorphism

$$\widetilde{Kh}^{*,j}(T_{3,4}) \cong \widetilde{Kh}^{*,j}(T_{3,2})[0,-4].$$

We present the E_1-pages for the remaining cases $j \in \{-12,-14,-16\}$ as tables with rows indexed by t and columns indexed by s. We only fill in the dimensions of non zero dimensional spaces.

	0	1	2	3	4
−6			1		
−7					1

(A) $j=-12$.

	0	1	2	3	4
−5	1				
−6		1	1		
−7				1	

(B) $j=-14$.

	0	1	2	3	4
−6		1			

(C) $j=-16$.

For $j=-16$, the sequence collapses at E_1. For $j=-12$, there is one potentially non zero differential

$$d_2^{2,-6} : E_2^{2,-6} \longrightarrow E_2^{4,-7}.$$

If it is non-zero, then it is an isomorphism since both spaces are one dimensional and it follows that $\widetilde{Kh}^{*,-12}(T_{3,4}) = 0$, which is not the result we should have. Thus $d_2^{2,-6} = 0$.

We complete our study with the case $j = -14$. There are two non-trivial diagonals, parametrized by $s+t = -4$ and $s+t = -5$, that potentially contribute to $\widetilde{Kh}^{-4,-14}(T_{3,4})$ and $\widetilde{Kh}^{-5,-14}(T_{3,4})$ respectively. From the explicit description of $\widetilde{Kh}^{*,*}(T_{3,4})$ we know that

$$\dim(\widetilde{Kh}^{-4,-14}(T_{3,4})) = \dim(\widetilde{Kh}^{-5,-14}(T_{3,4})) = 0.$$

Hence nothing can survive. There are two possible configurations of differentials that annihilate everything in the grid:

$$d_1^{1,-6} \text{ and } d_3^{0,-5} \text{ both non zero,}$$

or

$$d_2^{0,-5} \text{ and } d_2^{1,-6} \text{ both non zero.}$$

Both yield the same result and we do not claim to know which of the two configurations actually occurs. Nonetheless, in both cases the differentials are isomorphisms: the spectral sequence is nice. This concludes the example.

2. Relating families

This section is dedicated to a new approach for computing homology of torus links. The goal of this method is to provide, for any $N \geq 1$, a direct sum decomposition of $\widetilde{Kh}^{*,*}(T_{3,3N})$ and $\widetilde{Kh}^{*,*}(T_{3,3N+1})$ with $\widetilde{Kh}^{*,*}(T_{3,3N-1})$ as a summand. We will define two families of skein spectral sequences: one for $T_{3,3N}$ and one for $T_{3,3N+1}$. Using maps of skein spectral sequences induced by movies, we will show that the differentials are, for both families, entirely determined by the case $N = 1$ - which we have already studied in Example 2.26 (for $T_{3,3}$) and the previous section respectively (for $T_{3,4}$).

2.1. Relating $T_{3,3N}$ and $T_{3,3N-1}$. Throughout the first part of this section, we will consider a skein spectral sequence for each $T_{3,3N}$. Let $N \geq 1$ and ${}^jE(T_{3,3N})$ be the skein spectral sequence associated to the 2 circled crossings in Figure 16. Note that for $N = 1$, we recover the sequence of Example 2.26, which converges at E_1. The intermediate diagrams $\widetilde{D}_{(k)}$ for $k = 0, 1$ and $D_{(2)}$, are also given in Figure 16.

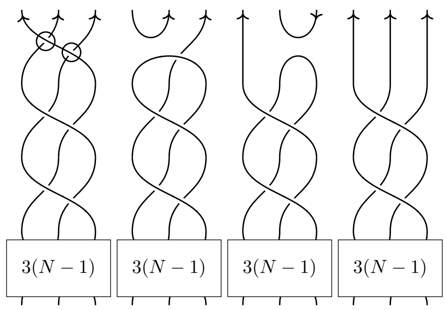


FIGURE 16. The choice of crossings and the intermediate diagrams in the spectral sequence for $T_{3,3N}$. From left to right: $D, \widetilde{D}_{(0)}, \widetilde{D}_{(1)}$ and $D_{(2)}$.

Their isotopy classes, as well as the number of negative crossings $n_-(\widetilde{D}_{(k)})$ of $\widetilde{D}_{(k)}$ with respect to the chosen orientations in Figure 16, can be described as follows:

$$\begin{aligned} &\widetilde{D}_{(0)} \sim U \sqcup U, && \text{and} \quad n_-(\widetilde{D}_{(0)}) = 2N,\\ &\widetilde{D}_{(1)} \sim U, && \text{and} \quad n_-(\widetilde{D}_{(1)}) = 2N-1,\\ &D_{(2)} \sim T_{3,3N-1}. && \end{aligned}$$

All crossings are negative so we use the spectral sequence of Corollary 2.27. We compute the shifts $\alpha_k = n_-(\widetilde{D}_{(k)}) - n_-(T_{3,3N})$ and $\beta_k = 3\alpha_k + 1$ associated to that sequence below:

$$\begin{array}{ll} \alpha_0 = 2N - 6N = -4N, & \beta_0 = -12N + 1, \\ \alpha_1 = 2N - 1 - 6N = -4N - 1, & \beta_1 = -12N - 3 + 1 = -12N + 2, \end{array}$$

For a generic $j \in \mathbb{Z}$, the E_1-page is then:

$$\begin{array}{rcl} {}^jE_1^{0,t} &=& \widetilde{Kh}^{t+4N,j+12N+1}(U \sqcup U), \\ {}^jE_1^{1,t} &=& \widetilde{Kh}^{t+4N+1,j+12N+2}(U), \\ {}^jE_1^{2,t} &=& \widetilde{Kh}^{t+2,j+2}(T_{3,3N-1}). \end{array}$$

For these spectral sequences, if $j \notin \{-12N-2, -12N\}$, only the third column contributes so it collapses at E_1. For these values of j, we have an isomorphism

$$\widetilde{Kh}^{*,j}(T_{3,3N}) \cong \widetilde{Kh}^{*,*}(T_{3,3N-1})[0,-2].$$

For the other values of j, we will show that the corresponding spectral sequence converges at E_1. In order to do that, we must first develop a way to relate these spectral sequences for increasing N. The idea underlying the strategy we propose is roughly: "if one adds a full twist to $T_{3,q}$, thus obtaining $T_{3,q+3}$, then the differentials in a skein spectral sequences for $T_{3,q}$ are the same as those in a sequence for $T_{3,q+3}$ with respect to the same crossings".

Let us be more precise. Given a pointed diagram (D, p) with a choice of m crossings and another diagram D', we have from Proposition 2.30 a very simple description of ${}^jE(D \sharp D')$, as the spectral sequence ${}^jE(D, D')$ with E_1-page

$$\bigoplus_{j=j_1+j_2} {}^{j_1}E(D) \otimes \widetilde{Kh}^{*,j_2}(D').$$

We specialize this spectral sequence to $D = T_{3,3N}$ (and later $D = T_{3,3N+1}$) and $D' = T_{3,3}$. Starting from $D \sharp D'$, we can produce movies with last frame $T_{3,3N+3}$ that realize the addition of a full twist. We define two maps of spectral sequences: the *fusion map*

$$\Sigma_L : {}^jE(T_{3,q} \otimes T_{3,3}) \cong {}^jE(T_{3,q} \sharp T_{3,3}) \longrightarrow {}^{j-2}E(T_{3,q+3}$$

induced by the 3-frames movie on the left in Figure 17 and the *right fusion map*

$$\Sigma_R : {}^jE(T_{3,q} \otimes T_{3,3}) \cong {}^jE(T_{3,q} \sharp T_{3,3}) \longrightarrow {}^{j-2}E(T_{3,q+3}$$

induced by 3-frames movie on the right in the same Figure.

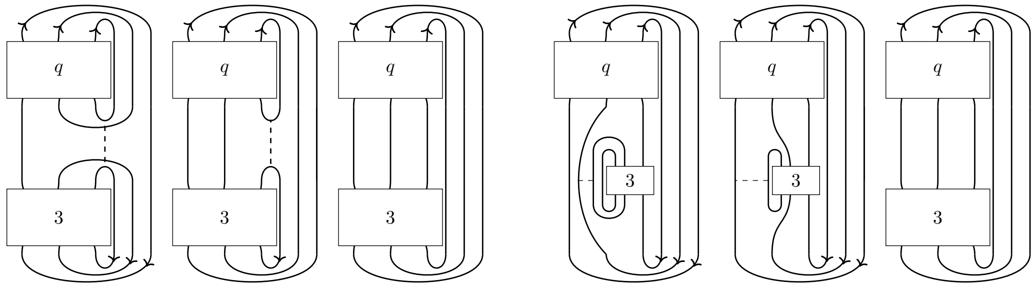
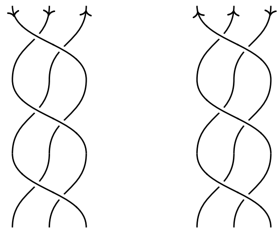


FIGURE 17. On the left, the usual fusion movie for 3-stranded torus links. On the right, the right fusion movie. The 1-handles are indicated by a dashed line. They will be used for $q = 3N, 3N + 1$.

Using Lemma 2.29, we are able to understand the restriction of the fusion and right fusion maps on each column.

The movies in Figure 17 are composition of 1-handle moves so Lemma 2.29 applies. The $\widetilde{D}_{(k)}$ may depend on orientation choices so care is needed when treating the restricted maps featured at the end of the lemma. We need two variations of $T_{3,3}$, denoted by P and P, as the closure of the braids pictured below. Note that for P', the closure will be different than the usual one: we close the strands with boundary $(0,1)$ and $(1,1)$ to the left, and the strand with boundary $(2,1)$ to the right (see Figure 17, bottom).



P $\quad$ P'

These two braids differ from $T_{3,3}$ by reversing the orientation of only one component. Any choice of component in $T_{3,3}$ has total linking number $2l = -4$, therefore we have by Proposition 2.10:

$$\widetilde{Kh}^{*,*}(P)[-4,-12] \cong \widetilde{Kh}^{*,*}(T_{3,3}) \cong \widetilde{Kh}^{*,*}(P')[-4,-12].$$

We have included the grid for $T_{3,3}$ on the left, and that for both P, P' on the right below.

−4	−3	−2	−1	0	i / j
				1	-4
					-6
		1			-8
1	1				-10
2					-12

0	1	2	3	4	i / j
				1	8
					6
		1			4
1	1				2
2					0

We turn now to our study of the restricted maps. We first consider the map for the first column. The corresponding movie induced by $\widetilde{D}_{(1)}$ is pictured in Figure 18.

Lemma 4.4. *For any $N \geq 1$, the map $\Phi^{N,2}$ induced by the movie in Figure 18*

$$\Phi^{N,2} : \widetilde{Kh}^{0,j_\pm}(U \sqcup U) \otimes \widetilde{Kh}^{0,2}(P) \longrightarrow \widetilde{Kh}^{0,j_\pm}(U \sqcup U)$$

where $j_\pm = \pm 1$ is onto.

Proof. The isotopy class of the diagrams in the movie does not depend on N. For any $N \geq 1$, denote by M_i^N the ith frame of the movie M. Then we have

$$M_1^N \sim P \sqcup U \sim P \sharp (U \sqcup U),\ M_2^N \sim U \sqcup U \sqcup U,\ M_2^N \sim U \sqcup U.$$

The map $\Phi^{N,2}$ is defined as the composition:

$$\Phi^{N,2} : \widetilde{Kh}^{*,*}(U) \otimes \widetilde{Kh}^{*,*}(P) \xrightarrow{\varphi_1} \widetilde{Kh}^{*,*-1}(U \sqcup U) \xrightarrow{\varphi_2} \widetilde{Kh}^{*,*-2}(U).$$

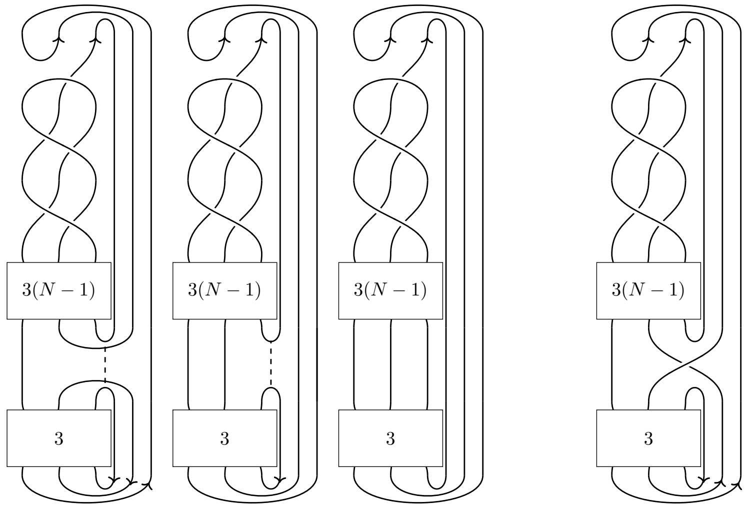


FIGURE 18. The movie that induces the restriction of Σ_L to the column $E_1^{0,*}(T_{3,3N})$. On the right, the completed diagram for the first 1-handle.

It is enough to show that both φ_1 and φ_2 are surjective. The second map φ_2 is surjective as it realizes a connected sum. The first map φ_1 is studied by completing the triple. We reverse the orientation of the unknot part and replace the strands by a positive crossing. The resulting diagram D^N is given in Figure 18 and for any $N \geq 1$, one checks easily that $D^N \sim T_{2,4} \sqcup U$. The dimensions of the associated homologies are related by the formula:

$$\dim(\widetilde{Kh}(P \sqcup U)) = \dim(\widetilde{Kh}(T_{3,3} \sqcup U)) = \dim(\widetilde{Kh}(U \sqcup U \sqcup U)) + \dim(\widetilde{Kh}(T_{2,4} \sqcup U)),$$

where the first equality is a consequence of Proposition 2.10. Therefore by Lemma 2.12, the induced map is surjective. For now, the map is surjective as a whole. We also need to check that the degrees match. The map is given by

$$\Phi^{N,2} : \widetilde{Kh}^{i,j}(U) \otimes \widetilde{Kh}^{i',j'}(P) \longrightarrow \widetilde{Kh}^{i'+i,j+j'-2}(U)$$

and the homology of $U \sqcup U$ is supported in bidegrees $(0, j_\pm = \pm 1)$. Therefore we have the system $i = 0, i + i' = 0$, so we necessarily have $i' = 0$. At $i' = 0$, we have $\widetilde{Kh}^{0,j'}(P) \neq 0$ if and only if $j' = 2$. For the quantum degrees, we have the system $j' = 2, j' + j - 2 = j_\pm$. Thus we have $j = j_\pm$ and the degrees match as claimed. This concludes the proof. □

As they are now, the map induced by the fusion map has domain of the form $A \otimes \widetilde{Kh}^{*,*}(P)$. The map we are interested in is the same, but with domain of the form $A \otimes \widetilde{Kh}^{*,*}(T_{3,3})$. Therefore, we must take into account the reversal of orientation that occurs on the full twist in Figure 4.4 when compared to Figure 17 (left hand movie). In order to understand the induced map at the E_1-page of our skein spectral sequence, we must also add the shifts related to the columns themselves.

Since both P and P' differ from $T_{3,3}$ by changing the orientation of one strand, we can *regrade* the map $\Phi^{N,2}$. We have

$$\widetilde{Kh}^{*,*}(P)[-4,-12] \cong \widetilde{Kh}^{*,*}(T_{3,3}) \cong \widetilde{Kh}^{*,*}(P')[-4,-12].$$

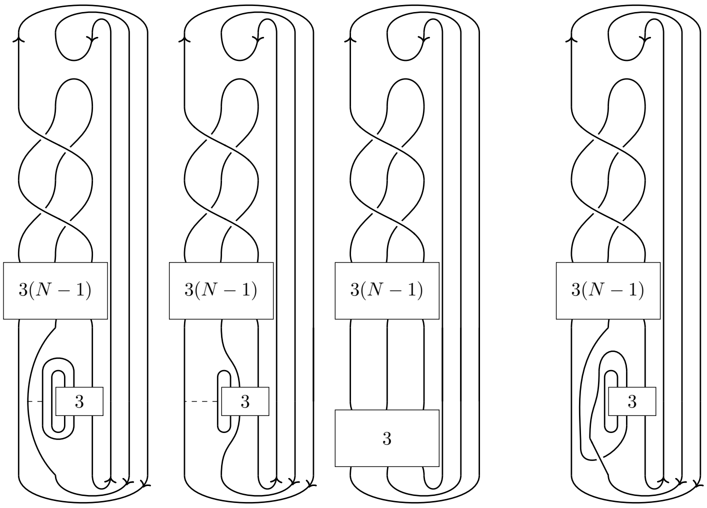


FIGURE 19. The movie that induces the restriction of Σ_R to the column $E_1^{1,*}(T_{3,3N})$. On the right, the completed diagram for the first 1-handle.

Let us re-grade the map

$$\Phi^{N,2} : \widetilde{Kh}^{0,j_\pm}(U \sqcup U) \otimes \widetilde{Kh}^{0,2}(P) \longrightarrow \widetilde{Kh}^{0,j_\pm}(U \sqcup U),$$

so that it coincides with the restrictions of Σ_L. We shift the P factor by $[-4, -12]$, the left hand side $U \sqcup U$ factor by $[-4N, -12N - 1]$, performing a total shift of $[-4(N+1), -12(N+1) - 1]$ then applied to the right hand side $U \sqcup U$ factor. The map with degrees then becomes

$$\begin{array}{c}\widetilde{Kh}^{-12N,-12N-1+j_\pm}(U \sqcup U)[-4N, -12N-1] \otimes \widetilde{Kh}^{-4,-10}(T_{3,3}) \\ \Phi^{N,2}\Big\downarrow \\ \widetilde{Kh}^{-4(N+1),-12(N+1)-1+j_\pm}(U \sqcup U)[-4(N+1), -12(N+1)-1].\end{array}$$

After regrading, we have $\Phi^{N,2} = (\Sigma_L)_1^{0,*}$. Then Lemma 4.4 tells us that $(\Sigma_L)_1^{0,*}$ is surjective.

We can now move on to the second column, i.e. we consider the map induced by the smoothing $\widetilde{D}_{(2)}$, pictured below.

LEMMA 4.5. *For any $N \geq 1$, the map $\Phi^{N,3}$ induced by the movie in Figure 19*

$$\Phi^{N,3} : \widetilde{Kh}^{0,0}(U) \otimes \widetilde{Kh}^{0,2}(P') \longrightarrow \widetilde{Kh}^{0,0}(U)$$

is onto.

Proof. First, we notice that the isotopy class of the diagrams in the movie does not depend on N. For any $N \geq 1$, denote by M_i^N the ith frame of the movie M. Then we have

$$M_1^N \sim P' \sim P' \sharp U, \; M_2^N \sim U \sqcup U, \; M_3^N \sim U.$$

The map $\Phi^{N,3}$ is defined as the composition:

$$\Phi^{N,3} : \widetilde{Kh}^{*,*}(U) \otimes \widetilde{Kh}^{*,*}(P') \xrightarrow{\varphi_1} \widetilde{Kh}^{*,*-1}(U \sqcup U) \xrightarrow{\varphi_2} \widetilde{Kh}^{*,*-2}(U).$$

It is enough to show that both φ_1 and φ_2 are surjective. The second map φ_2 is surjective as it realizes a connected sum. The first map φ_1 is studied by completing the triple. We reverse the orientation of the unknot part and replace the strands by a positive crossing. The resulting diagram D^N is given in Figure 19 and for any $N \geq 1$, one checks easily that $D^N \sim T_{2,4}$. The total dimensions of the associated homologies are related by the formula:

$$\dim(\widetilde{Kh}(P')) = \dim(\widetilde{Kh}(T_{3,3})) = \dim(\widetilde{Kh}(U \sqcup U)) + \dim(\widetilde{Kh}(T_{2,4})).$$

Therefore by Lemma 2.12, the induced map is surjective. We need to check that we can specialize the degrees as we claim. In terms of gradings the map is the following

$$\Phi^{N,3} : \widetilde{Kh}^{i,j}(U) \otimes \widetilde{Kh}^{i',j'}(P') \longrightarrow \widetilde{Kh}^{i'+i,j+j'-2}(U)$$

We know the homology of the unknot is concentrated in bidegree $(0,0)$ so we must have

$$i = j = 0, \quad i + i' = j + j' - 2 = 0.$$

Solving this system yields $i' = 0, j' = 2$ as claimed. □

The exact same re-grading process can be applied to $\Phi^{N,3}$, though we start with P' instead of P. Still P' differs from $T_{3,3}$ by a single orientation reversal. Adding the shifts of the column $E_1^{1,*}$ yields the map

$$\begin{array}{c}\widetilde{Kh}^{-12N,-12N-2}(U)[-12N,-12N-2] \otimes \widetilde{Kh}^{-4,-10}(T_{3,3}) \\ \Phi^{N,3}\big\downarrow \\ \widetilde{Kh}^{-12(N+1),-12(N+1)-2}(U)[-12(N+1),-12(N+1)-2],\end{array}$$

which coincides with the restriction $(\Sigma_R)_1^{1,*}$ of the right fusion map Σ_R on the second column. Lemma 4.5 then tells us that this map is surjective.

We are now one step away from relating $T_{3,3N}$ to $T_{3,3N-1}$.

PROPOSITION 4.6. *For $N \geq 1$, $j \in \{-12N-2, -12N\}$, the spectral sequence ${}^jE(T_{3,3N})$ converges at E_1.*

Proof. We proceed by induction on N. The case $N = 1$ has already been treated previously in Example 2.21. Fix $N \geq 2$ and let $j \in \{-12N-2, -12N\}$. We want to show, for both values of j, that all differentials d_r are zero for any $r \geq 1$ and we assume it is true for $N-1$.

For any $r \geq 1$, we have commutative diagrams

$$\begin{array}{ccc} {}^{j+12}E_r^{s,t}(T_{3,3(N-1)}) \otimes \widetilde{Kh}^{-4,-10}(T_{3,3}) & \xrightarrow{d_r^{s,t}\otimes 1} & {}^{j+12}E_r^{s+r,t+1-r}(T_{3,3(N-1)}) \otimes \widetilde{Kh}^{-4,-10}(T_{3,3}) \\ f_r^{s,t}\big\downarrow & & f_r^{s+1,t}\big\downarrow \\ {}^jE_r^{s,t}(T_{3,3N}) & \xrightarrow{d_r'^{s,t}} & {}^jE_r^{s+r,t+1-r}(T_{3,3N}) \end{array}$$

where f is either Σ_L or Σ_R and $s = 0,1$, which are both maps of spectral sequences.

We start with $s = 0$ and set $f = \Sigma_L$. For $r = 1$, we know by Lemma 4.4 that $\Phi^{N,2}$, which coincides with $f_1^{0,t}$, is surjective. From our hypothesis on $N-1$, we have that $d_1^{0,t} = 0$. Let $c \in {}^jE_1^{0,t}(T_{3,3N})$ and $b \in {}^{j+12}E_1^{0,t}(T_{3,3(N-1)})$ such that $f_1^{0,t}(b) = c$. Then we have

$$d_1'^{0,t}(c) = d_1'^{0,t}(f_1^{0,t}(b)) = f_1^{1,t}(d_1^{0,t}(b)) = 0,$$

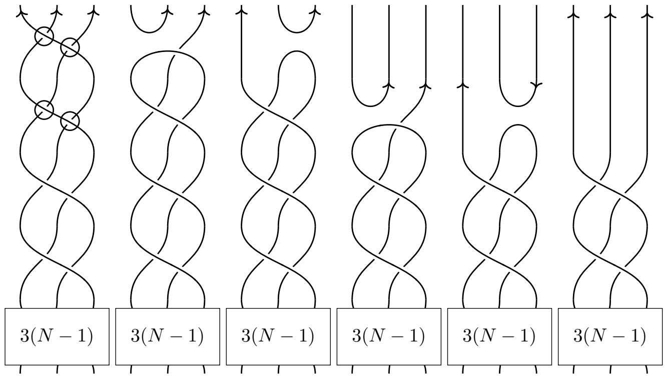


FIGURE 20. The diagrams for the skein spectral sequence for $E(T_{3,3N+1})$. From left to right: $D, \widetilde{D}_{(0)}, \widetilde{D}_{(1)}, \widetilde{D}_{(2)}, \widetilde{D}_{(3)}$ and $D_{(4)}$.

where the first equality follows from commutativity of the diagram above and the second from $d_1^{0,t} = 0$. Thus $d_1'^{0,t} = 0$ and $E_2^{0,t} = E_1^{0,t}$ since all incoming differentials are zero. Moreover, $f_2^{0,t} = f_1^{0,t}$ so $f_2^{0,t}$ is surjective.

For $r = 2$, we know that $d_2^{0,*} = 0$ and $f_2^{0,*}$ is surjective so the same argument applies. For $r \geq 3$, the target column is zero so $d'^{0,*} = 0$. Hence $d_r^{0,*} = 0$ for any $r \geq 1$.

For the column $s = 1$, the same proof for $r = 1$ applies by using $\Phi^{N,3}$ and $f = \Sigma_R$ with surjectivity given by Lemma 4.5. All incoming $d_1' : E_1^{0,*} \longrightarrow E_1^{1,*}$ are zero since they are outgoing d_1' from the first column, which we know to be zero from the case $s = 0$ above. In conclusion, all differentials at all pages are zero: the spectral sequence converges at E_1. □

We can now state and prove the main proposition of this paragraph.

PROPOSITION 4.7. *For any $N \geq 1$, there is an isomorphism of vector spaces*

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N}) &\cong \widetilde{Kh}^{*,*}(U \sqcup U)[-4N, -12N+1] \oplus \widetilde{Kh}^{*,*}(U)[-4N, -12N] \\ &\oplus \widetilde{Kh}^{*,*}(T_{3,3N-1})[0, -2].\end{aligned}$$

Proof. For any $N \geq 1$, the spectral sequences ${}^jE(T_{3,3N})$ converges at E_1, i.e. $E_1 = E_\infty$. Thus reconstructing $\widetilde{Kh}^{*,*}(T_{3,3N})$ with the E_∞-page yields immediately:

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N}) &\cong \widetilde{Kh}^{*,*}(U \sqcup U)[-4N, -12N+1] \oplus \widetilde{Kh}^{*,*}(U)[-4N, -12N] \\ &\oplus \widetilde{Kh}^{*,*}(T_{3,3N-1})[0, -2].\end{aligned}$$

This concludes the proof. □

2.2. Relating $T_{3,3N+1}$ and $T_{3,3N-1}$. We continue with the second family of spectral sequences we are interested in. Let $N \geq 1$ and ${}^jE(T_{3,3N+1})$ be the skein spectral sequence associated to the 4 circled crossings in Figure 20. Note that for $N = 1$, we recover the sequences of Example 4.3, whose differentials we know. The intermediate diagrams $\widetilde{D}_{(k)}$ for $k = 0, 1, 2, 3$ and $D_{(4)}$, are also given in Figure 20.

Their isotopy classes, as well as the number of negative crossings $n_-(\widetilde{D}_{(k)})$ of $\widetilde{D}_{(k)}$ with respect to the chosen orientations in Figure 20, can be described as follows:

$$\begin{array}{lll} \widetilde{D}_{(0)} \sim U, & \text{and} & n_-(\widetilde{D}_{(0)}) = 2N+1, \\ \widetilde{D}_{(1)} \sim U \sqcup U, & \text{and} & n_-(\widetilde{D}_{(1)}) = 2N, \\ \widetilde{D}_{(2)} \sim U \sqcup U, & \text{and} & n_-(\widetilde{D}_{(2)}) = 2N, \\ \widetilde{D}_{(3)} \sim U, & \text{and} & n_-(\widetilde{D}_{(3)}) = 2N-1, \\ D_{(4)} \sim T_{3,3N-1}. & & \end{array}$$

The shifts $\alpha_k = n_-(\widetilde{D}_{(k)}) - n_-(T_{3,3N+1})$ and $\beta_k = 3\alpha_k + 1$ associated to that sequence are computed below:

$$\begin{array}{ll} \alpha_1 = 2N+1-(6N+2) = -4N-1, & \beta_0 = -12N-3+1 = -12N-2, \\ \alpha_2 = 2N-(6N+2) = -4N-2, & \beta_1 = -12N-6+1 = -12N-5, \\ \alpha_3 = 2N-(6N+2) = -4N-2, & \beta_2 = -12N-6+1 = -12N-5, \\ \alpha_4 = 2N-1-(6N+2) = -4N-3, & \beta_3 = -12N-9+1 = -12N-5. \end{array}$$

For a generic $j \in \mathbb{Z}$, the columns of the E_1-page are:

$$\begin{array}{rcl} {}^jE_1^{0,t} & = & \widetilde{Kh}^{t+4N+1,j+12N+2}(U), \\ {}^jE_1^{1,t} & = & \widetilde{Kh}^{t+4N+2,j+12N+3}(U \sqcup U), \\ {}^jE_1^{2,t} & = & \widetilde{Kh}^{t+4N+2,j+12N+1}(U \sqcup U), \\ {}^jE_1^{3,t} & = & \widetilde{Kh}^{t+4N+3,j+12N+2}(U), \\ {}^jE_1^{4,t} & = & \widetilde{Kh}^{t+4,j+4}(T_{3,3N-1}). \end{array}$$

If $j \notin \{-12N-4, -12N-2, -12N\}$ then the columns $E^{0,*}, E^{1,*}, E^{2,*}, E^{3,*}$ are all zero so the sequences collapse at E_1. For any $N \geq 1$, we then have an isomorphism:

$$\widetilde{Kh}^{*,j}(T_{3,3N+1}) \cong \widetilde{Kh}^{*,j}(T_{3,3N-1})[0,-4].$$

For $j \in \{-12N-4, -12N-2, -12N\}$, we will get the following grids, where the stars mean that we don't know the dimension of the corresponding vector space: since we have made no

	0	1	2	3	4
$-4N-2$			1		$*$
$-4N-3$					$*$

(A) $j = -12N$.

	0	1	2	3	4
$-4N-1$	1				$*$
$-4N-2$		1	1		$*$
$-4N-3$				1	$*$

(B) $j = -12N-2$.

	0	1	2	3	4
$-4N-2$		1			$*$

(C) $j = -12N-4$.

assumptions on $\widetilde{Kh}^{*,*}(T_{3,3N-1})$, we cannot fill the entries for the 5th column. However if we can show that these spectral sequences are nice, a fact we know to be true for $N = 1$, then any differential incoming into the 5th column will be zero, so we won't need any extra assumptions on $\widetilde{Kh}^{*,*}(T_{3,3N-1})$.

We apply the same process as before: exhibit surjective maps at the E_1-pages and use them to control the behaviour of the differentials. We start with the column $s = 0$. The induced movie is that of Figure 21, and relies on the left fusion map Σ_L.

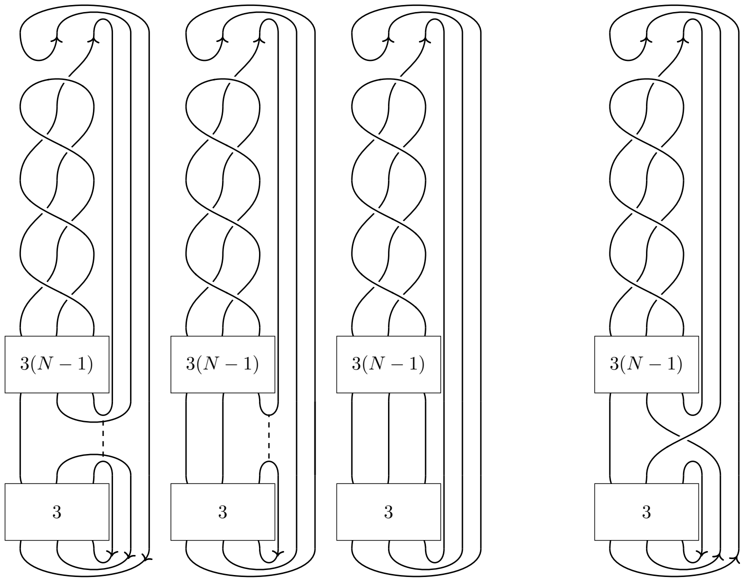


FIGURE 21. The movie that induces the restriction of Σ_L for the columns $E_1^{0,*}(T_{3,3N+1})$. On the right, the completed diagram for the first 1-handle.

LEMMA 4.8. *For any $N \geq 1$, the map $\Phi^{N,0}$ induced by the movie in Figure 21*

$$\Phi^{N,0} : \widetilde{Kh}^{0,0}(U) \otimes \widetilde{Kh}^{0,2}(P') \longrightarrow \widetilde{Kh}^{0,0}(U),$$

is onto.

Proof. First, we notice that the isotopy class of the diagrams in the movie does not depend on N. For any $N \geq 1$, denote by M_i^N the ith frame of the movie M. Then we have

$$M_1^N \sim P' \sim P' \sharp U,\ M_2^N \sim U \sqcup U,\ M_3^N \sim U.$$

The map $\Phi^{N,0}$ is defined as the composition:

$$\Phi^{N,0} : \widetilde{Kh}^{*,*}(U) \otimes \widetilde{Kh}^{*,*}(P') \xrightarrow{\varphi_1} \widetilde{Kh}^{*,*-1}(U \sqcup U) \xrightarrow{\varphi_2} \widetilde{Kh}^{*,*-2}(U).$$

The first map φ_1 is studied by completing the triple, while φ_2 is surjective as it realizes a connected sum. We reverse the orientation of the unknot part and replace the strands by a positive crossing. The resulting diagram D^N is given in Figure 21 and for any $N \geq 1$, one checks easily that $D^N \sim T_{2,4}$. The movie has the same first and last frames, as well as the same completion for the first 1-handle, as the one in Lemma 4.4 therefore it is also surjective, and the degrees also match by the same argument. □

This is regraded by applying the shifts corresponding to the first column, just as we did for the $T_{3,3N}$ case. It then becomes

$$\begin{array}{c} \widetilde{Kh}^{-12N,-12N-2}(U)[-12N,-12N-2] \otimes \widetilde{Kh}^{-4,-10}(T_{3,3}) \\ \Big\downarrow \Phi^{N,0} \\ \widetilde{Kh}^{-12(N+1),-12(N+1)-2}(U)[-12(N+1),-12(N+1)-2], \end{array}$$

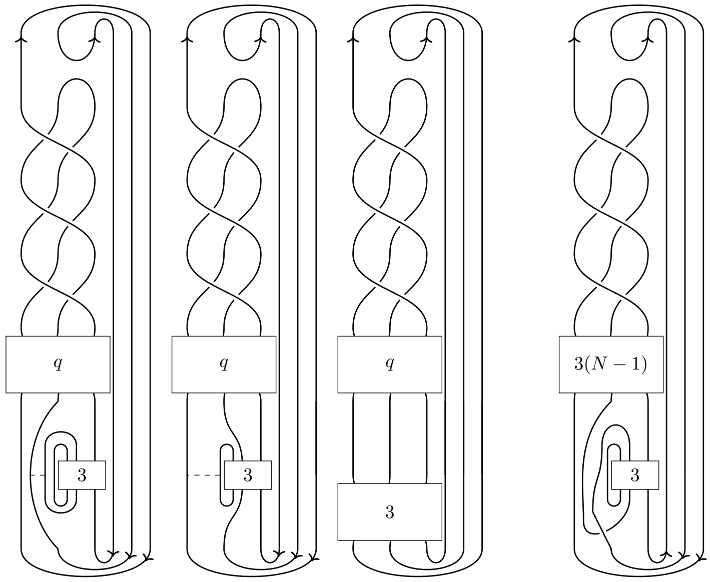


FIGURE 22. The movie that induces the restriction of Σ_R for the column $E_1^{1,*}(T_{3,3N+1})$. On the right, the completed diagram for the first 1-handle.

which coincides with the restriction $(\Sigma_L)_1^{0,*}$ of Σ_L. This map is surjective by Lemma 4.8.

LEMMA 4.9. *For any $N \geq 1$, the map $\Phi^{N,1}$ induced by the movie in Figure 22*

$$\Phi^{N,1} : \widetilde{Kh}^{0,j_\pm}(U \sqcup U) \otimes \widetilde{Kh}^{0,2}(P) \longrightarrow \widetilde{Kh}^{0,j_\pm}(U \sqcup U)$$

where $j_\pm = \pm 1$ is onto.

Proof. First, we notice that the isotopy class of the diagrams in the movie does not depend on N. For any $N \geq 1$, denote by M_i^N the ith frame of the movie M. Then we have

$$M_1^N \sim P \sqcup U \sim P \sharp (U \sqcup U),\ M_2^N \sim U \sqcup U \sqcup U,\ M_2^N \sim U \sqcup U.$$

The map $\Phi^{N,1}$ is defined as the composition:

$$\Phi^{N,1} : \widetilde{Kh}^{*,*}(P) \otimes \widetilde{Kh}^{*,*}(U \sqcup U) \xrightarrow{\varphi_1} \widetilde{Kh}^{*,*-1}(U \sqcup U \sqcup U) \xrightarrow{\varphi_2} \widetilde{Kh}^{*,*-2}(U \sqcup U).$$

We show that both φ_1 and φ_2 are surjective. The second map φ_2 is surjective as it realizes a connected sum. The first map φ_1 is studied by completing the triple. We reverse the orientation of top strand and replace the strands by a positive crossing. The resulting diagram D^N is given in Figure 22 and for any $N \geq 1$, one checks easily that $D^N \sim T_{2,4} \sqcup U$. The movie has the same first and last frames, as well as the same completion for the first 1-handle, as the one in Lemma 4.5, so it is surjective and the degrees have to match again. This concludes the proof. $\square$

The exact same process applied to $\Phi^{N,1}$ with the shifts of the column $E_1^{1,*}$ yields the map

$$\begin{array}{c}\widetilde{Kh}^{-12N,-12N-3+j_\pm}(U\sqcup U)[-4N,-12N-3]\otimes\widetilde{Kh}^{-4,-10}(T_{3,3})\\ \Phi^{N,1}\Big\downarrow\\ \widetilde{Kh}^{-4(N+1),-12(N+1)-3+j_\pm}(U\sqcup U)[-4(N+1),-12(N+1)-3],\end{array}$$

which coincides with the restriction of Σ_R. Thus the map $(\Sigma_R)_1^{1,*}$ is surjective by Lemma 4.9.

We are now one step away from relating $T_{3,3N+1}$ to $T_{3,3N-1}$.

LEMMA 4.10. *Let $N\geq 1$ be an integer. If $j\in\{-12N,-12N-4\}$ then ${}^jE(T_{3,3N+1})$ converges at E_1. If $j=-12N-2$, then ${}^jE(T_{3,3N+1})$ is nice. In this case the possible configurations are*

(1) $d_1^{1,-4N-2}$ and $d_3^{0,-4N-1}$ are both isomorphisms.
(2) $d_2^{0,-4N-1}$ and $d_2^{1,-4N-2}$ are both isomorphisms.

Proof. We proceed by induction on N. The case $N=1$ has already been treated previously in Example 4.3. Fix $N\geq 2$ and let $j\in\{-12N-2,-12N\}$. We want to show, for both values of j, that all differentials d_r are zero for any $r\geq 1$. We assume it is true for $N-1$.

For any $r\geq 1$, we have commutative diagrams

$$\begin{array}{ccc}{}^{j+12}E_r^{s,t}(T_{3,3(N-1)+1})\otimes\widetilde{Kh}^{-4,-10}(T_{3,3}) & \xrightarrow{d_r^{s,t}\otimes 1} & {}^{j+12}E_r^{s+r,t+1-r}(T_{3,3(N-1)+1})\otimes\widetilde{Kh}^{-4,-10}(T_{3,3})\\ f_r^{s,t}\Big\downarrow & & f_r^{s+1,t}\Big\downarrow\\ {}^jE_r^{s,t}(T_{3,3N+1}) & \xrightarrow{d_r'^{s,t}} & {}^jE_r^{s+r,t+1-r}(T_{3,3N}+1)\end{array}$$

where f is either Σ_L or Σ_R and $s=0,1,2,3$.

For $j=-12N$, we only need to show that the outgoing differentials from the E^2 column are zero. The argument of Lemma 4.7 applies when setting $f=\Sigma_R$, by using Lemma 4.4 for surjectivity again.

For $j=-12N-4$, the same argument is used for the column $s=1$, by setting $f=\Sigma_R$. This map coincides with $\Phi^{N,1}$, which is surjective by Lemma 4.9.

For $j=-12N-2$, we know by induction that the maps in the top row of the diagram are not necessarily zero. We treat in details one of the two possibilities in the statement of the proposition. We assume that for $N=1$, we aer in configuration (1), i.e. we have $d_1^{1,-6}$ and $d_3^{0,-5}$ both isomorphisms, and all the others are zero. We will show by induction that for any $N\geq 1$, the differentials $d_1^{1,-4N-2}$ and $d_3^{0,-4N-1}$ are both isomorphisms and all others are zero. Fix $N\geq 2$ and assume the result holds for $N-1$. We start with $s=0$ and set $f=\Sigma_R$. We know that the map $f_1^{1,*}$ coincides with $\Phi^{N,1}$, which is surjective by Lemma 4.9. By induction we know that $d_1^{1,-4(N-1)-2}$ is an isomorphism. Therefore $f_1^{2,*}\circ(d_1^{1,*}\otimes 1)$ is onto as the composition of two surjective maps. Moreover, using the commutative diagram above, we have

$$d_1'^{1,*}\circ f_1^{1,*}=f_1^{2,*}\circ(d_1^{1,*}\otimes 1)$$

and the right hand side map is surjective. It follows that $d_1'^{1,*}$ itself is onto. Finally, we know the domain and codomain of this differential are both one dimensional vector spaces, so $d_1'^{1,*}$ is an isomorphism. Consequently, at the E_2-page we have $E_2^{1,*}=0=E^{2,*}$.

For the two remaining columns $s=0,3$, we know by induction hypothesis that all outgoing d_1 and d_2 are zero for the rank $N-1$.

For $s = 0$, we set $f = \Sigma_L$. The map $f_1^{0,*}$ coincides with $\Phi^{N,0}$, which is surjective by Lemma 21.

For $s = 3$, we also set $f = \Sigma_R$ and the map $f_1^{3,*}$ coincides with $\Phi^{N,3}$ surjective by Lemma 4.5. The argument of Lemma 4.7 applies again: all outgoing d_1' and d_2' are zero for these two columns. Hence we have, for both $s = 2, 3$: $E_3^{s,*} = E_1^{s,*}$, and the maps $f_3^{0,*}$ and $f_3^{3,*}$ are still onto.

At the third page, by induction hypothesis, we have an isomorphism $d_3^{0,-4(N-1)-1}$. The argument for d_1 can be replicated by setting $f = \Sigma_L$, since $f_3^{3,*}$ is onto. This implies that

$$d_3'^{0,-4N-1} : \mathbb{Z}_2 \cong {}^{-12N-2}E_3^{0,-4N-1}(T_{3,3N+1}) \longrightarrow {}^{-12N-2}E_3^{3,-4N-3}(T_{3,3N+1}) \cong \mathbb{Z}_2$$

is non zero, and thus an isomorphism. At the fourth page we then have:

$$E_2^{0,*} = 0 = E^{3,*}.$$

Only the fifth column is non-zero so the sequence collapses at E_4. In particular it is nice.

If we assume we are in the configuration (2), where

$$d_2^{0,-4(N-1)-1} \text{ and } d_2^{1,-4(N-1)-2}$$

are both isomorphisms, the same argument to the one above can be applied. This concludes the proof. □

As a consequence of Lemma 4.10, we obtain a direct sum decomposition of $\widetilde{Kh}^{*,*}(T_{3,3N+1})$, the main result of this paragraph.

PROPOSITION 4.11. *For any $N \geq 1$, there is an isomorphism of vector spaces*

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N+1}) &\cong \widetilde{Kh}^{*,*}(U)[-4N, -12N] \oplus \widetilde{Kh}^{*,*}(U)[-4N-1, -12N-4] \\ &\quad \oplus \widetilde{Kh}^{*,*}(T_{3,3N-1})[0, -4].\end{aligned}$$

Proof. For any $N \geq 1$, the sequences for $j = -12N$ and $j = -12N - 4$ converge at E_1. For $j = -12N$, this yields an isomorphism

$$\begin{aligned}\widetilde{Kh}^{*,-12N}(T_{3,3N+1}) &\cong \widetilde{Kh}^{*,-12N}(U \sqcup U)[-4N, -12N-1] \oplus \widetilde{Kh}^{*,-12N}(T_{3,3N-1})[0, -4] \\ &\cong \widetilde{Kh}^{*,-12N}(U)[-4N, -12N] \oplus \widetilde{Kh}^{*,-12N}(T_{3,3N-1})[0, -4],\end{aligned}$$

where the second isomorphism is just a shift of the identification $\widetilde{Kh}^{0,1}(U \sqcup U) \cong \widetilde{Kh}^{0,0}(U)[0, 1]$. Similarly, for $j = -12N - 4$, we obtain:

$$\begin{aligned}\widetilde{Kh}^{*,-12N-4}(T_{3,3N+1}) &\cong \widetilde{Kh}^{*,-12N-4}(U \sqcup U)[-4N-1, -12N-3] \oplus \widetilde{Kh}^{*,-12N-4}(T_{3,3N-1})[0, -4] \\ &\cong \widetilde{Kh}^{*,-12N-4}(U)[-4N-1, -12N-4] \oplus \widetilde{Kh}^{*,-12N-4}(T_{3,3N-1})[0, -4],\end{aligned}$$

where the second isomorphism is just a shift of the identification $\widetilde{Kh}^{0,-1}(U \sqcup U) \cong \widetilde{Kh}^{0,0}(U)[0, -1]$. For the remaining case $j = -12N - 2$, the sequence for $N = 1$ is nice. Whichever is the correct configuration of non-zero differentials, all the entries in columns $0, 1, 2, 3$ disappear. This translates into an isomorphism:

$$\widetilde{Kh}^{*,-12N-2}(T_{3,3N+1}) \cong \widetilde{Kh}^{*,-12N-2}(T_{3,3N-1})[0, -4].$$

For $j \notin \{-12N-4, -12N-2, -12N\}$, we already had an isomorphism:

$$\widetilde{Kh}^{*,j}(T_{3,3N+1}) \cong \widetilde{Kh}^{*,j}(T_{3,3N-1})[0, -4].$$

If we combine these isomorphisms, we obtain the description

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N+1}) &\cong \widetilde{Kh}^{*,*}(U)[-4N,-12N] \oplus \widetilde{Kh}^{*,*}(U)[-4N-1,-12N-4] \\ &\oplus \widetilde{Kh}^{*,*}(T_{3,3N-1})[0,-4].\end{aligned}$$

This concludes the proof. □

The great property of Propositions 4.7 and 4.11 is that they do not rely on any explicit knowledge of $\widetilde{Kh}^{*,*}(T_{3,3N-1})$, even if some differentials in the spectral sequences are non-zero.

3. Computing $\widetilde{Kh}^{*,*}(T_{3,q})$.

In this final section of the chapter, we give an explicit description of the homology of $T_{3,q}$. To extract this form, we revert back to the inductive-style proofs of Gillam [**Gil12**], Stosic [**Sto09**] and Turner [**Tur08**]. However 2 out the 3 steps will be an immediate consequence of the direct sum descriptions of the previous section. The third step will rely on other spectral sequences, following Turner [**Tur08**].

We now have isomorphisms relating $\widetilde{Kh}^{*,*}(T_{3,3N+1})$ and $\widetilde{Kh}^{*,*}(T_{3,3N})$ with $\widetilde{Kh}^{*,*}(T_{3,3N-1})$, without any pre-existing knowledge on the latter. In order to produce an explicit description of $\widetilde{Kh}^{*,*}(T_{3,q})$, we still need to understand the step from $\widetilde{Kh}^{*,*}(T_{3,3N+1})$ to $\widetilde{Kh}^{*,*}(T_{3,3N+2})$. For a clearer statement, let us introduce a vector space that will be heavily featured in the computation. Let $K^{*,*}$ be the bigraded vector space defined by

$$K^{i,j} = \begin{cases} \mathbb{Z}_2 & \text{if } (i,j) \in \{(0,0),(-2,-4),(-3,-6),(-5,-10)\}. \\ 0 & \text{otherwise.} \end{cases}$$

This space is represented by the grid

−5	−4	−3	−2	−1	0	i \ j
					1	0
						-2
			1			-4
		1				-6
						-8
1						-10

and the corresponding quantum Poincaré polynomial is

$$P(t,q)(K) = 1 + q^{-4}t^{-2} + q^{-6}t^{-3} + q^{-10}t^{-5}.$$

In particular, there is a *de facto* isomorphism

$$K^{*,*} = K^{-5,-10} \oplus \widetilde{Kh}^{*,*}(T_{2,3})[0,2].$$

The δ-graded version K_*^* is given by

$$K_\delta^i = \begin{cases} \mathbb{Z}_2 & \text{if } \delta = 0 \text{ and } i \in \{0,-2,-3,-5\}. \\ 0 & \text{otherwise.} \end{cases}$$

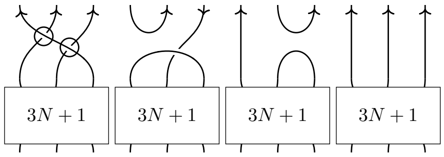


FIGURE 23. The diagrams for the skein spectral sequence for $T_{3,3N+2}$. From left to right: $D, \widetilde{D}_{(0)}, \widetilde{D}_{(1)}$ and $D_{(2)}$.

with grid

-5	-4	-3	-2	-1	0	i \ j
1		1	1		1	0

The δ-graded Poincaré polynomial is

$$P_\delta(t,q)(K) = 1 + t^{-2} + t^{-3} + t^{-5}.$$

Finally, we present the main result of this chapter, namely the description of $\widetilde{Kh}^{*,*}(T_{3,q})$. In a second separate statement, we present grids and Poincaré polynomials for the δ-graded versions.

THEOREM 4.12. *(i) For any $N \geq 1$, there is an isomorphism of vector spaces*

$$\widetilde{Kh}^{*,*}(T_{3,3N-1}) \cong \left(\frac{K^{*,*}}{K^{-5,-10}}\right)[-4(N-1), -12N+10] \oplus \left(\bigoplus_{0\leq i\leq N-2} K^{*,*}[-4i, -6N+4-6i]\right).$$

(ii) For any $N \geq 1$, there is an isomorphism of vector spaces

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N}) \cong\ & \widetilde{Kh}^{*,*}(U \sqcup U)[-4N, -12N+1] \oplus \widetilde{Kh}^{*,*}(U)[-4N, -12N] \\ & \oplus \left(\frac{K^{*,*}}{K^{-5,-10}}\right)[-4(N-1), -12N+8] \oplus \left(\bigoplus_{0\leq i\leq N-2} K^{*,*}[-4i, -6N+2-6i]\right).\end{aligned}$$

(iii) For any $N \geq 0$, there is an isomorphism of vector spaces

$$\widetilde{Kh}^{*,*}(T_{3,3N+1}) \cong \widetilde{Kh}^{*,*}(U)[-4N, -12N] \oplus \left(\bigoplus_{0\leq i\leq N-1} K^{*,*}[-4i, -6N-6i]\right).$$

Proof. The proof consists in an induction over q the number of twists, with three steps separated into 3 claims.

Claim 1: if (i) is true for $T_{3,3N-1}$, then (ii) is true for $T_{3,3N}$.

Claim 2: if (i) is true for $T_{3,3N-1}$, then (iii) is true for $T_{3,3N+1}$.

Claim 3: if (iii) is true for $T_{3,3N+1}$, then (i) is true for $T_{3,3N+2}$.

The first two claims will be consequences of Corollaries 4.7 and 4.11, while the third will require us to study a new family of spectral sequences. The base cases $T_{3,2}$, $T_{3,3}$ and $T_{3,4}$ have already been treated earlier and fit the pattern.

Proof of Claim 1. We have an isomorphism of vector spaces, courtesy of Corollary 4.7:

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N}) &\cong \widetilde{Kh}^{*,*}(U \sqcup U)[-4N,-12N+1] \oplus \widetilde{Kh}^{*,*}(U)[-4N,-12N] \\ &\quad \oplus \widetilde{Kh}^{*,*}(T_{3,3N-1})[0,-2].\end{aligned}$$

If (i) is true for $T_{3,3N-1}$, we obtain an isomorphism

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N}) &\cong \widetilde{Kh}^{*,*}(U \sqcup U)[-4N,-12N+1] \oplus \widetilde{Kh}^{*,*}(U)[-4N,-12N] \\ &\quad \oplus \left(\frac{K^{*,*}}{K^{-5,-10}}\right)[-4(N-1),-12N+8] \\ &\quad \oplus \left(\bigoplus_{0\leq i\leq N-2} K^{*,*}[-4i,-6N+8-6i]\right).\end{aligned}$$

Thus (ii) is true for $T_{3,3N}$.

Proof of Claim 2. We have an isomorphism of vector spaces, courtesy of Corollary 4.11:

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N+1}) &\cong \widetilde{Kh}^{*,*}(U)[-4N,-12N] \oplus \widetilde{Kh}^{*,*}(U)[-4N-1,-12N-4] \\ &\quad \oplus \widetilde{Kh}^{*,*}(T_{3,3N-1})[0,-4].\end{aligned}$$

If (i) is true for $T_{3,3N-1}$, we have an isomorphism

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N+1}) &\cong \widetilde{Kh}^{*,*}(U)[-4N,-12N] \oplus \widetilde{Kh}^{*,*}(U)[-4N-1,-12N-4] \\ &\quad \oplus \left(\frac{K^{*,*}}{K^{-5,-10}}\right)[-4(N-1),-12N+6] \\ &\quad \oplus \left(\bigoplus_{0\leq i\leq N-2} K^{*,*}[-4i,-6N-6i]\right).\end{aligned}$$

We can identify one more occurence of $K^{*,*}$ via the isomorphisms

$$\begin{aligned}\widetilde{Kh}^{*,*}(U)[-4N-1,-12N-4] \;\oplus\;& \left(\frac{K^{*,*}}{K^{-5,-10}}\right)[-4(N-1),-12N+6] \\ \cong\;& \left(\widetilde{Kh}^{*,*}(U)[-5,-10] \oplus \left(\frac{K^{*,*}}{K^{-5,-10}}\right)\right)[-4(N-1),-12N+6] \\ \cong\;& K^{*,*}[-4(N-1),-12N+6] \\ =\;& K^{*,*}[-4(N-1),-6N-6(N-1)].\end{aligned}$$

Therefore we have an isomorphism

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N+1}) &\cong \widetilde{Kh}^{*,*}(U)[-4N,-12N] \\ &\quad \oplus \left(\bigoplus_{0\leq i\leq N-1} K^{*,*}[-4i,-6N-6i]\right).\end{aligned}$$

And (iii) is true for $T_{3,3N+1}$.

Proof of Claim 3. We choose our crossings according to Figure 23. We describe the intermediate diagrams, as well as the number of negative crossings for our choices of orientations:

$$\begin{array}{lll} \widetilde{D}_{(0)} \sim U, & \text{and} & n_-(\widetilde{D}_{(0)}) = 2N+1, \\ \widetilde{D}_{(1)} \sim U, & \text{and} & n_-(\widetilde{D}_{(1)}) = 2N+1, \\ D_{(2)} \sim T_{3,3N+1}. & & \end{array}$$

Recall $\alpha_k = n_-(\widetilde{D}_{(k)}) - n_-(T_{3,3N+2})$ and $\beta_k = 3\alpha_k + 1$ so here:

$$\begin{array}{ll} \alpha_1 = 2N+1-(6N+4) = -4N-3, & \beta_0 = -12N-9+1 = -12N-8, \\ \alpha_2 = 2N+1-(6N+4) = -4N-3, & \beta_1 = -12N-9+1 = -12N-8, \end{array}$$

For a generic $j \in \mathbb{Z}$, the columns of the E_1-page are:

$$\begin{array}{rcl} {}^jE_1^{0,t} &=& \widetilde{Kh}^{t+4N+3,j+12N+8}(U), \\ {}^jE_1^{1,t} &=& \widetilde{Kh}^{t+4N+3,j+12N+6}(U), \\ {}^jE_1^{2,t} &=& \widetilde{Kh}^{t+2,j+2}(T_{3,3N+1}). \end{array}$$

If $j \notin \{-12N-8, -12N-6\}$ then the columns $E^{0,*}, E^{1,*}$ are zero so the sequence collapses at E_1 and we have an isomorphism:

$$\widetilde{Kh}^{*,j}(T_{3,3N+2}) \cong \widetilde{Kh}^{*,j}(T_{3,3N+1})[0,-2].$$

For $j \in \{-12N-8, -12N-6\}$, we start by making some observations. If (iii) is true for $T_{3,3N+1}$, then we know that:

$$\begin{array}{rcl} \widetilde{Kh}^{*,-12N-8}(T_{3,3N+1})[0,-2] &=& 0, \\ \widetilde{Kh}^{-4N-1,*}(T_{3,3N+1})[0,-2] &=& \mathbb{Z}_2, \text{ supported in quantum degree } j = -12N-6. \\ \widetilde{Kh}^{i,*}(T_{3,3N+1})[0,-2] &=& 0 \text{ if } i < -4N-2. \end{array}$$

For $j \in \{-12N-8, -12N-6\}$ we grids below. For $j = -12N-8$, the spectral sequence

	0	1	2
$-4N-3$	1		

(A) $j = -12N-8$.

	0	1	2
$-4N-3$		1	1

(B) $j = -12N-6$.

collapses at E_1, since there is only one non-zero column. Hence there is an isomorphism

$$\widetilde{Kh}^{-4N-3,-12N-8}(T_{3,3N+2}) \cong \widetilde{Kh}^{-4N-3,-12N-8}(U)[-4N-3,-12N-8].$$

We turn to the final grading $j = -12N-6$, see table (B) above. There is one potentially non-zero differential:

$$d_1^{1,-4N-3} : E_1^{1,-4N-3} \longrightarrow E_1^{2,-4N-3}.$$

If it is non zero, since the spaces are one dimensional it is an isomorphism and

$$\widetilde{Kh}^{-4N-1,*}(T_{3,3N+2}) = \widetilde{Kh}^{-4N-2,*}(T_{3,3N+2}) = 0.$$

This implies that the unique element in homological degree $-4N-3$, i.e the one with quantum grading $j = -12N-8$, survives through the Lee-Rasmussen spectral for Bar-Natan homology and generates a part of Bar-Natan homology. This contradicts the fact that Bar-Natan homology is supported in even degrees only. Therefore the differential

$$d_1^{1,-4N-3} : E_1^{1,-4N-3} \longrightarrow E_1^{2,-4N-3}.$$

is zero and both entries of the grid contribute to $\widetilde{Kh}^{*,*}(T_{3,3N+2})$. This gives a global isomorphism

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N+2}) \;\cong\; & \widetilde{Kh}^{*,*}(U)[-4N-3,-12N-8]\\ & \oplus\widetilde{Kh}^{*,*}(U)[-4N-2,-12N-6]\\ & \oplus\widetilde{Kh}^{*,*}(T_{3,3N+1})[0,-2].\end{aligned}$$

Using (iii) for $T_{3,3N+1}$, we can decompose the last summand further, yielding:

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N+2}) \;\cong\; & \widetilde{Kh}^{*,*}(U)[-4N-3,-12N-8]\\ & \oplus\widetilde{Kh}^{*,*}(U)[-4N-2,-12N-6]\\ & \oplus\widetilde{Kh}^{*,*}(U)[-4N,-12N-2]\\ & \oplus\left(\bigoplus_{0\le i\le N-1} K^{*,*}[-4i,-6N-2-6i]\right).\end{aligned}$$

The three unknot summands add up to a truncated $K^{*,*}[-4N,-12N-2]$. Finally we set $N' = N+1$ so $3N'-1 = 3N+2$, which gives:

$$\begin{aligned}\widetilde{Kh}^{*,*}(T_{3,3N'-1}) \;\cong\; & \left(\frac{K^{*,*}}{K^{-5,-10}}\right)[-4N',-12N'+10]\\ & \oplus\left(\bigoplus_{0\le i\le N'-2} K^{*,*}[-4i,-6N'+4-6i]\right).\end{aligned}$$

Thus (i) holds for $T_{3,3(N+1)-1}$. This complete the induction process and the proof. □

We can now present a more visual description of the δ-graded homology, in terms of grids (Figure 24) and δ-graded Poincaré polynomials.

THEOREM 4.13. *(i) For any $N \ge 1$, the δ-graded Poincaré polynomial of $T_{3,3N-1}$ is the following*

$$\begin{aligned}P_\delta(t,q)(T_{3,3N-1}) \;=\; & q^{-6N+4}(1+t^{-2}+t^{-3}+t^{-5})\left(\sum_{k=0}^{N-2} q^{2k}t^{-4k}\right)\\ & +q^{-4N+2}t^{-4N+4}(1+t^{-2}+t^{-3}).\end{aligned}$$

(ii) For any $N \ge 1$, the δ-graded Poincaré polynomial of $T_{3,3N}$ is the following

$$\begin{aligned}P_\delta(t,q)(T_{3,3N}) \;=\; & q^{-6N+2}(1+t^{-2}+t^{-3}+t^{-5})\left(\sum_{k=0}^{N-2} q^{2k}t^{-4k}\right)\\ & +q^{-4N}t^{-4N+4}(1+t^{-2}+t^{-3}+2t^{-4})+q^{-4N+2}t^{-4N}.\end{aligned}$$

(iii) For any $N \ge 0$, the δ-graded Poincaré polynomial $T_{3,3N+1}$ is the following

$$P_\delta(t,q)(T_{3,3N+1}) \;=\; q^{-6N}(1+t^{-2}+t^{-3}+t^{-5})\left(\sum_{k=0}^{N-1} q^{2k}t^{-4k}\right)+q^{-4N}t^{-4N}.$$

			$-4N+4$									-2	-1	0	i \ δ
1	1		1												$-4N+2$
				⋱	⋱	⋱									
					1		1	1		1					
									1		1	1		1	$-6N+4$

$N-2$ diagonals with the same pattern

$-4N$				$-4N+4$										-2	-1	0	i \ δ
1																	$-4N+2$
2	1	1		1													
			1		1	1		1									
						⋱	⋱	⋱									
							1		1	1		1					
											1		1	1		1	$-6N+2$

$N-2$ diagonals with the same pattern

	$-4N$				$-4N+4$									-2	-1	0	i \ δ
	1																$-4N$
1		1	1		1												
						⋱	⋱	⋱									
							1		1	1		1					
											1		1	1		1	$-6N$

$N-1$ diagonals with the same pattern

FIGURE 24. The grids for the δ-graded mod 2 reduced Khovanov homology of three stranded torus links. The grid on top describes the homology of the family $T_{3,3N-1}$ for $N \geq 1$, the middle grid describes that of the family $T_{3,3N}$ for $N \geq 1$ and the bottom grid gives the homology of the family $T_{3,3N+1}$ for $N \geq 0$. The entries of the grids are filled with the dimensions of the corresponding vector space $\widetilde{Kh}^i_\delta$ and an empty entry means that the dimension is zero.

REMARK 4.14. *A sum of the form* $\sum\limits_{k=0}^{i}$ *for* $i < 0$ *is set to be zero. For example, for the unknot* $U = T_{3,1}$*, we have the Poincaré polynomial*

$$P_\delta(q,t) = 1 = q^0t^0 = q^0(1+t^{-2}+t^{-3}+t^{-5})\left(\sum_{k=0}^{-1} q^{2k}t^{-4k}\right) + q^0t^0.$$

Hence our result gives the homology of any 3-stranded torus link, excpet $T_{3,0}$*, whose* δ*-graded Poincaré polynomial is given by*

$$P_\delta(q,t) = q^2 + 2 + q^{-2}.$$

Proof. We obtain the δ-graded description of the homologies from Theorem 4.12. The only detail to keep in mind is that one should also apply the change in grading to the shifts, i.e. a shift $[i,j]$ becomes a shift $[i, j-2i]$. Hence, we have

(i) For any $N \geq 1$, there is an isomorphism of vector spaces

$$\widetilde{Kh}^*_*(T_{3,3N-1}) \cong \left(\frac{K^*_*}{K^{-5}_0}\right)[-4(N-1), -4N+2] \oplus \left(\bigoplus_{0\leq i\leq N-2} K^*_*[-4i, -6N+4+2i]\right).$$

(ii) For any $N \geq 1$, there is an isomorphism of vector spaces

$$\begin{aligned}\widetilde{Kh}^*_*(T_{3,3N}) \cong\ & \widetilde{Kh}^*_*(U \sqcup U)[-4N, -4N+1] \oplus \widetilde{Kh}^*_*(U)[-4N, -4N] \\ & \oplus \left(\frac{K^*_*}{K^{-5}_0}\right)[-4(N-1), -4N] \oplus \left(\bigoplus_{0\leq i\leq N-2} K^{*,*}[-4i, -6N+2+2i]\right).\end{aligned}$$

(iii) For any $N \geq 0$, there is an isomorphism of vector spaces

$$\widetilde{Kh}^*_*(T_{3,3N+1}) \cong \widetilde{Kh}^*_*(U)[-4N, -4N] \oplus \left(\bigoplus_{0\leq i\leq N-1} K^*_*[-4i, -6N+2i]\right).$$

For the Poincaré polynomials, recall that

$$P_\delta(t,q)(K) = 1 + t^{-2} + t^{-3} + t^{-5}.$$

For a shifted version $K[i,\delta]$, the Poincaré polynomial is then

$$P_\delta(t,q)(K[i,\delta]) = q^\delta t^i(1+t^{-2}+t^{-3}+t^{-5}),$$

and for a sum of two copies, we have

$$\begin{aligned}P_\delta(t,q)(K[i,\delta]\oplus K[i',\delta']) &= P_\delta(t,q)(K[i,\delta]) + P_\delta(t,q)(K[i',\delta']) \\ &= (q^\delta t^i + q^{\delta'}t^{i'})(1+t^{-2}+t^{-3}+t^{-5}).\end{aligned}$$

The explicit value for the Poincaré polynomials follows immediately from the isomorphisms above, combined with the two rules for shifts and sums. This concludes the proof. □

CHAPTER 5

The algebra of torus links

In this chapter, we consider a limiting process on Khovanov homology. As we discussed in Chapter 2, if we resolve a crossing c of a diagram D to a 1-smoothing, we get a subcomplex $\widetilde{C}(D_1)$ of $\widetilde{C}(D)$, and a map i realizing the inclusion of the subcomplex. But what if one were to repeat this process? Include $\widetilde{C}(D)$ into the chain complex $\widetilde{C}(D')$ of another diagram D', and so on? This question is most interesting when applied to regular families of links, for example the family of torus links $T_{p,q}$ for fixed p and increasing q. Such considerations led Dunfield-Gukov-Rasmussen [**DGR06**] to conjecture the existence of a *stable homology* for torus links. This space, which we will denote by $\widetilde{Kh}^{*,*}(T_{p,\infty})$ was proved to be well-defined by Stošić [**Sto07**] and will be the main object of this chapter. The form it should take has been conjectured by Gorsky-Oblomkov-Rasmussen [**GOR13**]. These spaces have been shown to generalize the Jones Wenzl projectors by Cooper-Krushkal [**CK12**] and Rozansky [**Roz14**] independently. This chapter is divided into 2 sections. In a first section, we recall generalities about direct limits of vector spaces and chain complexes. We then define a directed system whose limit is $\widetilde{Kh}^{*,*}(T_{p,\infty})$ and describe it for the case $p = 2$. In a second section, we upgrade $\widetilde{Kh}^{*,*}(T_{p,\infty})$ to an algebra. Finally, we compute this algebra explicitly for $p = 2$.

1. Direct limits in Khovanov homology

In this section, for each $p \geq 2$, we construct a bigraded vector space $\widetilde{Kh}^{*,*}(T_{p,\infty})$ associated to the family of p-stranded negative torus links. The construction relies on the concept of directed system of chain complexes. Therefore, we begin by defining these objects, and then move on to $\widetilde{Kh}^{*,*}(T_{p,\infty})$.

Definition 5.1. A *directed system of vector spaces* $\langle V_i, f_{ji}\rangle$ is a family $\{V_i\}$ of vector spaces, indexed by a partially ordered set $(I, \leq)$, together with linear maps $f_{ji} : V_i \to V_j$ for each $i \leq j$ such that

(i) f_{ii} is the identity of V_i.
(ii) $f_{ki} = f_{kj} \circ f_{ji}$ for all $i \leq j \leq k$.

To any such system, one can associate another vector space defined by a quotient

$$V = \coprod_{i \in I} V_i / \sim .$$

The relation $\sim$ is defined as follows: if $x_i \in V_i, x_j \in V_i$, we have $x_i \sim x_j$ if there exists k such that $f_{ki}(x_i) = f_{kj}(x_j)$. The space V is called *direct limit*, denoted by $\varinjlim V_i$. Note that each V_i is equipped with a map $\varphi_i : V_i \to V$, the *canonical projection*, sending an element to its equivalence class.

If we require the V_i's to be chain complexes, and the f_{ji} to be chain maps then we have a *directed system of chain complexes*.

First we mention some general properties of the direct limit process. The direct limit is a functor from the category of directed systems of chain complexes to the category of chain complexes. Indeed, let $\langle V_i, f_{ji}\rangle$ and $\langle V_i', f_{ji}'\rangle$ be two directed systems of chain complexes, and $u_i : V_i \to V_i'$ a family of chain maps verifying $f_{ji}' \circ u_i = u_j \circ f_{ji}$, then there is an induced map, defined in the obvious way:

$$u : \varinjlim V_i \to \varinjlim V_i',$$

which commutes with the projections.

$$\begin{array}{ccc} V_i & \xrightarrow{u_i} & V_i' \\ \downarrow \varphi & & \downarrow \varphi' \\ \varinjlim V_i & \xrightarrow{u} & \varinjlim V_i'. \end{array}$$

In terms of general category theory, the direct limit is a *colimit*. In particular it is a universal object in the target category, i.e. it is unique, up to unique isomorphism.

From now on, we will consider our pre-ordered set $(I, \leq)$ to be $\mathbb{N}$ with its usual order. There are two kinds of directed systems that are particularly easy to understand, and that we will use later on: those whose maps f_{ji} are all isomorphisms or all zero. The lemma below follows directly from the definition of the limit.

LEMMA 5.1. *Let $\langle V_i, f_{ji}\rangle$ be a directed system of chain complexes over $\mathbb{N}$. Then the following hold.*

(1) If for all $i \in \mathbb{N}$, $f_{i+1,i} = 0$ then $V = 0$.
(2) If for all $i \in \mathbb{N}$, $f_{i+1,i}$ is an isomorphism then $V \cong V_0$.

Fundamentally, a directed system of chain complexes over $\mathbb{N}$ is an ordered sequence of vector spaces with some additional structure. There is an associated sequence of integers, given by their dimensions. If the sequence of dimensions converges, then any subsequence will also converge. So a natural question is whether direct limits mimic sequences of numbers well enough that this property also holds for directed systems. The answer is yes, as we prove below.

LEMMA 5.2. *Let $\langle V_i, f_{ji}\rangle$ be a directed system of chain complexes over $\mathbb{N}$ and V be its direct limit. Let i_n be a strictly increasing sequence of integers. Then the limit of the directed system*

$$\langle V_{i_n}, f_{i_m i_n}\rangle$$

is V.

Proof. Let $\langle V_i, f_{ji}\rangle$ be a directed system of chain complexes over $\mathbb{N}$ and V be its direct limit. Let i_n be a strictly increasing sequence of integers. Denote by V' the limit of the directed system

$$\langle V_{i_n}, f_{i_m i_n}\rangle.$$

We construct a chain map $h : V \longrightarrow V'$ as follows. Let $a \in V$. By definition of V, there exists $i \in \mathbb{N}$, $x \in V_i$ such that $a = [x]_V$. The sequence i_n is strictly increasing, so there exists $n \in \mathbb{N}$ such that $i_n > i$. We set $h(a) := [f_{i_n i}(x)]_{V'}$. First we show that this map is well-defined, i.e. that it depends neither on the choice of n nor on the choice of representative x. For the former, let $i_m > i$ be an integer. Without loss of generality we can assume $i_m > i_n$. Then, by property (ii) of the maps f_{ji} in Definition 5.1, we have an equality

$$f_{i_m i}(x) = (f_{i_m i_n} \circ f_{i_n i})(x)$$

and it follows immediately that $[f_{i_m i}(x)]_{V'} = [f_{i_n i}(x)]_{V'}$. For the second choice, i.e. that of representative of a, let $j \in \mathbb{N}$, $y \in V_j$ such that $a = [y]_V$. Without loss of generality, we can assume $j > i$. Then, by definition of V, there exists an integer $k > j > i$ such that $f_{ki}(x) = f_{kj}(y)$. Let $i_k > k$ be an integer in our sequence, then we have an equality

$$f_{i_k i}(x) = (f_{i_k k} \circ f_{ki})(x) = (f_{i_k k} \circ f_{kj})(y) = f_{i_k j}(y).$$

Thus h is well-defined.

Let us now show that h is an isomorphism. We begin with surjectivity of h. Let $b \in V'$. By definition of V', there exists $i_n \in \mathbb{N}$, $x \in V'_{i_n}$ such that $b = [x]_{V'}$. Since x is a representative of $[x]_V$, we have by definition of h that $h([x]_V) = [f_{i_n i_n} x]_{V'} = b$. The map h is thus onto. For injectivity, let $a \in V$ be such that $[f_{i_n i}(x)]_{V'} = h(a) = 0$, where x is a representative of a. By definition of V', there exists $i_m > i_n$ such that $f_{(i_m i)}(x) = (f_{(i_m i_n)} \circ f_{(i_n i)})(x) = 0$. Therefore we have $a = [x]_V = [0]_V = 0$ and h is injective. Hence h is an isomorphism. This concludes the proof. □

We now present two ways to construct directed systems from other directed systems.

Given a directed system of chain complexes, there is an associated directed system of vector spaces one can consider. It is defined by taking homology at each V_i, and use the induced maps f^*_{ji}. More precisely, we consider the directed system of vector spaces $\langle H_*(V_i), f^*_{ji}\langle$. We then obtain a limit vector space

$$V' := \varinjlim H_*(V_i).$$

It is natural to ask oneself whether we have an isomorphism

$$V' \cong H_*(V).$$

That is true, as we state below in Proposition 5.3.

Another possible construction is given by the tensor product. Let $\langle V_i, f_{ji}\rangle$ and $\langle W_i, g_{ji}\rangle$ be two directed systems of vector spaces. We define the tensor product of directed systems to be the directed system $\langle V_i \otimes W_i, f_{ji} \otimes g_{ji}\rangle$. In this context, it is also natural to ask oneself whether we have an isomorphism

$$\varinjlim (V_i \otimes W_i) \cong \varinjlim V_i \otimes \varinjlim W_i.$$

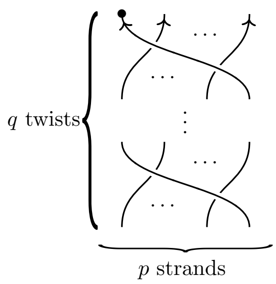
Again, the answer is yes. For both these constructions, we have grouped the answer in the following statement, with one additional useful property of the limit functor.

PROPOSITION 5.3. [**Bou70**] *The* $\varinjlim$ *functor has the following properties*

(i) The spaces $\varinjlim (V_i \otimes W_i)$ *and* $\varinjlim V_i \otimes \varinjlim W_i$ *are naturally isomorphic.*
(ii) The functors $\varinjlim$ *and* H_* *commute.*
(iii) The functor $\varinjlim$ *is exact.*

With this background in direct limits, we are now able to define the main object of this section, the limit space $\widetilde{Kh}^{*,*}(T_{p,\infty})$ associated to the family of negative torus links $T_{p,q}$ for fixed $p \geq 2$. In this, we follow Stošić [**Sto07**].

Let $D_{p,q}$ be the closure of the standard braid diagram for the negative torus link $T_{p,q}$, with upwards orientation, picture below. We set the basepoint to be at the coordinates $(1,1)$ of the braid and we will omit it from the notation henceforth.



$D_{p,q}$

Fix $p \geq 2$. For $q \geq 1$, the diagram $D_{p,q}$ is obtained from $D_{p,q+1}$ by 1-smoothing the top row of crossings. The action of 1-smoothing successively each of these $p-1$ crossings produces an inclusion of chain complexes, and we define the *ungraded* map

$$i_q : \widetilde{C}(D_{p,q}) \longrightarrow \widetilde{C}(D_{p,q+1})$$

as the composition of the inclusion obtained at each step. We iterate this process to obtain a sequence of inclusions:

$$\cdots \overset{i_{q-1}}{\hookrightarrow} \widetilde{C}(D_{p,q}) \overset{i_q}{\hookrightarrow} \widetilde{C}(D_{p,q+1}) \overset{i_{q+1}}{\hookrightarrow} \widetilde{C}(D_{p,q+2}) \overset{i_{q+2}}{\hookrightarrow} \cdots$$

This sequence of inclusions allow us to form a directed system of graded vector space

$$\langle \widetilde{C}(D_{p,q}), f_{jq} \rangle, \text{ where } f_{jq} = i_j \circ i_{j-1} \circ \cdots \circ i_q \text{ if } q < j, \text{ and } f_{qq} = 1.$$

A priori, these inclusions do not have bidegree $(0,0)$. For the limit to be a graded chain complex, we need to shift the various complexes involved to make sure they do have such a bidegree. Let us work with only one of these inclusions, and introduce back the gradings. With our choice of orientation for $D_{p,q+1}$, all the crossings we use are negative, thus for each of them the inclusion has bidegree $(0,-1)$. Their composition i_q then has bidegree $(0,-p+1)$:

$$i_q : \widetilde{C}^{*,*}(D_{p,q}) \longrightarrow \widetilde{C}^{*,*-p+1}(D_{p,q+1}).$$

By definition of the shifts, we have an equality

$$\widetilde{C}^{*,*-p+1}(D_{p,q+1}) = \widetilde{C}^{*,*}(D_{p,q+1})[0,p-1].$$

We start with $q = 1$. If we consider the shifted vector space $\widetilde{C}^{*,*}(D_{p,2})[0,p-1]$ as the target of i_1, the map

$$i_1 : \widetilde{C}^{*,*}(D_{p,1}) \longrightarrow \widetilde{C}^{*,*}(D_{p,2})[0,p-1]$$

has bidegree $(0,0)$. This process also changes the degree of i_2. We apply this shift to the rest of the directed system. The new map

$$i_2 : \widetilde{C}^{*,*}(D_{p,2})[0,p-1] \longrightarrow \widetilde{C}^{*,*}(D_{p,3})[0,p-1]$$

still has degree $(0,-p+1)$ - we applied the same shift on both sides. Therefore, we must shift the right hand vector space by an additional $[0,p-1]$, i.e. we have a degree $(0,0)$ map

$$i_2 : \widetilde{C}^{*,*}(D_{p,2})[0,p-1] \longrightarrow \widetilde{C}^{*,*}(D_{p,3})[0,(p-1)2].$$

This process repeats at each i_q, so we shift $\widetilde{C}^{*,*}(D_{p,q+1})$ by $[0,(p-1)q]$ in order for all the i_q to have bidegree $(0,0)$. The sequence of inclusions becomes:

$$\cdots \overset{i_{q-1}}{\hookrightarrow} \widetilde{C}(D_{p,q})[0,(p-1)(q-1)] \overset{i_q}{\hookrightarrow} \widetilde{C}(D_{p,q+1})[0,(p-1)q] \overset{i_{q+1}}{\hookrightarrow} \widetilde{C}(D_{p,q+2})[0,(p-1)(q+1)] \overset{i_{q+2}}{\hookrightarrow} \cdots$$

And we can factor these shifts into our directed system:

$$\langle \widetilde{C}(D_{p,q})[0,(p-1)(q-1)], f_{jq}\rangle, \text{ where } f_{jq} = i_j \circ i_{j-1} \circ \cdots \circ i_q \text{ if } q < j, \text{ and } f_{qq} = 1.$$

Now all the f_{jq} have bidegree $(0,0)$, as composition of maps with that bidegree. At the limit, this process yields a graded chain complex $(\widetilde{C}^{*,*}(D_{p,\infty}), d_\infty)$ and its homology is denoted by $\widetilde{Kh}^{*,*}(T_{p,\infty})$. Though this point of view is useful to discuss properties of $\widetilde{Kh}^{*,*}(T_{p,\infty})$, for obvious reasons of dimension, it is unsuited to computations. More practical is the induced directed system of homologies. We have the sequence of bigraded vector spaces:

$$\cdots \xrightarrow{i^*_{q-1}} \widetilde{Kh}(T_{p,q})[0,(p-1)(q-1)] \xrightarrow{i^*_q} \widetilde{Kh}(T_{p,q+1})[0,(p-1)q] \xrightarrow{i^*_{q+1}} \widetilde{Kh}(T_{p,q+2})[0,(p-1)(q+1)] \xrightarrow{i^*_{q+2}} \cdots$$

and they form a directed system via

$$\langle \widetilde{Kh}(D_{p,q})[0,(p-1)(q-1)], f^*_{jq}\rangle.$$

By property (ii) of Proposition 5.3, one has the relation

$$\widetilde{Kh}^{*,*}(T_{p,\infty}) = \varinjlim \widetilde{Kh}^{*,*}(T_{p,q})[0,(p-1)(q-1)].$$

There is a corresponding vector space for the δ-graded version, $\widetilde{Kh}^*_*(T_{p,\infty})$.

Let us begin our study with the case $p = 2$.

EXAMPLE 5.4. The vector space $\widetilde{Kh}^*_*(T_{2,\infty})$ can be explicitly described as

$$\widetilde{Kh}^i_\delta(T_{2,\infty}) = \begin{cases} \mathbb{Z}_2 & \text{if } \delta = 0 \text{ and } i \in \{0,-2,-3,-4,\dots\} \\ 0 & \text{otherwise.} \end{cases}$$

We have shown in Example 2.9 that

$$\widetilde{Kh}^i_\delta(T_{2,q}) = \begin{cases} \mathbb{Z}_2 & \text{if } \delta = -q+1 \text{ and } i \in \{0,-2,-3,\dots,-q\}. \\ 0 & \text{otherwise.} \end{cases}$$

Hence after a shift of $[0,q-1]$, we have

$$\widetilde{Kh}^i_\delta(T_{2,q})[0,q-1] = \begin{cases} \mathbb{Z}_2 & \text{if } \delta = 0 \text{ and } i \in \{0,-2,-3,\dots,-q\}. \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the shifted inclusions $i_q : \widetilde{Kh}^i_\delta(T_{2,q})[0,q-1] \longrightarrow \widetilde{Kh}^i_\delta(T_{2,q+1})[0,q]$ are all injective and have bi-degree $(0,0)$. Fix $i \in \{0,-2,-3,\cdots\}$. We consider the directed system

$$\langle \widetilde{Kh}^i_0(T_{2,q})[0,q-1], f_{q'q}\rangle$$

By definition it's limit is $\widetilde{Kh}^i_0(T_{2,\infty})$. The spaces $\widetilde{Kh}^i_0(T_{2,q})[0,q-1]$ are trivial if $-q > i$ and one dimensional if $-q \le i$. By Lemma 5.2, we can start our directed system at $q = -i$ without changing the limit. Then the maps $f_{q+1q} = i_q$ are all isomorphisms and by Lemma 5.1 (ii), we have an isomorphism

$$\widetilde{Kh}^i_0(T_{2,\infty}) \cong \widetilde{Kh}^i_0(T_{2,q})[0,q-1] = \mathbb{Z}_2.$$

If $i \notin \{0, -2, -3, \cdots\}$, then $\widetilde{Kh}^i_\delta(T_{2,q})[0, q-1]$ is zero dimensional for any $q \geq 1$ and any $\delta \in \mathbb{Z}$. In particular, the maps f_{q+1q} are all zero so by Lemma 5.1, we have $\widetilde{Kh}^i_\delta(T_{2,\infty}) = 0$. Thus we have

$$\widetilde{Kh}^i_\delta(T_{2,\infty}) = \begin{cases} \mathbb{Z}_2 & \text{if } \delta = 0 \text{ and } i \in \{0, -2, -3, -4, \dots\} \\ 0 & \text{otherwise,} \end{cases}$$

as claimed. This concludes the example.

Remark 5.5. *In the case $p = 2$, we also have that the projections*

$$\varphi_q : \widetilde{Kh}^i_\delta(T_{2,q})[0, q-1] \longrightarrow \widetilde{Kh}^i_\delta(T_{2,\infty})$$

are also all injective.

For $p = 2$, we have shown in Example 5.4 that $\widetilde{Kh}^{*,*}(T_{2,\infty})$ is non-trivial. This has been generalized for all p by Stošić. More precisely, he showed the following theorem, whose statement we adapt to our use of negative torus links.

Theorem 5.6. **[Sto07]** *Let p, q, i be integers such that $2 \leq p < q$ and $i > 3 - p - q$. Then for any $j \in \mathbb{Z}$, there is an isomorphism*

$$Kh^{i,j}(D_{p,q}) \cong Kh^{i,j-p+1}(D_{p,q-1}),$$

where $D_{p,q}$ is the standard diagram for $T_{p,q}$.

In order to understand these vector spaces, we will upgrade them to algebras, through the use of movies made of successive 1-handles.

2. Definition of the algebra and the 2-stranded links

This section's main focus is the definition of an algebra structure for $\widetilde{Kh}^{*,*}(T_{p,\infty})$ and its explicit description for the 2-stranded case. We begin with the definition of a family of maps induced by successive 1-handles, compatible with the limit process that defined $\widetilde{Kh}^{*,*}(T_{p,\infty})$.

2.1. Definition of the algebra structures. Consider two braids β, β' over p strands. There is a simple movie starting at the connected sum of their closures $\hat{\beta}, \hat{\beta}'$ and ending at the closure of their composition (as braids). It can be described as follows (Figure 26): start with the two closed braids on top of each other, i.e. with a diagram for their disjoint union. There is a natural choice of strands in $\hat{\beta}$ and $\hat{\beta}'$ such that a 1-handle realizes a connected sum. With this diagram for the connected sum, there is again a natural choice of two strands, one in $\hat{\beta}$ and one in $\hat{\beta}'$ that can be connected with a 1-handle (see Figure 26). We repeat this process until we run out of strands to connect: the last diagram is a diagram for the closure of $\beta \circ \beta'$. If we start at the connected sum, the movie is made of $p-1$ successive 1-handles, so it induces a map of bidegree $(0, -p+1)$:

$$\Sigma^p : \widetilde{Kh}^{*,*}(\widehat{\beta} \sharp \widehat{\beta'}) \longrightarrow \widetilde{Kh}^{*,*-p+1}(\widehat{\beta \circ \beta'})$$

If we pre-compose with the isomorphism S^* of Proposition 2.2, we obtain a map

$$\widetilde{\Sigma}^p : \widetilde{Kh}^{*,*}(\widehat{\beta}) \otimes \widetilde{Kh}^{*,*}(\widehat{\beta'}) \xrightarrow{S^*} \widetilde{Kh}^{*,*}(\widehat{\beta \sharp \beta'}) \xrightarrow{\Sigma^p} \widetilde{Kh}^{*,*-p+1}(\widehat{\beta \circ \beta'}),$$

which is the prototype map for the algebra structure for $\widetilde{Kh}^{*,*}(T_{p,\infty})$ we introduce in this section. We will specialize the two braids β and β' to be the standard braids that close into torus links. For example, if $p = 3$, and $\hat{\beta} = T_{3,q}, \hat{\beta}' = T_{3,q'}$, then $\widehat{\beta \circ \beta'} = T_{3,q+q'}$ and the movie is pictured in Figure 26.

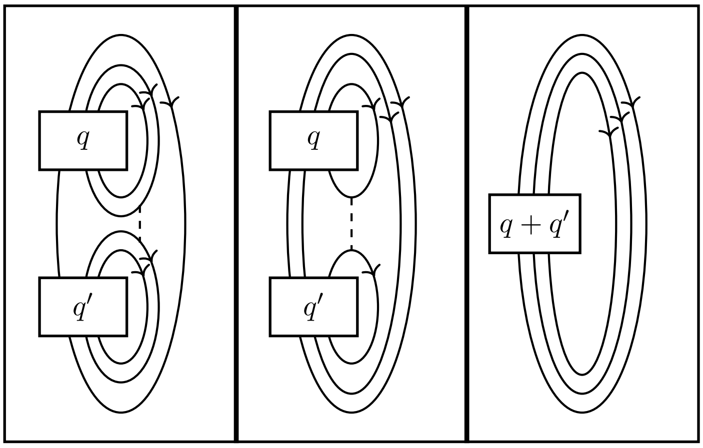


FIGURE 26. The fusion movie for $p = 3$

In this case the induced map has the form

$$\Sigma^3_{q,q'} : \widetilde{C}(D_{3,q} \sharp D_{3,q'}) \longrightarrow \widetilde{C}(D_{3,q+q'})$$

For any $p \geq 2$, the vector space $\widetilde{Kh}(T_{p,\infty})$ can be endowed with the structure of an algebra, which we describe now. Recall that any movie M made of successive 1-handles produces chain maps Φ^1_M, Φ_M such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{C}(D_1) & \longrightarrow & \widetilde{C}(D) \\ \downarrow \Phi^1_M & & \downarrow \Phi_M \\ \widetilde{C}(D'_1) & \longrightarrow & \widetilde{C}(D') \end{array}$$

where D_1 (resp. D'_1) is obtained from D (resp. D') by 1-smoothing of a crossing c and the horizontal arrows are the inclusions of the corresponding sub-complexes. We can iterate this process by smoothing more crossings, creating a ladder

$$\begin{array}{ccccc} \cdots \longrightarrow & \widetilde{C}(D_1) & \longrightarrow & \widetilde{C}(D) & \longrightarrow \cdots \\ & \downarrow \Phi^1_M & & \downarrow \Phi_M & \\ \cdots \longrightarrow & \widetilde{C}(D'_1) & \longrightarrow & \widetilde{C}(D') & \longrightarrow \cdots \end{array}$$

Define

$$\Sigma^p_{q,q'} : \widetilde{C}(D_{p,q} \sharp D_{p,q'}) \longrightarrow \widetilde{C}(D_{p,q+q'})$$

as the chain map with bidegree $(0, -p+1)$ induced by our movie over p strands. If we set $\Phi_M = \Sigma^p_{q,q'}$. $D_1 = T_{p,q} \sharp T_{p,q'}$ and $D = T_{p,q+1} \sharp T_{p,q'}$ in the ladder above, then in homology we get a commutative ladder

$$\begin{array}{ccccccc} \cdots \xrightarrow{i^*_{q-1}} & \widetilde{Kh}(T_{p,q} \sharp T_{p,q'}) & \xrightarrow{i_q} & \widetilde{Kh}(T_{p,q+1} \sharp T_{p,q'}) & \xrightarrow{i^*_{q+1}} \cdots \\ & \downarrow \Sigma^p_{q,q'} & & \downarrow \Sigma^p_{q+1,q'} & \\ \cdots \xrightarrow{i^*_{q+q'-1}} & \widetilde{Kh}(T_{p,q+q'}) & \xrightarrow{i_{q+q'}} & \widetilde{Kh}(T_{p,q+q'+1}) & \xrightarrow{i^*_{q+q'+1}} \cdots \end{array}$$

There is a similar ladder if q' increases instead of q. Now, consider the composite

$$\widetilde{\Sigma}^p_{q,q'} : \widetilde{Kh}^{i,j}(T_{p,q}) \otimes \widetilde{Kh}^{i',j'}(T_{p,q'}) \xrightarrow{S^*} \widetilde{Kh}^{i+i',j+j'}(T_{p,q} \sharp T_{p,q'}) \xrightarrow{\Sigma^p_{q,q'}} \widetilde{Kh}^{i+i',j+j'-p+1}(T_{p,q+q'})$$

In order to match the directed system which produces $\widetilde{Kh}^{*,*}(T_{p,\infty})$, let us shift $\widetilde{Kh}^{*,*}(T_{p,q})$ by $[0,(p-1)(q-1)]$ and $\widetilde{Kh}^{*,*}(T_{p,q'})$ by $[0,(p-1)(q'-1)]$, thus performing a total shift of $[0,(p-1)(q+q'-2)]$. If we shift the co-domain accordingly, by definition of the shifts we have

$$\widetilde{Kh}^{i+i',j+j'-p+1}(T_{p,q+q'})[0,(p-1)(q+q'-2)] = \widetilde{Kh}^{i+i',j+j'}(T_{p,q+q'})[0,(p-1)(q+q'-1)].$$

Hence the shifted version of the map $\widetilde{\Sigma}^p_{q,q'}$ has degree $(0,0)$:

$$\begin{array}{c}\widetilde{Kh}^{i,j}(T_{p,q})[0,(p-1)(q-1)] \otimes \widetilde{Kh}^{i',j'}(T_{p,q'})[0,(p-1)(q'-1)] \\ \Big\downarrow \widetilde{\Sigma}^p_{q,q'} \\ \widetilde{Kh}^{i+i',j+j'}(T_{p,q+q'})[0,(p-1)(q+q'-1)]\end{array}$$

To recover $\widetilde{Kh}^{*,*}(T_{p,\infty})$, we need to look properly at two directed systems that appear as the domain and co-domain of $\widetilde{\Sigma}^p_{q,q'}$. We study them below.

(1) For the domain, we have the sequence of maps induced by inclusions, where q increase by 1 at each step:

$$\cdots \xrightarrow{i_{q-1}\otimes i_{q-1}} \widetilde{Kh}^{i,j}(T_{p,q}) \otimes \widetilde{Kh}^{i',j'}(T_{p,q}) \xrightarrow{i_q\otimes i_q} \cdots$$

Therefore the directed system we consider is the tensor product of two copies of our original directed system

$$\langle \widetilde{Kh}(D_{p,q})[0,(p-1)(q-1)], f^*_{jq}\rangle.$$

By Proposition 5.3 (i), we have an natural isomorphism

$$\widetilde{Kh}^{*,*}(T_{p,\infty}) \otimes \widetilde{Kh}^{*,*}(T_{p,\infty}) \cong \varinjlim \widetilde{Kh}(D_{p,q})[0,(p-1)(q-1)] \otimes \widetilde{Kh}(D_{p,q})[0,(p-1)(q'-1)].$$

(2) For the co-domain, we set $m = 2q$, and consider the sequence of spaces, where m increases by 2 at each step, since q increases by 1 at each step:

$$\cdots \xrightarrow{i_{m-1}\circ i_{m-2}} \widetilde{Kh}^{i,j}(T_{p,m})[0,(p-1)(m-1)] \cdots \xrightarrow{i_{m+1}\circ i_m} \cdots$$

The associated directed system is thus, where m runs over odd positive integers:

$$\langle \widetilde{Kh}^{*,*}(T_{p,m})[0,(p-1)(m-1)], f^*_{jm}\rangle.$$

By Lemma 5.2, we have an isomorphism:

$$\widetilde{Kh}^{*,*}(T_{p,\infty}) \cong \varinjlim \widetilde{Kh}(T_{p,m})[0,(p-1)(m-1)].$$

It is not clear yet that the family of maps $\widetilde{\Sigma}^p_{q,q}$ produce a map at the limit: we still need to show that they give a map of directed systems. Let us briefly use the notation $A_q := \widetilde{Kh}^{*,*}(T_{p,q+1})[0,(p-1)(q-1)]$ and consider the diagram in Figure 27. The horizontal faces commute, just by definition of the maps involved. The vertical faces of the cubes where one parameter is fixed also commute: these are the squares in our previous ladders. Hence the middle square, the one we are interested in, where both parameters q and q' increase, also commutes.

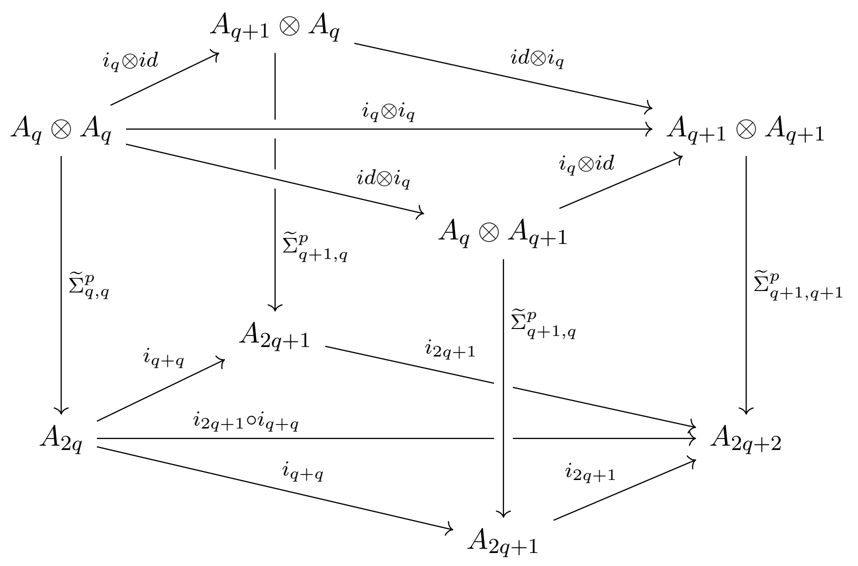


FIGURE 27. The maps $\widetilde{\Sigma}^p_{q,q}$ are maps of directed systems.

The commutativity makes $\widetilde{\Sigma}^p_{q,q}$ into a map of directed systems with degree (0,0):

$$\widetilde{\Sigma}^p_{q,q} : \langle A_q \otimes A_q, i_q \otimes i_q \rangle \longrightarrow \langle A_{2q}, i_{2q+1} \circ i_{qq} \rangle$$

The direct limit functor, together with property (i) of Proposition 5.3, produces a map:

$$\widetilde{\Sigma}^p_\infty : \widetilde{Kh}^{i,j}(T_{p,\infty}) \otimes \widetilde{Kh}^{i',j'}(T_{p,\infty}) \longrightarrow \widetilde{Kh}^{i+i',j+j'}(T_{p,\infty}),$$

which satisfies the relation $\widetilde{\Sigma}^p_\infty \circ (\varphi_q \otimes \varphi_q) = \varphi_{2q} \circ \widetilde{\Sigma}^p_{q,q}$, where φ_q is the canonical projection sending $x \in A_q$ to its equivalence class in $\widetilde{Kh}^{*,*}(T_{p,\infty})$.

We will refer to the maps $\widetilde{\Sigma}^p_\infty$ and their finite counterparts $\widetilde{\Sigma}^p_{q,q'}$ as *fusion maps.* We are now ready to state and prove the main theorem of this section, namely that the fusion maps give additional structure to the limit space $\widetilde{Kh}^{*,*}(T_{p,\infty})$.

THEOREM 5.7. *For each $p \geq 2$, the fusion map*

$$\widetilde{\Sigma}^p_\infty : \widetilde{Kh}^{*,*}(T_{p,\infty}) \otimes \widetilde{Kh}^{*,*}(T_{p,\infty}) \longrightarrow \widetilde{Kh}^{*,*}(T_{p,\infty})$$

endows the stable space $\widetilde{Kh}^{*,*}(T_{p,\infty})$ with the structure of a commutative bi-graded unitary algebra.

Proof. In order to show that $\widetilde{Kh}^{*,*}(T_{p,\infty})$ is an algebra, we need only show that the map $\widetilde{\Sigma}^p_\infty$ is left and right distributive. These two properties follow immediately from the fact that $\widetilde{\Sigma}^p_\infty$ is a linear map with domain the tensor product of two copies of $\widetilde{Kh}^{*,*}(T_{p,\infty})$. Also, since the map $\widetilde{\Sigma}^p_\infty$ has bi-degree $(0,0)$, our structure will be bi-graded.

For the associativity recall the main property of the canonical projections $\varphi_i = \varphi_j \circ f_{ji}$ if $j \geq i$ and that by definition, a map of directed systems commutes with the f_{ji}. Since any

$a \in \widetilde{Kh}^{*,*}(T_{p,\infty})$ arises as the equivalence class of some $x \in \widetilde{Kh}^{*,*}(T_{p,q})$ via the projections, then the associativity

$$\widetilde{\Sigma}^p_\infty \circ (\widetilde{\Sigma}^p_\infty \otimes id) = \widetilde{\Sigma}^p_\infty \circ (id \otimes \widetilde{\Sigma}^p_\infty)$$

will follow from the equality

$$\widetilde{\Sigma}^p_\infty \circ (\widetilde{\Sigma}^p_\infty \otimes id) \circ (\varphi_q \otimes \varphi_q \otimes \varphi_q) = \widetilde{\Sigma}^p_\infty \circ (id \otimes \widetilde{\Sigma}^p_\infty) \circ (\varphi_q \otimes \varphi_q \otimes \varphi_q),$$

which we show now. Using the properties recalled above, we have a sequence of equalities, for the left hand side

$$\begin{aligned}
\widetilde{\Sigma}^p_\infty \circ (\widetilde{\Sigma}^p_\infty \otimes id) \circ (\varphi_q \otimes \varphi_q \otimes \varphi_q) &= \widetilde{\Sigma}^p_\infty \circ (\varphi_{2q} \otimes \varphi_q) \circ (\widetilde{\Sigma}^p_{q,q} \otimes id) \\
&= \widetilde{\Sigma}^p_\infty \circ (\varphi_{2q} \otimes (\varphi_{2q} \circ f_{2qq})) \circ (\widetilde{\Sigma}^p_{q,q} \otimes id) \\
&= \widetilde{\Sigma}^p_\infty \circ (\varphi_{2q} \otimes \varphi_{2q}) \circ (id \otimes f_{2qq})) \circ (\widetilde{\Sigma}^p_{q,q} \otimes id) \\
&= \varphi_{4q} \circ \widetilde{\Sigma}^p_{2q,2q} \circ (id \otimes f_{2qq})) \circ (\widetilde{\Sigma}^p_{q,q} \otimes id) \\
&= \varphi_{4q} \circ f_{4q3q} \circ \widetilde{\Sigma}^p_{2q,q} \circ (\widetilde{\Sigma}^p_{q,q} \otimes id) \\
&= \varphi_{3q} \circ \widetilde{\Sigma}^p_{2q,q} \circ (\widetilde{\Sigma}^p_{q,q} \otimes id).
\end{aligned}$$

For the left hand side, we apply the same steps to obtain:

$$\widetilde{\Sigma}^p_\infty \circ (id \otimes \widetilde{\Sigma}^p_\infty) \circ (\varphi_q \otimes \varphi_q \otimes \varphi_q) \quad = \quad \varphi_{3q} \circ \widetilde{\Sigma}^p_{q,2q} \circ (id \otimes \widetilde{\Sigma}^p_{q,q}).$$

Recall from Lemma 1.17 that two movies that differ from an exchange of distant 1-handles induce the same map in homology. Therefore, for any choice of q, q', q'', we have an equality

$$\widetilde{\Sigma}^p_{q,q'+q''} \circ (id \otimes \widetilde{\Sigma}^p_{q',q''}) = \widetilde{\Sigma}^p_{q+q',q''} \circ (\widetilde{\Sigma}^p_{q,q'} \otimes id).$$

In particular, if we set $q = q' = q''$, we then have

$$\widetilde{\Sigma}^p_{q,2q} \circ (id \otimes \widetilde{\Sigma}^p_{q,q}) = \widetilde{\Sigma}^p_{2q,q} \circ (\widetilde{\Sigma}^p_{q,q} \otimes id).$$

Therefore we have the equality

$$\widetilde{\Sigma}^p_\infty \circ (\widetilde{\Sigma}^p_\infty \otimes id) \circ (\varphi_q \otimes \varphi_q \otimes \varphi_q) = \widetilde{\Sigma}^p_\infty \circ (id \otimes \widetilde{\Sigma}^p_\infty) \circ (\varphi_q \otimes \varphi_q \otimes \varphi_q),$$

and the associativity follows. We are left with proving the existence of a neutral element and the commutativity.

For the unit, we consider $D = T_{p,1}$. We aim to show that the generator of $\widetilde{Kh}^{0,0}(D,p)$ maps via φ_1 to a unit in $\widetilde{Kh}^{*,*}(T_{p,\infty})$. Since $\widetilde{Kh}^{i,j}(D) = 0$ for $(i,j) \neq (0,0)$, we can restrict the fusion map to

$$\widetilde{\Sigma}^p_{1,q} : \widetilde{Kh}^{0,0}(T_{p,1}) \otimes \widetilde{Kh}^{i',j'}(T_{p,q}) \longrightarrow \widetilde{Kh}^{i',j'-p+1}(T_{p,q+1}).$$

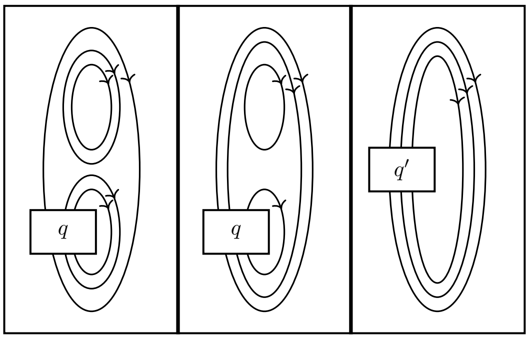
Therefore, we can also restrict our attention to the chain complex $\widetilde{C}^{0,*}(D) \otimes \widetilde{C}^{*,*}(T_{p,q})$. At homological degree $i = 0$, $\widetilde{C}^{i,*}(D)$ arises from 1-smoothing all the crossings in D. We set $A = \chi_D$, and the smoothing s_A is a collection of p concentric circles. Recall we set the basepoint to have coordinates $(1,1)$, so it lies on the outermost circle. We consider the element $x_\bullet \in \widetilde{C}^{0,*}(D)$. It has quantum degree given by

$$q(x_\bullet) = |s_A| - 2 + |A| + n_+(D) - 2n_-(D) + 1 = p - 2 + (p-1) + 0 - 2(p-1) + 1 = 0.$$

Note that here we have added "+1" to the original formula for the quantum grading, in order to take into account the shift in the definition $\widetilde{C}^{*,*}(D,p) = \ker(x_\bullet)^{*,*}[0,1]$. Any other monomial has strictly lower quantum degree, since it has more variables. Therefore $\widetilde{Kh}^{0,0}(D) = \mathbb{Z}_2\{x_\bullet\}$. For $[x_\bullet]$ to be a unit, we show that, for any $q \geq 1$, the fusion map satisfies

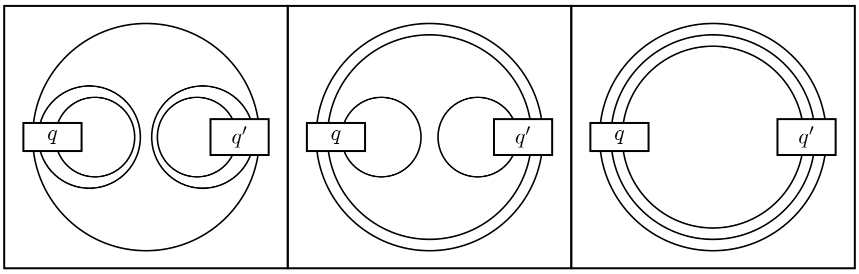
$$\widetilde{\Sigma}^p_{1,q}(x_\bullet \otimes x_\gamma v) = x_\bullet v.$$

We rely on the movie below, which starts at the disjoint union of $T_{p,q}$ and $p-1$ copies of an unknot and ends at $T_{p,q}$. This movie is induced by the fusion movie through 1-smoothing all the crossings in D.



Since $S(x_\bullet \otimes x_\gamma v) = x_\bullet v$, we need only show that $x_\bullet v$ is fixed by $\Sigma^p_{1,q}$. The fusion map is a succession of merging of circles, therefore it acts on the chain complex with products $m_{A,B}$. Since $x_\bullet$ is the only variable in the generator, with respect to the circles in the top half, it follows that $\Sigma^p_{1,q}(x_\bullet v) = x_\bullet v$ by definition of the maps $m_{A,B}$.

For the commutativity, we consider the picture below, which is symmetric in q and q'.



This movie represents the fusion map and is symmetric in q and q'. Therefore the fusion map does not depend on whether the q' factor is on top or at the bottom: we have commutativity. This concludes the proof. □

Given $a, b \in \widetilde{Kh}^{*,*}(T_{p,\infty})$, we define their *product* by $a \cdot b := \widetilde{\Sigma}^p_\infty(a \otimes b)$.

2.2. **2-stranded torus links.** We study the algebra structure associated to the family of 2-stranded torus links. The final result of this thesis, Theorem 5.10, will rely on our explicit knowledge of $\widetilde{Kh}^*_*(T_{2,\infty})$ (described in Example 5.4). Lemma 5.8 will be the key to understanding the algebra structure. We will use the δ-graded version of Khovanov homology throughout this section.

LEMMA 5.8. *For any $q \geq 3$, the fusion maps induced by the movie above:*

(i) $\widetilde{\Sigma}^2_{2,q} : \widetilde{Kh}^*_*(T_{2,2}) \otimes \widetilde{Kh}^*_*(T_{2,q}) \longrightarrow \widetilde{Kh}^*_*(T_{2,2+q})$

(ii) $\widetilde{\Sigma}^2_{3,q} : \widetilde{Kh}^*_*(T_{2,3}) \otimes \widetilde{Kh}^*_*(T_{2,q}) \longrightarrow \widetilde{Kh}^*_*(T_{2,3+q})$

are surjective.

REMARK 5.9. *For both (i) and (ii), the result for $q = 3$ has already been shown in Examples 2.15 and 2.19. The proof will follow the same argument.*

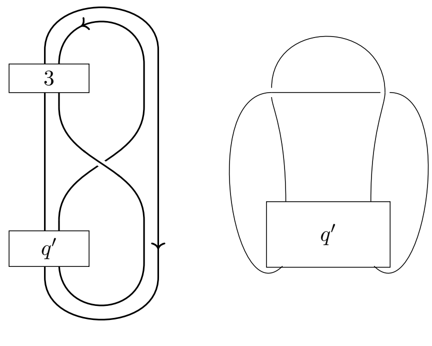
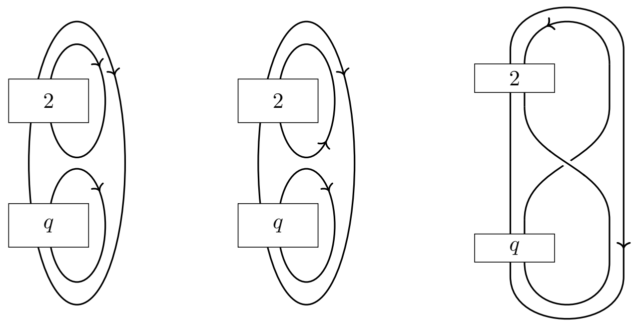


FIGURE 28. Two diagrams for the alternating knot $L_{q'}$ in the proof of lemma 5.8.

Proof. The maps in both cases are induced by oriented 1-handle moves. Therefore we shall first "complete the triples". For the first map, the movie begins with $F = T_{2,2} \sharp T_{2,q}$ and $F' = T_{2,q+2}$ and the steps are given below.

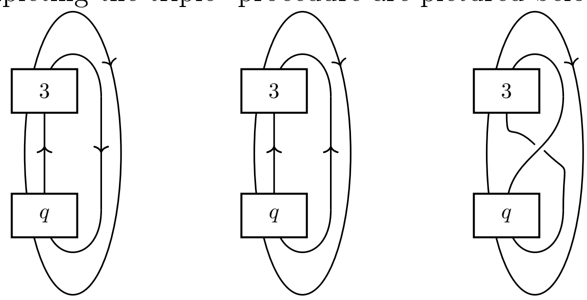


The two strands of F involved in the "completing the triple" procedure don't belong to the same component, so we modify F. The triple is completed by a diagram D equivalent to a negative torus link $T_{2,q-2}$. The dimensions if these homologies are related by the formula:

$$\dim(\widetilde{Kh}_*^*(T_{2,2} \sharp T_{2,q})) = 2q = (q-2) + (q+2) = \dim \widetilde{Kh}_*^*(T_{2,q-2}) + \dim(\widetilde{Kh}_*^*(T_{2,q+2})).$$

By Proposition 2.12, the map $\Sigma^2_{2,q'}$ must be surjective.

The second map is induced by a movie starting at $F = T_{2,3} \sharp T_{2,q}$ and ending at $F' = T_{2,q+3}$. The steps of the "completing the triple" procedure are pictured below (for q odd).



If q is even, then the strands in F involved in the procedure belong to different components, so we modify F. If q is odd, one should modify F' instead. In both cases, the triple is completed by a diagram D equivalent to the knot $L_{q'-2}$ of Figure 28, which is alternating thus $\widetilde{Kh}$-thin.

Moreover, F' is also $\widetilde{Kh}$-thin, supported in $\delta_{F'} = -q-2$. Finally, by Corollary 2.4, F is $\widetilde{Kh}$-thin, supported in

$$\delta_F = -2 + (-q+1) = -q-1 = \delta_{F'} + 1.$$

Thus, by Lemma 2.18, the map is surjective. This concludes the proof. □

THEOREM 5.10. *There is a bi-graded algebra isomorphism:*

$$\widetilde{Kh}^*_*(T_{2,\infty}) \cong \mathbb{Z}_2[x,y]/(x^3 = y^2).$$

The degrees of the generators are given by $|x| = (-2,0)$, $|y| = (-3,0)$.

Proof. We already know $\widetilde{Kh}^{*,*}(T_{2,\infty})$ as a bigraded vector space from Example 5.4. It is described as

$$\widetilde{Kh}^i_\delta(T_{2,\infty}) = \begin{cases} \mathbb{Z}_2 & \text{if } \delta = 0 \text{ and } i \in \{0,-2,-3,-4,\dots\} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for each $i \in \mathbb{Z}$, the space is either 0 or 1-dimensional. Denote by a_i the generator of $\widetilde{Kh}^{-i}_*(T_{2,\infty})$, whenever it is non zero. As an algebra, there are at least 2 generators a_2, a_3 in homological degrees $-2,-3$ respectively for $\widetilde{Kh}^*_*(T_{2,\infty})$, and a_0 is the neutral element. We aim to show the following.

(1) For any $n \geq 5$, $a_n = a_2 \cdot a_{n-2}$.
(2) For any $n \geq 5$, $a_n = a_3 \cdot a_{n-3}$.
(3) The relation $a_4 = a_2^2$ holds in $\widetilde{Kh}^{*,*}(T_{2,\infty})$.

The canonical projections

$$\varphi_q : \widetilde{Kh}^*_*(T_{2,q})[0,q-1] \longrightarrow \widetilde{Kh}^*_*(T_{2,\infty})$$

are all injective (Example 5.4), so we can work with the fusion maps at a finite level. We identify the generators of $\widetilde{Kh}^*_*(T_{2,q})$ with the a_i's. Let $n \geq 5$. By Lemma 5.8 (i), the map below is surjective

$$\widetilde{\Sigma}^2_{2,n-2} : \widetilde{Kh}^*_*(T_{2,2}) \otimes \widetilde{Kh}^*_*(T_{2,n-2}) \longrightarrow \widetilde{Kh}^*_*(T_{2,n}).$$

We set the homological degree on the right hand side to be $-n$, so that $\widetilde{Kh}^*_*(T_{2,n}) = \mathbb{Z}_2\{a_n\}$. Hence there exists $i_1, i_2 \in \mathbb{Z}$ with $i_1 + i_2 = -n$, such that the restricition

$$\widetilde{\Sigma}^2_{2,n-2} : \widetilde{Kh}^{i_1}_*(T_{2,2}) \otimes \widetilde{Kh}^{i_2}_*(T_{2,n-2}) \longrightarrow \widetilde{Kh}^{-n}_*(T_{2,n})$$

is surjective. There is a unique choice of i_1 and i_2 such that $\widetilde{Kh}^{i_1}_*(T_{2,2}) \otimes \widetilde{Kh}^{i_2}_*(T_{2,n-2})$ is non zero, namely $i_1 = -2$ and $i_2 = -n+1$. We plug these values in the map and obtain that

$$\widetilde{\Sigma}^2_{2,n-2} : \mathbb{Z}_2\{a_2\} \otimes \mathbb{Z}_2\{a_{n-2}\} \longrightarrow \mathbb{Z}_2\{a_n\}.$$

is surjective. In terms of image, we then obtain (1):

$$a_n = \widetilde{\Sigma}^2_{2,n-2}(a_2 \otimes a_{n-2}) = a_2 \cdot a_{n-2},$$

as claimed. For $n \geq 5$, the same process can be applied to the map

$$\widetilde{\Sigma}^2_{3,n-3} : \widetilde{Kh}^*_*(T_{2,3}) \otimes \widetilde{Kh}^*_*(T_{2,n-3}) \longrightarrow \widetilde{Kh}^*_*(T_{2,n})$$

which is surjective by Lemma 5.8 (ii). This provides the identification (2)

$$a_n = a_3 \cdot a_{n-3}.$$

We move on to (3), i.e we show that $a_4 = a_2^2$. We follow a similar process and consider the surjective map

$$\widetilde{\Sigma}^2_{3,2} : \widetilde{Kh}^*_*(T_{2,3}) \otimes \widetilde{Kh}^*_*(T_{2,2}) \longrightarrow \widetilde{Kh}^*_*(T_{2,5}).$$

We restrict the right-hand side to $i = -4 = i_1 + i_2$. The only possible choice is $i_1 = i_2 = -2$. Thus the identification (3):

$$a_4 = a_2^2.$$

Hence our algebra has exactly 2 generators, namely a_2 and a_3. From (1),(2) and (3), we derive the relation

$$a_2^3 = a_2 \cdot a_2^2 = a_2 \cdot a_4 = a_6 = a_3^2.$$

Consider the bidegree $(0,0)$ algebra homomorphism defined by

$$\begin{array}{rcl} g : \mathbb{Z}_2[x,y]/(x^2 = y^3) & \longrightarrow & \widetilde{Kh}^*_*(T_{2,\infty}) \\ x & \longmapsto & a_2 \\ y & \longmapsto & a_3 \end{array}$$

Since $\widetilde{Kh}^{*,*}(T_{2,\infty})$ is generated, as an algebra by a_2 and a_3, the map g is surjective. Let us show that it is also injective. For any monogeneous element v in $\mathbb{Z}_2[x,y]/(x^2 = y^3)$, there exists $n \in \mathbb{N}$, $m \in \{0,1\}$ such that $v = x^n y^m$. Let $w = x^{n'} y^{m'}$ be such that $g(v) = g(w)$. Since g has bidegree $(0,0)$, we must have an equality of the degrees $(2n + 3m, 0) = (2n' + 3m')$. This equation, taken modulo 2, yields $m = m' (\mod 2)$. Since both $m, m' \in \{0,1\}$, we must have $m = m'$. Consequently, we also have $n = n'$ and thus $v = w$, i.e the map g is injective. Thus it is an isomorphism. This concludes the proof. □

2.3. The Gorsly-Oblomkov-Rasmussen Conjecture. We conclude this chapter by comparing our result with the Gorsky-Oblomkov-Rasmussen conjecture [**GOR13**] (whose statement we adapt to our conventions). Consider the polynomial ring in variables $x_1, \dots x_{p-1}$ and an equal number of odd variables $\chi_1, \dots, \chi_{p-1}$ bi-graded as

$$|x_k| = (-2k, 2k-2), \qquad |\chi_k| = (-2k-1, 2k-2).$$

The differential d is given by the formula

$$d(x_k) = 0, \qquad d(\chi_k) = \sum_{i=1}^{k-1} x_i x_{k-i}.$$

CONJECTURE 1 (Gorsky-Oblomkov-Rasmussen,[**GOR13**]). *The stable homology $\widetilde{Kh}^*_*(T_{p,\infty})$ is isomorphic to the homology of $\mathbb{Z}_2[x_1, \dots, x_{p-1}] \otimes \Lambda[\chi_1, \dots, \chi_{p-1}]$ with respect to d.*

In the case $p = 2$, we consider the ring

$$\mathbb{Z}_2[x_1, \dots, x_{p-1}] \otimes \Lambda[\chi_1, \dots, \chi_{p-1}] = \mathbb{Z}_2[x_1, \chi_1]/(\chi_1^2 = 0).$$

The differential is given by

$$d(x_1) = 0, \qquad d(\chi_1) = 0.$$

The conjectured algebra is then given by

$$H_c(T_{2,\infty}) = \mathbb{Z}_2[x_1, \chi_1]/(\chi_1^2 = 0).$$

The result from Theorem 5.10 is different, we have $x_1^3 = \chi_1^2$ with these notations. So the conjectured algebra structure does not match ours, even if the underlying vector spaces are the same.

This discrepancy can actually be clarified as the algebras $\widetilde{Kh}_*^*(T_{2,\infty})$ and $H_c(T_{2,\infty})$ can be related. Indeed, the algebra $\widetilde{Kh}_*^*(T_{2,\infty})$ is a filtered algebra, and the associated graded algebra is $H_c(T_{2,\infty})$ as we show now.

Consider the filtration on $\widetilde{Kh}_*^*(T_{2,\infty})$ defined as

$$0 \subset \mathcal{F}^0(\widetilde{Kh}_*^*(T_{2,\infty})) = \{u \in \widetilde{Kh}_*^*(T_{2,\infty}) \mid u \text{ has even homological degree } \} \subset \widetilde{Kh}_*^*(T_{2,\infty}).$$

The bidegree is additive under product, so $\mathcal{F}^0(\widetilde{Kh}_*^*(T_{2,\infty}))$ is sub-algebra. Thence $\widetilde{Kh}_*^*(T_{2,\infty})$ is a filtered algebra and the associated graded algebra is given by

$$\mathcal{G}_2 = \mathcal{F}^0(\widetilde{Kh}_*^*(T_{2,\infty})) \bigoplus \frac{\widetilde{Kh}_*^*(T_{2,\infty})}{\mathcal{F}^0(\widetilde{Kh}_*^*(T_{2,\infty}))}.$$

The second summand contains the elements with odd homological degree and their squares must be zero. By definition of $\mathcal{G}_2$, there is an isomorphism of vector spaces $\mathcal{G}_2 \cong \widetilde{Kh}_*^*(T_{2,\infty})$. Moreover, we have

$$\mathcal{G}_2 \cong H_c(T_{2,\infty}).$$

Thus the graded algebra associated to the filtered algebra $\widetilde{Kh}_*^*(T_{2,\infty})$ is the algebra conjectured by Gorsky, Oblomkov and Rasmussen.

CHAPTER 6

Outlook

In this very informal chapter, we discuss the general case of $\widetilde{Kh}^{*,*}(T_{p,\infty})$. More precisely, we propose a strategy to generalize the results of Theorem 5.10. This Chapter is divided into two sections. In a first we propose a possible direction of study for $\widetilde{Kh}^{*,*}(T_{3,\infty})$. In a second section, we outline a strategy for the general case and point out what features of the proof of the previous section survive, which don't and how one might still be able to give a description of $\widetilde{Kh}^{*,*}(T_{p,\infty})$.

1. The approach to $\widetilde{Kh}^{*,*}(T_{3,\infty})$.

In this informal section, we point out key features that could help us describe $\widetilde{Kh}^{*,*}(T_{3,\infty})$.

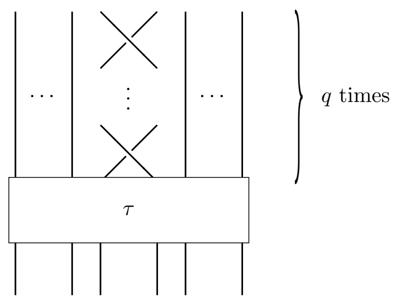
In order to compute $\widetilde{Kh}^{*,*}(T_{3,\infty})$, here are the main steps we could follow:

(1) For any (n,n)-tangle we can define a vector space $\widetilde{Kh}(\tau,2,l)$ defined as follows. For any fixed $l \in \{1,\dots,n-1\}$, we define a family of (n,n)-tangles $\{\tau^{l,q}\}_{q\geq 0}$ in terms of composition of tangles:

$$\tau^{l,q} = \underbrace{\sigma_l \circ \cdots \circ \sigma_l}_{q \text{ times}} \circ \tau.$$

The corresponding diagram is the depicted below.

$\tau^{l,q} =$



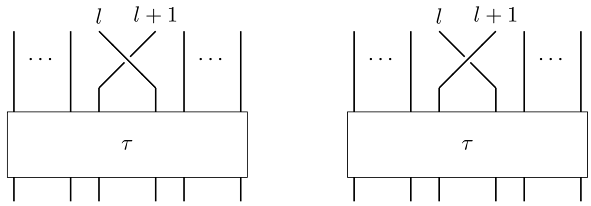
By smoothing crossings, as we did to define the limit space $\widetilde{Kh}^{*,*}(T_{p,\infty})$, we obtain a directed system of vector spaces:

$$\langle \widetilde{Kh}^{*,*}(D^q), f^*_{jq}\rangle, \text{ where } f_{jq} = i_{j-1} \circ \cdots \circ i_q, \text{ if } q < j, \text{ and } f_{qq} = 1.$$

We denote it's limit by $\widetilde{Kh}(\tau,2,l)$ and call it the *(2,l)-limit.*

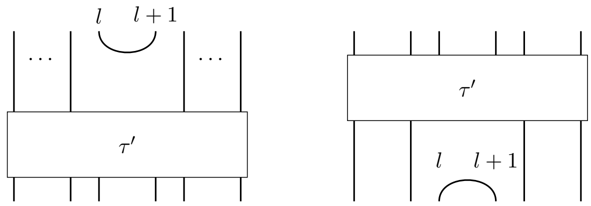
Depending on the sign of the crossings, we may not be able to obtain a graded vector space. However, if we consider oriented tangles that are *admissible*, i.e. such that the two strands are both oriented in the same direction, then the all the maps f_{ji} have bidegree

$(0,0)$ and the limit is bigraded. This limit has two main properties: it can *absorb* crossings. More precisely, if $\tau^A = \sigma_l \circ \tau$, and $\tau^B = \sigma^{-1} \circ \tau$ are the tangles pictured below,



τ^A τ^B

then their 2-limit coincides with that of τ. Moreover, it *annihilates cups and caps*, i.e. if τ is either one of the diagrams below



$\tau = \varepsilon_l \circ \tau'$ $\tau = \tau' \circ \varepsilon_l^{-1}$

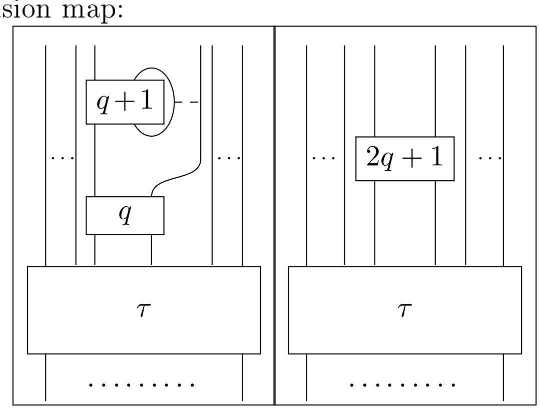
then we have $\widetilde{Kh}(\tau, 2, l) = 0$.

Finally, using the exactness of the limit functor, if c is a crossing in τ an admissible tangle such that τ_0 and τ_1 are also admissible, then we obtain a limit short exact sequence

$$0 \longrightarrow \widetilde{C}^{*,*}(\tau_1, l)[w_+, 3w_+ - 1] \xrightarrow{i} \widetilde{C}^{*,*}(\tau, l) \xrightarrow{\pi} \widetilde{C}^{*,*}(\tau_0, l)[w_-, 3w_- + 1] \longrightarrow 0,$$

where w_+ and w_- are the shifts associated to the original exact triple $(\hat{\tau_1}, \hat{\tau}, \hat{\tau_0})$.

(2) Under the admissibility conditions, we can endow the 2-limit with a structure of $\widetilde{Kh}^{*,*}(T_{2,\infty})$-module via a fusion map:



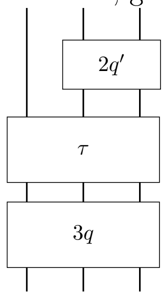
Indeed, at the limit this movie produces a map

$$\Sigma_l^* : \widetilde{Kh}^{*,*}(T_{2,\infty}) \otimes \widetilde{Kh}^{*,*}(\tau, 2, l) \longrightarrow \widetilde{Kh}^{*,*}(\tau, 2, l).$$

This can be checked to be a left action by adapting the proof of Theorem 5.7.

In this context, the skein total exact triangle for the 2-limit obtained from the limit short exact sequence is a triangle of modules over $\widetilde{Kh}^{*,*}(T_{2,\infty})$.

(3) Using the explicit description of $\widetilde{Kh}^{*,*}(T_{3,q})$ (Theorem 4.13) one can compute $\widetilde{Kh}^{*,*}(T_{3,\infty})$ as a vector space. This vector space can be identified with an infinitely generated $\widetilde{Kh}^{*,*}(T_{2,\infty})$-module by using a limit over two parameters, given by the family of tangles below.

$\tau_{q,q'} =$ 

If we let K^*_* denote the $\widetilde{Kh}^{*,*}(T_{2,\infty})$-module $\widetilde{Kh}^*_*(T_{2,\infty})/(a_2^2 = a_3^2 = 0)$, then we should obtain $\widetilde{Kh}^*_*(T_{3,\infty})$ as an infinetely generated $\widetilde{Kh}^{*,*}(T_{2,\infty})$-module:

$$\widetilde{Kh}^*_*(T_{3,\infty}) \cong \bigoplus\left(\bigoplus_{i=0}^{\infty} K^*_*[-4i, 2i]\right).$$

(4) Finally, using the maps $\Phi^{N,2}$ of Proposition 4.4, we expect to find a polynomial generator $z \in \widetilde{Kh}_2^{-4}(T_{3,\infty})$ that connects consecutive copies of K^*_*, thus obtaining $\widetilde{Kh}^{*,*}(T_{3,\infty})$ as a polynomial algebra over $\widetilde{Kh}^{*,*}(T_{2,\infty})/(a_2^2 = a_3^2 = 0)$. This would mean there is a bi-graded algebra isomorphism

$$\widetilde{Kh}^*_*(T_{3,\infty}) \cong \mathbb{Z}_2[x, y, z]/(x^2 = y^2 = 0),$$

where $|x| = (-2, 0), |y| = (-3, 0)$ and $|z| = (-4, 2)$. And that is actually the statement of the Gorsky-Oblomkov-Rasmussen conjecture for $p = 3$.

2. An approach to the general case.

In this second section, we propose a strategy to treat the general case of the p-stranded torus links, following the steps described in the previous section. For each of them, we explain if, how and why it can or cannot be extended to the general family of p-stranded torus links. We will also explain the limits of the strategy we propose.

(1) It is straightforward to generalize the various notions of limits into (k, l)-limits, where k is the number of strands that get twisted, and l is the position of the leftmost strand that is affected by the twisting. One obtains first an ungraded vector space. In particular for a (p, p)-tangle, the $(p, 1)$-limit corresponds to the p-limit. The properties of annihilation of cups/caps and absorption of crossings should also generalize. The admissibility conditions extend in the obvious way. So we can re-introduce gradings for admissible tangles. The skein spectral sequence can also be adapted to this setting, when the intermediate diagrams are either admissible.
(2) Under the new admissibility conditions, a fusion movie provides (k, l)-limits with a structure of module over $\widetilde{Kh}^{*,*}(T_{k,\infty})$. This module structure extends to $\widetilde{Kh}^{*,*}(T_{p,\infty})$ as induced by 1-handles, and should be compatible with the algebra structure. We have a short exact sequence of limit complexes, and thus the skein spectral sequence becomes a spectral sequence of $\widetilde{Kh}^{*,*}(T_{k,\infty})$-modules.

(3) Let $\tau_{p,pN+1}$ be the standard braid that closes into $T_{p,pN+1}$. In terms of braids, it is given by

$$\tau_{p,pN+1} = \underbrace{(\sigma_1 \circ \cdots \circ \sigma_l) \circ \cdots \circ (\sigma_1 \circ \cdots \circ \sigma_l)}_{pN+1 \text{ times}}$$

We can exhibit a family of spectral sequences $E(\tau_{p,pN+1}, p-1, 2)$ by choosing the p occurences of σ_1 in $\tau_{p,pN+1}$ and, with the absorption, this spectral sequence has columns given by

$$\begin{array}{lcl} E_1^{0,*} & = & \widetilde{Kh}^{*,*}(T_{p-1,\infty})\{b_N\}, \\ E_1^{1,*} & = & \widetilde{Kh}^{*,*}(T_{p-1,\infty})\{a_N\}[-1,0], \\ E_1^{n,*} & = & 0, \text{ for } 2 \leq n \leq p, \\ E_1^{p+1,*} & = & \widetilde{Kh}^{*,*}(\tau_{p,p(N-1)+1}, p-1, 2), \end{array}$$

where a_N and b_N are formal variable introduced to take care of grading shifts. A simple computation of these shifts show that the generator b_N of the E_1^0 column verifies

$$d_1(b_N) = 0.$$

Therefore, using the module structure, $d_1 = 0$. With our description of the columns, we also know that $d_n = 0$ for $n \leq p-1$. Hence we can jump to the E_p-page immediately. Additionally, we know that the sequence converges at the E_{p+2}-page. Therefore we are left with trying to understand d_p and d_{p+1}. However, this spectral sequence does not have, in general a nice convergence. In fact it is not true for $p = 3$, which is why we had to use 4 crossings to compute $\widetilde{Kh}^{*,*}(T_{3,3N+1})$ and not just only two.

(4) This step is the last one we can provide *without* any data of any kind. We consider our usual kind of directed systems, over N, but for the E_p-pages

$$\langle E_p(\tau_{p,pN+1}, p-1, 2), f_{jN} \rangle$$

with f_{jN} the map of spectral sequences induced by 1-smoothing. The limit $E_p(\tau_{p,\infty}, p-1, 2)$ is explicit:

$$\begin{array}{rcl} E_p(\tau_{p,\infty}, p-1, 2) & \cong & \bigoplus_{l\in\mathbb{Z}} \left(\widetilde{Kh}^{*,*}(T_{p-1,\infty})\{b_l\}[0, (p-1)pNl]\right) \\ & & \bigoplus_{l\in\mathbb{Z}} \left(\widetilde{Kh}^{*,*}(T_{p-1,\infty})\{a_l\}[-1, (p-1)pNl]\right) \\ & & \oplus\, \widetilde{Kh}^{*,*}(T_{p-1,\infty})\{1\}. \end{array}$$

It gives a spectral sequence that converges to $\widetilde{Kh}^{*,*}(T_{p,\infty})$ as a $\widetilde{Kh}^{*,*}(T_{p-1,\infty})$-module. We can do slightly better: the skein spectral sequence is natural with respect to 1-handle moves so if we use our fusion maps, we actually have a spectral sequence of algebras.

The remaining steps all use the fact that we have data to lean on. In general, there is no such data yet. But this problem can be partially solved. There is a homology theory, related to Khovanov homology, that is always explicit: Bar-Natan homology. Moreover, maps induced by 1-handle in Bar-Natan homology are very well understood, see Rasmussen [**Ras10**]. With this facts in mind, one can think of the next steps given below.

(5) Extend the limits to Bar-Natan homology and obtain a graded vector space $\widetilde{BN}^*(\tau, k, l)$. This graded vector space should be explicitly given in terms of linking numbers. The corresponding fusion maps in this theory should also be easy to understand.

(6) We expect a_1 to be a polynomial generator, in the sense that the action of a_1 on the E^0 and E^1 columns should provides an identification

$$a_{l+1} = a_1 \cdot a_l, \quad b_{l+1} = a_1 \cdot b_l.$$

The proof should use the explicit form of $\widetilde{BN}^*(\tau, p-1, 2)$, and the functoriality of the Lee-Rasmussen spectral sequence. Moreover, we should have b_1^2 be an element of $\widetilde{Kh}^{*,*}(T_{p-1,\infty})\{1\}$ otherwise $E_p(\tau_{p,\infty}, p-1, 2)$ would contain more copies of $\widetilde{Kh}^{*,*}(T_{p-1,\infty})$. This leads to an isomorphism of algebras

$$E_p(\tau_{p,\infty}, p-1, 2) \cong \frac{\widetilde{Kh}^{*,*}(T_{p-1,\infty})[a_1, b_1]}{b_1^2 = \alpha},$$

for some $\alpha \in \widetilde{Kh}^{*,*}(T_{p-1,\infty})\{1\}$, recovering Hogancamp's result [**Hog15**].

(7) We still have differentials d_p and d_{p+1}, making $E_p(\tau_{p,\infty}, p-1, 2)$ into a differential graded algebra. More precisely, this means that there exists a differential d on $E_p(\tau_{p,\infty}, p-1, 2)$ such that

$$\widetilde{Kh}^{*,*}(T_{p,\infty}) \cong H_*(E_p(\tau_{p,\infty}, p-1, 2), d).$$

The isomorphism would be an isomorphism of algebras.

(8) The generator a_1 appears first from $T_{p,p}$ and appears in Bar-Natan homology also. We expect it to survive all the way to $\widetilde{Kh}^{*,*}(T_{p,\infty})$, or equivalently $d(a_1) = 0$.

In order to completely describe $\widetilde{Kh}^{*,*}(T_{p,\infty})$, we still need to understand $d(b_1)$. This is much harder and we cannot rely on Bar-Natan homology - that is already the case for $p = 3, 4$. The reason we cannot use Bar-Natan homology is the one we mentionned in Chapter 4 Section 3. We should have, in terms of this homology

$$d_{BN}(b_1) \neq 0.$$

However, d_{BN} splits into two pieces

$$d_{BN} = d + \beta.$$

With d_{BN} non zero, we could have either $d(b_1) \neq 0$ or $\beta(b_1) = a_1$. In the first case, b_1 does not survive and in the second it does. The methods developped in this thesis do not allow us to choose between the two possibilities. There is still some hope though. There exists another spectral sequence that compute Khovanov homology and that contains different information than the skein spectral sequence: the Everitt-Turner spectral sequence [**ET09**]. We hope that using this sequence would allow us to understand $d(b_1)$.

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