Minimal bipartite dimers and higher genus Harnack curves

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Abstract

This paper completes the comprehensive study of the dimer model on infinite minimal graphs with Fock's weights [Foc15] initiated in [BCdT20]: the latter article dealt with the elliptic case, *i.e.*, models whose associated spectral curve is of genus 1, while the present work applies to models of arbitrary genus. This provides a far-reaching extension of the genus 0 results of [Ken02, KO06], from isoradial graphs with critical weights to minimal graphs with weights defining an arbitrary spectral data. For any minimal graph with Fock's weights, we give an explicit local expression for a two-parameter family of inverses of the associated Kasteleyn operator. In the periodic case, this allows us to prove local formulas for all ergodic Gibbs measures, thus providing an alternative description of the measures constructed in [KOS06]. We also compute the corresponding slopes, exhibit an explicit parametrization of the spectral curve, identify the divisor of a vertex, and build on [KO06, GK13] to establish a correspondence between Fock's models on periodic minimal graphs and Harnack curves endowed with a standard divisor.

1 Introduction

This paper is a follow up to [BCdT20]. The latter aimed at giving a comprehensive study of the dimer model on infinite minimal graphs with Fock's *elliptic* weights, *i.e.*, with underlying Riemann surface of genus 1. This extended the *rational* (genus 0) results of [Ken02, KO06], since the rational case can be interpreted as a degeneration of the elliptic one [BCdT20, Section 8.1]. We now turn to the general case, and consider Fock's weights in any genus $g > 0$ [Foc15].

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As in many questions involving compact Riemann surfaces, moving from $g = 1$ to $g \geq 1$ is by no means a trivial extension. Moreover, for the dimer model to have a probabilistic meaning, it needs to be parametrized by a specific type of abstract, compact Riemann surface know as an *M-curve* [Har76, Mik00], which we need to thoroughly understand. Therefore, a significant part of this paper, namely the whole of Section 2, is devoted to establishing the results on M-curves necessary for the study of the corresponding dimer models: we describe their period matrix, Abel-Jacobi map, Riemann theta functions and prime form.

Before turning to the statistical mechanics implications, let us recall the context of Fock's work [Foc15] and ours, that is, the rich interplay between dimer models on the one hand, real algebraic geometry and complex analysis on the other. In their seminal paper [KOS06], Kenyon, Okounkov and Sheffield show that the spectral curve of a dimer model on a $\mathbb{Z}^2$-periodic, bipartite graph is of a very special type, namely a *Harnack curve*. In the subsequent articles [KO06, GK13], the authors define the *spectral data* of such a dimer model as the spectral curve $\mathcal{C}$ together with a divisor consisting of one point on each of the ovals of $\mathcal{C}$. Furthermore, they show that given any Harnack curve and any such *standard divisor*, there exists a dimer model realizing this spectral data, and that the dimer model can be chosen on a *minimal* graph [Thu17, GK13]. We refer the reader to Section 3.1 for the definition of minimal graphs and of the related notion of *train-tracks*. Let us point out that throughout this article and unless otherwise stated, the graphs are locally finite, embedded in the plane, with faces consisting of bounded topological discs; in particular, they are infinite.

The articles [KO06, GK13] contain an explicit construction of a periodic minimal graph from the spectral curve, a characterisation of genus 0 spectral curves as coming from isoradial graphs with critical weights [Ken02], but no determination of the actual dimer model in general. In other words, these powerful results do not answer the following question: *given a Harnack curve $\mathcal{C}$, can we explicitly construct a dimer model on a periodic bipartite graph whose spectral curve is $\mathcal{C}$?*

The remarkable contribution of Fock [Foc15] consists in filling this gap. More precisely, Fock starts with an arbitrary complex curve $\mathcal{C}$ (not necessarily Harnack) and an arbitrary divisor of the appropriate degree (not necessarily a standard divisor), and constructs an explicit "dimer model" on a periodic minimal graph whose spectral data is the curve $\mathcal{C}$ together with this divisor. The quotation marks are due to the fact that in Fock's construction, dimer weights are complex. From our statistical mechanics perspective, an important question is to understand in which setting the "dimer model" is indeed a probabilistic model, *i.e.*, has positive edge-weights. This is the first main contribution of the present paper and the content of Proposition 31, whose proof heavily relies on the study of M-curves of Section 2. Let us briefly introduce the tools required to explain this statement, referring the reader to the relevant parts of Sections 2 and 3 for more complete definitions.

Fix a compact Riemann surface Σ of positive genus together with an element t of its Jacobian variety $\mathrm{Jac}(\Sigma)$ and a theta characteristic $\binom{\delta'}{\delta''} \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$. Let G be a bipartite

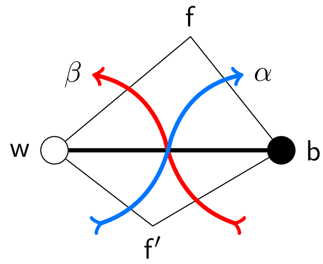

Figure 1: An arbitrary edge wb of G with the two adjacent faces f, f' depicted as dual vertices. The four corresponding edges of the quad-graph $\mathsf{G}^\diamond$ are also drawn, together with the two incident train-tracks with angles $\alpha, \beta \in \Sigma$. The discrete Abel map $\boldsymbol{d}$ satisfies the local rule $\boldsymbol{d}(\mathsf{f}') - \boldsymbol{d}(\mathsf{f}) = \alpha - \beta \in \mathrm{Pic}^0(\Sigma) \simeq \mathrm{Jac}(\Sigma)$.

graph (not necessarily periodic), and let $\boldsymbol{\alpha}\colon \mathcal{T} \to \Sigma$ be a map assigning to each train-track of G an element of Σ called its *angle*. Let $\boldsymbol{d}\colon \{\text{faces of } \mathsf{G}\} \to \mathrm{Jac}(\Sigma)$ be the *discrete Abel map*, uniquely defined up to an additive constant by the local rule described in Figure 1. *Fock's adjacency operator* is represented by an infinite matrix K whose rows are indexed by white vertices of G, columns by black ones, and whose non-zero entries correspond to edges of G and are given by: for every edge wb of G as in Figure 1,

$$\mathsf{K}_{\mathsf{w},\mathsf{b}} = \frac{E(\alpha,\beta)}{\theta[^{\delta'}_{\delta''}](t + \boldsymbol{d}(\mathsf{f}))\,\theta[^{\delta'}_{\delta''}](t + \boldsymbol{d}(\mathsf{f}'))}\,, \tag{1}$$

where E is the prime form of Σ and $\theta[^{\delta'}_{\delta''}]$ the theta function with theta characteristic $\binom{\delta'}{\delta''}$.

Remark 1. In the genus 0 case [Ken02], the weight of the edge wb only depends on the angles α, β of the train-tracks crossing wb, and is therefore referred to as a *local weight*. In the general case defined above, the situation is just about as favorable, but not quite. Indeed, the weight $\mathsf{K}_{\mathsf{w},\mathsf{b}}$ depends on α, β, but also on the value of $\boldsymbol{d}$ at neighbouring faces. The map $\boldsymbol{d}$ is non-local, but is defined via a local rule: it should be thought of as a discrete primitive of the angle map $\boldsymbol{\alpha}$ with initial condition given by t. Despite this subtle difference, we make a slight abuse of terminology and still refer to the weights defined in Equation (1) as *local weights*.

An *M-curve* is a compact Riemann surface of genus g endowed with an anti-holomorphic involution σ whose set of fixed points is given by $g + 1$ circles, called *real components* (see Figure 2). Proposition 31 can now be stated as follows.

Proposition 2. *Let us assume that*

(i) the surface Σ is an M-curve (of genus $g > 0$ and period matrix Ω);

(ii) the element t of $\mathrm{Jac}(\Sigma)$ is real, i.e., *belongs to $(\mathbb{R}/\mathbb{Z})^g + \Omega\delta$ for some $\delta \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$;*

(iii) the theta characteristic $\binom{\delta'}{\delta''}$ satisfies $\delta' = \delta$;

(iv) the graph G *is minimal;*

(v) the image of $\boldsymbol{\alpha}$ *is contained in a real (oriented) component* A_0 *of* Σ*, and* $\boldsymbol{\alpha}\colon \mathcal{T} \to A_0$ *is monotone with respect to the natural cyclic orders on* $\mathcal{T}$ *and on* A_0*.*

Then, Fock's adjacency operator (1) *is a Kasteleyn operator,* i.e., *defines a dimer model on the graph* G*.*

In the periodic setting, these constraints can be heuristically explained as follows. As we will see in more detail below, the abstract Riemann surface Σ serves as a parametrization domain for the spectral curve $\mathcal{C}$; this latter curve being Harnack, it has the maximal number of real components. As for the element $t \in \mathrm{Jac}(\Sigma)$, it corresponds via the Abel-Jacobi map and the identification $\Sigma \simeq \mathcal{C}$ to the divisor of the model, which consists of one point on each of the ovals of $\mathcal{C}$; such divisors are mapped to real elements of the Jacobian variety. Finally, the train-tracks of G correspond to the "points at infinity" of $\mathcal{C}$, which according to the definition of a Harnack curve, must be arranged in a natural cyclic order; this forces the angle map $\boldsymbol{\alpha}$ to be monotone. As proved in [BCdT21], the minimality of G then ensures that this (global) monotonicity implies the corresponding (local) monotonicity around each face of G. This can finally be translated into the Kasteleyn condition using properties of the prime form restricted to the real component of an M-curve.

To a certain extent, the arguments given above show that conditions (*i*), (*ii*) and (*v*) are actually necessary for K to be a Kasteleyn operator, at least in the periodic setting. The fact that the graph G needs to be minimal for the theory to apply is discussed at the end of Section 5.2, see also [BCdT21, Theorem 31]. Finally, there is little hope for Proposition 2 to hold without condition (*iii*) because of the last point of Lemma 18.

On the other hand, one shows that all theta characteristics $\binom{\delta'}{\delta''} \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$ yield gauge equivalent models, so we set $\delta' = \delta'' = 0$ for definiteness in our final definition, see the third and fourth points of Remark 30, and Definition 29.

Note that Fock originally defined his operator for periodic graphs, and some of the heuristic arguments above only hold in this restricted situation. However, Proposition 2 is valid for any minimal graph, allowing us to harness the power of Kasteleyn theory in this very general setting. Before doing so, let us mention that even with the constraints listed in Proposition 2, the dimer models given by Fock's weights span all periodic dimer models, when considered from the point of view of their spectral data. Indeed, we prove the following result, see Theorem 49 for a full statement.

Theorem 3. *For any Harnack curve* $\mathcal{C}$ *and standard divisor* D*, there exists an M-curve* Σ*, a periodic minimal graph* G*, a monotone angle map* $\boldsymbol{\alpha}$ *and a real element* t *of* $\mathrm{Jac}(\Sigma)$*, such that the associated Fock operator* K *is periodic, and the spectral data of the corresponding dimer model coincides with* $(\mathcal{C}, D)$*.*

Together with [GK13, Theorem 7.3], this implies that two periodic dimer models on the same minimal graph G arising from the same M-curve Σ, the same angle map $\boldsymbol{\alpha}$, and elements $t, t' \in (\mathbb{R}/\mathbb{Z})^g$ are gauge equivalent if and only if $t = t'$, see Remark 50.

We have another result worth mentioning on the topic of spectral curves. Consider the dimer model on a periodic, minimal graph G with Fock's weights given by parameters Σ, t and $\boldsymbol{\alpha}$ as in Proposition 2. In Proposition 45, we provide an explicit birational parametrization $\Sigma \to \mathcal{C}$ of the spectral curve. This allows us to transport several notions, such as the standard divisor of the model, from the spectral curve to the underlying abstract M-curve, thus extending these notions beyond the periodic setting. Most notably, the phase diagram of the dimer model can be tranported from $\mathcal{C}$ to Σ. This is related to probabilistic questions which we now address.

Before turning to our results, let us describe the setting and motivations. Consider a dimer model on a periodic bipartite graph G. Kenyon, Okounkov and Sheffield [She05, KOS06] prove that there is a two-parameter family of ergodic Gibbs measures, indexed by the *slope*, and that the set of allowed slopes coincides with the *Newton polygon* $N(\mathsf{G})$. Moreover, they provide an explicit expression for this family of measures by taking the weak limit of the Boltzmann measures on a toroidal exhaustion, with weights modified by magnetic field coordinates. They also prove that the dimer model has three phases, liquid, solid and gaseous, and that the phase diagram is given by the amoeba $\mathcal{A}$ of the spectral curve. Using different techniques, Kenyon [Ken02] establishes an explicit expression for the maximal entropy Gibbs measure in the case of isoradial graphs with critical weights, which has the remarkable property of being *local*: this means that edge probabilities can be computed using geometric information of paths joining these edges. Note that by uniqueness, we know that the expressions of [KOS06] and [Ken02] are equal, but this is only explicitly understood since [BdTR17]. Local expressions have now been obtained for dimer models related to the Ising model [BdT11, BdTR19], to rooted spanning forests [BdTR17], and for the two parameter family of Gibbs measures of the dimer model with Fock's elliptic weights [BCdT20]. Note that these are non trivial extensions of the result of [Ken02], two of the main difficulties being to find the appropriate extension of the discrete exponential functions of Mercat [Mer04] and to define suitable paths of integration. Having local expressions for Gibbs measures opens the way to computing precise asymptotics, and to constructing Gibbs measures for general, possibly non-periodic graphs. This latter application requires an additional argument, however, and such extensions to non-periodic graphs have only been obtained in some specific rational and elliptic cases [dT07, BdT11, BdTR17, BdTR19].

These results yield the following question: *can we obtain an explicit local expression for the two-parameter family of ergodic Gibbs measures of the dimer model on periodic, bipartite graphs?* We give a positive answer for all dimer models with Fock's weights on minimal periodic graphs.

Recall that, by Theorem 3 and the general theory of [KO06, GK13], any dimer model on a periodic minimal graph is gauge-equivalent to a model with Fock's weights (see Remark 50). In that sense, our result extends the theory initiated by Kenyon twenty years ago [Ken02], originally valid for one measure on isoradial graphs, to the full set of ergodic Gibbs measures of *any* dimer model on a periodic minimal graph.

We now state this result in two steps. Let us fix a minimal graph G together with parameters $\Sigma, t, \boldsymbol{\alpha}$ as in Proposition 2, and consider the associated Kasteleyn operator K defined in Equation (1). Let Σ^+ denote the upper half of the M-curve Σ, see Figure 2, and set $\mathcal{D} = \Sigma^+ \setminus \boldsymbol{\alpha}(\mathcal{T})$. The first step is an explicit, *local* expression for a two parameter family of inverses $(A^{u_0})_{u_0 \in \mathcal{D}}$ of Fock's Kasteleyn operator K, see Definition 37 and Theorem 40 for details.

Theorem 4. *For every $u_0 \in \mathcal{D}$, consider the operator A^{u_0} defined as follows: for every black vertex b and white vertex w of G, set*

$$\mathsf{A}^{u_0}_{\mathsf{b},\mathsf{w}} = \frac{1}{2i\pi} \int_{\sigma(u_0)}^{u_0} g_{\mathsf{b},\mathsf{w}} \,,$$

where $g_{\mathsf{b},\mathsf{w}}$ is the meromorphic 1-form on Σ with explicit local expression given in Section 3.4, and the integration path in Σ from $\sigma(u_0)$ to u_0 is defined in Section 3.5. Then, the operator A^{u_0} is an inverse of the Kasteleyn operator K.

Remark 5.

1. The terminology *local* is used in the same sense as in Remark 1: when $g \geq 1$, there is some non-local information, all encoded in the discrete Abel map $\boldsymbol{d}$.
2. The cornerstone of the proof is Fay's celebrated identity [Fay73], see Section 2.5.3. This identity is also the reason why (and in some precise sense, equivalent to the fact that) the dimer model with Fock's weights is invariant under natural local transformations, see Section 5.2.

We are now ready to state our result for Gibbs measures on periodic minimal graphs. This is a combination of Theorem 51 and Corollary 52.

Theorem 6. *For every $u_0 \in \mathcal{D}$, consider the measure $\mathbb{P}^{u_0}$ whose expression on cylinder sets is given as follows: for every set $\{\mathsf{e}_1 = \mathsf{w}_1\mathsf{b}_1, \ldots, \mathsf{e}_k = \mathsf{w}_k\mathsf{b}_k\}$ of distinct edges of G,*

$$\mathbb{P}^{u_0}(\mathsf{e}_1, \ldots, \mathsf{e}_k) = \Big(\prod_{j=1}^{k} \mathsf{K}_{\mathsf{w}_j,\mathsf{b}_j}\Big) \times \det_{1\leq i,j\leq k}\Big(\mathsf{A}^{u_0}_{\mathsf{b}_i,\mathsf{w}_j}\Big)\,.$$

This defines an ergodic Gibbs measure on dimer configurations of G.

Moreover, the measures $(\mathbb{P}^{u_0})_{u_0 \in \mathcal{D}}$ form the two-parameter family of ergodic Gibbs measures of [KOS06], where $u_0 \in \mathcal{D} \subset \Sigma$ is related to the magnetic field coordinates in $\mathcal{A}$ via the composition of the explicit parametrization $\Sigma \to \mathcal{C}$ from Proposition 45 with the amoeba map $\mathcal{C} \to \mathcal{A}$.

Finally, if u_0 belongs to the real component A_0 of Σ (resp. to the complement of A_0 in the real locus of Σ, to the interior of $\mathcal{D}$), then the corresponding dimer model is in a solid (resp. gaseous, liquid) phase.

Note that if u_0 and u_1 belong to the same connected component of the real locus of $\mathcal{D}$, then the operators A^{u_0} and A^{u_1} coincide, yielding identical measures $\mathbb{P}^{u_0} = \mathbb{P}^{u_1}$.

As mentioned above, this set of ergodic Gibbs measures is also naturally parametrized by slopes, and it is natural to wonder whether a simple expression can be given for the slope of the ergodic Gibbs measure $\mathbb{P}^{u_0}$. This is done in Section 4.5, but we shall not attempt to summarize these results here. Let us only mention that, in our setting, the identification of the g distinct slopes corresponding to the gaseous phases uses as its main ingredient the Riemann bilinear relation, see Corollary 57. Once again, this illustrates the very rich interplay between statistical physics and complex analysis on compact Riemann surfaces at work in this theory.

Outline of the paper

- Section 2 gathers all the results of complex analysis needed for our study of minimal bipartite dimers. After introducing abstract M-curves in Section 2.1, we recall the definition of several classical objects for an arbitrary compact Riemann surface Σ and study their special properties when Σ is an M-curve: first the period matrix in Section 2.2, then the Abel-Jacobi map in Section 2.3, the Riemann theta functions in Section 2.4, and finally the prime form in Section 2.5.

- In Section 3, after briefly recalling some background material on minimal bipartite dimers (Section 3.1), the discrete Abel map and monotone angle maps (Section 3.2), we prove Proposition 2 in Section 3.3. In Section 3.4, we construct explicit forms in the kernel of K, which we use in Section 3.5 to construct a two-parameter family of inverses of K, proving Theorem 4.

- Section 4 deals with the case of $\mathbb{Z}^2$-periodic models. After the preliminary Section 4.1, we study the periodicity of K in Section 4.2. In Section 4.3, we give an explicit parametrization of the spectral curve by the abstract M-curve Σ, and prove Theorem 3. In Section 4.4, we study the full set of ergodic Gibbs measures, proving Theorem 6. Finally, we derive explicit formulas for the slopes of the Gibbs measures in Section 4.5, and study the surface tension and free energy in Section 4.6.

- The more informal Section 5 deals with miscellaneous additional features of our theory: the construction of Gibbs measures beyond the periodic case in Section 5.1, the invariance of the model under local transformations in Section 5.2, and its relation to known models in Section 5.3. This concluding section also contains various future perspectives.

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2 Compact Riemann surfaces and M-curves

This section contains all the results in complex analysis that are needed for our study of dimers on minimal graphs. More precisely, we recall classical statements about Riemann surfaces, referring to [Jos06, FK92, Mum07a, Mum07b] for proofs and details, and explain what more can be said in the case of M-curves.

We start in Section 2.1 by recalling the definition of this special class of compact Riemann surfaces, and provide several examples. In Section 2.2, we briefly summarise the theory of period matrices, whose entries are showed to be purely imaginary in the case of M-curves (Lemma 11). Section 2.3 deals with the Abel-Jacobi map, whose behaviour for M-curves is described in Lemma 15. In Section 2.4, we recall the definition of the Riemann theta functions along with their well-known general properties (Lemma 17), and lesser-known behaviour for purely imaginary period matrices (Lemma 18). Finally, Section 2.5 deals with the general theory of prime forms, with Lemmas 25 and 26 containing the results needed in the case of M-curves.

2.1 Abstract M-curves

Recall that an *anti-holomorphic involution* on a Riemann surface Σ is a smooth involution $\sigma\colon \Sigma \to \Sigma$ whose induced map $\sigma_*\colon T\Sigma \to T\Sigma$ satisfies $\sigma_* \circ J = -J \circ \sigma_*$, where J denotes the almost-complex structure on Σ. The points of Σ that are fixed by σ are said to be *real*.

One easily shows that if σ is an anti-holomorphic involution on a compact orientable surface of genus g, then its set of fixed points consists of at most $g+1$ topological circles. (This is Harnack's theorem, whose proof follows from an Euler characteristic argument.)

Definition 7. An *(abstract) M-curve* is a compact Riemann surface Σ endowed with an anti-holomorphic involution σ whose set of fixed points is given by $g+1$ topological circles, where g is the genus of Σ.

The M in M-curve stands for 'maximal'. We now give some examples.

Example 8. Any genus 0 Riemann surface is isomorphic to the Riemann sphere, which is trivially an M-curve with respect to complex conjugation. This case being well-known,

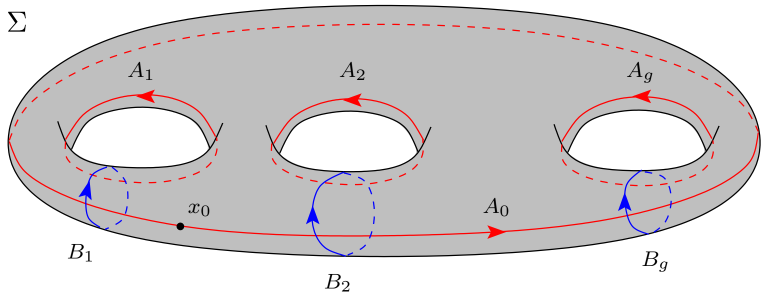

Figure 2: The surface Σ together with the cycles $A_0, A_1, \dots, A_g$ (in red) and $B_1, \dots, B_g$ (in blue). In this picture, the anti-holomorphic involution σ should be understood as the reflection across the horizontal plane containing $A_0, A_1, \dots, A_g$.

we assume from now on that g is positive. Note however that this rational case can be recovered as a degeneration of the elliptic case, *i.e.* $g = 1$, as explained in [BCdT20, Section 8.1].

Example 9. A Riemann surface of genus 1 is isomorphic to a torus $\mathbb{T}(\tau) = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ of modular parameter τ with $\Im(\tau) > 0$. The complex conjugation admits as real locus the curve $\mathbb{R}/\mathbb{Z}$, together with the curve $\mathbb{R}/\mathbb{Z} + \frac{\tau}{2}$ if and only if τ is purely imaginary. Therefore, $\mathbb{T}(\tau)$ is an M-curve if and only if τ is purely imaginary. This case is treated extensively in [BCdT20].

Example 10. By definition, (the toric closure of) a Harnack curve in $(\mathbb{C}^*)^2$ is an M-curve with respect to the anti-holomorphic involution given by $\sigma(z, w) = (\overline{z}, \overline{w})$.

2.2 The period matrix

This section is devoted to the study of the period matrix of an abstract M-curve.

We fix a real point x_0 in an M-curve Σ, denote by A_0 the corresponding real circle, and number the remaining ones as $A_1, \dots, A_g$. Note that the real locus necessarily separates Σ into two connected surfaces with boundary; we fix an orientation of the real locus so that the oriented boundary of one of these surfaces, denoted by Σ^+, is equal to $A_0 - (A_1 + \dots + A_g)$. Finally, we use the same symbol A_j for the oriented cycle in Σ and its homology class in $H_1(\Sigma; \mathbb{Z})$.

Note that there are homology classes $B_1, \dots, B_g \in H_1(\Sigma; \mathbb{Z})$ with $\sigma_*(B_i) = -B_i$ and such that $\{A_1, \dots, A_g, B_1, \dots, B_g\}$ forms a basis of $H_1(\Sigma; \mathbb{Z})$ satisfying the equalities

$$A_i \wedge A_j = 0, \quad B_i \wedge B_j = 0, \quad A_i \wedge B_j = \delta_{i,j},$$

for all $1 \leq i, j \leq g$, where $\wedge$ denotes the intersection form. This is illustrated in Figure 2.

The complex vector space of holomorphic differential forms has dimension g. Let us denote by $\vec{\omega} = (\omega_1, \dots, \omega_g)$ the basis of this space determined by

$$\forall\ 1 \leq i, j \leq g, \quad \int_{A_i} \omega_j = \delta_{i,j}\,.$$

We let Ω be the matrix with entries $\Omega_{ij} := \int_{B_j} \omega_i$. This is the "interesting part" of the *period matrix* $\begin{pmatrix} I_g & \Omega \end{pmatrix}$ of Σ in the basis $\vec{\omega}$. By the general theory, Ω is a symmetric matrix whose imaginary part is positive definite, and the columns of the period matrix are linearly independent over $\mathbb{R}$. They generate a full rank lattice $\Lambda = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$ in $\mathbb{C}^g$.

In the setting of M-curves, the entries of Ω are purely imaginary. This is the subject of the following lemma.

Lemma 11. *If Σ is an abstract M-curve with anti-holomorphic involution σ, then the following holds.*

1. *For all $1 \leq j \leq g$, we have the equality $\sigma^*\omega_j = \overline{\omega}_j$.*
2. *For all $0 \leq j \leq g$, the subspace $TA_j \subset T\Sigma$ is fixed pointwise by σ_*.*
3. *The entries of Ω are purely imaginary.*

Proof. To show the first point, consider the 1-forms defined by $\omega'_j := \overline{\sigma^*\omega_j}$ for $1 \leq j \leq g$. We now check that these are holomorphic forms. By definition, if a form ω is holomorphic, it is a $(1,0)$-form with $\overline{\partial}\omega = 0$. It follows that $\sigma^*\omega$ is a $(0,1)$-form with $\partial(\sigma^*\omega) = 0$. Indeed, the involution σ being anti-holomorphic, we have

$$\sigma^*\omega(J(v)) = \omega(\sigma_*(J(v)) = \omega(-J(\sigma_*(v))) = -i\omega(\sigma_*(v)) = -i\sigma^*\omega(v)$$

for all $v \in T\Sigma$, checking the first claim. The second follows from the naturality of σ^* via

$$\partial(\sigma^*\omega) = d(\sigma^*\omega) = \sigma^*(d\omega) = \sigma^*(\overline{\partial}\omega) = \sigma^*(0) = 0\,.$$

The fact that $\omega' = \overline{\sigma^*\omega}$ is a $(1,0)$-form with $\overline{\partial}\omega' = 0$ now follows easily, proving that $\omega'_j = \overline{\sigma^*\omega_j}$ is a holomorphic form. Next, observe that these holomorphic forms satisfy

$$\int_{A_i} \omega'_j = \int_{A_i} \overline{\sigma^*\omega_j} = \overline{\int_{\sigma_* A_i} \omega_j} = \overline{\int_{A_i} \omega_j} = \overline{\delta_{i,j}} = \delta_{i,j},$$

for all $1 \leq i, j \leq g$. Since these properties characterize the basis of holomorphic forms, this shows the equality $\omega'_j = \omega_j$, and the first point.

To prove the second one, simply observe that an element of TA_j is the form $\gamma'(0)$ with γ a parametrization of A_j. Since this curve is fixed pointwise by σ, we have $\sigma \circ \gamma = \gamma$ and $\sigma_*(\gamma'(0)) = \frac{d}{dt}(\sigma \circ \gamma)(0) = \gamma'(0)$.

The third point follows from the first one via

$$\overline{\int_{B_i} \omega_j} = \int_{B_i} \overline{\omega_j} = \int_{B_i} \sigma^*\omega_j = \int_{\sigma_* B_i} \omega_j = \int_{-B_i} \omega_j = -\int_{B_i} \omega_j\,.$$

This concludes the proof. $\square$

Example 12. As mentioned above, a Riemann surface of genus 1 is isomorphic to a torus $\mathbb{T}(\tau) = \mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ of modular parameter τ with $\Im(\tau) > 0$. It is an M-curve if and only if τ is purely imaginary. In such a case, the imaginary axis $\tau\mathbb{R}/\tau\mathbb{Z}$ can be chosen as the cycle representing the class B_1. The basis of holomorphic forms is then given by $\omega_1 = dz$, since its integral along $A_1 = \mathbb{R}/\mathbb{Z} + \frac{\tau}{2}$ is equal to 1. Along $B_1 = \tau\mathbb{R}/\tau\mathbb{Z}$, the integral is τ, so the period matrix is simply given by $\begin{pmatrix} 1 & \tau \end{pmatrix}$.

Example 13. In the case of a Harnack curve, there is a very concrete way to describe $\vec{\omega}$ and compute Ω, explained in the proof of Proposition 6 of [KO06], see also the proof of Theorem 3 of [CL18] for the general setting.

We need an additional lemma, part of which is known to hold for Harnack curves, see the end of Section 2 of [KO06]. We now show that it is valid in the more general setting of M-curves (with a simpler proof).

Lemma 14. *Let Σ be an M-curve, with real circles $A_0, A_1, \dots, A_g$ and associated basis of holomorphic forms $\vec{\omega} = (\omega_1, \dots, \omega_g)$ as above. Then, for any $1 \le i \le g$ and $\alpha \neq \beta \in A_0$, resp. A_i, we have $\int_\alpha^\beta \omega_i > 0$, where the integration path follows the orientation of A_0, resp. A_i.*

Proof. Fix $1 \le i \le g$. For any $1 \le j \le g$ with $j \neq i$, we have $\int_{A_j} \omega_i = 0$, and the form ω_i is real on A_j. Therefore, it can be written $\omega_i = df$ in a neighborhood of A_j, with f a complex-valued function taking real values on A_j. This form not being identically zero, the function f is non-constant, and ω_i admits at least two zeros on A_j. Furthermore, being a holomorphic differential form, it admits exactly $2g-2$ zeros (counted with multiplicity). In conclusion, for all $1 \le j \le g$ with $j \neq i$, the form ω_i admits exactly 2 simple zeros on A_j, and no zero elsewhere. In particular, it has no zero on A_i and on A_0, so the integration along these real components is monotone. Finally, by definition of the orientation of A_0, the integration along these full loops is given by

$$\int_{A_0} \omega_i = \sum_{j=1}^{g} \int_{A_j} \omega_i = \int_{A_i} \omega_i = 1 > 0\,.$$

This implies both claims. $\square$

2.3 The Abel-Jacobi map

In this section, we briefly recall the definition of the Abel-Jacobi map associated with an arbitrary compact Riemann surface, before explaining its special features in the case of an M-curve in Lemma 15.

Recall that a *divisor* on Σ is a formal linear combination of points of Σ with integer coefficients. The set of divisors on Σ is endowed with a natural grading $\mathrm{Div}(\Sigma) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Div}^n(\Sigma)$, where the degree of a divisor is the sum of its integer coefficients. A divisor D is said to be *principal* if it represents the zeros and poles of a non-zero meromorphic function f on Σ, *i.e.*, if it is of the form

$$D = \sum_z \mathrm{ord}_f(z) z.$$

Two divisors are said to be *linearly equivalent* if their difference is a principal divisor. Since the number of zeros and poles of a non-zero meromorphic function coincide, *i.e.*, the degree of a principal divisor vanishes, the set of linear equivalence classes of divisors forms a $\mathbb{Z}$-graded Abelian group, denoted by $\mathrm{Pic}(\Sigma) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Pic}^n(\Sigma)$.

Abel's theorem [Jos06, Theorem 5.9.1] implies that there is an injection from $\mathrm{Pic}^0(\Sigma)$ to the *Jacobian variety* $\mathrm{Jac}(\Sigma) = \mathbb{C}^g/\Lambda$ of Σ through the so-called *Abel-Jacobi map*

$$D = \sum_i (y_i - x_i) \longmapsto \sum_i \int_{x_i}^{y_i} \vec{\omega} \in \mathbb{C}^g.$$

The decomposition of $D \in \mathrm{Pic}^0(\Sigma)$ is not unique, and the right-hand side depends on the choice of paths between x_i and y_i on Σ. However, two possible results differ by an element of Λ, so the formula displayed above gives a well-defined map $\Phi\colon \mathrm{Pic}^0(\Sigma) \to \mathrm{Jac}(\Sigma)$.

Jacobi's inversion theorem [Jos06, Theorem 5.9.2] states that this map induces an isomorphism of Abelian groups $\mathrm{Pic}^0(\Sigma) \simeq \mathrm{Jac}(\Sigma)$. More concretely, the Abel-Jacobi map can be inverted as follows: given $\lambda \in \mathrm{Jac}(\Sigma)$ and a fixed point $x_0 \in \Sigma$, one can find a divisor $D = \sum_{i=1}^g x_i$ of degree g with only positive coefficients (the x_i's may not be distinct) such that $\Phi(\sum_{i=1}^g (x_i - x_0)) = \lambda$. Following standard practice, we use the same notation for (the equivalence class of) a degree 0 divisor and for its corresponding element in $\mathrm{Jac}(\Sigma)$.

Note that one can define a map $\Phi\colon \mathrm{Pic}(\Sigma) \to \mathrm{Jac}(\Sigma)$ by first sending $\mathrm{Pic}^n(\Sigma)$ to $\mathrm{Pic}^0(\Sigma)$ via $D \mapsto D - nx_0$ and then applying the Abel-Jacobi map. In particular, this gives a well-defined map $\bigsqcup_{n>0} \Sigma^n \to \mathrm{Jac}(\Sigma)$. By abuse of notation, we simply denote it by Φ, even though it does depend on the choice of x_0.

In the case of Harnack curves, the real torus $A_1 \times \cdots \times A_g$ is known to inject into $\mathrm{Jac}(\Sigma)$ and form one of its real components, see [KO06], end of Section 2. We now show that this still holds in the more general setting of M-curves.

Lemma 15. *Let* $\Phi\colon \bigsqcup_{n>0} \Sigma^n \to \mathrm{Jac}(\Sigma)$ *be the Abel-Jacobi map associated to an M-curve* Σ*, defined with respect to a fixed real point* $x_0 \in A_0$*. Let* $(e_1, \ldots, e_g)$ *denote the canonical basis of* $\mathbb{C}^g$ *(and of* $\mathbb{Z}^g$*), and set* $\mathbf{1} = e_1 + \cdots + e_g$*.*

1. *The real locus of* $\mathrm{Jac}(\Sigma)$ *is equal to* $(\mathbb{R}/\mathbb{Z})^g \oplus \Omega(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$*,* i.e.*, it consists of* 2^g *real tori of dimension* g *indexed by* $(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$*.*

2. *For every* $1 \le i \le g$*, the map* $\Phi\colon \Sigma \to \mathrm{Jac}(\Sigma)$ *sends the real component* A_i *to a cycle of homology class* $e_i \in \mathbb{Z}^g$ *inside the real torus indexed by* $\frac{e_i}{2} \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$*, strictly increasing in the* e_i*-direction.*

3. *The map $\Phi\colon \Sigma \to \mathrm{Jac}(\Sigma)$ sends the component A_0 to a cycle of homology class $\mathbf{1} \in \mathbb{Z}^g$ inside the real torus indexed by $0 \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$, strictly increasing in the e_i-direction for all $1 \leq i \leq g$.*

4. *The restriction of $\Phi\colon \Sigma^g \to \mathrm{Jac}(\Sigma)$ to $A_1 \times \cdots \times A_g$ defines a homeomorphism onto the real torus indexed by $\frac{1}{2}\mathbf{1} \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$.*

Proof. To check the first point, consider an element $t \in \mathrm{Jac}(\Sigma) = \mathbb{C}^g/\Lambda$, represented by $x + y \in \mathbb{C}^g$ with $x \in \mathbb{R}^g$ and $y \in i\mathbb{R}^g$. This element t of $\mathrm{Jac}(\Sigma)$ is real if and only if the difference $(x + y) - (\overline{x + y}) = 2y$ belongs to $\Lambda = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$, which is equivalent to $y \in \Omega\frac{1}{2}\mathbb{Z}^g$. In conclusion, the real locus of $\mathrm{Jac}(\Sigma)$ is indeed given by

$$(\mathbb{R}^g \oplus \tfrac{1}{2}\Omega\mathbb{Z}^g)/\Lambda = (\mathbb{R}/\mathbb{Z})^g \oplus \Omega(\tfrac{1}{2}\mathbb{Z}/\mathbb{Z})^g\,.$$

We now check that the real components $A_1, \ldots, A_g$ are mapped to this real locus of $\mathrm{Jac}(\Sigma)$. Indeed, fix any $P_i \in A_i$. A path from x_0 to P_i can be chosen as a first path $\gamma_0 \subset A_0$ from x_0 to the intersection of A_0 with B_i, then a path $\beta_i \subset B_i$ (following the orientation of B_i) to the intersection of B_i with A_i, and a path $\gamma_i \subset A_i$ to P_i. As in the proof of Lemma 11, we can compute

$$(\Phi(P_i) - \overline{\Phi(P_i)})_j = \int_{\gamma_0 - \sigma_*\gamma_0} \omega_j + \int_{\beta_i - \sigma_*\beta_i} \omega_j + \int_{\gamma_i - \sigma_*\gamma_i} \omega_j = \int_{B_i} \omega_j = \Omega_{ji}\,.$$

Hence, we see that $\Phi(P_i) - \overline{\Phi(P_i)}$ belongs to $\Omega\mathbb{Z}^g$, and so $\Phi(P_i)$ and $\overline{\Phi(P_i)}$ define the same element of $\mathrm{Jac}(\Sigma)$. More precisely, since $\Phi(P_i) - \overline{\Phi(P_i)} = \Omega e_i$, we see that A_i is mapped inside the real torus indexed by $\frac{e_i}{2} \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$. Moreover, going once around A_i replaces $\Phi(P)$ with $\Phi(P) + \int_{A_i} \vec{\omega} = \Phi(P) + e_i$. Hence, the component A_i is mapped to a cycle in this real torus with homology class $e_i \in \mathbb{Z}^g = H_1((\mathbb{R}/\mathbb{Z})^g; \mathbb{Z})$. The fact that it is strictly increasing in the e_i-direction is a reformulation of Lemma 14, and the second point is proved.

With our choice of base point x_0 in A_0, the real component A_0 also clearly embeds into the real locus of the Jacobian via Φ, and its image $\Phi(A_0)$ contains the origin. Hence, the component A_0 embeds in the real torus indexed by $0 \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$. Note also that since the homology class of A_0 is given by $A_1 + \cdots + A_g$, going once around the component A_0 replaces $\Phi(P)$ by $\Phi(P) + \mathbf{1}$. Therefore, $\Phi(A_0)$ is a cycle in this real torus with homology class $\mathbf{1} \in \mathbb{Z}^g = H_1((\mathbb{R}/\mathbb{Z})^g; \mathbb{Z})$. The monotonicity follows from Lemma 14, showing the third point.

Understanding an element of $A_1 \times \cdots \times A_g$ as a divisor $P_1 + \cdots + P_g$ with $P_i \in A_i$, the last point now follows from the third one: indeed, the restriction of Φ to such divisors defines a map from the real torus $A_1 \times \cdots \times A_g$ to the real torus $(\mathbb{R}/\mathbb{Z})^g + \Omega\frac{1}{2}\mathbf{1}$, a map of degree 1, hence surjective. The injectivity follows from the monotonicity, and the homeomorphism from compactness. □

We conclude this section by recalling the classical *Riemann bilinear relation*, in the form stated in [Fay73, Eq. (7)], but accounting for the different normalisation of ω_k. (This result is also known as a *reciprocity law*, see e.g. [Jos06, Theorem 5.3.1 ii].) Let ω_D be a differential 1-form of the third kind (that is, a meromorphic differential 1-form having only simple poles [Jos06, Section 5.3]) with zero period along the A-cycles and simple poles at β_j with integer residue $r_j \in \mathbb{Z}^*$, for $1 \leq j \leq n$.

Note that the corresponding degree 0 divisor $D = \sum_{j=1}^{n} r_j \beta_j$ splits as $D = D^+ - D^-$, where

$$D^+ = \sum_{j:\, r_j > 0} r_j \beta_j \quad \text{and} \quad D^- = \sum_{j:\, r_j < 0} (-r_j) \beta_j$$

are *effective divisors* (that is, with positive coefficients). Then, for any $1 \leq k \leq g$, we have the equality

$$\int_{B_k} \omega_D = 2i\pi \int_{D^-}^{D^+} \omega_k \,, \tag{2}$$

where the paths of integration for the right-hand side between pairs of points of D^- and D^+ are paths in the surface Σ cut along $\{A_i, B_i : 1 \leq i \leq g\}$, see Figure 7.

2.4 Theta functions

In this section, we recall the definition of the Riemann theta functions, following the conventions of [Mum07a, Chapter II] (see also [FK92, Chapter VI]), and state their basic properties in Lemma 17. In the case of a purely imaginary period matrix, more subtle properties are proved in Lemma 18.

The *Riemann theta function* $\theta(z|\Omega)$ associated with a Riemann surface Σ is a higher-dimensional analog of the classical Jacobi theta functions [Law89]. For $z \in \mathbb{C}^g$, set

$$\theta(z|\Omega) = \sum_{n \in \mathbb{Z}^g} e^{i\pi(n\cdot\Omega n + 2n\cdot z)},$$

where $\cdot$ represent the canonical scalar product in $\mathbb{C}^g$. For $\binom{\delta'}{\delta''} \in (\frac{1}{2}\mathbb{Z})^{2g}$, the *theta function with characteristic* $\binom{\delta'}{\delta''}$, denoted by $\theta[\begin{smallmatrix}\delta'\\ \delta''\end{smallmatrix}]$, is defined by

$$\theta[\begin{smallmatrix}\delta'\\ \delta''\end{smallmatrix}](z|\Omega) = \sum_{n \in \mathbb{Z}^g} e^{i\pi[(n+\delta')\cdot\Omega(n+\delta') + 2(n+\delta')\cdot(z+\delta'')]}.$$

Example 16.

1. The theta function with characteristic $\binom{0}{0}$ is the Riemann theta function θ defined above.

2. When $g = 1$ and $\Omega = \tau$, $\theta(z|\Omega)$ coincides with the Jacobi function $\theta_3(z\pi|\tau)$, see [Law89, Equation (1.2.13)]. The theta functions corresponding to the four characteristics $\binom{0}{0}, \binom{0}{\frac{1}{2}}, \binom{\frac{1}{2}}{0}, \binom{\frac{1}{2}}{\frac{1}{2}}$ are the rescaled versions of $\theta_3, \theta_4, \theta_2, -\theta_1$ respectively.

The following elementary identities between theta functions are well-known, see e.g. [FK92].

Lemma 17.

1. *For all* $\binom{\delta'}{\delta''} \in (\frac{1}{2}\mathbb{Z})^{2g}$*, we have the equality*

$$\theta[{}^{\delta'}_{\delta''}](z|\Omega) = e^{i\pi(\delta'\cdot\Omega\delta'+2\delta'\cdot(z+\delta''))}\theta(z+\Omega\delta'+\delta''|\Omega).$$

2. *For all* $m, n \in \mathbb{Z}^g$*, we have*

$$\theta[{}^{\delta'}_{\delta''}](z+m+\Omega n|\Omega) = e^{-i\pi n\cdot(2z+2\delta''+\Omega n)}e^{2i\pi\delta'\cdot m}\theta[{}^{\delta'}_{\delta''}](z|\Omega)\,.$$

In particular, the function θ *is periodic in the* $\mathbb{Z}^g$ *directions, and quasi-periodic in the* $\Omega\mathbb{Z}^g$ *directions:*

$$\theta(z+m+\Omega n|\Omega) = e^{-i\pi n\cdot(2z+\Omega n)}\theta(z|\Omega).$$

3. *For all* $\gamma', \gamma'' \in \frac{1}{2}\mathbb{Z}^g$*, we have*

$$\theta[{}^{\delta'}_{\delta''}](z+\gamma''+\Omega\gamma'|\Omega) = e^{-i\pi\gamma'\cdot(2z+2\delta''+2\gamma''+\Omega\gamma')}\theta[{}^{\delta'+\gamma'}_{\delta''+\gamma''}](z|\Omega).$$

4. *For all* $m, n \in \mathbb{Z}^g$*, we have*

$$\theta[{}^{\delta'+n}_{\delta''+m}](z|\Omega) = e^{2i\pi\delta'\cdot m}\theta[{}^{\delta'}_{\delta''}](z|\Omega) = \pm\theta[{}^{\delta'}_{\delta''}](z|\Omega)\,.$$

This justifies the notation $\binom{\delta'}{\delta''} \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$*, that we will now use even though in practice, we always work with fixed representatives in* $\frac{1}{2}\mathbb{Z}^{2g}$*.*

5. *For every* $z \in \mathbb{C}^g$*, we have*

$$\theta[{}^{\delta'}_{\delta''}](-z|\Omega) = (-1)^{2\delta'\cdot 2\delta''}\theta[{}^{\delta'}_{\delta''}](z|\Omega),$$

implying that $\theta[{}^{\delta'}_{\delta''}]$ *is even, resp. odd, if and only if* $2\delta'\cdot 2\delta''$ *is even, resp. odd.*

As showed in Lemma 11, the matrix Ω associated with an M-curve is purely imaginary. We will need the following properties of the corresponding theta functions. The second one is of great importance to our work.

Lemma 18. *Let us assume that* Ω *is purely imaginary.*

1. *For all* $z \in \mathbb{C}^g$*, we have*

$$\theta[{}^{\delta'}_{\delta''}](\bar{z}|\Omega) = \overline{\theta[{}^{\delta'}_{\delta''}](z|\Omega)}\,.$$

In particular, $\theta[{}^{\delta'}_{\delta''}](z|\Omega)$ *is real for* $z \in \mathbb{R}^g$*.*

2. *If* z *belongs to* $\mathbb{R}^g$*, then* $\theta[{}^{0}_{\delta''}](z|\Omega)$ *is strictly positive.*

3. *When* $\delta' \neq 0$*,* $\theta[{}^{\delta'}_{\delta''}](z|\Omega)$ *takes strictly positive and negative values on* $\mathbb{R}^g$*.*

Proof.

1. Let us check the equality

$$\overline{\theta[^{\delta'}_{\delta''}](z|\Omega)} = (-1)^{2\delta'\cdot 2\delta''}\theta[^{\delta'}_{\delta''}](-\overline{z}|\Omega)$$

which, together with the last point of Lemma 17, implies the first point of the statement. Using that Ω is purely imaginary, we have

$$\begin{aligned}\overline{\theta[^{\delta'}_{\delta''}](z|\Omega)} &= \sum_{n\in\mathbb{Z}^g} e^{i\pi[(n+\delta')\cdot\Omega(n+\delta')+2(n+\delta')\cdot(-\overline{z}-\delta'')]} \\ &= \sum_{n\in\mathbb{Z}^g} e^{i\pi[(n+\delta')\cdot\Omega(n+\delta')+2(n+\delta')\cdot(-\overline{z}+\delta'')]}e^{-4i\pi(n+\delta')\cdot\delta''},\end{aligned}$$

and the proof is concluded using that δ'' belongs to $(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$.

2. By Point 1 of Lemma 17, it is enough to prove the statement for $\delta'' = 0$, *i.e.*, for the Riemann theta function θ. By Point 2 of Lemma 17 and the first point above, for any $t > 0$ and Ω purely imaginary, the real function $z \mapsto \theta(z|t\Omega)$ is well defined on the torus $(\mathbb{R}/\mathbb{Z})^g$. Furthermore, it converges as $t \to 0$, in the sense of distributions, to the Dirac distribution on the torus whose Fourier coefficients are all equal to 1. Moreover, it satisfies the heat equation

$$\frac{\partial}{\partial t}f = -\frac{1}{2}\Delta f\,,$$

where $\Delta = -\frac{1}{2i\pi}\sum_{j,k}\Omega_{j,k}\frac{\partial^2}{\partial z_j\partial z_k}$ is a positive definite Laplace operator (in a non-orthonormal system of coordinates) on the torus. This means that θ is the fundamental solution of this heat equation. By the maximum principle, we conclude that $\theta(z|t\Omega)$ is strictly positive for any z and any $t > 0$, therefore in particular for $t = 1$. (From a more probabilistic point of view, the function $(z, z', t) \mapsto \theta(z'-z|t\Omega)$ is the transition kernel of a non isotropic Brownian motion on the torus with no drift and a diffusivity matrix given by $\frac{1}{2i\pi}\Omega$, in other words, of the linear image of a standard Brownian motion by a matrix A such that $AA^T = \frac{1}{2i\pi}\Omega$.)

3. To show the last point, let us fix $\delta' \neq 0$. By Point 2 of Lemma 17, we have

$$\theta[^{\delta'}_{\delta''}](z+m|\Omega) = e^{2i\pi\delta'\cdot m}\theta[^{\delta'}_{\delta''}](z|\Omega).$$

Now since $\delta' \neq 0$, it has at least one coefficient equal to $\frac{1}{2}$, say the i^{th}; take m_i to be equal to zero except at position i where it is equal to 1. Then,

$$\theta[^{\delta'}_{\delta''}](z+m_i|\Omega) = -\theta[^{\delta'}_{\delta''}](z|\Omega).$$

Since $\theta[^{\delta'}_{\delta''}]$ is real-valued on $\mathbb{R}^g$ by the first point above, and not identically zero, this concludes the proof. $\square$

From now on, the matrix Ω being fixed once and for all, we simply write $\theta(z|\Omega)$ as $\theta(z)$.

We conclude this section by recalling a fundamental result, known as *Riemann's theorem*, following [Mum07a, Theorem 3.1, p. 149]. Consider a fixed point $x_0 \in \Sigma$, and a lift $\widetilde{x}_0$ in the universal cover $\widetilde{\Sigma}$ of Σ. For any $e \in \mathbb{C}^g$, the function $f_e \colon \widetilde{\Sigma} \to \mathbb{C}$ given by $f_e(\widetilde{x}) = \theta(e + \int_{\widetilde{x}_0}^{\widetilde{x}} \vec{\omega})$ does not induce a well-defined function on Σ, because of θ being only quasi-periodic. However, its zeros form a periodic subset of $\widetilde{\Sigma}$ which has a well-defined projection on Σ. A precise description of this set, called the *theta divisor of* Σ, is given as follows. There exists an element $\Delta \in \mathbb{C}^g$ (depending on the choice of x_0), such that for any e, if f_e is not identically equal to 0, then it admits g zeros $x_1, \ldots, x_g \in \Sigma$ which satisfy the following equality in $\operatorname{Jac}(\Sigma)$:

$$\sum_{j=1}^{g} \int_{x_0}^{x_j} \vec{\omega} = -e + \Delta\,. \tag{3}$$

Moreover, the theta divisor $x_1 + \cdots + x_g$ is uniquely determined by this condition, see [Mum07a, Corollary 3.2, p. 153]. Note that the points x_j may be not distinct; in that case, they correspond to zeros with higher multiplicity, so that the total degree of the theta divisor is g.

More can be said in the case of an M-curve Σ with fixed point $x_0 \in A_0$ and $e \in \mathbb{R}^g$.

Lemma 19. *Let Σ be an M-curve, and let x_0 be an element of A_0. Then, for every $e \in \mathbb{R}^g$, each of the zeros x_j of the function f_e belongs to a different A_i with $1 \leq i \leq g$. Thus, these zeros are distinct and satisfy $x_j \in A_j$ for all $1 \leq j \leq g$ up to relabeling. Consequently, the constant Δ belongs to $\mathbb{R}^g + \Omega\frac{1}{2}\mathbf{1}$.*

Proof. Let us fix an element $x_0 \in A_0$ and a real vector $e \in \mathbb{R}^g$. For $1 \leq j \leq g$, let $\delta_j \in \mathbb{R}^g$ denote the vector whose coordinates are all zero, except the j^{th} one, which is equal to $\frac{1}{2}$. Finally, let $\widetilde{x}_0 \in \widetilde{\Sigma}$ be an arbitrary lift of $x_0 \in A_0 \subset \Sigma$. We now show that the function $f_e \colon \widetilde{\Sigma} \to \mathbb{C}$ given by $f_e(\widetilde{x}) = \theta\left(e + \int_{\widetilde{x}_0}^{\widetilde{x}} \vec{\omega}\right)$ vanishes at least once on any given lift $\widetilde{A}_j \subset \widetilde{\Sigma}$ of the real component $A_j \subset \Sigma$.

Let $\gamma_j \colon \mathbb{R} \to \widetilde{A}_j$ be a lift of a parametrization of A_j by $\mathbb{R}/\mathbb{Z}$. By Lemma 15, Point 2, the integral

$$\int_{\widetilde{x}_0}^{\gamma_j(t)} \vec{\omega}$$

belongs to $\mathbb{R}^g + \Omega(\delta_j + \mathbb{Z}^g)$ for every $t \in \mathbb{R}$. By continuity, it belongs to $\mathbb{R}^g + \Omega(\delta_j + v)$ for some fixed $v \in \mathbb{Z}^g$. Moreover, this same Point 2 of Lemma 15 implies the equality

$$\int_{\widetilde{x}_0}^{\gamma_j(t+1)} \vec{\omega} = \int_{\widetilde{x}_0}^{\gamma_j(t)} \vec{\omega} + 2\delta_j$$

for all $t \in \mathbb{R}$. Hence, by Lemma 17, Point 2, the function

$$h_j : t \mapsto \theta\left[\begin{smallmatrix}\delta_j\\0\end{smallmatrix}\right]\Big(e - \Omega(\delta_j + v) + \int_{\widetilde{x}_0}^{\gamma_j(t)} \vec{\omega}\Big)$$

satisfies $h_j(t+1) = -h_j(t)$ for all $t \in \mathbb{R}$. Finally, this function is real by Lemma 18, Point 1. Therefore, there is a point $\widetilde{x}_j = \gamma_j(t_j)$ in $\widetilde{A}_j$ such that $h_j(\widetilde{x}_j) = 0$. We then conclude by Lemma 17, Points 1 and 2, that

$$|f_e(\widetilde{x}_j)| = \left|\theta\Big(e + \int_{\widetilde{x}_0}^{\widetilde{x}_j} \vec{\omega}\Big)\right| = \left|\theta[{}^{\delta_j}_{0}]\Big(e - \Omega(\delta_j + v) + \int_{\widetilde{x}_0}^{\widetilde{x}_j} \vec{\omega}\Big)\right| = |h_j(\widetilde{x}_j)| = 0.$$

This point $\widetilde{x}_j$ is thus a zero of f_e, which projects on Σ to a point $x_j \in A_j$ contributing to the theta divisor of Σ. The claim follows from the fact that this divisor is of degree g.

The constant Δ belonging to $\mathbb{R}^g + \Omega\frac{1}{2}\mathbf{1}$ is now a direct consequence of Point 4 of Lemma 15, together with Equation (3) applied to $e = 0$. □

2.5 Prime form

This section deals with the so-called *prime form* associated with an arbitrary compact Riemann surface Σ. In Section 2.5.1, we start by studying the general theory of sections of line bundles, a necessary formalism for the precise definition of the prime form given in Section 2.5.2. When Σ is an M-curve, this form exhibits special properties that are stated and proved in Lemmas 25 and 26. Finally, Section 2.5.3 recalls Fay's trisecant identity, which plays a crucial role in the rest of our work.

2.5.1 Sections of line bundles and automorphic forms

The aim of this preliminary paragraph is to prove a statement that is most probably standard: holomorphic sections of a fixed holomorphic line bundle on a Riemann surface Σ of genus $g > 0$ can be understood as *automorphic forms* on the universal cover $\widetilde{\Sigma}$ of Σ, *i.e.*, holomorphic functions with quasi-periodicity under the action of $\pi_1(\Sigma)$ prescribed by the line bundle.

To check this fact, fix a holomorphic line bundle $p\colon L \to \Sigma$ and denote by $\pi\colon \widetilde{\Sigma} \to \Sigma$ the universal cover of Σ. This gives rise to the holomorphic line bundle $\pi^*p\colon E \to \widetilde{\Sigma}$ induced by π. Since we assume $g > 0$, the Riemann surface $\widetilde{\Sigma}$ is isomorphic to the open disc or the complex plane. Since it is non-compact, the Weierstrass theorem implies that $H^1(\widetilde{\Sigma}, \mathcal{O}^*)$ vanishes, so all holomorphic line bundles on $\widetilde{\Sigma}$ are trivial. In particular, there is an isomorphism $\varphi\colon \widetilde{\Sigma} \times \mathbb{C} \to E$ such that $\pi^*p \circ \varphi$ is equal to the projection $pr\colon \widetilde{\Sigma} \times \mathbb{C} \to \widetilde{\Sigma}$. In a nutshell, we have the commutative diagram

$$\begin{tikzcd} \widetilde{\Sigma} \times \mathbb{C} \arrow[r, "\varphi"] \arrow[dr, "pr"'] & E \arrow[r, "\Pi"] \arrow[d, "\pi^*p"] & L \arrow[d, "p"] \\ & \widetilde{\Sigma} \arrow[r, "\pi"] & \Sigma, \end{tikzcd}$$

where $\Pi\colon E \to L$ is the $\pi_1(\Sigma)$-covering induced by p.

Now, consider a holomorphic section $\psi\colon \Sigma \to L$, which by definition satisfies $p \circ \psi = id$. By the lifting property of the $\pi_1(\Sigma)$-covering $\Pi\colon E \to L$, there is a map $\widetilde{\psi}\colon \widetilde{\Sigma} \to E$ such that $\Pi \circ \widetilde{\psi} = \psi \circ \pi$, which is uniquely determined by its value on a lift $\widetilde{x}_0 \in \widetilde{\Sigma}$ of a base point $x_0 \in \Sigma$. Hence, we have

$$\pi \circ \pi^* p \circ \widetilde{\psi} = p \circ \Pi \circ \widetilde{\psi} = p \circ \psi \circ \pi = \pi \,,$$

and we see that $\widetilde{\psi}$ is a holomorphic section of $\pi^* p$ up to action of the group $\pi_1(\Sigma)$ of the covering π. Moreover, one can choose the value $\widetilde{\psi}(\widetilde{x}_0) \in E$ so that $(\pi^* p \circ \widetilde{\psi})(\widetilde{x}_0) = \widetilde{x}_0$. By continuity, we have $\pi^* p \circ \widetilde{\psi} = id$, and $\widetilde{\psi}$ is a holomorphic section of $\pi^* p$. The composition $\varphi^{-1} \circ \widetilde{\psi}$ now defines a holomorphic section of the trivial bundle $pr\colon \widetilde{\Sigma} \times \mathbb{C} \to \widetilde{\Sigma}$, *i.e.*, it is of the form $(\varphi^{-1} \circ \widetilde{\psi})(\widetilde{x}) = (\widetilde{x}, f_\psi(\widetilde{x}))$ for some holomorphic function

$$f_\psi\colon \widetilde{\Sigma} \longrightarrow \mathbb{C}$$

uniquely determined by ψ.

Moreover, its quasi-periodicity properties under the action of $\pi_1(\Sigma)$ are uniquely determined by the holomorphic line bundle $p\colon L \to \Sigma$. They are of the following form: for $\widetilde{x} \in \widetilde{\Sigma}$ and $\gamma \in \pi_1(\Sigma)$, we have

$$f_\psi(\gamma \cdot \widetilde{x}) = p_\gamma(\widetilde{x}) f_\psi(\widetilde{x})$$

for some holomorphic map $p_\gamma\colon \widetilde{\Sigma} \to \mathbb{C}^*$ which only depends on γ and on the line bundle $L \to \Sigma$. This is the *factor of automorphy* of the automorphic form f_ψ.

Example 20. Consider the case of a spin structure $L \to \Sigma$ with $\Sigma = \mathbb{T}(\tau)$. It is a square root of the canonical line bundle, which is nothing but the trivial bundle. Since $\pi_1(\Sigma) = \mathbb{Z} + \mathbb{Z}\tau$ acts by translations on $\widetilde{\Sigma} = \mathbb{C}$, the discussion above shows that holomorphic sections of $L \to \Sigma$ (*i.e.*, spinors) can be understood as holomorphic maps f on $\widetilde{\Sigma} = \mathbb{C}$ with quasi-periodicity

$$f(x+1) = p_1(x) f(x) \quad \text{and} \quad f(x+\tau) = p_\tau(x) f(x)$$

for some $p_1, p_\tau\colon \mathbb{C} \to \mathbb{C}^*$ with $p_1^2 = p_\tau^2 = 1$. This gives 4 different spin structures corresponding to the 4 possible signs $p_1, p_\tau \in \{\pm 1\}$. Note that the choice $p_1 = p_\tau = 1$ leads to constant spinors, while the other choices only allow for identically zero holomorphic sections. This is coherent with the fact that the dimension of the space of spinors has the same parity as the spin structure [Ati71]. In the present case, the only odd spin structure corresponds to the trivial line bundle.

2.5.2 The prime form

We now give a definition with all necessary details for our purposes, but very little more, referring the reader to [Mum07b] for additional information.

Given an arbitrary compact Riemann surface Σ, let us fix a *non-degenerate theta characteristic*, *i.e.*, a theta characteristic such that the corresponding theta function satisfies

$$\mathrm{d}_z\theta[^{\delta'}_{\delta''}](0) \neq 0\,.$$

Such a theta characteristic is known to hold by the Lefschetz embedding theorem. Note that $\binom{\delta'}{\delta''}$ must be odd, and thus also satisfy $\theta[^{\delta'}_{\delta''}](0) = 0$.

By the general theory of spin structures on Riemann surfaces [Ati71], this theta characteristic corresponds to a spin structure, understood as a line bundle L whose square is isomorphic to the canonical line bundle K. An explicit holomorphic section of this bundle L can be constructed as follows. Consider the holomorphic form

$$\zeta = \sum_{i=1}^{g} \frac{\partial\theta[^{\delta'}_{\delta''}]}{\partial z_i}(0)\omega_i = \mathrm{d}_z\theta[^{\delta'}_{\delta''}](0)\cdot\vec{\omega}\,,$$

which is nothing but the differential of the function $y \mapsto \theta[^{\delta'}_{\delta''}](y-x){:=}\theta[^{\delta'}_{\delta''}](\int_x^y \vec{\omega})$, in the variable $y \in \Sigma$, evaluated at $y = x$. All the zeros of ζ are double zeros [Fay73, Corollary 1.3]. Therefore, interpreting ζ as a holomorphic section of K, it admits a square root $\xi[^{\delta'}_{\delta''}]$ which is a section of the line bundle L. Note that $\xi[^{\delta'}_{\delta''}]$ depends on the theta characteristic in two ways: in the construction of ζ, and in the choice of the square root.

Definition 21. The *prime form* is the form E defined by

$$E(x,y) = \frac{\theta[^{\delta'}_{\delta''}](y-x)}{\xi[^{\delta'}_{\delta''}](x)\xi[^{\delta'}_{\delta''}](y)}\,.$$

for x, y in the universal cover $\widetilde{\Sigma}$ of Σ.

Let us give an explicit example.

Example 22. If Σ has genus 1, it is isomorphic to a torus $\mathbb{T}(\tau)$ with $\Im(\tau) > 0$. There is a unique odd theta characteristic, namely $\binom{\frac{1}{2}}{\frac{1}{2}}$, which is always non-degenerate. Hence, the numerator of the prime form is given by $\theta[^{\frac{1}{2}}_{\frac{1}{2}}](y-x) = -\theta_1(\pi(y-x))$, see Example 16. As for the denominator, recall that the canonical line bundle of the torus is trivial. Furthermore, the square root L of $K = 1$ corresponding to the odd theta characteristic is the trivial bundle $L = 1$ (recall Example 20). The constant value of $\xi[^{\frac{1}{2}}_{\frac{1}{2}}]$ is given by a square root of $-\pi\theta_1'(0)$, leading to the explicit formula $E(x,y) = \frac{\theta_1(\pi(y-x))}{\pi\theta_1'(0)}$.

The prime form is the key ingredient for constructing meromorphic functions on Σ. Indeed, if $x_1,\ldots,x_k,y_1,\ldots,y_k$ are points on Σ with $x_i \neq y_j$ for all i,j and such that $\sum_{i=1}^k(x_i - y_i)$ is a principal divisor, then all meromorphic functions with this divisor are of the form

$$g(x) = c \times \prod_{i=1}^{k} \frac{E(x_i,x)}{E(y_i,x)}\,,$$

with c a complex number.

We will make use of the following standard properties of the prime form, valid for any $x, y \in \widetilde{\Sigma}$ (see e.g. [Mum07b, p. 3.210]):

- $E(x,y) = 0$ if and only if x and y project to the same point on Σ;
- these are first order zeros;
- When x and y are close to each other,

$$E(x,y) = \frac{z(y) - z(x)}{\sqrt{dz(x)}\sqrt{dz(y)}} \left(1 + O(z(x) - z(y))^2\right),$$

  where z is a local coordinate such that $\zeta = dz$ in a connected open set containing x and y.
- $E(x,y) = -E(y,x)$.

The following result is also well-known, but most references only consider equalities up to signs. Since signs do play an important role in our setting, we include the proof for completeness.

Proposition 23. *The prime form does not depend on the choice of the non-degenerate theta characteristic.*

Proof. Let us start by studying the quasi-periodicity of $E(x,y)$. If $y' \in \widetilde{\Sigma}$ is obtained from y by adding the corresponding lift of the cycle A_j, we have

$$\theta[{}^{\delta'}_{\delta''}](y' - x) = \theta[{}^{\delta'}_{\delta''}]((y - x) + \int_{A_j} \vec{\omega}) = \theta[{}^{\delta'}_{\delta''}]((y - x) + e_j) = e^{2i\pi\delta'_j}\theta[{}^{\delta'}_{\delta''}](y - x)$$

by Lemma 17. On the other hand and by definition, we have $\xi[{}^{\delta'}_{\delta''}](y') = p_j[{}^{\delta'}_{\delta''}](y)\xi[{}^{\delta'}_{\delta''}](y)$, with $p_j[{}^{\delta'}_{\delta''}](y)^2 =: p_j^2(y)$ encoding the A_j-quasi-periodicity of any holomorphic form. Recall however that the sign of this square root does depend on the theta characteristic, hence the heavy notation. Fixing an arbitrary square root $p_j(y)$, we see that

$$E(x, y') = \pm(-1)^{2\delta'_j} p_j^{-1}(y) E(x,y)\,, \tag{4}$$

with p_j independent of the theta characteristic, while the sign might a priori depend on it. (Note that even in the genus 1 case, we have $E(x,y') = -E(x,y)$, recall Example 22; therefore, and unlike claimed in many references, the prime form is in general not invariant along A-cycles.) Similarly, if $y'' \in \widetilde{\Sigma}$ is obtained from y by adding the lift of the cycle B_j, then we have

$$E(x, y'') = \pm(-1)^{2\delta''_j} q_j^{-1}(y) e^{2i\pi(-\frac{\Omega_{jj}}{2} - \int_x^y \omega_j)} E(x,y)\,, \tag{5}$$

with $q_j^2(y)$ encodes the B_j-quasi-periodicity of any holomorphic form. The crucial point here is that, once again, the factor appearing does not depend on the theta characteristic, at least up to sign.

Let us now consider prime forms E and E' obtained from two non-singular theta characteristics. For any fixed $x \in \widetilde{\Sigma}$, set $f_x(y) := \frac{E(x,y)}{E'(x,y)}$. By the fundamental properties of E listed above, both the numerator and denominator have the same zeros, namely a simple zero at each element of $\widetilde{\Sigma}$ with same image in Σ as x. Therefore, this quotient defines a (non-vanishing) holomorphic function on $\widetilde{\Sigma}$. Moreover, by the equalities displayed above, the A and B periods of f_x are trivial up to a sign. Therefore, the function f_x induces a well-defined holomorphic function on the cover Σ' of Σ given by the homomorphism $\pi_1(\Sigma) \to (\mathbb{Z}/2\mathbb{Z})^{2g}$ mapping the cycles A_j and B_j to distinct elements of the canonical basis of $(\mathbb{Z}/2\mathbb{Z})^{2g}$. This cover being finite, the Riemann surface Σ' is compact, and f_x constant. The normalization of E and E' ensures that they have the same asymptotic behavior near the diagonal, and hence that this constant is equal to 1. This completes the proof. $\square$

Remark 24. The statement above implies that the factor of automorphy for the prime form does not depend on $\binom{\delta'}{\delta''}$. It means that there is a bijection between the collection of signs $\{(-1)^{2\delta'_j}, (-1)^{2\delta''_j}\}_{1\leq j\leq g}$ and the collection of $\pm$ coming from the square roots of $p_j^2(y)$ and $q_j^2(y)$ we need to consider for the line bundle L, and if we choose correctly the sign of "reference" square roots $p_j(y)$ and $q_j(y)$, these collections of signs can be taken to be equal.

We now come back to the setting of M-curves. Note that the anti-holomorphic involution $\sigma\colon \Sigma \to \Sigma$ lifts to an anti-holomorphic involution on the universal cover $\widetilde{\Sigma}$, that we also denote by σ.

Lemma 25. *If Σ is an M-curve, then the associated prime form satisfies*

$$\forall\, x, y \in \widetilde{\Sigma}, \quad E(\sigma(x), \sigma(y)) = \overline{E(x,y)}\,.$$

Proof. First note that by Points 1 and 3 of Lemma 11 together with Point 1 of Lemma 18, we have

$$\theta[^{\delta'}_{\delta''}]\Big(\int_{\sigma(x)}^{\sigma(y)} \vec{\omega}\Big) = \theta[^{\delta'}_{\delta''}]\Big(\int_x^y \sigma^*\vec{\omega}\Big) = \theta[^{\delta'}_{\delta''}]\Big(\overline{\int_x^y \vec{\omega}}\Big) = \overline{\theta[^{\delta'}_{\delta''}]\Big(\int_x^y \vec{\omega}\Big)}\,.$$

Furthermore, the theta characteristic being odd and the period matrix purely imaginary, the number $\frac{\partial}{\partial z_j}\theta[^{\delta'}_{\delta''}](0)$ is easily seen to be real. An additional use of the first point of Lemma 11 then leads to the identity $\sigma^*\zeta = \overline{\zeta}$, which implies that $\xi[^{\delta'}_{\delta''}](\sigma(x)) = \pm\overline{\xi[^{\delta'}_{\delta''}](x)}$, with the global sign independent of x. Together with the equality displayed above, this implies the statement. $\square$

We now study the restriction of the prime form to $\widetilde{A}_0 \times \widetilde{A}_0$, where $\widetilde{A}_0$ denotes the universal covering of A_0 given by an arbitrary connected component of the preimage of A_0 in $\widetilde{\Sigma}$. More precisely, we compute the phase of the $(-1/2,-1/2)$-form $E(x,y)$ for $x,y \in \widetilde{A}_0$, evaluated at the tangent vectors (v_x, v_y), where v denotes the velocity vector field of (any regular parametrization of) the oriented curve $\widetilde{A}_0$.

For lifts $x,y \in \widetilde{A}_0$ of different elements of A_0, we write $\lfloor y-x \rfloor \in \mathbb{Z}$ for the unique integer such that the inequalities

$$x + \lfloor y-x \rfloor \mathbf{1} < y < x + (\lfloor y-x \rfloor + 1)\mathbf{1}$$

hold in $\widetilde{A}_0$, where $x+\mathbf{1}$ denotes the image of x in $\widetilde{A}_0$ via the generator of the infinite cyclic covering group. (This slight abuse of notation is motivated by the third point of Lemma 15.)

Lemma 26. *There exists $\varphi \in \{\pm 1, \pm i\}$ such that for any lifts $x,y \in \widetilde{A}_0$ of different elements of A_0, the phase of $E(x,y)$ evaluated at the velocity vectors (v_x,v_y) is equal to $(-1)^{\lfloor y-x \rfloor}\varphi$.*

Proof. Recall that in our setting, the period matrix Ω is purely imaginary. Given an odd theta characteristic $\binom{\delta'}{\delta''}$, this easily implies that the number $\frac{\partial}{\partial z_j}\theta[\substack{\delta' \\ \delta''}](0)$ is real. Furthermore, given any real element $P \in \Sigma$, the first two points of Lemma 11 imply

$$\overline{\omega_j(v)} = \sigma^*\omega_j(v) = \omega_j(\sigma_*(v)) = \omega_j(v)$$

for all $1 \leq j \leq g$, where v denotes the velocity vector field of the real component containing P. As a consequence, the holomorphic form $\xi[\substack{\delta' \\ \delta''}]^2 = \sum_j \frac{\partial}{\partial z_j}\theta[\substack{\delta' \\ \delta''}](0)\,\omega_j$ takes real values on A_0, when evaluated along the corresponding velocity vector field. By the first point of Lemma 18, $\theta[\substack{\delta' \\ \delta''}](y-x)$ is also real, so $E(x,y)$ evaluated at (v_x,v_y) is either real or purely imaginary. In other words, the corresponding phase $f(x,y)$ is ± 1 or $\pm i$.

We now use the crucial fact that the prime form vanishes only if both variables are lifts of the same element of A_0. This implies in particular that $f(x,y)$ is constant for all $x < y < x+\mathbf{1}$, say equal to $\varphi(x)$ for some map $\varphi\colon \widetilde{A}_0 \to \{\pm 1, \pm i\}$. This map being continuous, it is constant. Hence, for $x,y \in \widetilde{A}_0$ with $x < y < x+\mathbf{1}$, the fact that the prime form is skew-symmetric now implies

$$f(x,y) = \varphi(x) = \varphi(y) = f(y, x+\mathbf{1}) = -f(x+\mathbf{1}, y)\,.$$

The equality $f(x,y) = (-1)^{\lfloor y-x \rfloor}\varphi$ easily follows. $\square$

2.5.3 Fay's identity

We will make use of the following three versions of Fay's identity [Fay73]. They are easy consequences of the standard version formulated by Mumford [Mum07b, p. 3.214], that we now recall without proof.

Theorem 27 (Fay's identity)**.** *For any* $z \in \mathbb{C}^g$ *and* $a, b, c, d \in \widetilde{\Sigma}$*, we have*

$$\theta(z+c-a)\theta(z+d-b)E(c,b)E(a,d) + \theta(z+c-b)\theta(z+d-a)E(c,a)E(d,b) \\ = \theta(z+c+d-a-b)\theta(z)E(c,d)E(a,b)\,. \quad (6)$$

The first variation is the $n = 3$ case of [Foc15, Lemma 1], and can be obtained as follows. Divide Equation (6) by $\theta(z)\theta(z+d-a)\theta(z+d-b)E(c,a)E(c,b)E(c,d)$ and set $a = \alpha$, $b = \beta$, $c = u$, $d = \gamma$ in $\widetilde{\Sigma}$ and $z = s - \gamma \in \widetilde{\mathrm{Pic}^0(\Sigma)} = \mathbb{C}^g$. This yields the equality

$$\frac{\theta(s+u-\alpha-\beta)E(\alpha,\beta)}{E(\alpha,u)E(\beta,u)\theta(s-\alpha)\theta(s-\beta)} + \frac{\theta(s+u-\beta-\gamma)E(\beta,\gamma)}{E(\beta,u)E(\gamma,u)\theta(s-\beta)\theta(s-\gamma)} \\ + \frac{\theta(s+u-\gamma-\alpha)E(\gamma,\alpha)}{E(\alpha,u)E(\gamma,u)\theta(s-\alpha)\theta(s-\gamma)} = 0, \quad (7)$$

for all u, α, β, γ in $\widetilde{\Sigma}$, and all s in the universal cover $\widetilde{\mathrm{Pic}^1(\Sigma)}$ of $\mathrm{Pic}^1(\Sigma)$.

To obtain the second version, simply pass the second and third terms in Equation (7) on the right-hand side. This yields the equation

$$\frac{\theta(s+u-\alpha-\beta)E(\alpha,\beta)}{E(\alpha,u)E(\beta,u)\theta(s-\alpha)\theta(s-\beta)} = F^{s,\gamma}(u;\beta) - F^{s,\gamma}(u;\alpha)\,, \quad (8)$$

where

$$F^{s,\gamma}(u;\alpha) = \frac{\theta(s+u-\alpha-\gamma)E(\gamma,\alpha)}{E(\alpha,u)E(\gamma,u)\theta(s-\alpha)\theta(s-\gamma)}.$$

Note that γ does not appear in the left-hand side of Equation (8) and can be chosen arbitrarily to define F. By carefully letting γ tend to u in the definition of $F^{s,\gamma}(u;\alpha)$, one obtains the following version of Equation (8):

$$\theta(u-s)\frac{\theta(s+u-\alpha-\beta)E(\alpha,\beta)}{E(\alpha,u)E(\beta,u)\theta(s-\alpha)\theta(s-\beta)} \\ = \omega_{\beta-\alpha}(u) + \sum_{j=1}^{g}\left(\frac{\partial \log\theta}{\partial z_j}(s-\alpha) - \frac{\partial \log\theta}{\partial z_j}(s-\beta)\right)\omega_j(u)\,, \quad (9)$$

where $\omega_{\beta-\alpha}(u) = \mathrm{d}_u \log\frac{E(u,\beta)}{E(u,\alpha)}$ is the unique meromorphic 1-form with 0 integral along A-cycles, and two simple poles: at β with residue 1, and at α with residue -1. See [Fay73, Proposition 2.10] for a derivation of this variant.

The third version, which can be found at the very end of [Foc15], is simply obtained by setting $F_t(a,b) := \theta(a+b-t)E(a,b)$ in Equation (6) with $z = a+b-t$, yielding

$$F_t(a,b)F_t(c,d) + F_t(a,d)F_t(b,c) + F_t(a,c)F_t(d,b) = 0 \quad (10)$$

for all $a, b, c, d \in \widetilde{\Sigma}$ and $t \in \widetilde{\mathrm{Pic}^2(\Sigma)}$.

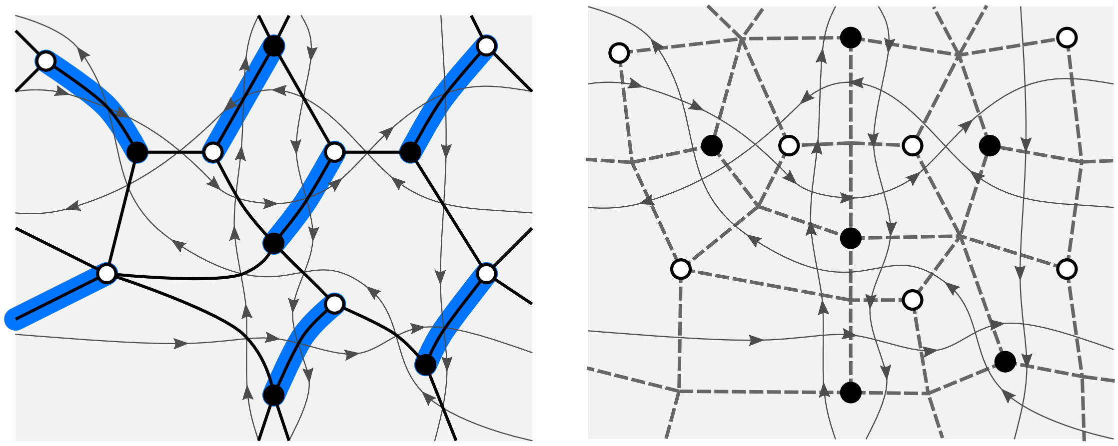

Figure 3: Left: piece of a minimal graph with its associated oriented train-tracks (in grey), and the corresponding portion of a dimer configuration (in blue). Right: the same train-tracks together with the associated graph $\mathsf{G}^\diamond$ (in dashed lines).

Remark 28. It should be noted that in Equation (6), the theta function θ can be replaced with any theta function $\theta[^{\delta'}_{\delta''}]$ with theta characteristic. This is a consequence of the first point of Lemma 17. The same holds true for all the versions of Fay's identity displayed above.

3 Fock's Kasteleyn operators and their inverses

In this section, we define our dimer models on an arbitrary minimal graph G, and initiate their study. More precisely, we start in Section 3.1 by briefly recalling the necessary combinatorial concepts, namely those of train-tracks and minimal graphs, together with the definition of the dimer model. Section 3.2 also deals with background material, *i.e.*, the definition of the *discrete Abel map* of [Foc15], and of the parameter spaces X_{G} of [BCdT21]. In Section 3.3, we finally give the definition of the models via the corresponding adjacency operators of [Foc15], but restricting the parameters to ensure that the resulting edge-weights are positive. Section 3.4 deals with explicit functions in the kernel of these operators inspired by [Foc15], functions that are used in Section 3.5 to define a two-parameter family of inverses of each of these operators.

3.1 Dimer models on minimal graphs

In an attempt to be reasonably self-contained, we now recall the fundamental concepts of train-tracks and minimal graphs, as well as the definition of the dimer model and of the associated Kasteleyn matrices. However, this article being a sequel to [BCdT20], we favor brevity over rigor and completeness, referring the reader to Sections 2.1 and 2.2 of [BCdT20] for details.

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a locally finite graph embedded in the plane with faces being bounded topological discs; in particular, the graph G is infinite. If $\mathsf{G}^* = (\mathsf{V}^*, \mathsf{E}^*)$ stands for the dual embedded graph, then the associated *quad-graph* $\mathsf{G}^\diamond$ is defined from the vertex set $\mathsf{V} \sqcup \mathsf{V}^*$ by joining a primal vertex $\mathsf{v} \in \mathsf{V}$ and a dual vertex $\mathsf{f} \in \mathsf{V}^*$ with an edge each time v lies on the boundary of the face corresponding to f. Note that $\mathsf{G}^\diamond$ embeds in the plane with (possibly degenerate) quadrilaterals faces (see Figure 1). Following [Ken02, KS05], we define a *train-track* of G as a maximal chain of adjacent quadrilaterals of $\mathsf{G}^\diamond$ such that when one enters a quadrilateral, one exits through the opposite edge, see Figure 3.

Let us now assume that G is *bipartite*, *i.e.*, that V admits a partition $\mathsf{B} \sqcup \mathsf{W}$ into black and white vertices such that no edge of E connects two vertices of the same color. In this case, train-tracks can be consistently oriented, say, with black vertices on the right and white vertices on the left of the path, see again Figure 1. We let $\mathcal{T}$ denote the set of consistently oriented train-tracks of the bipartite graph G. A bipartite, planar graph G is said to be *minimal* [Thu17, GK13] if its train-tracks do not self-intersect, and no pair of oriented train-tracks intersect twice in the same direction. This implies that train-tracks do not form loops, and that G has neither multiple edges, nor degree 1 vertices. In particular, a minimal graph is a simple graph.

We now quickly recall basic facts on dimer models, referring to [Ken04] for details.

A *dimer configuration* of a graph G is a collection $\mathsf{M} \subset \mathsf{E}$ such that every vertex is incident to exactly one edge of M, see Figure 3. If G is finite and endowed with a positive edge-weight function $\nu = (\nu_\mathsf{e})_{\mathsf{e} \in \mathsf{E}}$, then the *dimer Boltzmann measure* $\mathbb{P}$ on the set of dimer configurations of G is defined by

$$\mathbb{P}(\mathsf{M}) = \frac{\prod_{\mathsf{e} \in \mathsf{M}} \nu_\mathsf{e}}{Z(\mathsf{G}, \nu)},$$

where $Z(\mathsf{G}, \nu) = \sum_\mathsf{M} \prod_{\mathsf{e} \in \mathsf{M}} \nu_\mathsf{e}$ is the *dimer partition function.* When the graph G is infinite and planar, this notion is replaced by that of *Gibbs measure*, see e.g. [KOS06].

Two dimer models on G defined via edge-weights ν and ν' are called *gauge equivalent* if there is a positive vertex-function σ such that $\nu'_{\mathsf{xy}} = \sigma_\mathsf{x}\, \nu_{\mathsf{xy}}\, \sigma_\mathsf{y}$ holds for each $\mathsf{e} = \mathsf{xy} \in \mathsf{E}$. If G is finite, then ν and ν' yield the same Boltzmann measure. When G is planar and bipartite, two dimer models on G are gauge equivalent if and only if the corresponding edge-weights define equal *face weights*, where faces weights are the alternating product of edge-weights around each given bounded face.

One of the most fundamental tools for studying the dimer model is the *Kasteleyn matrix*, named after [Kas61], see also [TF61], and extended by Kuperberg [Kup98] as follows. Let us fix a finite, planar and bipartite graph G. Consider a weighted adjacency matrix K of G twisted by a phase, *i.e.*, a matrix K with $\mathsf{K}_{\mathsf{w},\mathsf{b}} = \omega_{\mathsf{wb}} \nu_{\mathsf{wb}}$ and ω_{wb} any modulus 1 complex number. Let us assume that for any bounded face f of degree $2m$ of G, the phase ω satisfies the following *Kasteleyn condition*:

$$\prod_{j=1}^{m} \frac{\omega_{\mathsf{w}_j \mathsf{b}_j}}{\omega_{\mathsf{w}_j \mathsf{b}_{j+1}}} = (-1)^{m+1}\,,$$

assuming the notation of Figure 4. Then, the dimer partition function and Boltzmann measure can be computed from K and its inverse K^{-1}, see [Ken97].

3.2 The discrete Abel map and the parameter space X_{G}

Let Σ be an M-curve. Recall that it admits an oriented real component denoted by A_0, which contains the base point x_0. We assign to each oriented train-track $T \in \mathcal{T}$ of G an element α_T of A_0, referred to as its *angle*. (This terminology originates from the elliptic case, where A_0 is naturally identified with $\mathbb{R}/\mathbb{Z}$.)

Following Fock [Foc15], we define a function $\boldsymbol{d}$ from the set of vertices of $\mathsf{G}^\diamond$ into $\mathrm{Pic}(\Sigma)$, as follows. Choose a face f_0 and set $\boldsymbol{d}(\mathsf{f}_0) = 0$. Then, along an edge of $\mathsf{G}^\diamond$ crossing a train-track T with angle α_T, we formally add α_T to the value of $\boldsymbol{d}$ if we arrive at a black vertex or leave a white vertex (see Figure 1). In this way, the degree of $\boldsymbol{d}(\mathsf{x}) \in \mathrm{Div}(\Sigma)$ is equal to

$$\deg \boldsymbol{d}(\mathsf{x}) = \begin{cases} 1 & \text{if } \mathsf{x} \text{ is a black vertex of } \mathsf{G}, \\ 0 & \text{if } \mathsf{x} \text{ is a face of } \mathsf{G}, \\ -1 & \text{if } \mathsf{x} \text{ is a white vertex of } \mathsf{G}. \end{cases}$$

In particular, for any face f of G, the element $\boldsymbol{d}(\mathsf{f})$ belongs to $\mathrm{Pic}^0(\Sigma)$. By Lemma 15, its image by the Abel-Jacobi map belongs to $(\mathbb{R}/\mathbb{Z})^g \subset \mathrm{Jac}(\Sigma)$.

As it turns out, only a special class of angle assignments $T \mapsto \alpha_T$ gives rise to probabilistic models. It can be described as follows, see [BCdT21] for more detail.

Let us call two non-closed oriented planar curves *parallel* (resp. *antiparallel*) if they intersect infinitely many times in the same direction (resp. in opposite directions), or if they are disjoint and cross a topological disc in the same direction (resp. in opposite directions). Consider a triple of oriented train-tracks of G, pairwise non-parallel. Let B be a compact disk outside of which these train-tracks do not meet, apart from possible anti-parallel ones, and order this triple of elements of $\mathcal{T}$ cyclically according to the outgoing points of the corresponding oriented curves in the circle ∂B. This gives a well-defined partial cyclic order on $\mathcal{T}$, see [BCdT21, Section 2.3]. Note that A_0 is an oriented topological circle, and therefore endowed with a total cyclic order as well, which allows for the following definition.

We define X_{G} as the set of maps $\boldsymbol{\alpha} \colon \mathcal{T} \to A_0$ that are monotone, in the sense that they preserve the cyclic order, and such that non-parallel train-tracks have distinct images. One of the main results of [BCdT21] is that if G is minimal, then X_{G} is included in the space of *minimal immersions* of G, and coincides with it if G is minimal and periodic, see in particular [BCdT21, Theorem 23, and Corollary 29].

3.3 Fock's Kasteleyn operators

To define a version of Fock's adjacency operator satisfying Kasteleyn's condition, let us fix a minimal graph G, an M-curve Σ and an angle map $\boldsymbol{\alpha} \in X_{\mathsf{G}}$.

We now fix an arbitrary lift $\widetilde{\boldsymbol{\alpha}}\colon \mathcal{T} \to \widetilde{A}_0$ of $\boldsymbol{\alpha}$, *i.e.*, lifts $\widetilde{\alpha}_T \in \widetilde{A}_0$ of the angles $\alpha_T \in A_0$, where $\widetilde{A}_0 \subset \widetilde{\Sigma}$ denotes the universal cover of A_0. Recall from Lemma 15 that the Abel-Jacobi map defines an embedding of A_0 in $(\mathbb{R}/\mathbb{Z})^g$, and therefore an embedding of $\widetilde{A}_0$ in $\mathbb{R}^g \subset \widetilde{\mathrm{Jac}(\Sigma)} = \mathbb{C}^g$. We define a lift $\widetilde{\boldsymbol{d}}\colon \mathsf{V}(\mathsf{G}^\diamond) \to \mathrm{Div}(\widetilde{\Sigma})$ of the discrete Abel map $\boldsymbol{d}$ by setting $\widetilde{\boldsymbol{d}}(\mathsf{f}_0) = 0$, and computing the values at every vertex iteratively by adding and subtracting the lifts $\widetilde{\alpha}_T$ of the crossed train-tracks, with the same local rule as $\boldsymbol{d}$. In particular, if b (resp. w) and f are separated by a train-track with angle α (resp. β), one has $\widetilde{\boldsymbol{d}}(\mathsf{b}) = \widetilde{\boldsymbol{d}}(\mathsf{f}) + \widetilde{\alpha}$ (resp. $\widetilde{\boldsymbol{d}}(\mathsf{w}) = \widetilde{\boldsymbol{d}}(\mathsf{f}) - \widetilde{\beta}$). Note that for any face f of G, the divisor $\widetilde{\boldsymbol{d}}(\mathsf{f})$ has degree 0, and its image by the Abel-Jacobi map $\mathrm{Div}^0(\widetilde{\Sigma}) \to \mathbb{C}^g$ belongs to $\mathbb{R}^g$.

Definition 29. *Fock's adjacency operator* K is the complex weighted adjacency operator of the graph G, indexed by elements $t \in (\mathbb{R}/\mathbb{Z})^g \subset \mathrm{Jac}(\Sigma)$, with non-zero coefficients given as follows: for every edge wb crossed by train-tracks with angles α, β in A_0 as in Figure 1, we have

$$\mathsf{K}_{\mathsf{w},\mathsf{b}} = \frac{E(\widetilde{\alpha}, \widetilde{\beta})}{\theta(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f}))\, \theta(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f}'))}\,, \tag{11}$$

where $\widetilde{t} \in \mathbb{R}^g$ is a lift of $t \in (\mathbb{R}/\mathbb{Z})^g$.

Here are several remarks on this definition.

Remark 30.

1. Since we are working in the universal cover, the coefficient $\mathsf{K}_{\mathsf{w},\mathsf{b}}$ can be understood as an honest complex-valued function of the lifted angles $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{A}_0$ (recall Section 2.5.1). On the other hand, it does *not* project to a well-defined function of $\alpha, \beta \in A_0$. Indeed, while the denominator gives a function on $A_0 \times A_0$, the numerator is only well-defined on the universal cover $\widetilde{A}_0 \times \widetilde{A}_0$: replacing $\widetilde{\alpha} \in \widetilde{A}_0$ by $\widetilde{\alpha} + \mathbf{1}$, *i.e.*, going once around A_0, leads to

$$E(\widetilde{\alpha} + \mathbf{1}, \widetilde{\beta}) = \frac{\xi[\begin{smallmatrix}\gamma'\\ \gamma''\end{smallmatrix}](\widetilde{\alpha})}{\xi[\begin{smallmatrix}\gamma'\\ \gamma''\end{smallmatrix}](\widetilde{\alpha} + \mathbf{1})} e^{2i\pi\gamma'\cdot\mathbf{1}} E(\widetilde{\alpha}, \widetilde{\beta}) = -\lambda E(\widetilde{\alpha}, \widetilde{\beta})$$

for some $\lambda > 0$, by Lemma 26, and similarly for β. (Here, we use the notation $\left(\begin{smallmatrix}\gamma'\\ \gamma''\end{smallmatrix}\right)$ for the odd theta characteristic appearing in the prime form, to distinguish it from the theta characteristic $\left(\begin{smallmatrix}\delta'\\ \delta''\end{smallmatrix}\right)$ possibly appearing in the denominator, see points 3–5 below.) However, the corresponding face weights are well-defined, *i.e.*, these factors cancel up to gauge equivalence.

2. By the second point of Lemma 17, the entry $\mathsf{K}_{\mathsf{w},\mathsf{b}}$ does not depend on the choice of the lift $\widetilde{t} \in \mathbb{R}^g$ of $t \in (\mathbb{R}/\mathbb{Z})^g$.

3. By the third point of Lemma 17, we have the identity $\theta[\begin{smallmatrix}\delta'\\ 0\end{smallmatrix}](z) = \theta[\begin{smallmatrix}\delta'\\ \delta''\end{smallmatrix}](z - \delta'')$. Hence, up to a translation of $\widetilde{t}$ by an element of $\frac{1}{2}\mathbb{Z}^g$, we can choose δ'' arbitrarily in the theta characteristic of the denominator of (1). We set $\delta'' = 0$ in (11) for definiteness.

4. As stated in Proposition 2, it is possible to define Kasteleyn operators indexed by an arbitrary real element of $\mathrm{Jac}(\Sigma)$, *i.e.*, by any element of the form $t + \Omega\delta$ with $t \in (\mathbb{R}/\mathbb{Z})^g$ and $\delta \in (\frac{1}{2}\mathbb{Z})^g$, as long as the theta characteristic $[\begin{smallmatrix}\delta' \\ \delta''\end{smallmatrix}]$ of the denominator satisfies $\delta' = \delta$. Indeed, the gauge equivalence class of the resulting operator only depends on $t \in (\mathbb{R}/\mathbb{Z})^g$, and we set $\delta = \delta' = 0$ for definiteness.

   To check this claim, let us fix $\delta \in (\frac{1}{2}\mathbb{Z})^g$ and write K' for the Kasteleyn operator defined as in (11), but with $\widetilde{t}$ replaced by $\widetilde{t} + \Omega\delta$ and θ replaced by $\theta[\begin{smallmatrix}\delta \\ 0\end{smallmatrix}]$. By the third point of Lemma 17, we get

$$\mathsf{K}'_{\mathsf{w},\mathsf{b}} = e^{2i\pi\delta\cdot\Omega\delta}\, e^{2i\pi(2\delta)\cdot\widetilde{t}}\, e^{2i\pi\delta\cdot(\widetilde{\boldsymbol{d}}(\mathsf{b})+\widetilde{\boldsymbol{d}}(\mathsf{w}))}\,\mathsf{K}_{\mathsf{w},\mathsf{b}}\,,$$

   yielding the statement.

   This is no surprise, as we know from [KO06] that for periodic graphs, the gauge-equivalence classes are parametrized by one point on each of the ovals, *i.e.*, by a g-dimensional torus, see also the third point of Remark 50.

5. Note that another natural choice of theta characteristic is given by $\delta' = \delta'' = \frac{1}{2}\mathbf{1}$, leading to

$$\mathsf{K}_{\mathsf{w},\mathsf{b}} = \frac{E(\widetilde{\alpha},\widetilde{\beta})}{\theta[\begin{smallmatrix}\frac{1}{2}\mathbf{1} \\ \frac{1}{2}\mathbf{1}\end{smallmatrix}](\widetilde{t}' + \widetilde{\boldsymbol{d}}(\mathsf{f}))\,\theta[\begin{smallmatrix}\frac{1}{2}\mathbf{1} \\ \frac{1}{2}\mathbf{1}\end{smallmatrix}](\widetilde{t}' + \widetilde{\boldsymbol{d}}(\mathsf{f}'))}$$

   indexed by $t' \in (\mathbb{R}/\mathbb{Z})^g + \Omega\frac{1}{2}\mathbf{1}$. This generalizes the genus 1 case of [BCdT20].

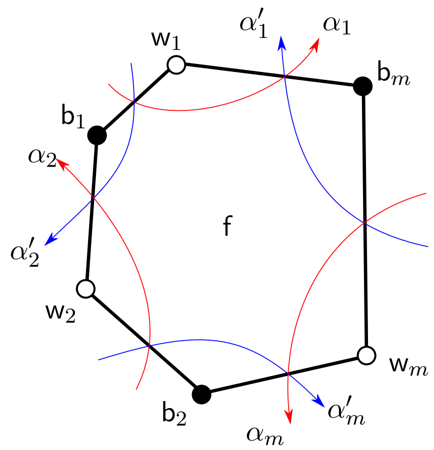

Figure 4: Train-tracks around a typical face f of degree $2m$, with white vertices $\mathsf{w}_1,\dots,\mathsf{w}_m$ and black vertices $\mathsf{b}_1,\dots,\mathsf{b}_m$ on its boundary.

The following result establishes the fact that we can harness Kasteleyn's theory. Note that even though it extends the elliptic case treated in [BCdT20, Proposition 12],the proof given below is significantly simpler then the one appearing in [BCdT20].

Proposition 31. *For any lift $\widetilde{\boldsymbol{\alpha}}$ of the map $\boldsymbol{\alpha} \in X_{\mathsf{G}}$ and any $t \in (\mathbb{R}/\mathbb{Z})^g \subset \mathrm{Jac}(\Sigma)$, Fock's adjacency operator K is a Kasteleyn operator.*

Proof. Let us study the validity of the Kasteleyn condition around an arbitrary face f as in Figure 4. By definition, we have

$$\mathsf{K}_{\mathsf{w},\mathsf{b}} = \frac{E(\widetilde{\alpha}, \widetilde{\beta})}{\theta(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f}))\, \theta(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f}'))}\,,$$

with $\widetilde{t}, \widetilde{\boldsymbol{d}}(\mathsf{f})$ and $\widetilde{\boldsymbol{d}}(\mathsf{f}')$ elements of $\mathbb{R}^g$. Hence, by the third point of Lemma 11 and the second point of Lemma 18, the denominator is strictly positive. Since the prime form is skew-symmetric, the phase of the face weight around f is therefore equal to the phase of

$$(-1)^m \frac{E(\widetilde{\alpha}_1, \widetilde{\alpha}'_2)E(\widetilde{\alpha}_2, \widetilde{\alpha}'_3) \cdots E(\widetilde{\alpha}_m, \widetilde{\alpha}'_1)}{E(\widetilde{\alpha}_1, \widetilde{\alpha}'_1)E(\widetilde{\alpha}_2, \widetilde{\alpha}'_2) \cdots E(\widetilde{\alpha}_m, \widetilde{\alpha}'_m)}\,.$$

Note that this phase is well-defined for $\alpha_j, \alpha'_j \in A_0$, as replacing $\widetilde{\alpha}_j$ by $\widetilde{\alpha}_j + \mathbf{1}$ makes a sign appear at the numerator and denominator, and similarly for α'_j.

By Lemma 26, we know that the phase of $E(x, y)$ is equal to $(-1)^{\lfloor y-x \rfloor} \varphi$ for some fixed $\varphi \in \{\pm 1, \pm i\}$. Therefore, we are left with the proof that the integer

$$\sum_{j=1}^{m} \lfloor \widetilde{\alpha}'_{j+1} - \widetilde{\alpha}_j \rfloor + \sum_{j=1}^{m} \lfloor \widetilde{\alpha}'_j - \widetilde{\alpha}_j \rfloor$$

is odd, where α'_{m+1} stands for α'_1. Since $\boldsymbol{\alpha}$ belongs to X_{G} and G is minimal, Lemma 8 of [BCdT21] states that $\boldsymbol{\alpha}$ is monotone and injective on the cyclically ordered set of oriented train-track strands adjacent to the vertex w_j. This implies the modulo 2 equality $\lfloor \widetilde{\alpha}'_{j+1} - \widetilde{\alpha}_j \rfloor + \lfloor \widetilde{\alpha}'_j - \widetilde{\alpha}_j \rfloor = \lfloor \widetilde{\alpha}'_{j+1} - \widetilde{\alpha}'_j \rfloor$ for all $j = 1, \ldots, m$; indeed, observe that this equality does not depend on the lifts, and holds for any choice of the form $\widetilde{\alpha}_j < \widetilde{\alpha}'_j < \widetilde{\alpha}'_{j+1} < \widetilde{\alpha}_j + \mathbf{1}$. Furthermore, since $\boldsymbol{\alpha}$ belongs to X_{G} and G is minimal, Lemma 9 of [BCdT21] states that $\boldsymbol{\alpha}$ is also monotone and non-constant on the cyclically ordered set of oriented train-track strands turning counterclockwise around f (appearing in blue in Figure 4). This implies that the integer $\sum_{j=1}^{m} \lfloor \widetilde{\alpha}'_{j+1} - \widetilde{\alpha}'_j \rfloor$ is odd; indeed, observe that its parity does not depend on the lifts, and that it is equal to 1 for any choice of the form $\widetilde{\alpha}'_1 < \widetilde{\alpha}'_2 < \cdots < \widetilde{\alpha}'_m < \widetilde{\alpha}'_1 + \mathbf{1}$. This concludes the proof. □

3.4 Kernel of the Kasteleyn operators

As before, we consider an M-curve Σ with fixed parameter $t \in (\mathbb{R}/\mathbb{Z})^g \subset \mathrm{Jac}(\Sigma)$, a minimal graph G, and an angle map $\boldsymbol{\alpha} \in X_{\mathsf{G}}$. The aim of this section is to define a meromorphic form $g_{\mathsf{x},\mathsf{y}}$ on Σ, with x, y arbitrary vertices of the quad-graph $\mathsf{G}^\diamond$, providing elements in the kernel of the operator K, see also [Foc15]. These forms play a crucial role in the definition and study of the *divisor of a vertex*, see Proposition 36, and of the inverses of the Kasteleyn operators, see Section 3.5.

To do so, and as in Section 3.3, we also fix a lift $\widetilde{t} \in \mathbb{R}^g$ of t and a lift $\widetilde{\boldsymbol{\alpha}} \colon \mathcal{T} \to \widetilde{A}_0$ of $\boldsymbol{\alpha}$, which induces a lift $\widetilde{\boldsymbol{d}}$ of the discrete Abel map. We first define a function $g_{\mathsf{x},\mathsf{y}}$ on $\widetilde{\Sigma}$, starting with x, y being adjacent vertices in $\mathsf{G}^\diamond$. One of these vertices is a vertex f of G^*, while the other one is a (white w or black b) vertex of G. Depending on these two cases, and following the notation of Figure 1, we set:

$$g_{\mathsf{f},\mathsf{w}}(\widetilde{u}) = g_{\mathsf{w},\mathsf{f}}(\widetilde{u})^{-1} = \frac{\theta(\widetilde{t} + (\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})))}{E(\widetilde{\beta}, \widetilde{u})},$$
$$g_{\mathsf{b},\mathsf{f}}(\widetilde{u}) = g_{\mathsf{f},\mathsf{b}}(\widetilde{u})^{-1} = \frac{\theta(-\widetilde{t} + (\widetilde{u} - \widetilde{\boldsymbol{d}}(\mathsf{b})))}{E(\widetilde{\alpha}, \widetilde{u})}$$

for $\widetilde{u} \in \widetilde{\Sigma}$, noting that the divisors $\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})$ and $\widetilde{u} - \widetilde{\boldsymbol{d}}(\mathsf{b})$ both have degree zero and hence can be considered naturally inside $\mathbb{C}^g$ via the Abel-Jacobi map $\mathrm{Div}^0(\widetilde{\Sigma}) \to \mathbb{C}^g$. By the second point of Lemma 17, the function $g_{\mathsf{x},\mathsf{y}}$ does not depend on the choice of the lift $\widetilde{t} \in \mathbb{R}^g$ of $t \in (\mathbb{R}/\mathbb{Z})^g$.

When x and y are not necessarily neighbors, consider a path $\mathsf{x} = \mathsf{x}_1, \ldots, \mathsf{x}_n = \mathsf{y}$ in $\mathsf{G}^\diamond$ and set:

$$g_{\mathsf{x},\mathsf{y}}(\widetilde{u}) = \prod_{j=1}^{n-1} g_{\mathsf{x}_j,\mathsf{x}_{j+1}}(\widetilde{u}).$$

The result is well defined, *i.e.*, does not depend on the choice of the path, because the product along any closed path is easily seen to be equal to 1. For example, if b and w are neighbors in G as in Figure 1, we get

$$g_{\mathsf{b},\mathsf{w}}(\widetilde{u}) = g_{\mathsf{b},\mathsf{f}}(\widetilde{u}) g_{\mathsf{f},\mathsf{w}}(\widetilde{u}) = \frac{\theta(\widetilde{t} + (\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})))\theta(-\widetilde{t} + (\widetilde{u} - \widetilde{\boldsymbol{d}}(\mathsf{b})))}{E(\widetilde{\alpha}, \widetilde{u}) E(\widetilde{\beta}, \widetilde{u})}.$$

Due to its quasi-periodicity properties, the divisor of $g_{\mathsf{x},\mathsf{y}}$ is well-defined on Σ for any vertices x, y of $\mathsf{G}^\diamond$. Furthermore, when x and y are vertices of G, the function $g_{\mathsf{x},\mathsf{y}}$ defined on $\widetilde{\Sigma}$ projects to a well-defined form on Σ, as follows.

Lemma 32. *Let b, b' (resp. w, w') be two black (resp. white) vertices. Then:*

- *the meromorphic function $g_{\mathsf{b},\mathsf{w}}$ on $\widetilde{\Sigma}$ projects to a meromorphic 1-form on Σ (*i.e.*, a section of the canonical bundle);*
- *the meromorphic functions $g_{\mathsf{b},\mathsf{b}'}$ and $g_{\mathsf{w},\mathsf{w}'}$ on $\widetilde{\Sigma}$ project to meromorphic functions on Σ.*

Moreover, these 1-forms and functions are real, in the sense that they satisfy:

$$\sigma^* g_{\mathsf{b},\mathsf{w}} = \overline{g_{\mathsf{b},\mathsf{w}}}, \quad \sigma^* g_{\mathsf{b},\mathsf{b}'} = \overline{g_{\mathsf{b},\mathsf{b}'}}, \quad \sigma^* g_{\mathsf{w},\mathsf{w}'} = \overline{g_{\mathsf{w},\mathsf{w}'}}.$$

Proof. To show the first point, fix two vertices b,w and consider a path $\mathsf{x}_0,\mathsf{x}_1,\dots,\mathsf{x}_{2n}$ in $\mathsf{G}^\diamond$ from b to w. The vertices x_1 and x_{2n-1} represent faces f and f' of G, which may coincide if b and w are neighbors. Denote by α (resp. β) the angle associated to the train-track separating b from f (resp. w from f'). For $1\leq k\leq n-1$, denote by α_k and β_k the angles of the two train-tracks associated to the two edges of this path incident with x_{2k}, so that $g_{\mathsf{b},\mathsf{w}}$ can be written as

$$g_{\mathsf{b},\mathsf{w}}(\widetilde{u})=\frac{\theta(\widetilde{t}+\widetilde{\boldsymbol{d}}(\mathsf{f}')+(\widetilde{u}-\widetilde{\beta}))\,\theta(-\widetilde{t}-\widetilde{\boldsymbol{d}}(\mathsf{f})+(\widetilde{u}-\widetilde{\alpha}))}{E(\widetilde{\alpha},\widetilde{u})E(\widetilde{\beta},\widetilde{u})}\prod_{k=1}^{n-1}\frac{E(\widetilde{\alpha}_k,\widetilde{u})}{E(\widetilde{\beta}_k,\widetilde{u})}\,. \tag{12}$$

The divisor of g on $\widetilde{\Sigma}$ is $\pi_1(\Sigma)$-periodic, and projects to a divisor on Σ. We now compute this divisor from the product form above, making use of Riemann's theorem, see Section 2.4, to understand the zeros of the first two factors.

By Riemann's theorem applied to $f_e(u)=\theta(e+\int_{x_0}^u\vec{\omega})$ with e equal to $t+\boldsymbol{d}(\mathsf{f}')+(x_0-\beta)$ and $-t-\boldsymbol{d}(\mathsf{f})+(x_0-\alpha)$ respectively, there exist elements $x_1,\dots,x_g$ and $y_1,\dots,y_g$ of Σ such that for all $1\leq j\leq g$,

$$\theta(t+\boldsymbol{d}(\mathsf{f}')+(x_j-\beta))=\theta(-t-\boldsymbol{d}(\mathsf{f})+(y_j-\beta))=0\,.$$

Therefore, the divisor of $g_{\mathsf{b},\mathsf{w}}$, representing its zeros and poles, is well defined as an algebraic sum of points in Σ (and not only in $\widetilde{\Sigma}$), and is given by

$$D=\sum_{j=1}^{g}(x_j+y_j)-\alpha-\beta+\sum_{k=1}^{n-1}(\alpha_k-\beta_k)\,.$$

Now we use Relation (3) to get the following equalitites in $\mathrm{Jac}(\Sigma)$:

$$\sum_{j=1}^{g}(x_j-x_0)=\Delta-t-\boldsymbol{d}(\mathsf{f}')+(\beta-x_0),\quad \sum_{j=1}^{g}(y_j-x_0)=\Delta+t+\boldsymbol{d}(\mathsf{f})+(\alpha-x_0)\,.$$

Moreover, the definition of $\boldsymbol{d}$ yields the equality $\boldsymbol{d}(\mathsf{f})-\boldsymbol{d}(\mathsf{f}')=\sum_{k=1}^{n-1}(\beta_k-\alpha_k)$. Thus, the divisor D of $g_{\mathsf{b},\mathsf{w}}$ satisfies the equality

$$D-2(g-1)x_0=2\Delta$$

in $\mathrm{Jac}(\Sigma)$, which by [Mum07a, Corollary 3.11, p. 166] implies that D is linearly equivalent to the canonical divisor. By standard arguments, $g_{\mathsf{b},\mathsf{w}}$ is a meromorphic 1-form, and the first point is proved.

We now briefly sketch an alternative proof of this first statement, using the viewpoint of Section 2.5.1 and the notation of the proof of Proposition 23. When u is moving along the cycle A_j, we get a factor of automorphy $p_j^2(u)$, and when u is moving along B_j, the "extra" factors coming from the theta functions and the prime forms cancel exactly, and only remains the factor of automorphy $q_j^2(u)$. Hence, the map $g_{\mathsf{b},\mathsf{w}}$ on $\widetilde{\Sigma}$ transforms exactly like meromorphic 1-forms lifted to the universal cover, and therefore projects onto Σ as a meromorphic 1-form.

The second point can be proved in the same way: use the product form for $g_{\mathsf{b},\mathsf{b}'}$ and $g_{\mathsf{w},\mathsf{w}'}$ together with Riemann's theorem to check that the corresponding divisors are principal.

Finally, the last point is a direct consequence of the fact that the Riemann matrix Ω is purely imaginary (Lemma 11), together with Point 1 of Lemma 18 and Lemma 25. $\square$

These forms provide non-zero vectors in the kernel of K, as follows.

Lemma 33 ([Foc15])**.** *Fix $\widetilde{u} \in \widetilde{\Sigma}$.*

- *For any vertex x of $\mathsf{G}^\diamond$, $(g_{\mathsf{b},\mathsf{x}}(\widetilde{u}))_{\mathsf{b}\in\mathsf{B}}$ is in the right kernel of K. Equivalently, for any white vertex $\mathsf{w} \in \mathsf{W}$, we have $\sum_{\mathsf{b}\sim\mathsf{w}} \mathsf{K}_{\mathsf{w},\mathsf{b}}\, g_{\mathsf{b},\mathsf{x}}(\widetilde{u}) = 0$.*
- *For any vertex x of $\mathsf{G}^\diamond$, $(g_{\mathsf{x},\mathsf{w}}(\widetilde{u}))_{\mathsf{w}\in\mathsf{W}}$ is in the left kernel of K. Equivalently, for any black vertex $\mathsf{b} \in \mathsf{B}$, we have $\sum_{\mathsf{w}\sim\mathsf{b}} g_{\mathsf{x},\mathsf{w}}(\widetilde{u})\mathsf{K}_{\mathsf{w},\mathsf{b}} = 0$.*

Proof. This is a consequence of Fay's trisecant identity in its telescopic form (8), as noted in the periodic context by Fock [Foc15], see also [BCdT20, Proposition 16] for details in the elliptic case. $\square$

We now come to the study of poles and zeros of $g_{\mathsf{x},\mathsf{y}}$. Recall that for any vertices x,y of $\mathsf{G}^\diamond$, these poles and zeros give a well-defined divisor of $g_{\mathsf{x},\mathsf{y}}$ on Σ. In [BCdT20, Section 3.4], this divisor was studied in the elliptic case. We now adapt and generalize this discussion to the present case.

Lemma 34. *Suppose that the graph G is minimal and that the angle map $\boldsymbol{\alpha}$ belongs to X_{G}. Then, for any vertices x,y of $\mathsf{G}^\diamond$, there exists a partition of A_0 into two intervals, such that one contains no poles of $g_{\mathsf{x},\mathsf{y}}$ and the other no zeros.*

Proof. Recall that $\widetilde{t}$ belongs to $\mathbb{R}^g$. Moreover, for $u \in A_0$, the divisors $\widetilde{u} - \widetilde{\boldsymbol{d}}(\mathsf{b})$ and $\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})$ are mapped by the Abel-Jacobi to $\mathbb{R}^g \oplus \Omega\mathbb{Z}^g$ (recall Lemma 15). As a consequence, the arguments of the θ functions appearing in the factors of $g_{\mathsf{x},\mathsf{y}}$ belong to $\mathbb{R}^g \oplus \Omega\mathbb{Z}^g$. By Point 2 of Lemma 17 and of Lemma 18, this function does not vanish on $\mathbb{R}^g \oplus \Omega\mathbb{Z}^g$. Therefore, the two “theta” factors appearing in the expression of $g_{\mathsf{b},\mathsf{w}}$ do not contribute to the zeros or poles of that 1-form on A_0: those come from the remaining factors expressed with the prime form in the numerator and denominator, respectively. The statement now follows from Section 3.4 of [BCdT20]: indeed, the proof of Lemma 19 of [BCdT20] can now be applied verbatim, as it only relies on the partial cyclic order of train-tracks of the minimal graph G and the properties of the angle map $\boldsymbol{\alpha} \in X_{\mathsf{G}}$. $\square$

As a consequence, we can extend the terminology of [BCdT20, Definition 10]: we define the *sector associated to* $g_{\mathsf{b},\mathsf{w}}$, denoted by $s_{\mathsf{b},\mathsf{w}}$, to be the part of the partition of A_0 containing the poles of $g_{\mathsf{b},\mathsf{w}}$. If it has no zeros on A_0 (which happens when b and w are neighbors), then $s_{\mathsf{b},\mathsf{w}}$ is defined to be the arc from α to β in the positive direction of A_0, with the convention of Figure 1.

The 1-forms $g_{\mathsf{b},\mathsf{w}}$ also allow us to define the *divisor of a vertex*, thus extending this notion due to Kenyon-Okounkov [KO06] from the periodic to the general case, see Section 4.3.

To do so, we fix a white vertex w and assume that the parameter $t \in (\mathbb{R}/\mathbb{Z})^g$ is *generic*, in the sense that the function $\widetilde{u} \mapsto \theta(\widetilde{t} + (\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})))$ is not identically zero on $\widetilde{\Sigma}$.

Definition 35. Let w be a white vertex of G, and let t be generic. Then, the *divisor of* w, denoted by $\operatorname{div}(\mathsf{w})$, is the element of $\operatorname{Div}(\Sigma)$ given by the divisor of the function $\widetilde{u} \mapsto \theta(\widetilde{t} + (\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})))$ on $\widetilde{\Sigma}$.

This divisor can be given a more concrete description, as follows. Let us fix a point $x_0 \in A_0 \subset \Sigma$, and denote by $\Delta \in \mathbb{C}^g$ the associated constant given by Riemann's theorem (3).

Proposition 36. *For any white vertex* w *of* G*,* $\operatorname{div}(\mathsf{w})$ *is a divisor of degree* g *whose class in* $\operatorname{Pic}(\Sigma)$ *is determined by the following equality in* $\operatorname{Pic}^0(\Sigma) = \operatorname{Jac}(\Sigma)$*:*

$$(\operatorname{div}(\mathsf{w}) - gx_0) + (\boldsymbol{d}(\mathsf{w}) + x_0) = \Delta - t\,. \tag{13}$$

Moreover, $\operatorname{div}(\mathsf{w})$ *is given by the common zeros of the* 1*-forms* $\{g_{\mathsf{b},\mathsf{w}}\}_{\mathsf{b}\in\mathsf{B}}$*, which consist of one point on each of the real components* $A_1, \dots, A_g$ *of* Σ*. Finally, the assignment* $t \mapsto \operatorname{div}(\mathsf{w})$ *defines a bijection from* $(\mathbb{R}/\mathbb{Z})^g \subset \operatorname{Jac}(\Sigma)$ *to the product* $A_1 \times \cdots \times A_g$*.*

Proof. Let us fix a white vertex w of G. The zeros of $\theta(\widetilde{t} + (\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})))$ can be computed using Riemann's theorem (3) as in the proof of Lemma 32, easily leading to Equation (13). Now, recall that t belongs to $(\mathbb{R}/\mathbb{Z})^g$ by definition, and so does $\boldsymbol{d}(\mathsf{w}) + x_0$ by the third point of Lemma 15. On the other hand, we know by Lemma 19 that the constant Δ, which only depends on x_0, belongs to $(\mathbb{R}/\mathbb{Z})^g + \Omega\frac{1}{2}\mathbf{1}$. Finally, the fourth point of Lemma 15 ensures that the Abel-Jacobi map defines a homeomorphism from the product $A_1 \times \cdots \times A_g$ onto that particular real torus, thus showing that $\operatorname{div}(\mathsf{w})$ consists of one simple zero on each of the real components $A_1, \dots, A_g$ of Σ. Moreover, Equation (13) defines a bijection between $A_1 \times \cdots \times A_g$ and the elements t of $(\mathbb{R}/\mathbb{Z})^g \subset \operatorname{Jac}(\Sigma)$.

By definition, the form $g_{\mathsf{b},\mathsf{w}}$ contains the factor $\theta(\widetilde{t} + (\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})))$ for any $\mathsf{b} \in \mathsf{B}$. Hence, we are left with the proof that the family $\{g_{\mathsf{b},\mathsf{w}}\}_{\mathsf{b}\in\mathsf{B}}$ does not contain any additional common zero or pole. Focusing on the black vertices adjacent to w, and following the notation of Figure 1, we have

$$g_{\mathsf{b},\mathsf{w}}(\widetilde{u}) = \frac{\theta(\widetilde{t} + (\widetilde{u} + \widetilde{\boldsymbol{d}}(\mathsf{w})))\theta(-\widetilde{t} + (\widetilde{u} - \widetilde{\boldsymbol{d}}(\mathsf{w}) - \widetilde{\alpha} - \widetilde{\beta}))}{E(\widetilde{\alpha}, \widetilde{u})E(\widetilde{\beta}, \widetilde{u})}\,.$$

Since G is minimal and $\boldsymbol{\alpha}$ belongs to X_{G}, the train-track angles are cyclically ordered around w, see [BCdT21, Lemma 8]. In particular, the functions $\{g_{\mathsf{b},\mathsf{w}}\}_{\mathsf{b}\sim\mathsf{w}}$ do not have common poles unless w is of degree 2. In this latter case, a similar argument shows that $\{g_{\mathsf{b},\mathsf{w}}\}_{\mathsf{b}\in\mathsf{B}_1}$ do not have common poles. Furthermore, a common zero of the second "theta factor" above would contradict Riemann's theorem. Indeed, imagine that u is a common zero to the second theta factor in $g_{\mathsf{b},\mathsf{w}}$ and $g_{\mathsf{b}',\mathsf{w}}$ for two consecutive black vertices b and b' around w, which we first assume to have degree at least 3. In other words, u is a common zero of f_e and $f_{e'}$, with $e, e' \in \mathbb{R}^g$ given by

$$e = e_0 - \int_{\widetilde{x}_0}^{\widetilde{\alpha}} \vec{\omega} - \int_{\widetilde{x}_0}^{\widetilde{\beta}} \vec{\omega}, \quad e' = e_0 - \int_{\widetilde{x}_0}^{\widetilde{\beta}} \vec{\omega} - \int_{\widetilde{x}_0}^{\widetilde{\gamma}} \vec{\omega}, \quad \text{where } e_0 = -\widetilde{t} - \widetilde{\boldsymbol{d}}(\mathsf{w}) - \widetilde{x}_0\,.$$

If e and e' are not degenerate, then f_e and $f_{e'}$ are not identically zero and, by Lemma 19, they both have one zero on A_j for each $1 \leq j \leq g$. If we denote these zeros by x_j and x'_j, respectively, then they satisfy the relation (3) in $\mathrm{Jac}(\Sigma)$, which takes the form

$$\Delta = e + \sum_{j=1}^{g} \int_{x_0}^{x_j} \vec{\omega} = e' + \sum_{j=1}^{g} \int_{x_0}^{x'_j} \vec{\omega}\,.$$

Let us assume without loss of generality that the common zero of f_e and $f_{e'}$ is $u = x_1 = x'_1$. Then, if we add to the previous equality the vector

$$-e_1 := -e_0 + \int_{x_0}^{\alpha} \vec{\omega} + \int_{x_0}^{\beta} \vec{\omega} + \int_{x_0}^{\gamma} \vec{\omega} - \int_{x_0}^{u} \vec{\omega}\,,$$

we get

$$-e_1 + \Delta = \int_{x_0}^{\gamma} \vec{\omega} + \sum_{j=2}^{g} \int_{x_0}^{x_j} \vec{\omega} = \int_{x_0}^{\alpha} \vec{\omega} + \sum_{j=2}^{g} \int_{x_0}^{x'_j} \vec{\omega}\,.$$

Since Equation (3) uniquely determines the theta divisor (if e_1 is not degenerate), this means that the divisor of f_{e_1} is described by the two g-tuples of points: $\gamma, x_2, \dots, x_g$ on one hand, and $\alpha, x'_2, \dots, x'_g$ on the other hand. But this is impossible: indeed, α is not equal to γ since w has degree at least 3, and $\alpha \in A_0$ cannot be equal to $x_j \in A_j$ since they belong to different real components of Σ. If w has degree 2, then a similar argument shows that the second "theta" factors in $\{g_{\mathsf{b},\mathsf{w}}\}_{\mathsf{b}\in\mathsf{B}_1}$ do not have a common zero. This concludes the proof. $\square$

3.5 Inverses of the Kasteleyn operators

Once again, we fix an M-curve Σ with an element $t \in (\mathbb{R}/\mathbb{Z})^g \subset \mathrm{Jac}(\Sigma)$, a minimal graph G, and an angle map $\boldsymbol{\alpha}\colon \mathcal{T} \to A_0$ in the parameter space X_{G}. The aim of this section is to define a two-parameter family of inverses for the associated operator K.

To do so, we need some preliminary definitions. Recall from Section 2.2 that Σ cut along the cycles $A_0, \dots, A_g$ consists in two compact oriented surfaces with boundary; we denote by Σ^+ the one whose oriented boundary contains A_0 endowed with the fixed orientation, see Figure 2. Define $\mathcal{D}$ as the subset of Σ given by

$$\mathcal{D} = \Sigma^+ \setminus \boldsymbol{\alpha}(\mathcal{T})\,.$$

Given b, w and u_0 in the interior of Σ^+, we define a path $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ in Σ as an oriented simple path from $\sigma(u_0)$ to u_0, intersecting A_0 once in the complement of the sector $s_{\mathsf{b},\mathsf{w}}$ (recall Section 3.4), disjoint from $A_1 \cup \cdots \cup A_g$ and such that $\sigma(\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}) = -\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$.

When u_0 lies on the boundary of Σ^+ (*i.e.*, when $\sigma(u_0) = u_0$), we define $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ as the natural limit of $\mathsf{C}^{u}_{\mathsf{b},\mathsf{w}}$ when u approaches u_0 in Σ^+, namely as a closed loop based at u_0 with the following properties:

- if u_0 belongs to A_0, then $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ is a homotopically trivial closed contour on Σ, maybe enclosing some poles of $g_{\mathsf{b},\mathsf{w}}$;
- if u_0 belongs to A_j for some $1 \leq j \leq g$, then $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ is homologous to B_j.

Note that these properties do not determine the path $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ uniquely, even up to continuous deformation in $\Sigma \setminus \boldsymbol{\alpha}(\mathcal{T})$. However, the resulting operator turns out not to depend on this choice.

We now define a family of operators $(\mathsf{A}^{u_0})_{u_0 \in \mathcal{D}}$, and prove in Theorem 40 that they are indeed inverses of the Kasteleyn operator K.

Definition 37. For every $u_0 \in \mathcal{D}$, we define the linear operator A^{u_0} mapping functions on white vertices (with finite support for definiteness) to functions on black vertices by its entries: for every pair (b, w) of black and white vertices of G, we set

$$\mathsf{A}^{u_0}_{\mathsf{b},\mathsf{w}} = \frac{1}{2i\pi} \int_{\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}} g_{\mathsf{b},\mathsf{w}} \,, \tag{14}$$

where the path of integration $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ is as given above, and the meromorphic 1-form $g_{\mathsf{b},\mathsf{w}}$ as defined in Section 3.4.

We recall from Lemma 32 that $g_{\mathsf{b},\mathsf{w}}$ is a meromorphic 1-form, with poles in $s_{\mathsf{b},\mathsf{w}}$ by construction, so the integral is invariant if we deform the path $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ in $\Sigma \setminus s_{\mathsf{b},\mathsf{w}}$. Note also that if u_0 and u_1 belong to the same connected component of $A_0 \setminus \boldsymbol{\alpha}(\mathcal{T})$, the operators A^{u_0} and A^{u_1} coincide.

Using the same argument of contour deformation as in [BCdT20, Lemma 24], we obtain the following alternative expression for the coefficents of A^{u_0}.

Lemma 38. *Let H^{u_0} be a meromorphic function on $\Sigma \setminus \mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ with a discontinuity jump of $+1$ when crossing $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ from right to left, and let $\gamma^{u_0}_{\mathsf{b},\mathsf{w}}$ be a collection of contours, homologically trivial in Σ, surrounding all the poles of $g_{\mathsf{b},\mathsf{w}} H^{u_0}$ exactly once counterclockwise. Then, we have the equality*

$$\mathsf{A}^{u_0}_{\mathsf{b},\mathsf{w}} = \frac{1}{2i\pi} \oint_{\gamma^{u_0}_{\mathsf{b},\mathsf{w}}} H^{u_0}\, g_{\mathsf{b},\mathsf{w}} \,. \tag{15}$$

Remark 39.

1. The function H^{u_0} is well defined up to addition of a meromorphic function on Σ. By a careful choice of that meromorphic function, it might be assumed that H^{u_0} has no pole on A_0. It is also possible if needed to ensure that all poles of H^{u_0} are simple.

2. If u_0 belongs to A_0, then $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ bounds a disk, and H^{u_0} can simply be chosen to be the indicator function of this disk. If u_0 belongs to A_j for $1 \leq j \leq g$, then H^{u_0} can be understood as a determination, which depends on b and w, of the multivalued function on Σ given by the projection of a meromorphic function on the infinite cyclic cover of Σ determined by the loop A_j. Finally, if $u_0 \neq \sigma(u_0)$, then H^{u_0} can be understood as a determination of the multivalued function on $\Sigma \setminus \{u_0, \sigma(u_0)\}$ given by the projection of a meromorphic function on the infinite cyclic cover determined by the loops around u_0 and $\sigma(u_0)$. In any case, even though the function H^{u_0} depends on b and w, it can be chosen so that its poles (and residues against a 1-form) do not depend on these vertices, hence their absence in the notation.

3. An explicit form for H^{u_0} is given in Remark 42 below.

We are finally ready to state and prove the main result of this section.

Theorem 40. *For any $u_0 \in \mathcal{D}$, the operator A^{u_0} is an inverse of the operator K.*

Proof. The proof follows the same lines as the one of [BCdT20, Theorem 26], which in turn is inspired from [Ken02]. We need to check that we have $(\mathsf{K}\mathsf{A}^{u_0})_{\mathsf{w},\mathsf{w}'} = \delta_{\mathsf{w},\mathsf{w}'}$ for every pair of white vertices w, w' and $(\mathsf{A}^{u_0}\mathsf{K})_{\mathsf{b},\mathsf{b}'} = \delta_{\mathsf{b},\mathsf{b}'}$ for any pair of black vertices b, b'. We write the proof of the first equality in detail; the second can be checked in a similar way.

Let us first assume that w and w' are distinct, and use Definition 37 together with Lemma 34. By [BCdT20, Lemma 23], the intersection of the complements in A_0 of the sectors $\{s_{\mathsf{b},\mathsf{w}'}\}_{\mathsf{b}\sim\mathsf{w}}$ is non-empty. Therefore, the paths $\{\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}'}\}_{\mathsf{b}\sim\mathsf{w}}$ can be chosen to coincide with a single path $\mathsf{C}^{u_0}_{\mathsf{w},\mathsf{w}'}$. This leads to the equality

$$(\mathsf{K}\mathsf{A}^{u_0})_{\mathsf{w},\mathsf{w}'} = \sum_{\mathsf{b}\sim\mathsf{w}} \mathsf{K}_{\mathsf{w},\mathsf{b}} \frac{1}{2i\pi} \int_{\mathsf{C}^{u_0}_{\mathsf{w},\mathsf{w}'}} g_{\mathsf{b},\mathsf{w}'} = \frac{1}{2i\pi} \int_{\mathsf{C}^{u_0}_{\mathsf{w},\mathsf{w}'}} \left(\sum_{\mathsf{b}\sim\mathsf{w}} \mathsf{K}_{\mathsf{w},\mathsf{b}}\, g_{\mathsf{b},\mathsf{w}'} \right),$$

which vanishes by Lemma 33.

We now turn to the case where w and w' coincide and use Lemma 38, which yields

$$(\mathsf{K}\mathsf{A}^{u_0})_{\mathsf{w},\mathsf{w}} = \frac{1}{2i\pi} \sum_{\mathsf{b}\sim\mathsf{w}} \oint_{\gamma^{u_0}_{\mathsf{b},\mathsf{w}}} H^{u_0}\, \mathsf{K}_{\mathsf{w},\mathsf{b}}\, g_{\mathsf{b},\mathsf{w}}\,.$$

Let us fix a black vertex b adjacent to w, and denote by α and β the angles of the incident train-tracks and by f, f' the adjacent faces, as in Figure 1. We compute explicitly the corresponding integral by residues. The residues contributing to the integral come in two classes: on the one hand, those coming from poles of H^{u_0}, and on the other hand, those from singularities of $\mathsf{K}_{\mathsf{w},\mathsf{b}}g_{\mathsf{b},\mathsf{w}}$.

Recall from Remark 39 that, even though H^{u_0} depends on b,w, its poles do not. The residue computation for the contribution of the poles of H^{u_0}

$$\sum_{v \text{ pole of } H^{u_0}} \mathrm{Res}_v \left(H^{u_0}\, \mathsf{K}_{\mathsf{w},\mathsf{b}}g_{\mathsf{b},\mathsf{w}}\right)$$

can be carried to the universal cover $\widetilde{\Sigma}$. In particular, if all the poles of H^{u_0} are simple, it will result in a linear combination of evaluations of $g_{\mathsf{b},\mathsf{w}}$ at lifts $\widetilde{v}$ of the poles of H^{u_0}. When summing over all black vertices adjacent to the white vertex w, this linear combination of functions g vanishes as $\mathsf{b} \mapsto g_{\mathsf{b},\mathsf{w}}(\widetilde{v})$ is in the kernel of K by Lemma 33.

Let us now turn to the remaining residues at poles of $\mathsf{K}_{\mathsf{w},\mathsf{b}}g_{\mathsf{b},\mathsf{w}}$. From Fay's identity in the form of Equation (9), we see that the meromorphic 1-form $\mathsf{K}_{\mathsf{w},\mathsf{b}}g_{\mathsf{b},\mathsf{w}}$ has the following decomposition

$$\mathsf{K}_{\mathsf{w},\mathsf{b}}\, g_{\mathsf{b},\mathsf{w}} = \omega_{\beta-\alpha} + \sum_{j=1}^{g} c^j_{\mathsf{w},\mathsf{b}}\omega_j,$$

with

$$c^j_{\mathsf{w},\mathsf{b}} = \frac{\partial \log\theta}{\partial z_j}(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f})) - \frac{\partial \log\theta}{\partial z_j}(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f}')),$$

on which we read that it has two simple poles: at β with residue $+1$, and at α with residue -1. Therefore, since we assumed that H^{u_0} has no pole on A_0, we have:

$$\mathrm{Res}_{u=\alpha}\, H^{u_0}\omega_{\beta-\alpha} = -H^{u_0}(\alpha), \quad \mathrm{Res}_{u=\beta}\, H^{u_0}\omega_{\beta-\alpha} = H^{u_0}(\beta).$$

Since the angle map $\boldsymbol{\alpha}$ belongs to the space X_{G} and G is minimal, Lemma 8 of [BCdT21] states that its restriction is monotone and injective on the (cyclically ordered) set of oriented train-track strands adjacent to the vertex w. In other words, the angles $\alpha, \beta, \dots$ of the train-track strands w wind once around A_0. By construction, the corresponding increments $H^{u_0}(\beta) - H^{u_0}(\alpha), \dots$ sum to $+1$, yielding

$$(\mathsf{K}\mathsf{A}^{u_0})_{\mathsf{w},\mathsf{w}} = \sum_{\mathsf{b}\sim\mathsf{w}} \mathsf{K}_{\mathsf{w},\mathsf{b}}\mathsf{A}^{u_0}_{\mathsf{b},\mathsf{w}} = 1\,.$$

This concludes the proof. □

From the mere existence of at least an inverse of K, we get the following simple but useful lemma.

Lemma 41. *Let f a function on black vertices of G with finite support, such that $\mathsf{K}f$ is identically zero on white vertices. Then f is identically zero.*

Proof. Let A^{u_0} be one of the inverses of K given above. Since all the occurring sums have a finite number of nonzero terms, we can write $f = (\mathsf{A}^{u_0}\mathsf{K})\, f = \mathsf{A}^{u_0}\,(\mathsf{K}f) = \mathsf{A}^{u_0}0 = 0$. □

Remark 42. A first step towards the construction of H^{u_0} is to define a function with a jump along a path joining u_0 to $\sigma(u_0)$. Instead of working directly on the surface Σ, we pass to its universal cover $\widetilde{\Sigma}$. Let $\widetilde{u}_0 \in \widetilde{\Sigma}$ be an arbitrary lift of $u_0 \in \Sigma$, and let $\sigma(\widetilde{u}_0) \in \widetilde{\Sigma}$ denote a lift of $\sigma(u_0) \in \Sigma$ such that u_0 and $\sigma(\widetilde{u}_0)$ belong to one fundamental domain.

The expected discontinuity can be obtained by taking the logarithm of an expression having a zero at $\widetilde{u}_0$ and a pole at $\sigma(\widetilde{u}_0)$. Hence, a natural first candidate is given by

$$H^{u_0}_{\mathrm{pre}}(x) = \frac{1}{2i\pi} \log \frac{E(\widetilde{u}_0, x)}{E(\sigma(\widetilde{u}_0), x)}$$

for every $x \in \widetilde{\Sigma}$ whose orbit under the action of the fundamental group does not meet a path C connecting $\sigma(\widetilde{u}_0)$ to $\widetilde{u}_0$. (We can take C to be a lift to $\widetilde{\Sigma}$ of $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ for a given pair (b, w) but any continuous deformation will do.) This ensures that we work with a consistent determination of the logarithm. This function has the desired behaviour of jumping by $+1$ when crossing the path C. Moreover, $H^{u_0}_{\mathrm{pre}}$ is quasi-periodic: if $x' \in \widetilde{\Sigma}$ (resp. x'') is obtained from x by the action of a loop in $\pi_1(\Sigma)$ corresponding to A_j (resp. B_j), then we have by Equations (4) and (5)

$$H^{u_0}_{\mathrm{pre}}(x') = H^{u_0}_{\mathrm{pre}}(x), \quad H^{u_0}_{\mathrm{pre}}(x'') = H^{u_0}_{\mathrm{pre}}(x) + \int_{\mathsf{C}} \omega_j.$$

We can compensate this defect of periodicity along the B-cycles by noting that for $y \in \widetilde{\Sigma}$, we have

$$\frac{\partial \log \theta}{\partial z_k}(x' - y) = \frac{\partial \log \theta}{\partial z_k}(x - y), \quad \frac{\partial \log \theta}{\partial z_k}(x'' - y) = \frac{\partial \log \theta}{\partial z_k}(x - y) - 2i\pi \delta_{j,k}\,.$$

Hence, we fix $y \in \widetilde{\Sigma}$ and set

$$H^{u_0}(x) = \frac{1}{2i\pi} \log \frac{E(\widetilde{u}_0, x)}{E(\sigma(\widetilde{u}_0), x)} + \frac{1}{2i\pi} \sum_{j=1}^{g} \frac{\partial \log \theta}{\partial z_j}(x - y) \times \int_{\mathsf{C}} \omega_j\,, \tag{16}$$

which projects to a well-defined function on Σ deprived from the projection of C, and satisfies the desired conditions.

In the genus 1 case (recall Examples 12 and 22), the choice $y = \frac{1}{2} \in \mathbb{C} = \widetilde{\mathbb{T}}$ gives the function H^{u_0} of [BCdT20, Section 4.3].

4 The periodic case

This section deals with the special case where the bipartite planar graph G is $\mathbb{Z}^2$-periodic. We start in Section 4.1 by recalling the properties of train-tracks in this case; we also introduce the space $X^{per}_{\mathsf{G}} \subset X_{\mathsf{G}}$ of periodic angle maps, and the Newton polygon $N(\mathsf{G})$. In Section 4.2, we show that $\boldsymbol{\alpha} \in X^{per}_{\mathsf{G}}$ induces a periodic operator K if and only if its image by some natural map $\varphi\colon X_{\mathsf{G}}^{per} \to N(\mathsf{G})$ lies in $\mathbb{Z}^2$. In Section 4.3, we assume that K is periodic and use the functions $g_{\mathsf{x},\mathsf{y}}$ of Section 3.4 to give an explicit parametrization of the spectral curve for the corresponding periodic dimer model; we also identify the divisor of a vertex, and show that dimer models with Fock's weights can realize any such "spectral data". We then describe the set of ergodic Gibbs measures of this model in Section 4.4, and use the map φ to express the corresponding slopes in Section 4.5. Finally, in Section 4.6, we give an explicit local formula for the free energy and the surface tension of this model.

4.1 Preliminaries

The aim of this preliminary section is to quickly recall the specificities of oriented train-tracks and angle maps in the periodic case, referring to [BCdT20, Section 4.1] for details.

In the whole of this section, we assume that the bipartite planar graph G is $\mathbb{Z}^2$*-periodic*, *i.e.*, that the group $\mathbb{Z}^2$ acts freely by translation on colored vertices, edges and faces. We fix a basis of $\mathbb{Z}^2$, allowing to identify a *horizontal* direction (along the vector $(1,0)$) and a *vertical* one (along the vector $(0,1)$). The graph G has a natural toroidal exhaustion $(\mathsf{G}_n)_{n\geq 1}$, where $\mathsf{G}_n := \mathsf{G}/n\mathbb{Z}^2$. We use similar notation for the toroidal exhaustions of the dual graph G^*, of the quad-graph $\mathsf{G}^\diamond$, and of the train-tracks $\mathcal{T}$.

Fix a face f of G and draw two simple dual paths in the plane, denoted by γ_x and γ_y, joining f to $\mathsf{f}+(1,0)$ and $\mathsf{f}+(0,1)$ respectively, and intersecting only at f. They project onto the torus to two simple closed loops in G_1^*, also denoted by γ_x and γ_y, winding around the torus and intersecting only at f. Their homology classes $[\gamma_x]$ and $[\gamma_y]$ form a basis of the first homology group of the torus $H_1(\mathbb{T},\mathbb{Z})$, and allow for its identification with $\mathbb{Z}^2$. Every oriented train-track $T\in\mathcal{T}$ projects to an oriented closed curve on the torus, so the corresponding homology class $[T]\in H_1(\mathbb{T},\mathbb{Z})$ can be written as $[T] = h_T[\gamma_x]+v_T[\gamma_y]$ with h_T and v_T coprime integers. This allows to define a partial cyclic order on $\mathcal{T}$ by using the natural cyclic order of coprime elements of $\mathbb{Z}^2$ around the origin, an order which coincides with the partial cyclic order on $\mathcal{T}$ defined in Section 3.2. By construction, this cyclic order induces a cyclic order on $\mathcal{T}_1 = \mathcal{T}/\mathbb{Z}^2$. Note also that two oriented train-tracks $T,T'\in\mathcal{T}$ are parallel (resp. anti-parallel) as defined in Section 3.2 if and only if $[T]=[T']$ (resp. $[T]=-[T']$) in $H_1(\mathbb{T},\mathbb{Z})$.

Recall that X_{G} denotes the set of monotone maps $\boldsymbol{\alpha}\colon\mathcal{T}\to A_0$ assigning different images to non-parallel train-tracks. Following [BCdT20], we denote by X_{G}^{per} the set of $\mathbb{Z}^2$-periodic elements of X_{G}:

$$X_{\mathsf{G}}^{per} = \{\boldsymbol{\alpha}\in X_{\mathsf{G}} \,|\, \alpha_{T+(m,n)} = \alpha_T \text{ for all } T\in\mathcal{T} \text{ and } (m,n)\in\mathbb{Z}^2\}\,.$$

Since disjoint curves on the torus have either identical or opposite homology classes, this space can be described as

$$X_{\mathsf{G}}^{per} = \{\boldsymbol{\alpha}\colon \mathcal{T}_1\to A_0 \,|\, \boldsymbol{\alpha} \text{ is monotone and } \alpha_T\neq\alpha_{T'} \text{ for } [T]\neq[T']\}\,.$$

By construction, the sum of all oriented closed curves $T \in \mathcal{T}_1$ bounds a 2-chain in the torus, so its homology class vanishes and we have $\sum_{T\in\mathcal{T}_1}[T] = 0$. As a consequence, the collection of vectors $([T])_{T\in\mathcal{T}_1}$ in $\mathbb{Z}^2$, ordered cyclically, and drawn so that the initial point of a vector $[T]$ is the end point of the previous vector, gives a convex polygon well-defined up to translations. This polygon is referred to as the *geometric Newton polygon* of G [GK13] and denoted by $N(\mathsf{G})$. The space X_{G}^{per} can now be described combinatorially as the set of order-preserving maps from oriented boundary edges of $N(\mathsf{G})$ to A_0 mapping distinct vectors to distinct images.

4.2 Periodicity of the Kasteleyn operator

From now on, we assume that the graph G is minimal and $\mathbb{Z}^2$-periodic. We further suppose that G is *non-degenerate*, in the sense that its geometric Newton polygon $N(\mathsf{G})$ has positive area. The aim of this section is to understand for which maps $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$ the corresponding Kasteleyn operator K defined in Equation (11) is periodic. This criterion in expressed in terms of a natural map $\varphi\colon X_{\mathsf{G}}^{per} \to N(\mathsf{G})$ that also proves useful in Section 4.5.

Note that the periodicity of G and of $\boldsymbol{\alpha}$ is not sufficient to ensure the periodicity of the operator K. Indeed, this operator makes use of the $\mathrm{Pic}(\Sigma)$-valued discrete Abel map $\boldsymbol{d}$ defined in Section 3.2, which might have horizontal and vertical *periods*. More precisely, we have that for every vertex x of $\mathsf{G}^{\diamond}$ and $(m,n) \in \mathbb{Z}^2$, the equality

$$\boldsymbol{d}(\mathsf{x} + (m,n)) = \boldsymbol{d}(\mathsf{x}) + \sum_{T\in\mathcal{T}_1} (mv_T - nh_T)\alpha_T \tag{17}$$

holds in $\mathrm{Pic}(\Sigma)$, where recall that $[T] = (h_T, v_T) \in \mathbb{Z}^2$ denotes the homology class of T.

Consider the map

$$\varphi\colon X_{\mathsf{G}}^{per} \longrightarrow \mathbb{C}^g$$

defined as follows. Let us enumerate by $T_1, \dots, T_r$ the elements of $\mathcal{T}_1$ respecting the cyclic order, and let $P_1, \dots, P_r \in \mathbb{C}$ denote the integer points on the boundary of $N(\mathsf{G})$ numbered so that $P_{j+1} - P_j = [T_j]$ (where P_{r+1} stands for P_1). Given a map $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$, set

$$\varphi(\boldsymbol{\alpha}) = \sum_{j=1}^{r} P_j \int_{\alpha_{j-1}}^{\alpha_j} \vec{\omega} \in \mathbb{C}^g\,, \tag{18}$$

where α_j stands for α_{T_j}, and the integration path follows the orientation of A_0.

Proposition 43. *For any $1 \le i \le g$, the image of the coordinate $\varphi_i\colon X_{\mathsf{G}}^{per} \to \mathbb{C}$ of φ is equal to the interior of $N(\mathsf{G})$. Moreover, a periodic map $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$ induces a periodic Kasteleyn operator K if and only if $\varphi(\boldsymbol{\alpha})$ lies in $(\mathbb{Z}^2)^g$. In such a case, the g integer points in the interior of $N(\mathsf{G})$ given by $\varphi(\boldsymbol{\alpha})$ are distinct.*

Proof. Knowing Lemma 14, the beginning of the proof follows quite closely the analogous result in the genus 1 case, see [BCdT20, Proposition 39]. Proving that, when $\varphi(\boldsymbol{\alpha})$ lies in

$(\mathbb{Z}^2)^g$, its g coordinates correspond to distinct integer points of the interior of the Newton polygon $N(\mathsf{G})$ is new (this is not relevant when $g = 1$) and non-trivial. In particular this shows that, when $\varphi(\boldsymbol{\alpha})$ lies in $(\mathbb{Z}^2)^g$, the interior of $N(\mathsf{G})$ contains at least g integer points.

Let us fix $\boldsymbol{\alpha} \in X^{per}_{\mathsf{G}}$ and consider its image by φ_i for an arbitrary $1 \leq i \leq g$. First observe that since $\boldsymbol{\alpha}$ belongs to X_{G}, we have

$$\sum_{j=1}^{r} \int_{\alpha_{j-1}}^{\alpha_j} \omega_i = \int_{A_0} \omega_i = \sum_{k=1}^{g} \int_{A_k} \omega_i = \sum_{k=1}^{g} \delta_{k,i} = 1\,.$$

By Lemma 14, we also have that $\int_{\alpha_{j-1}}^{\alpha_j} \omega_i \geq 0$. Therefore, $\varphi_i(\boldsymbol{\alpha})$ is a convex combination of the vertices $P_1, \ldots, P_r$, and hence an element of the convex hull $N(\mathsf{G})$ of these vertices, so we have the inclusion of the image of φ_i into $N(\mathsf{G})$.

Now, let us write $\overline{X}^{per}_{\mathsf{G}}$ for the set of non-constant monotone maps $\boldsymbol{\alpha} \colon \mathcal{T}_1 \to A_0$ ($\overline{X}^{per}_{\mathsf{G}}$ is the set X^{per}_{G} without the condition that train-tracks with different homology classes need to have distinct images), and denote by $\Delta = \{\boldsymbol{\beta} = (\beta_j)_j \in [0,1]^r \mid \sum_{j=1}^{r} \beta_j = 1\}$ the standard simplex of dimension $r-1$. Observe that φ_i can be described as the restriction to X^{per}_{G} of the composition

$$\overline{X}^{per}_{\mathsf{G}} \xrightarrow{\delta_i} \Delta \xrightarrow{p} N(\mathsf{G})\,,$$

with $\delta_i(\boldsymbol{\alpha}) = (\int_{\alpha_{j-1}}^{\alpha_j} \omega_i)_j$ and $p(\boldsymbol{\beta}) = \sum_j \beta_j P_j$. Since p is an affine surjective map, any point in the interior of $N(\mathsf{G})$ is the image under p of an element of the interior of Δ, *i.e.*, an element $\boldsymbol{\beta} \in \Delta$ with no vanishing coordinate. Therefore, we have

$$\delta_i^{-1}(p^{-1}(\operatorname{int} N(\mathsf{G}))) \subset \delta_i^{-1}(\operatorname{int} \Delta) \subset \{\boldsymbol{\alpha} \in \overline{X}^{per}_{\mathsf{G}} \mid \boldsymbol{\alpha} \text{ injective}\} \subset X^{per}_{\mathsf{G}}\,,$$

thus checking the inclusion of the interior of $N(\mathsf{G})$ into $\varphi_i(X^{per}_{\mathsf{G}})$.

To prove the opposite inclusion, consider an arbitrary element x of $N(\mathsf{G}) \setminus \operatorname{int} N(\mathsf{G})$, and let us write F for the biggest face of $N(\mathsf{G})$ containing x in its interior: concretely, $F = x$ if x is a vertex of $N(\mathsf{G})$, and F is the boundary edge of $N(\mathsf{G})$ containing x otherwise. By definition, we have $p^{-1}(x) = \{\boldsymbol{\beta} \in \Delta \mid \sum_j \beta_j P_j = x\}$. Fix a reference frame for $\mathbb{R}^2$ with origin at x and first coordinate axis orthogonal to F. Then, the first coordinate of the equation $\sum_j \beta_j P_j = x$ leads to $\beta_j = 0$ for all j such that P_j does not belong to F. Since $N(\mathsf{G})$ has positive area, we have $\beta_j = 0$ for some vertex P_j of $N(\mathsf{G})$. By Lemma 14, such an element of Δ can only be realized as $\delta_i(\boldsymbol{\alpha})$ with $\alpha_j = \alpha_{j-1}$. Since P_j is a vertex of $N(\mathsf{G})$, we have $[T_j] \neq [T_{j-1}]$, so $\boldsymbol{\alpha}$ does not belong to X^{per}_{G}. This shows the inclusion of $\varphi_i(X^{per}_{\mathsf{G}})$ into the interior of $N(\mathsf{G})$, and thus the equality of these two sets.

Since $\boldsymbol{\alpha}$ is assumed to be periodic, the operator K itself is periodic if and only if the $\operatorname{Pic}^0(\Sigma)$-valued discrete Abel map $\boldsymbol{d}$ on faces is periodic by Lemma 15. By Equation (17), this holds if and only if

$$\sum_{T \in \mathcal{T}_1} [T] \alpha_T = \sum_{T \in \mathcal{T}_1} \left(\begin{smallmatrix} h_T \\ v_T \end{smallmatrix}\right) \alpha_T = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \in (\mathbb{R}/\mathbb{Z})^{2g}\,.$$

This is equivalent to requiring that the following element of $\mathbb{R}^{2g}$ belongs to $\mathbb{Z}^{2g}$:

$$\sum_{j=1}^{r}[T_j]\int_{x_0}^{\alpha_j}\vec{\omega}=\sum_{j=1}^{r}(P_{j+1}-P_j)\int_{x_0}^{\alpha_j}\vec{\omega}=\sum_{j=1}^{r}n_jP_j-\varphi(\boldsymbol{\alpha})\,, \tag{19}$$

with $n_j = \int_{\alpha_{j-1}}^{\alpha_j}\vec{\omega} - (\int_{x_0}^{\alpha_j}\vec{\omega} - \int_{x_0}^{\alpha_{j-1}}\vec{\omega})$, and x_0 the reference point of A_0 chosen in Section 2.3. Since n_j belongs to $\mathbb{Z}^g$ and P_j to $\mathbb{Z}^2$ for all j, this is equivalent to requiring that $\varphi(\boldsymbol{\alpha})$ belongs to $\mathbb{Z}^{2g}$. This concludes the proof of the second statement.

To show the last statement, let us fix $k \neq l$ and consider the holomorphic 1-form $\omega = \omega_l - \omega_k$. Since for all $i \neq k, l$, the integral along A_i of ω_k and ω_l is zero, then so is the integral of ω. By the same argument as in the proof of Lemma 14, the form ω has at least 2 zeros on each such A_i, that is at least $2(g-2)$ zeros (counted with multiplicity).

We now turn to the behavior of ω on A_0. As this form is real and has vanishing integral along A_0, it can be written in a tubular neighborhood of A_0 as $\omega = df$ with f a non-constant real-valued function. Therefore, ω has at least two (distinct) zeros on A_0. On the other hand, as the divisor of ω has degree $2g-2$, it cannot have more. Let us call these two zeros β and γ, corresponding respectively to the minimum and maximum of f along A_0. Let $1 \leq j_0 \leq j_1 \leq r$ be the indices such that

$$\alpha_{j_0-1} < \beta \leq \alpha_{j_0}, \quad \alpha_{j_1-1} < \gamma \leq \alpha_{j_1}.$$

We suppose for the moment that we are in the generic situation where β and γ are distinct from the α_j's.

By means of contradiction, let us now assume that the k^{th} and l^{th} coordinates of $\varphi(\boldsymbol{\alpha})$ coincide, *i.e.*, that we have

$$\sum_{j=1}^{r}P_j\int_{\alpha_{j-1}}^{\alpha_j}\omega_k=\sum_{j=1}^{r}P_j\int_{\alpha_{j-1}}^{\alpha_j}\omega_l\,. \tag{20}$$

This can be written as

$$\sum_{j=0}^{r}{}' P_j\int_{\alpha_{j-1}}^{\alpha_j}\omega=0\,, \tag{21}$$

where the prime in the sum means that we drop indices j for which $\alpha_{j-1}=\alpha_j$. Since the Newton polygon has positive area, it has at least three corners (or extremal points), and since $\boldsymbol{\alpha}$ belongs to X_{G}^{per}, the angles associated to the two train-tracks with homology given by the two edges attached to such a corner are different. Therefore, the sum with a prime has at least a number of terms equal to the number of corners, which is at least three.

When $\alpha_{j-1} \neq \alpha_j$, then

$$\int_{\alpha_{j-1}}^{\alpha_j}\omega \text{ is } \begin{cases}\text{strictly positive} & \text{if } j_0 < j < j_1 \text{ in cyclic order,}\\ \text{strictly negative} & \text{if } j_1 < j < j_0.\end{cases} \tag{22}$$

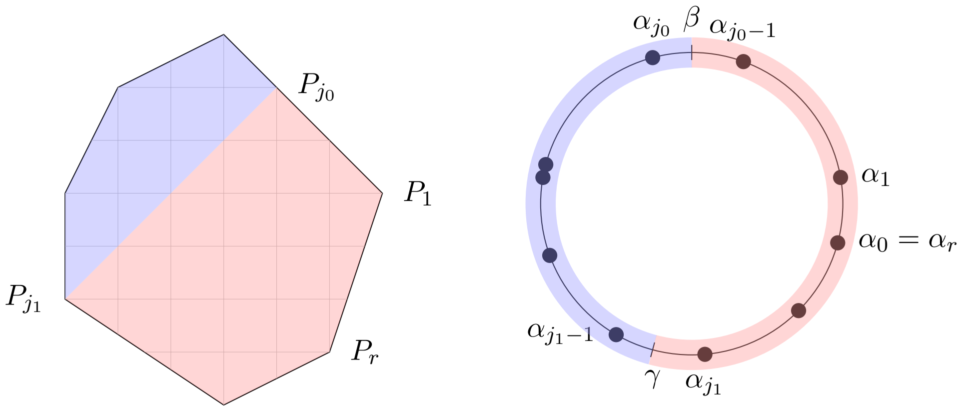

Figure 5: Left: the Newton polygon $N(\mathsf{G})$ and its subdivision induced by the convex hulls of $\{P_{j_0}, P_{j_0+1}, \ldots, P_{j_1}\}$ (in blue) and of $\{P_{j_1}, P_{j_1+1}, \ldots, P_{j_0}\}$ (in red). Right: a schematic representation of A_0 and the angles associated to train-tracks with homology classes given by the boundary of $N(\mathsf{G})$. Along the blue (resp. red) arc from β to γ (resp. from γ to β), ω is positive (resp. negative).

We also split the integral from α_{j_0-1} to α_{j_0} (resp. from α_{j_1-1} to α_{j_1}) into a negative part from α_{j_0-1} to β (resp. from γ to α_{j_1}), and a positive part from β to α_{j_0} (resp. from α_{j_1-1} to γ).

Passing terms with negative coefficients on the right-hand side of the equal sign in (21), one gets:

$$\begin{aligned} P_{j_0} \int_{\beta}^{\alpha_{j_0}} \omega + {\sum_{j_0<j<j_1}}' P_j \int_{\alpha_{j-1}}^{\alpha_j} \omega + P_{j_1} \int_{\alpha_{j_1-1}}^{\gamma} \omega \\ = P_{j_1} \left(- \int_{\gamma}^{\alpha_{j_1}} \omega \right) + {\sum_{j_1<j<j_0}}' P_j \left(\int_{\alpha_{j-1}}^{\alpha_j} \omega \right) + P_{j_0} \left(\int_{\alpha_{j_0-1}}^{\beta} \omega \right). \end{aligned} \tag{23}$$

Note that at least one of the two sums is not empty, because of the number of corners being at least 3. The sums of the scalar coefficients on both sides are non-zero and equal, as the total integral of ω along A_0 is zero. If we divide both sides by this sum, this equation can be interpreted as defining a point Q of the Newton polygon, written once as a convex combination of $P_{j_0}, P_{j_0+1}, \ldots, P_{j_1}$, and once as a convex combination of $P_{j_1}, P_{j_1+1}, \ldots, P_{j_0}$. As these two collections of vertices of $N(\mathsf{G})$ are vertices of two subpolygons forming a subdivision of $N(\mathsf{G})$, it means that this point Q is along the segment $[P_{j_0}, P_{j_1}]$ shared by the two subpolygons, see Figure 5.

This segment being part of the boundary of both subpolygons, all coefficients in front of corners different from P_{j_0} and P_{j_1} are equal to 0. Since there is at least one such corner, there exists a $j \neq j_0, j_1$ such that $\int_{\alpha_{j-1}}^{\alpha_j} \omega = 0$, contradicting (22). Therefore Equation (20) cannot hold, and the conclusion of the lemma follows.

The non-generic case where either β or γ is equal to one of the α_j's can be treated similarly: the point P_{j_0} or P_{j_1} appears only on one side of Equation (23). (The discussion changes a little bit depending on whether these points are corners or not.) $\square$

4.3 The spectral data

The aim of this section is to understand the spectral data associated to periodic dimer models with Fock's weights. After recalling the necessary prerequisites, we give an explicit parametrization of the spectral curve, following and extending the discussion of [BCdT20, Section 5.4], see Proposition 45. Then, we identify the divisor of a vertex as defined by Kenyon-Okounkov [KO06] with its counterpart from Definition 35, see Proposition 47. Finally, in Theorem 49, we show that any Harnack curve endowed with a standard divisor can be explicitly realized as the spectral data of a dimer model of this class.

We start by recalling what is meant by the *spectral data* of a dimer model [KO06]. For this part of the discussion, let us suppose that G is any planar, periodic, bipartite weighted graph (not necessarily minimal) and that K is the corresponding Kasteleyn operator. Following [KOS06], we define the finite matrix $\mathsf{K}(z,w)$ for any $(z,w)\in(\mathbb{C}^*)^2$ as the action in a natural basis of K on (z,w)-quasiperiodic functions on G, *i.e.*, functions f satisfying

$$f(\mathsf{x}+(m,n)) = z^m w^n f(\mathsf{x})$$

for any (black or white) vertex x and any $(m,n)\in\mathbb{Z}^2$. The *characteristic polynomial* $P(z,w)$ is the determinant of $\mathsf{K}(z,w)$. The *Newton polygon* of P, denoted by $N(P)$, is the convex hull of lattice points $(i,j)\in\mathbb{Z}^2$ such that z^iw^j appears as a monomial in P. It actually coincides (up to translations) with its geometric counterpart $N(\mathsf{G})$ defined in Section 4.1, see [GK13, Theorem 3.12].

The *spectral curve* $\mathcal{C}$ is the zero locus of the characteristic polynomial:

$$\mathcal{C} = \{(z,w)\in(\mathbb{C}^*)^2 \ : \ P(z,w)=0\}.$$

Following the convention of [BCdT20], we define the *amoeba* $\mathcal{A}$ of the curve $\mathcal{C}$ as the image of $\mathcal{C}$ through the map $(z,w)\mapsto(-\log|w|,\log|z|)$. By [KO06, KOS06], $\mathcal{C}$ is a *Harnack curve* [Mik00] (also known as a *simple Harnack curve*, see e.g. [Bru15]), which is equivalent to saying that the amoeba map $\mathcal{C}\to\mathcal{A}$ is at most 2-to-1 [MR01]. The *real locus* of the curve $\mathcal{C}$ is the set of points that are invariant under complex conjugation deprived from its isolated nodes $\mathcal{C}_{\mathrm{sing}}$, which are the only singularities a Harnack curve admits [Mik00, Bru15].

Being Harnack is a very strong condition on $\mathcal{C}$, and on the associated amoeba $\mathcal{A}$. For example, if $\mathcal{C}$ is a genus g Harnack curve with Newton polygon $N(P)$, then the boundary of $\mathcal{A}$ (which is the image of the real locus of $\mathcal{C}$ under the amoeba map) consists of g ovals together with one unbounded component producing a tentacle for each boundary edge of $N(P)$. Moreover, each interior point of $N(P)$ with integer coordinates corresponds to an oval or to an isolated node, and each tentacle's asymptotic direction coincides with the vector in $\mathbb{Z}^2$ given by the corresponding boundary edge of $N(P)$.

Remark 44. As stated above, for a Harnack curve $\mathcal{C}$ with given Newton polygon $N(P)$, each interior point of $N(P)$ with integer coordinates corresponds either to an oval of $\mathcal{C}$ or to an element of $\mathcal{C}_{\text{sing}}$. In our setting, the situation is very explicit: as explained in Proposition 43, any $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$ inducing a periodic Kasteleyn operator defines g distinct integer points in the interior of $N(\mathsf{G})$ via $\varphi(\boldsymbol{\alpha}) \in (\mathbb{Z}^2)^g$. These are precisely the points that give rise to the ovals, see Corollary 57 and the proof of Proposition 45 below.

The spectral curve is only the first part of the *spectral data* introduced by Kenyon and Okounkov in [KO06], see also [GK13]: the second consists of a divisor on $\mathcal{C}$, whose definition we now recall. To do so, let us fix an element w in the set W_1 of white vertices of $\mathsf{G}_1 = \mathsf{G}/\mathbb{Z}^2$. The kernel and cokernel of

$$\mathsf{K}(z,w)\colon \mathbb{C}^{\mathsf{B}_1} \longrightarrow \mathbb{C}^{\mathsf{W}_1}$$

are 1-dimensional for every smooth $(z,w) \in \mathcal{C}$, see e.g. the proof of [CT79, Theorem 2.2], thus defining line bundles over $\mathcal{C} \setminus \mathcal{C}_{\text{sing}}$. The class in $\operatorname{Coker} \mathsf{K}(z,w)$ of the indicator function $\mathbf{1}_{\mathsf{w}} \in \mathbb{C}^{\mathsf{W}_1}$ defines a section of the cokernel line bundle, whose divisor $(\mathsf{w}) \in \operatorname{Div}(\mathcal{C})$ is called the *divisor of the vertex* w. In concrete terms, the equations of (w) are the (w,b)-cofactors of $\mathsf{K}(z,w)$ for $\mathsf{b} \in \mathsf{B}_1$. In the case of periodic graphs, this yields a second definition of the divisor of a vertex, see Definition 35. We prove that the two coincide in Proposition 47. Remarkably, this divisor is a so-called *standard divisor*, *i.e.*, it consists of the sum of one point on each of the g ovals of $\mathcal{C}$, see [KO06, Theorem 1].

We now come back to our setting: the graph G is minimal and periodic, Σ is an M-curve, t is a real element of $\operatorname{Jac}(\Sigma)$, and the angle map $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$ is such that the corresponding Kasteleyn operator K is periodic. Recall that by Proposition 43, this implies in particular that the Newton polygon $N(\mathsf{G})$ has at least g interior integer points.

We progress towards an explicit parametrization of the associated spectral curve $\mathcal{C}$. Let us fix an arbitrary vertex x_0 of G. For any $\widetilde{u} \in \widetilde{\Sigma}$, the function $\mathsf{x} \mapsto g_{\mathsf{x},\mathsf{x}_0}(\widetilde{u})$ is $(z(\widetilde{u}), w(\widetilde{u}))$-quasiperiodic with

$$z(\widetilde{u}) = g_{\mathsf{x}_0+(1,0),\mathsf{x}_0}(\widetilde{u}), \quad w(\widetilde{u}) = g_{\mathsf{x}_0+(0,1),\mathsf{x}_0}(\widetilde{u})\,.$$

These quantities are easily seen not to depend on x_0. By Lemma 33, and as already observed in [Foc15], the pair $(z(\widetilde{u}), w(\widetilde{u}))$ belongs to the spectral curve $\mathcal{C}$ for all $\widetilde{u} \in \widetilde{\Sigma}$ not corresponding to a $u \in \boldsymbol{\alpha}(\mathcal{T})$. Using the definition of $g_{\mathsf{x},\mathsf{y}}$ together with Equation (17) and the second point of Lemma 17, we get the following explicit expressions in terms of train-track angles and homology classes:

$$z(\widetilde{u}) = \prod_{T \in \mathcal{T}_1} E(\widetilde{\alpha}_T, \widetilde{u})^{-v_T}, \quad w(\widetilde{u}) = \prod_{T \in \mathcal{T}_1} E(\widetilde{\alpha}_T, \widetilde{u})^{h_T}\,. \tag{24}$$

By Lemma 32, the maps z and w project to meromorphic functions on Σ, thus defining a holomorphic map $\psi\colon \Sigma \setminus \boldsymbol{\alpha}(\mathcal{T}) \to \mathcal{C}$. This map is not injective in general, as it may send two conjugated elements of Σ to an isolated node in $\mathcal{C}_{\mathrm{sing}}$. However, we have the following result.

Proposition 45. *The map $\psi\colon \Sigma \to \mathcal{C}$ given by $\psi(u) = (z(u), w(u))$ is an explicit birational parametrization of the spectral curve $\mathcal{C}$, mapping $A_1, \ldots, A_g$ to the ovals of $\mathcal{C}$ and A_0 to the unbounded real component of $\mathcal{C}$, implying in particular that $\mathcal{C}$ has geometric genus g. More precisely, its restriction*

$$\psi\colon \Sigma \setminus \{\boldsymbol{\alpha}(\mathcal{T}) \cup \psi^{-1}(\mathcal{C}_{\mathrm{sing}})\} \to \mathcal{C} \setminus \mathcal{C}_{\mathrm{sing}}$$

is a biholomorphic parametrization of the spectral curve deprived from its singularities.

Proof. The map ψ being meromorphic, it parametrizes an open set of an irreducible component of the spectral curve $\mathcal{C}$. Since this curve is Harnack [KO06], it is irreducible. Therefore, the map ψ is a parametrization of the whole spectral curve.

By Equation (24) and Lemma 25, we have $\psi(\sigma(u)) = \overline{\psi(u)}$ for all $u \in \Sigma$. This implies that ψ maps the real locus $A_0 \cup A_1 \cup \cdots \cup A_g$ of Σ to the real locus of $\mathcal{C}$. Since $z(u)$ and $w(u)$ have zeros and poles on A_0, this real component of Σ is mapped to the unbounded real component of $\mathcal{C}$, so the remaining real components $A_1, \ldots, A_g$ are mapped to the ovals of $\mathcal{C}$ or to its isolated real nodes (recall Remark 44). This latter case is excluded, as it would imply that the holomorphic map z is constant along some A_j, and hence constant. By Corollary 57, each of these ovals A_j gives rise to a different slope, and is therefore mapped to a distinct oval of $\mathcal{C}$. Finally, note that since $\boldsymbol{\alpha}$ belongs to X_{G}^{per}, the cyclic ordering of $\boldsymbol{\alpha}(\mathcal{T}) \subset A_0$ coincides with the cyclic ordering of the tentacles of $\mathcal{C}$. We are now in the setting of [Bru15, Theorem 10], and can therefore conclude: if ψ was not birational, then the curve $\psi(\Sigma) = \mathcal{C}$ would not be reduced, which is impossible as the isolated nodes are its only possible singularities. $\square$

Before turning to the divisor, we need a preliminary lemma whose proof is postponed to Section 4.4.

Lemma 46. *Let $Q(z, w)$ be the adjugate matrix of $\mathsf{K}(z, w)$. For every b and w on G_1 and every $u \in \Sigma \setminus \boldsymbol{\alpha}(\mathcal{T})$, we have*

$$Q(z(u), w(u))_{\mathsf{b},\mathsf{w}} \lambda(u) = g_{\mathsf{b},\mathsf{w}}(u)\,,$$

where λ is the meromorphic 1-form on Σ given by $\lambda = \frac{dz}{zw\partial_w P(z,w)} = -\frac{dw}{zw\partial_z P(z,w)}$.

We now turn to the identification of the divisor of a vertex of Definition 35 with the homonymous notion of Kenyon-Okounkov.

Proposition 47. *For any white vertex w, the divisors $\psi^{-1}((\mathsf{w}))$ and $\operatorname{div}(\mathsf{w})$ coincide.*

Proof. Let w be an arbitrary white vertex, let (w) be the associated divisor on $\mathcal{C}$, and let $\psi^{-1}((\mathsf{w}))$ be the corresponding divisor on Σ. By definition, it is given by the common zeros in $\Sigma \setminus \boldsymbol{\alpha}(\mathcal{T})$ of $Q(z(u), w(u))_{\mathsf{b},\mathsf{w}}$ for all black vertices $\mathsf{b} \in \mathsf{B}_1$. By Lemma 46, this adjugate matrix satisfies

$$Q(z(u), w(u))_{\mathsf{b},\mathsf{w}} \lambda(u) = g_{\mathsf{b},\mathsf{w}}(u)$$

for all b and $u \in \Sigma \setminus \boldsymbol{\alpha}(\mathcal{T})$, with $\lambda = \frac{dz}{zw\partial_w P(z,w)} = -\frac{dw}{zw\partial_z P(z,w)}$. Note that the poles of λ coming from $\frac{dz}{z}$ or $\frac{dw}{w}$ lie in $\boldsymbol{\alpha}(\mathcal{T})$, and therefore do not contribute to the divisor. Therefore, the possible poles of λ outside $\boldsymbol{\alpha}(\mathcal{T})$ come from the remaining factors $\frac{1}{w\partial_w P(z,w)}$ and $\frac{1}{z\partial_z P(z,w)}$, and hence correspond to singular points of $\mathcal{C}$ where the divisor is not defined. As a consequence, the divisor $\psi^{-1}((\mathsf{w}))$ is determined by the common zeros in $\Sigma \setminus \boldsymbol{\alpha}(\mathcal{T})$ of $g_{\mathsf{b},\mathsf{w}}$ for all $\mathsf{b} \in \mathsf{B}_1$. The conclusion now follows from Proposition 36. □

Remark 48. It should be noted that the arguments of Propositions 36 and 47 provide an independent proof of the following fact: given a dimer model on a minimal periodic graph with Fock's weights, the divisor of a vertex is standard. We hope that this discussion, in particular Equation (13), helps clarifying the discrepancy between the viewpoints of Kenyon-Okounkov [KO06] and Goncharov-Kenyon [GK13], who consider standard divisors of degree g, and of Fock [Foc15], who deals with holomorphic line bundles of degree $g-1$.

In their seminal work [KO06], Kenyon and Okounkov not only prove that the spectral curve of a dimer model is Harnack and comes equipped with a standard divisor; they also show that every Harnack curve with triangular Newton polygon endowed with such a divisor can be realized by a dimer model. This result is extended to arbitrary Newton polygons by Goncharov and Kenyon in [GK13]. The main aim of Fock in [Foc15] is to give an explicit form to this inverse map: given a smooth curve $\mathcal{C}$ (non-necessarily Harnack), he constructs an explicit "dimer model" (not-necessarily with real edge-weights) whose spectral curve is $\mathcal{C}$.

We are in the position to give a modified version of his result, now restricted to Harnack curves, following and completing the discussion of [BCdT20, Section 5.4].

Theorem 49. *Fix a Harnack curve $\mathcal{C}$ endowed with a standard divisor D. Then, there exists an abstract M-curve Σ, a periodic minimal graph G, a map $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$ and an element t of $(\mathbb{R}/\mathbb{Z})^g \subset \mathrm{Jac}(\Sigma)$ such that the associated Fock operator K is periodic, and such that the spectral data of the corresponding dimer model coincides with $(\mathcal{C}, D)$ (up to a scale change $(z,w) \mapsto (\lambda z, \mu w)$ with $\lambda, \mu \in \mathbb{R}^*$, and fixing a vertex w). Moreover, the assignment $t \mapsto D$ defines a bijection from $(\mathbb{R}/\mathbb{Z})^g$ to the set of standard divisors on $\mathcal{C}$.*

Proof. The curve $\mathcal{C}$ being Harnack, it has the maximal number of real components, *i.e.*, it is an M-curve. Hence, there exists an abstract M-curve Σ and a birational map $\psi\colon \Sigma \to \mathcal{C}$ such that $\psi(\sigma(u)) = \overline{\psi(u)}$ for $u \in \Sigma$. By definition of a Harnack curve, it has a single unbounded real component, and we denote by A_0 the corresponding real component of Σ. Let us write $\psi(u) = (z(u), w(u))$ for the coordinates of the parametrization ψ of Σ. The maps z, w being meromorphic functions on Σ, their respective divisors are of the form $-\sum_j v_j\alpha_j$ and $\sum_j h_j\alpha_j$ for some finite set of elements $\{\alpha_j\}_j$ of Σ and integers $\{v_j\}_j, \{h_j\}_j$ so that $\sum_j v_j = \sum_j h_j = 0$. Since an element $\alpha_j \in \Sigma$ with $v_j \neq 0$ or $h_j \neq 0$ corresponds via ψ to an element of the complement of $\mathcal{C}$ in its (toric) closure, and since such elements lie in the closure of the unbounded real component of $\mathcal{C}$, we have $\alpha_j \in A_0$ for all j. By the discussion in Section 2.5.2, there exist constants $\lambda, \mu \in \mathbb{C}^*$ such that

$$z(u) = \lambda \prod_j E(u,\alpha_j)^{-v_j}, \quad w(u) = \mu \prod_j E(u,\alpha_j)^{h_j}$$

for all $u \in \Sigma$. The equality $\psi(\sigma(u)) = \overline{\psi(u)}$ together with Lemma 25 imply that λ and μ belong to $\mathbb{R}^*$, and can therefore be assumed to be ± 1 via a global scaling.

Allowing for the same element α_j to appear multiple times, it can be assumed that v_j and h_j are coprime for all j. Then, the equalities $\sum_j v_j = \sum_j h_j = 0$ ensure that there exists a minimal graph $\mathsf{G}_1 \subset \mathbb{T}$ with oriented train-tracks $\mathcal{T} = \{T_j\}_j$ satisfying $[T_j] = (h_j, v_j) \in H_1(\mathbb{T};\mathbb{Z})$ for all j, see e.g. [GK13, Theorem 2.5] for the proof.

Let $\boldsymbol{\alpha}\colon \mathcal{T} \to A_0$ be defined by $\boldsymbol{\alpha}(T_j) = \alpha_j$. By construction, the cyclic order on the boundary edges of $N(\mathsf{G})$ agrees with the partial cyclic order on $\mathcal{T}$ given by homology classes, which in turn coincides with the cyclic order of the tentacles of the amoeba of $\mathcal{C}$. By definition of a Harnack curve (recall the discussion before Remark 44), this cyclic order coincides with the cyclic order on $\{\alpha_j\}_j \subset A_0$. Moreover, two train-tracks with different homology classes correspond to distinct elements in A_0. In conclusion, the map $\boldsymbol{\alpha}$ belongs to the space X_{G}^{per}, as reinterpreted in Section 4.1.

Let K be Fock's adjacency operator corresponding to the M-curve Σ, to the universal cover $\mathsf{G} \subset \mathbb{R}^2$ of $\mathsf{G}_1 \subset \mathbb{T}$, to $\boldsymbol{\alpha}$, and to any $t \in (\mathbb{R}/\mathbb{Z})^g \subset \mathrm{Jac}(\Sigma)$. This operator is Kasteleyn by Proposition 31, so it defines a dimer model on G with positive edge-weights. By construction, the curve $\mathcal{C}$ is given by the elements (z,w) of $(\mathbb{C}^*)^2$ such that the kernel of $\mathsf{K}(z,w)$ is non-trivial, so it is the spectral curve of this dimer model. Also, the divisors $\sum_j v_j\alpha_j$ and $\sum_j h_j\alpha_j$ are principal by construction. By Equation (17), this ensures that the discrete Abel map $\boldsymbol{d}\colon \{\text{faces of } \mathsf{G}\} \to \mathrm{Pic}^0(\Sigma) \simeq \mathrm{Jac}(\Sigma)$ is $\mathbb{Z}^2$-periodic, and so is the operator K.

The last statement is now a direct consequence of Propositions 36 and 47. $\square$

We close this section with several remarks.

Remark 50.

1. It is natural to wonder to which extent the minimal graph G is completely determined by the curve $\mathcal{C}$. This question is answered by [GK13, Theorem 2.5]: two minimal graphs defining the same spectral curve $\mathcal{C}$ are not necessarily the same, but they define the same associated Newton polygon and therefore, they are related by a sequence of explicit local transformations. In [BCdT20, Section 7], see also Section 5.2 below, we check that the models defined by Fock's operators are invariant under these local transformations.

2. Another natural question is whether every periodic Kasteleyn operator on a minimal graph G with spectral curve $\mathcal{C}$ is gauge-equivalent to Fock's Kasteleyn operators for some $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$ and some $t \in (\mathbb{R}/\mathbb{Z})^g$. The answer is positive, and is a consequence of Theorem 49 together with [GK13, Theorem 7.3].

3. Two dimer models on the same graph but coming from different M-curves, or different angle maps, define different spectral curves, and are therefore not gauge-equivalent. Moreover, by Theorem 49 and [GK13, Theorem 7.3], we have the following result: two periodic dimer models on the same minimal graph G arising from the same M-curve, the same angle map, and elements $t, t' \in (\mathbb{R}/\mathbb{Z})^g$ are gauge equivalent if and only if $t = t'$.

4. Theorem 49 shows that any spectral data can be realized by a periodic minimal graph. Does this statement hold for a smaller class of (periodic, bipartite) graphs? A natural candidate is given by the set of bipartite *isoradial* graphs [Ken02], which can be described as the bipartite graphs whose train-tracks do not self-intersect and meet at most once [KS05]. However, it is unknown whether every convex polygon can be realized as the Newton polygon of such an isoradial graph.

5. As defined by Goncharov-Kenyon [GK13], the spectral data of a dimer model consists not only of the couple $(\mathcal{C}, D)$, but also of a *parametrization* ν of the divisor at infinity of the spectral curve. More concretely, ν should be thought of as a total cyclic ordering of the boundary points of $\mathcal{C}$, see [KO06]. Via the spectral transform, such a parametrization is realized by the set $\mathcal{T}_1$ of train-tracks of G_1.

   In our context, recall that X_{G}^{per} consists of maps $\boldsymbol{\alpha}\colon \mathcal{T}_1 \to A_0$ which preserve the partial cyclic ordering on $\mathcal{T}_1$. Therefore, one can very well fix a full spectral data $(\mathcal{C}, D, \nu)$ in the statement of Theorem 49: indeed, the specification of ν simply fixes a total cyclic ordering on $\mathcal{T}_1$ compatible with its intrinsic partial ordering, and hence restricts the possible angle maps $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$ to those which preserve this total ordering.

4.4 Ergodic Gibbs measures

For any planar, bipartite, periodic weighted graph G (not necessarily minimal), the set of ergodic Gibbs measures on dimer configurations was completely characterized in the work of Kenyon, Okounkov and Sheffield [KOS06]: they form a two-parameter family $(\mathbb{P}^B)$ indexed by $B = (B_x, B_y) \in \mathbb{R}^2$. All these measures are determinantal and have an explicit expression in terms of the periodic Kasteleyn operator K and its companion $\mathsf{K}(z, w)$: for any B, the probability of occurrence of k distinct edges $\mathsf{e}_1 = \mathsf{w}_1\mathsf{b}_1, \ldots, \mathsf{e}_k = \mathsf{w}_k\mathsf{b}_k$ is given by

$$\mathbb{P}^B(\mathsf{e}_1, \ldots, \mathsf{e}_k) = \left(\prod_{j=1}^{k} \mathsf{K}_{\mathsf{w}_j,\mathsf{b}_j}\right) \det_{1\leq i,j\leq k} A^B_{\mathsf{b}_i,\mathsf{w}_j}\,. \tag{25}$$

Here, the operator A^B has entries given by the following formula: if w and b are in the same fundamental domain and (m,n) belongs to $\mathbb{Z}^2$, then

$$A^B_{\mathsf{b}+(m,n),\mathsf{w}} = \iint_{\mathbb{T}_B} \mathsf{K}(z,w)^{-1}_{\mathsf{b},\mathsf{w}}\, z^m w^n \frac{dz}{2i\pi z}\frac{dw}{2i\pi w} = \iint_{\mathbb{T}_B} \frac{Q(z,w)_{\mathsf{b},\mathsf{w}}}{P(z,w)} z^m w^n \frac{dz}{2i\pi z}\frac{dw}{2i\pi w}\,,$$

with $Q(z,w)$ the adjugate matrix of $\mathsf{K}(z,w)$ and $\mathbb{T}_B = \{(z,w)\in(\mathbb{C}^*)^2\ ;\ |z| = e^{B_y}, |w| = e^{-B_x}\}$.

The phase diagram of this family is described by the amoeba of the characteristic polynomial $P(z,w)$, see [KOS06, Theorem 4.1]:

- If U is a connected component of the complement of the interior of the amoeba, then all the values of B inside U give the same measure:
  - if U is unbounded, the measure is called *solid* or *frozen*: every edge has a deterministic state;
  - if U is bounded, the measure is called *gaseous* (or *smooth* in the more general terminology of random surfaces): correlation between edges decay exponentially fast.
- Any B in the interior of the amoeba gives a different measure $\mathbb{P}^B$. These measures are called *liquid* (or *rough* in the terminology of random surfaces): the covariance of two edges at distance n decays like n^{-2}.

We now directly relate these operators A^B with our operators A^{u_0}, indexed by the subset $\mathcal{D} = \Sigma^+ \setminus \boldsymbol{\alpha}(\mathcal{T})$ of Σ.

Theorem 51. *For any $B = (B_x, B_y)$ in the amoeba of $\mathcal{C}$, let u_0 be the unique element of $\mathcal{D}$ such that $\log|z(u_0)| = B_y$ and $\log|w(u_0)| = -B_x$. Then, the operators A^B and A^{u_0} coincide.*

This result together with Proposition 45 and [KOS06, Theorem 4.1] immediately yield the following alternative presentation of the Gibbs measures and of the associated phase diagram.

Corollary 52. *Fix a periodic minimal graph G, an M-curve Σ, an element $\boldsymbol{\alpha}\in X^{per}_{\mathsf{G}}$ and a real element $t\in\mathrm{Jac}(\Sigma)$, and consider the dimer model on G with corresponding Kasteleyn operator K which we assume to be periodic. Then, the set of ergodic Gibbs measures is given by the measures $(\mathbb{P}^{u_0})_{u_0\in\mathcal{D}}$ whose expression on cylinder sets is given as follows: for any set $\{\mathsf{e}_1 = \mathsf{w}_1\mathsf{b}_1,\dots,\mathsf{e}_k = \mathsf{w}_k\mathsf{b}_k\}$ of distinct edges of G,*

$$\mathbb{P}^{u_0}(\mathsf{e}_1,\dots,\mathsf{e}_k) = \Big(\prod_{j=1}^k \mathsf{K}_{\mathsf{w}_j,\mathsf{b}_j}\Big)\times \det_{1\le i,j\le k}\Big(\mathsf{A}^{u_0}_{\mathsf{b}_i,\mathsf{w}_j}\Big)\,. \tag{26}$$

Furthermore, the model is solid (resp. gaseous, liquid) if u_0 belongs to A_0 (resp. to $A_1\cup\ldots\cup A_g$, to the interior of $\mathcal{D}$). □

Before turning to the proof of Theorem 51, let us emphasize the key property of Corollary 52.

Remark 53. Although it is far from obvious from their expression (25), the local statistics (26) for the Gibbs measures $(\mathbb{P}^{u_0})_{u_0\in\mathfrak{D}}$ are *local*, in the sense that $\mathbb{P}^{u_0}(\mathsf{e}_1,\dots,\mathsf{e}_k)$ only depends on the graph G in a neighborhood of $\mathsf{e}_1\cup\cdots\cup\mathsf{e}_k$, see however Remark 1. As an example of application, let us compute the probability of occurrence of a single edge $\mathsf{e}=\mathsf{wb}$ using the notation of Figure 1.

Using Corollary 52 and the definition of A^{u_0}, see Equation (14), we have:

$$\mathbb{P}^{u_0}(\mathsf{e}) = \frac{1}{2i\pi}\int_{\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}} \mathsf{K}_{\mathsf{w},\mathsf{b}}\, g_{\mathsf{b},\mathsf{w}}.$$

Now, using Fay's identity in the form of (9) to compute the product $\mathsf{K}_{\mathsf{w},\mathsf{b}}g_{\mathsf{b},\mathsf{w}}$, we obtain

$$\mathbb{P}^{u_0}(\mathsf{e}) = \frac{1}{2i\pi}\int_{\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}} \omega_{\beta-\alpha} + \frac{1}{2i\pi}\sum_{j=1}^{g}\Big(\frac{\partial\log\theta}{\partial z_j}(\widetilde{t}+\widetilde{\boldsymbol{d}}(\mathsf{f})) - \frac{\partial\log\theta}{\partial z_j}(\widetilde{t}+\widetilde{\boldsymbol{d}}(\mathsf{f}'))\Big)\int_{\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}}\omega_j,$$

where recall that $\omega_{\beta-\alpha} = \mathrm{d}_u\log\frac{E(u,\beta)}{E(u,\alpha)}$ is the unique meromorphic 1-form with 0 integral along A-cycles, and two simple poles: at β with residue 1, and α with residue -1. Using the definition of the contours in the different phases, see Section 3.5, we can push the computation further.

• Gaseous phases. Then $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ is homologous to B_k for some $1\le k\le g$. Using Riemann's bilinear relation (2) with the differential of the third kind $\omega_{\beta-\alpha}$, and the fact that $\int_{B_k}\omega_j = \Omega_{j,k}$, we obtain

$$\mathbb{P}^{u_0}(\mathsf{e}) = \int_\alpha^\beta \omega_k + \frac{1}{2i\pi}\sum_{j=1}^{g}\Omega_{j,k}\Big(\frac{\partial\log\theta}{\partial z_j}(\widetilde{t}+\widetilde{\boldsymbol{d}}(\mathsf{f})) - \frac{\partial\log\theta}{\partial z_j}(\widetilde{t}+\widetilde{\boldsymbol{d}}(\mathsf{f}'))\Big),$$

where the path of integration from α to β lies in the surface Σ cut along $\{A_j,B_j : 1\le j\le g\}$, see Figure 7.

Because in the surgery to cut open Σ, we take $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ as a realization of B_k, and since by definition, the contour $\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}$ does not intersect A_0 in the oriented arc from α to β, the integration from α to β of ω_k is really an integral along A_0 in the positive direction.

• Solid phases. Then, since we are integrating holomorphic 1-forms on closed contours bounding disks in Σ, the integrals $\int_{\mathsf{C}^{u_0}_{\mathsf{b},\mathsf{w}}}\omega_j$ are all equal to 0. The first integral is non zero if and only if u_0 is such that the cyclic order $[\alpha,u_0,\beta]$ on A_0 is preserved. In this case, it is equal to the residue at β which is 1 by definition of $\omega_{\beta-\alpha}$, so we find

$$\mathbb{P}^{u_0}(\mathsf{wb}) = \mathbf{1}_{\{\text{the cyclic relation }[\alpha,u_0,\beta]\text{ holds in }A_0\}}.$$

• Liquid phase. We use the explicit form of $\omega_{\beta-\alpha}$, and obtain

$$\mathbb{P}^{u_0}(\mathsf{wb}) = \frac{1}{2i\pi}\log\frac{E(\sigma(u_0),\alpha)}{E(u_0,\alpha)}\frac{E(u_0,\beta)}{E(\sigma(u_0),\beta)} + \frac{1}{2i\pi}\sum_{j=1}^{g}\Big(\frac{\partial\log\theta}{\partial z_j}(\widetilde{t}+\widetilde{\boldsymbol{d}}(\mathsf{f})) - \frac{\partial\log\theta}{\partial z_j}(\widetilde{t}+\widetilde{\boldsymbol{d}}(\mathsf{f}'))\Big)\int_{\mathsf{C}^{u_0}_{\mathsf{w},\mathsf{b}}}\omega_j.$$

Note that specializing to the case $g=1$, we recover the computation of Proposition 43 of [BCdT20].

Theorem 51 is a generalization of [BCdT20, Theorem 34] which deals with the elliptic case. We follow the same strategy for the proof: we do not use a uniqueness argument for inverses of K with some growth property, but perform a direct computation to partially evaluate the double integral defining $A^{B}_{\mathsf{b}+(m,n),\mathsf{w}}$ by taking a residue, then make a change of variable to transform the remaining integral as an integral on the surface Σ.

Before doing that, let us associate to any closed, oriented, dual path γ on G_1 the following function : for $\widetilde{u}\in\widetilde{\Sigma}$, set

$$J_\gamma(\widetilde{u}) = \sum_{\mathsf{e}=\mathsf{wb}}(\mathsf{e}\wedge\gamma)\mathsf{K}_{\mathsf{w},\mathsf{b}}g_{\mathsf{b},\mathsf{w}}(\widetilde{u}), \tag{27}$$

where $\mathsf{e}\wedge\gamma\in\mathbb{Z}$ denotes the algebraic intersection number of the oriented edge $\mathsf{e}=\mathsf{wb}$ with the oriented curve γ. By Lemma 32, the function J_γ on $\widetilde{\Sigma}$ projects to a meromorphic 1-form on Σ. We now relate J_γ with z and w given by Equation (24).

Proposition 54. *For any closed oriented dual path γ on G_1, we have the equality*

$$J_\gamma = -d\log z_\gamma\,,$$

where z_γ stands for $z^{h_\gamma}w^{v_\gamma}$ if $[\gamma]=(h_\gamma,v_\gamma)\in\mathbb{Z}^2=H_1(\mathbb{T};\mathbb{Z})$.

Proof. These two quantities define meromorphic 1-forms on Σ. To prove they are equal, it is enough to show that they have the same singular parts, and the same periods along the cycles A_j, $1\le j\le g$. More precisely, we prove that these two differential forms

- have 0 integral along A-cycles;
- have no pole outside of $\{\alpha_T\ ;\ T\in\mathfrak{T}_1,\ T\wedge\gamma\neq 0\}$;
- admit α_T as a simple pole with residue $-T\wedge\gamma$ if $T\wedge\gamma$ does not vanish.

Let us start with J_γ. First, we rewrite Fay's identity in the form of Equation (9) in terms of K and g for two neighboring vertices b and w with the convention of Figure 1. It yields the following unique decomposition of the meromorphic 1-form $\mathsf{K}_{\mathsf{w},\mathsf{b}}g_{\mathsf{b},\mathsf{w}}$ as a sum of a holomorphic part (*i.e.*, a linear combination of the ω_j's) and a meromorphic part with zero integral along the A-cycles:

$$\mathsf{K}_{\mathsf{w},\mathsf{b}} g_{\mathsf{b},\mathsf{w}} = \omega_{\beta-\alpha} + \sum_{j=1}^{g} \left(\frac{\partial \log \theta}{\partial z_j}(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f})) - \frac{\partial \log \theta}{\partial z_j}(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f}')) \right) \omega_j \,. \tag{28}$$

Note that the coefficient of ω_j is the difference of the same function evaluated at f and f'. By definition, the 1-form J_γ is the weighted sum of these contributions, with weights given by the algebraic intersections of γ with edges $\mathsf{e} = \mathsf{wb}$.

The first point now follows from the fact that for any face f, the term $\frac{\partial \log \theta}{\partial z_j}(\widetilde{t} + \widetilde{\boldsymbol{d}}(\mathsf{f}))$ appears in the sum defining J_γ with coefficient $\sum_{\mathsf{e}\subset\partial\mathsf{f}}(\mathsf{e} \wedge \gamma) = \partial\mathsf{f} \wedge \gamma = 0$ since γ is a closed oriented path and $\partial\mathsf{f}$ a closed oriented path that bounds; hence, the total coefficient of ω_j in that sum is 0. To check the second and third points, note that the only poles come from train-tracks intersecting γ. Fix a train-track T_α of $\mathcal{T}_1$ with angle α. Every intersection of T_α with γ corresponds to an edge. The contribution of this intersection to $T_\alpha \wedge \gamma$ can be positive (as on Figure 6), and the residue of $\omega_{\beta-\alpha}$ is -1, or it can be negative, and the residue is $+1$. Summing all these contributions, we see that the total residue of J_γ at α is $-T_\alpha \wedge \gamma$.

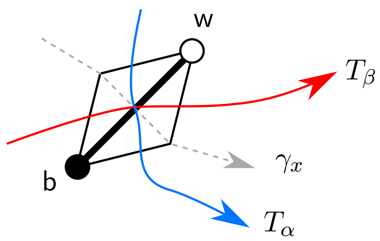

Figure 6: Intersection of the two train-tracks corresponding to an edge and γ_x when the white vertex is on the left of γ_x.

We now turn to $d \log z_\gamma$. To check the first point, note that z_γ is real along the A-cycles, and does not vanish as all its zeros and poles lie on A_0. Therefore, the argument of z_γ is constant along any A_j, so the integral of $d \log z_\gamma$ is equal to 0. To prove the second and third points, we use Equation (24) to express z_γ in terms of angles associated to train-tracks. It follows that the 1-form $d \log z_\gamma$ has a simple pole at α_T if the train-track T intersects γ, and that the associated residue is the degree of that point in the divisor of z_γ, which is given by $h_T v_\gamma - v_T h_\gamma = T \wedge \gamma$. This concludes the proof. □

Remark 55. For $\gamma = \gamma_y$, we can automatically keep track of the edges of G_1 intersecting γ_y with the correct signs and multiplicities, by using $\mathsf{K}(z, w)$ as a kind of generating function for those. Because of the definition of the variable w, selecting these intersections with the correct signs boils down to applying the $w\partial_w$ differential operator to $\mathsf{K}(z, w)$. Thus we have, see also Equation (27) of [BCdT20],

$$J_{\gamma_y}(u) = -\sum_{\mathsf{b},\mathsf{w}} w(u)\partial_w \mathsf{K}(z(u), w(u))_{\mathsf{w},\mathsf{b}} g_{\mathsf{b},\mathsf{w}}(u).$$

Likewise, for $\gamma = \gamma_x$, we have

$$J_{\gamma_x}(u) = +\sum_{\mathsf{b},\mathsf{w}} z(u)\partial_z \mathsf{K}(z(u), w(u))_{\mathsf{w},\mathsf{b}} g_{\mathsf{b},\mathsf{w}}(u).$$

We are finally ready to give the proofs of Lemma 46 relating $Q(z,w)_{\mathsf{b},\mathsf{w}}$ and $g_{\mathsf{b},\mathsf{w}}$, and of Theorem 51 on the equality of the operators A^B and A^{u_0}.

Proof of Lemma 46. Let $Q(z,w)$ be the adjugate matrix of $\mathsf{K}(z,w)$. We need to show the equality

$$Q(z(u), w(u))_{\mathsf{b},\mathsf{w}}\lambda(u) = g_{\mathsf{b},\mathsf{w}}(u) \tag{29}$$

for every b and w on G_1 and every $u \in \Sigma \setminus \boldsymbol{\alpha}(\mathcal{T})$, with λ the meromorphic 1-form on Σ given by $\lambda = \frac{dz}{zw\partial_w P(z,w)} = -\frac{dw}{zw\partial_z P(z,w)}$. From the fact that the adjugate matrix $Q(z,w)$ satisfies

$$Q(z,w)\mathsf{K}(z,w) = P(z,w)\,\mathrm{Id}, \tag{30}$$

we know that it has rank at most 1 on the spectral curve. It is a product of a vector in the right kernel of $\mathsf{K}(z,w)$ times one in the left kernel of $\mathsf{K}(z,w)$. Hence, there exists a non-zero meromorphic 1-form λ on Σ such that Equation (29) is satisfied for every b and w on G_1. Moreover, differentiating (30) with respect to w, evaluating on the spectral curve and taking the trace yields, see also [BCdT20, Lemma 37]:

$$\begin{aligned}\lambda(u)\partial_w P(z(u), w(u)) &= \sum_{\mathsf{b},\mathsf{w}} \partial_w \mathsf{K}(z(u), w(u))_{\mathsf{w},\mathsf{b}} g_{\mathsf{b},\mathsf{w}}(u) \\ &= -\frac{1}{w(u)} J_{\gamma_x}(u) = \frac{dz}{zw}(u),\end{aligned}$$

where in the last line we use Remark 55 and Proposition 54. This implies the first formula for λ, while the second can be obtained in the same way by exchanging the roles of z and w and dealing carefully with signs, or noticing that on the spectral curve, the following equality holds:

$$\partial_z P(z(u), w(u))dz(u) + \partial_w P(z(u), w(u))dw(u) = d(P(z(u), w(u)) = 0. \qquad \square$$

Proof of Theorem 51. We just sketch the proof, as it is analogous to that of [BCdT20, Theorem 34]. Consider a white vertex w of G_1. By Lemma 41 applied to $f = A^B_{\cdot,\mathsf{w}} - A^{u_0}_{\cdot,\mathsf{w}} \in \mathbb{C}^{\mathsf{B}}$, it is enough to show that for any black vertex b of G_1, the equality

$$A^B_{\mathsf{b}+(m,n),\mathsf{w}} = A^{u_0}_{\mathsf{b}+(m,n),\mathsf{w}}$$

holds for all $(m,n) \in \mathbb{Z}^2$, except possibly for a finite number of values. Therefore, we now assume that n is large enough. The other situations are treated similarly by exchanging the roles of z and w, and/or replacing z and w by their inverses.

We now evaluate by the residue theorem one of the two integrals defining $A^B_{\mathsf{b}+(m,n)}$: for every z on the circle of radius e^{B_y}, the only poles of

$$w \mapsto \frac{Q(z,w)_{\mathsf{b},\mathsf{w}}}{P(z,w)} w^{n-1}$$

inside the disk $\{|w| < e^{-B_x}\}$ come from zeros of $P(z,\cdot)$, and we write

$$\int_{|w|=e^{-B_x}} \frac{Q(z,w)_{\mathsf{b},\mathsf{w}}}{P(z,w)} w^{n-1} \frac{dw}{2i\pi} = \sum_{j=1}^{d_{z,B_x}} \frac{Q(z,w_j(z))_{\mathsf{b},\mathsf{w}}}{\partial_w P(z,w_j(z))} w_j(z)^{n-1},$$

where $w_1(z), \dots, w_{d_{z,B_x}}(z)$ represent the poles of $w \mapsto P(z,w)$ inside the circle of radius e^{-B_x}. When z varies along the circle of radius e^{B_y}, the points $(z, w_j(z))$ describe paths which can be pulled back by ψ as paths on the surface Σ. This collection of paths consists of

- a certain number of closed loops with trivial homology along the A-cycles
- a path connecting $\sigma(u_0)$ to u_0 if B is in the interior of the amoeba. If B is on the boundary, then this path is also a closed loop.

This family of paths and loops on Σ can be deformed without crossing any $\{\alpha_T\}_{T\in\mathcal{T}_1}$, to become the path $\mathsf{C}^{u_0}_{\mathsf{b}+(m,n),\mathsf{w}}$.

Performing the change of variable from z to u in the remaining integral defining $A^B_{\mathsf{b}+(m,n),\mathsf{w}}$ and using Lemma 46, we have

$$\begin{aligned}
A^B_{\mathsf{b}+(m,n),\mathsf{w}} &= \int_{|z|=e^{B_y}} \sum_{j=1}^{d_{z,B_x}} \frac{Q(z,w_j(z))_{\mathsf{b},\mathsf{w}}}{\partial_w P(z,w_j(z))} w_j(z)^{n-1} z^{m-1} \frac{dz}{2i\pi} \\
&= \frac{1}{2i\pi} \int_{\mathsf{C}^{u_0}_{\mathsf{b}+(m,n),\mathsf{w}}} \frac{Q(z(u),w(u))_{\mathsf{b},\mathsf{w}}}{z(u)w(u)\partial_w P(z(u),w(u))} z^m(u) w^n(u) dz(u) \\
&= \frac{1}{2i\pi} \int_{\mathsf{C}^{u_0}_{\mathsf{b}+(m,n),\mathsf{w}}} g_{\mathsf{b},\mathsf{w}}(u) z(u)^m w(u)^n \\
&= \frac{1}{2i\pi} \int_{\mathsf{C}^{u_0}_{\mathsf{b}+(m,n),\mathsf{w}}} g_{\mathsf{b}+(m,n),\mathsf{w}}(u) = A^{u_0}_{\mathsf{b}+(m,n),\mathsf{w}}\,.
\end{aligned}$$

This concludes the proof. □

4.5 Slope of the Gibbs measures $\mathbb{P}^{u_0}$

The discussion of this section follows [BCdT20, Section 5.6]. Therefore, we only develop the new aspects coming from higher genus and often refer the reader to this article for details.

Let us fix $u_1 \in A_0 \setminus \boldsymbol{\alpha}(\mathcal{T}_1)$ and denote by M_1 the dimer configuration on which the solid Gibbs measure $\mathbb{P}^{u_1}$ is concentrated. Also, let P_1 be the integer point on the boundary of $N(\mathsf{G})$ corresponding to the interval of $A_0 \setminus \boldsymbol{\alpha}(\mathcal{T}_1)$ containing u_1.

For any dimer configuration M, the corresponding *height difference* (relative to M_1) between two faces f and f' of G is defined as

$$h(\mathsf{f}') - h(\mathsf{f}) = \sum_{\mathsf{e}=\mathsf{wb}} (\mathsf{e} \wedge \gamma) \left(\mathbf{1}_{\{\mathsf{e} \in \mathsf{M}\}} - \mathbf{1}_{\{\mathsf{e} \in \mathsf{M}_1\}} \right) , \tag{31}$$

where γ is an oriented dual path connecting f to f', and $\mathsf{e} \wedge \gamma$ its algebraic intersection number with the oriented edge $\mathsf{e} = \mathsf{wb}$. This quantity is well defined and does not depend on the choice of γ because M and M_1, viewed as 1-forms on G, have the same divergence at any vertex.

The *slope* (s^{u_0}, t^{u_0}) of the Gibbs measure $\mathbb{P}^{u_0}$ is the expected horizontal and vertical height change [KOS06], *i.e.*, the expectation of Expression (31) for f' equal to $\mathsf{f} + (1,0)$ and $\mathsf{f} + (0,1)$, or in other words, for $\gamma = \gamma_x$ and $\gamma = \gamma_y$, respectively.

Applying Corollary 52 in the case $k = 1$ to Expression (31) as in [BCdT20, Theorem 38], we get the equalities

$$s^{u_0} = \frac{1}{2i\pi} \int_{\mathsf{C}_{u_1}^{u_0}} J_{\gamma_x}, \quad t^{u_0} = \frac{1}{2i\pi} \int_{\mathsf{C}_{u_1}^{u_0}} J_{\gamma_y} ,$$

where $\mathsf{C}_{u_1}^{u_0}$ is an oriented path in Σ connecting $\sigma(u_0)$ to u_0, crossing A_0 once at u_1, disjoint from $A_1 \cup \cdots \cup A_g$, such that $\sigma(\mathsf{C}_{u_1}^{u_0}) = -\mathsf{C}_{u_1}^{u_0}$. By Proposition 54, this can be rewritten as

$$s^{u_0} = -\frac{1}{2i\pi} \int_{\mathsf{C}_{u_1}^{u_0}} \mathrm{d}\log z, \quad t^{u_0} = -\frac{1}{2i\pi} \int_{\mathsf{C}_{u_1}^{u_0}} \mathrm{d}\log w \,. \tag{32}$$

In other words, s^{u_0} and t^{u_0} are (up to a multiplication by $-\pi$) continuous determinations of the arguments of $z(u_0)$ and $w(u_0)$ respectively. Up to a proper normalization, they correspond to a unique point in the *coamoeba* of $\mathcal{C}$. Note that these formulas can be seen as a refinement of [KOS06, Theorem 5.6], where the equalities are only valid up to a sign and modulo π. They are also related to [KOS06, Proposition 3.2], since the part of the coamoeba of $\mathcal{C}$ parametrized by Σ^+ is in 1-to-1 correspondence with the Newton polygon of P [Pas16].

Since the magnetic field is given by the log of the modulus of $z(u_0)$ and $w(u_0)$, the pair slope/magnetic field is realizing (half of) the *amoeba-to-coamoeba mapping* for the Harnack curve $\mathcal{C}$, which has been described and computed explicitly by Passare [Pas16]. In terms of dimer models, this translates to the fact that the slope and the magnetic field are dual variables when performing Legendre transform between the *free energy*, represented by the Ronkin function of the characteristic polynomial, and the *surface tension* [KOS06].

We now give explicit formulas for the slopes of the solid and gaseous phases. For solid phases, Corollary 39 from [BCdT20] is valid in the current more general context without modification, as it only relies on the connection between the divisor for $z(u)$ and the homology class of $T \in \mathcal{T}_1$.

Corollary 56 (slopes of solid phases, [BCdT20], Corollary 39)**.** *Suppose that u_0 belongs to one of the connected components of $A_0 \setminus \boldsymbol{\alpha}(\mathcal{T}_1)$. Then, we have*

$$(s^{u_0}, t^{u_0}) = \sum_{T \in \mathcal{T}_1 : [u_0, \alpha_T, u_1]} (v_T, -h_T)\,,$$

where the sum is over all $T \in \mathcal{T}_1$ such that the cyclic order relation $[u_0, \alpha_T, u_1]$ holds in A_0. In particular, the points $P_1 + (t^{u_0}, -s^{u_0})$ indexed by the connected components of $A_0 \setminus \boldsymbol{\alpha}(\mathcal{T}_1)$ are the integer boundary vertices of $N(\mathsf{G})$. □

For gaseous phases of the model, we have the following correspondence with the g marked interior lattice points of $N(\mathsf{G})$, recall Proposition 43. Note that the main ingredient in the proof is the Riemann bilinear relation (2).

Corollary 57 (slopes of gaseous phases)**.** *Suppose that u_0 belongs to A_k for some $1 \leq k \leq g$. Then, the slope (s^{u_0}, t^{u_0}) of the corresponding Gibbs measure is related to the k^{th} component of $\varphi(\boldsymbol{\alpha})$ by*

$$P_1 + (t^{u_0}, -s^{u_0}) = \varphi_k(\boldsymbol{\alpha})\,.$$

Proof. We give the details for the imaginary part of the identity, which corresponds to the horizontal slope s^{u_0}. The computation of the real part, corresponding to the vertical slope t^{u_0}, is similar, and therefore left to the reader.

Since u_0 belongs to A_k, the contour of integration $\mathsf{C}_{u_1}^{u_0}$ is a loop homologous to B_k in Σ. However, because of the possible presence of singularities of z, it can a priori not be moved outside the interval of $A_0 \setminus \boldsymbol{\alpha}(\mathcal{T}_1)$ containing u_1. For this reason, we consider realizations of the cycles $B_1, \dots, B_g$ crossing A_0 in that same fixed interval.

According to Proposition 54 and its proof, the differential form $J_{\gamma_x} = -d\log z$ is the unique differential of the third kind with a simple pole at every α_T for $T \in \mathcal{T}_1$ such that $v_T \neq 0$, and residue v_T. Therefore, we can take this form for ω_D in the Riemann bilinear relation (2), with corresponding divisor $D = \sum_{T \in \mathcal{T}_1} v_T \alpha_T$. This yields

$$s^{u_0} = \frac{1}{2i\pi} \int_{\mathsf{C}_{u_1}^{u_0}} J_{\gamma_x} = \frac{1}{2i\pi} \int_{B_k} \omega_D = \int_{D^-}^{D^+} \omega_k\,, \tag{33}$$

where the integration paths from $D^- = \sum_{T : v_T < 0} (-v_T)\alpha_T$ to $D^+ = \sum_{T : v_T > 0} v_T \alpha_T$ lie in the surface Σ cut along the cycles A_ℓ, B_ℓ. Note that in this surface with boundary, the cycle A_0 is represented by g oriented segments. Because of the assumption that all the B_ℓ's intersect A_0 in the same interval of $A_0 \setminus \boldsymbol{\alpha}(\mathcal{T}_1)$, all the α_T's are in the same segment. We label them $\alpha_1, \dots, \alpha_r$ in the increasing order along this oriented segment of A_0 (see Figure 7), and write $T_1, \dots, T_r$ for the corresponding train-tracks. In

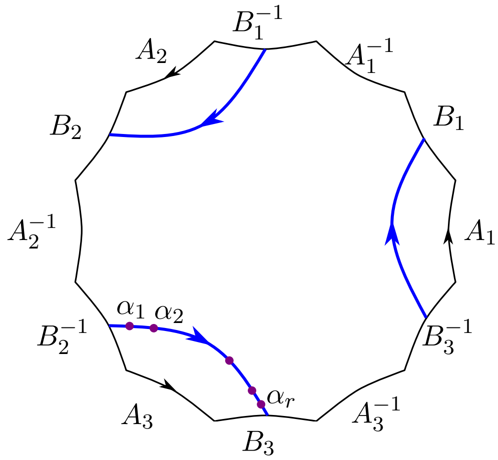

Figure 7: The surface Σ cut along cycles representing a basis of its homology, chosen in such a way that all cycles B_ℓ cut A_0 in the same interval of $A_0 \setminus \boldsymbol{\alpha}(\mathcal{T})$. The arcs in blue are the segments of A_0 after the cuts, one of which contains all of $\boldsymbol{\alpha}(\mathcal{T})$.

conclusion, the integral of ω_k from D^- to D^+ is given by integrals along a single segment of A_0, with orientation from α_j with $v_{T_j} < 0$ to α_ℓ with $v_{T_\ell} > 0$.

Let us now study Equation (19), where we take $x_0 \in A_0$ just before α_1. The k^{th} component of $n_j \in \mathbb{Z}^g$ is easily seen to be equal to δ_{j1}, so the vertical coordinate of the k^{th} component of Equation (19) yields

$$\sum_{j=1}^{r} v_{T_j} \int_{x_0}^{\alpha_j} \omega_k = \Im\left(P_1 - \varphi_k(\boldsymbol{\alpha})\right) ,$$

where the integrals are along the positive orientation of A_0. Comparing this equation with the right-hand side of (33), and carefully taking into account the different orientation constraints for these integrals, we obtain the equality $s^{u_0} = \Im\left(P_1 - \varphi_k(\boldsymbol{\alpha})\right)$. $\square$

4.6 Surface tension and free energy

For $(s,t) \in N(G)$, the *surface tension* $\tau(s,t)$ of the dimer model is defined as the exponential growth rate of the partition function $Z_n^{(s,t)}$ of dimer configurations on $\mathsf{G}_n = \mathsf{G} \setminus n\mathbb{Z}^2$ whose horizontal and vertical height change around the torus are conditioned to be $\lfloor ns \rfloor$ and $\lfloor nt \rfloor$:

$$\tau(s,t) = -\lim_{n\to\infty} \frac{1}{n^2} \log Z_n^{(s,t)} ,$$

see [KOS06, Section 3.2.4]. Sheffield [She05] proved that τ is a strictly convex function over $N(G)$. Using the correspondence between (s,t) and u_0 given by Equation (32), one can now see τ as a function $\underline{\tau}(u_0) = \tau(s^{u_0}, t^{u_0})$ on $\mathcal{D} = \Sigma^+ \setminus \boldsymbol{\alpha}(\mathcal{T})$.

The *free energy* $F(B_x, B_y)$ of the dimer model is the Legendre dual of τ

$$F(B_x, B_y) = \max_{(s,t)} \bigl(sB_x + tB_y - \tau(s,t)\bigr) .$$

It is well defined up to an anchoring in $\mathbb{Z}^2$ of the Newton polygon $N(G)$ to fix the additive constant in the definition of the height, which corresponds to a change in the linear part in (B_x, B_y). This is done for example by choosing a "frozen" point u_1 in $\mathcal{D} \cap A_0$ as the reference configuration to measure the height function.

As the surface tension, the free energy can be seen as a function on $\mathcal{D}$:

$$\underline{F}(u) = F(-\log|w(u)|, \log|z(u)|) .$$

Because of the duality relation between the height and the magnetic field, the differential of $\underline{\tau}$ is given by

$$\begin{aligned} d\underline{\tau}(u) &= \frac{\partial \tau}{\partial s} ds^u + \frac{\partial \tau}{\partial t} dt^u = B_x(u) ds^u + B_y(u) dt^u \\ &= \frac{1}{\pi}\bigl(\log|w(u)| d\arg z(u) - \log|z(u)| d\arg w(u)\bigr) . \end{aligned}$$

Here and later in this section, the arguments are measured along continuous paths in $\mathcal{D}$ starting from the fixed point u_1.

When $u_0 = u_1$, the measure concentrates on a single periodic configuration M_1, and therefore, the surface tension is explicitly obtained as

$$\underline{\tau}(u_1) = -\sum_{\mathsf{wb} \in \mathsf{M}_1} \log\bigl|\mathsf{K}_{\mathsf{w},\mathsf{b}}\bigr| .$$

Then one can obtain the expression for $\underline{\tau}(u_0)$ for any other $u_0 \in \mathcal{D}$ by integration:

$$\underline{\tau}(u_0) = \underline{\tau}(u_1) + \frac{1}{\pi}\int_{u_1}^{u_0} \log|w(u)| d\arg z(u) - \log|z(u)| d\arg w(u) .$$

Taking advantage of the fact that $z(u)$ and $w(u)$ are expressed as products over train-tracks of G_1, recall Equation (24), one can rewrite the 1-form

$$\log|w(u)| d\arg z(u) - \log|z(u)| d\arg w(u) = -\sum_{S}\sum_{T}(h_S v_T - v_S h_T) k_{\alpha_S}(u) d\ell_{\alpha_T}(u) ,$$

where $k_\alpha(u) = \log|E(\widetilde{\alpha}, \widetilde{u})|$ and $\ell_\alpha(u) = \arg E(\widetilde{\alpha}, \widetilde{u})$. The integer $h_S v_T - v_S h_T$ is nothing but the algebraic intersection number of the train-tracks S and T. Therefore, each edge $\mathsf{e} \in \mathsf{E}_1$ contributes exactly twice to this double sum: if e is at the intersection of the train-tracks T_α and T_β, then it contributes once when $(S,T) = (T_\alpha, T_\beta)$, and once when $(S,T) = (T_\beta, T_\alpha)$. Following the convention of Figure 1, this leads to

$$\log|w(u)| d\arg z(u) - \log|z(u)| d\arg w(u) = \sum_{\mathsf{e} \in \mathsf{E}_1} \bigl(k_\beta(u) d\ell_\alpha(u) - k_\alpha(u) d\ell_\beta(u)\bigr) .$$

This yields the following *local formula* for the surface tension, in the sense that it consists of a sum of terms associated to edges of the fundamental domain.

Proposition 58. *Let u_1 be a point on $\mathcal{D} \cap A_0$ describing a frozen phase, and let M_1 be the corresponding dimer configuration on G_1, repeated in a periodic way on G. For any $u_0 \in \mathcal{D}$, the surface tension $\underline{\tau}(u_0)$ is given by*

$$\underline{\tau}(u_0) = -\sum_{\mathsf{wb} \in \mathsf{M}_1} \log \left| \mathsf{K}_{\mathsf{w},\mathsf{b}} \right| + \frac{1}{\pi} \sum_{\mathsf{e} \in \mathsf{E}_1} \int_{u_1}^{u_0} k_\beta(u) d\ell_\alpha(u) - k_\alpha(u) d\ell_\beta(u) \,. \quad \square$$

Remark 59.

1. Note that the term associated to a given edge e in the sum over E_1 is genuinely local, as it only depends on the two parameters α and β of the train-tracks crossing that edge. All the dependency on t and the non-locality associated to $\boldsymbol{d}$ is contained in the constant $\underline{\tau}(u_1)$.

2. *Per se*, the functions k_α and ℓ_β are not well-defined on Σ, so one needs to be slightly cautious when manipulating them. However, changing the lift $\widetilde{u}$ in their definition would add to the integrand $k_\beta(u) d\ell_\alpha(u) - k_\alpha(u) d\ell_\beta(u)$ a term which is of the form $F(\widetilde{u}; \alpha) - F(\widetilde{u}; \beta)$, and hence contributes 0 when summing over $\mathsf{e} \in \mathsf{E}_1$.

3. One can take advantage of telescopic contributions to add (or subtract) $k_\alpha(u) d\ell_\alpha(u) - k_\beta(u) d\ell_\beta(u)$ to the integrand, and so replace it by $(k_\alpha(u) \mp k_\beta(u))(d\ell_\alpha(u) \pm d\ell_\beta(u))$.

We now derive a formula in the same spirit for $\underline{F}$.

Corollary 60. *For any u_0 in $\mathcal{D}$, the free energy $\underline{F}(u_0)$ is given by*

$$\underline{F}(u_0) = \sum_{\mathsf{wb} \in \mathsf{M}_1} \log \left| \mathsf{K}_{\mathsf{w},\mathsf{b}} \right| + \sum_{\mathsf{e} \in \mathsf{E}_1} \frac{1}{\pi} \int_{u_1}^{u_0} \ell_\beta(u) dk_\alpha(u) - \ell_\alpha(u) dk_\beta(u) \,.$$

Proof. For any u_0 in $\mathcal{D}$, Legendre duality yields the following relation between $\underline{F}(u_0)$ and $\underline{\tau}(u_0)$:

$$\underline{F}(u_0) = s^{u_0} B_x(u_0) + t^{u_0} B_y(u_0) - \underline{\tau}(u_0) \,.$$

As above, we now rewrite the quantity $s^{u_0} B_x(u_0) + t^{u_0} B_y(u_0)$ as a sum over edges of E_1:

$$\begin{aligned} s^{u_0} B_x(u_0) + t^{u_0} B_y(u_0) &= \frac{1}{\pi} \big(\arg z(u_0) \log |w(u_0)| - \arg w(u_0) \log |z(u_0)| \big) \\ &= \frac{1}{\pi} \sum_S \sum_T (h_S v_T - v_S h_T) k_{\alpha_S}(u_0) l_{\alpha_T}(u_0) \\ &= \frac{1}{\pi} \sum_{\mathsf{e} \in \mathsf{E}_1} k_\alpha(u_0) \ell_\beta(u_0) - k_\beta(u_0) \ell_\alpha(u_0) \,. \end{aligned}$$

Finally, for any α and β, integration by parts yields

$$k_\alpha(u_0) \ell_\beta(u_0) - \int_{u_1}^{u_0} k_\alpha(u) d\ell_\beta(u) = \int_{u_1}^{u_0} \ell_\beta(u) dk_\alpha(u). \qquad \square$$

Remark 61.

1. The choice of the reference frozen phase appears in ℓ_α, which is defined as a continuous argument computed along a path from that reference point u_1.
2. Using the fact that $k_\alpha, d\ell_\alpha$ (resp. dk_α, ℓ_α) are symmetric (resp. antisymmetric) with respect to σ, the integration from u_1 to u_0 can also be expressed as $\frac{1}{2}$ times an integral from $\sigma(u_0)$ to u_0 along the path $\mathsf{C}_{u_1}^{u_0}$, symmetric with respect to σ and passing through the sector of $A_0 \backslash \boldsymbol{\alpha}(\mathcal{T})$ containing u_1. One can then replace $\ell_\alpha(u) = \arg E(\widetilde{\alpha}, \widetilde{u})$ by $\frac{1}{i} \log E(\widetilde{\alpha}, \widetilde{u})$.

The free energy is also given [KOS06] by the *Ronkin function* $R(B_x, B_y)$ of the characteristic polynomial P (up to a linear factor in (B_x, B_y) depending on the anchoring of $N(P)$ and its relation to u_1), *i.e.*, by

$$R(B_x, B_y) = \iint_{\substack{|z|=e^{B_y} \\ |w|=e^{-B_x}}} \log\bigl|P(z,w)\bigr| \frac{dz}{2i\pi z} \frac{dw}{2i\pi w} .$$

Note that the coordinates are 'rotated' by 90 degrees when compared with the original definition because of our choice of conventions, as for the amoeba. One can indeed check that the two expressions match by comparing the formula from Corollary 60 with computations of R when an explicit parametrization for the spectral curve $\mathcal{C}$ is known, see for example Theorem 7.5 from [BZ20] and references therein.

5 Additional features, and perspectives

This final and slightly informal section deals with miscellaneous additional results, together with upcoming work. We start in Section 5.1 by explaining that under some natural hypothesis, the construction of Gibbs measures extends beyond the periodic case, following and generalising [BCdT20, Section 6.1]. In Section 5.2, we check that [BCdT20, Section 7] extends without modification: the model is invariant under local transformations, and this invariance is a consequence of (and in some precise sense, equivalent to) Fay's identity; a possible extension of our results beyond minimal graphs is also discussed. Finally, in Section 5.3, we relate these models on specific classes of minimal graphs to known models, delaying their detailed study to future publications.

5.1 Beyond the periodic case

It is natural to wonder whether some results of Section 4, in particular the classification of Gibbs measures of Corollary 52, extend to arbitrary minimal graphs. We are not able to fully answer this question, but the discussion of [BCdT20, Section 6.1] applies, leading to the following result.

Let us assume that the minimal graph G and angle map $\boldsymbol{\alpha} \in X_{\mathsf{G}}$ satisfy the following condition: any finite simply connected subgraph $\mathsf{G}_0 \subset \mathsf{G}$ extends to a periodic minimal graph G', with the restriction of $\boldsymbol{\alpha}$ to the train-tracks of G_0 extending to an element $\boldsymbol{\alpha}'$ of $X_{\mathsf{G}'}$.

Theorem 62. *Consider a minimal graph G and an element $\boldsymbol{\alpha} \in X_{\mathsf{G}}^{per}$ satisfying the assumption above. Fix an M-curve Σ, a real element $t \in \mathrm{Jac}(\Sigma)$, and consider the dimer model on G with corresponding Kasteleyn operator K. Then, for every $u_0 \in \mathcal{D}$, the operator A^{u_0} defines a Gibbs measure $\mathbb{P}^{u_0}$ whose expression on cylinder sets is given as follows: for any set $\{\mathsf{e}_1 = \mathsf{w}_1\mathsf{b}_1, \dots, \mathsf{e}_k = \mathsf{w}_k\mathsf{b}_k\}$ of distinct edges of G,*

$$\mathbb{P}^{u_0}(\mathsf{e}_1, \dots, \mathsf{e}_k) = \Big(\prod_{j=1}^{k} \mathsf{K}_{\mathsf{w}_j,\mathsf{b}_j}\Big) \times \det_{1\leq i,j\leq k}\Big(\mathsf{A}^{u_0}_{\mathsf{b}_i,\mathsf{w}_j}\Big)\,.$$

Once again, the remarkable property of these Gibbs measures is that they are local, in the sense that the probability of occurrence of any set of edges only depends on the weighted graph near these edges. The computation of the probability of occurrence of a single edge done in Remark 53 also holds in the more general setting of Theorem 62. This is an illustration of the strength of local formulas, which allow for explicit computations also in the case of non-periodic graphs.

Remark 63. We believe that the condition stated above holds for any minimal graph G and map $\boldsymbol{\alpha} \in X_{\mathsf{G}}$. Proving that this is indeed the case would not only imply that Theorem 62 holds for any minimal graph G. As another consequence, the *t-embedding* [KLRR18, CLR20] determined by Fock's Kasteleyn operator would define an embedding of the dual graph G^* (for u_0 in the interior of Σ^+), a fact that is currently known to hold only for infinite periodic graphs, and for finite graphs with outer face of degree 4.

5.2 Invariance under moves, and going beyond minimal graphs

Dimer configurations behave in a controlled way under several local transformations of bipartite graphs. A natural family of such moves was introduced by Kuperberg and studied by Propp [Pro03] under the name of *urban renewal*. An equivalent set of moves was considered by Goncharov and Kenyon [GK13] and called *shrinking/expanding of a 2-valent vertex* and *spider move*, see Figure 8.

These moves play a crucial role in the theory. As was shown in [GK13, Theorem 2.5], the work of Thurston [Thu17] implies that any two periodic minimal graphs with the same Newton polygon are related by a finite sequence of these local transformations (recall Remark 50 above). It is therefore natural to wonder how the dimer models studied in the present article behave under these moves.

The answer is the content of [BCdT20, Section 7], which extends verbatim from the elliptic setting to the general case of arbitrary genus. We now give a brief summary of these results.

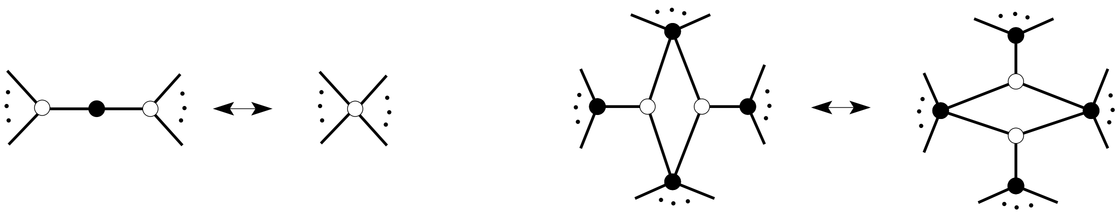

Figure 8: Shrinking/expanding of a 2-valent (black) vertex, and spider move (with black boundary vertices).

First of all, one easily checks that given a finite, bipartite, planar graph G (not necessarily minimal) with Kasteleyn operator K (not necessarily Fock's), the associated partition function is invariant under shrinking/expanding of a 2-valent black vertex b with adjacent vertices $\mathsf{w}_1, \mathsf{w}_2$ if and only K satisfies the equality $\mathsf{K}_{\mathsf{w}_1,\mathsf{b}} + \mathsf{K}_{\mathsf{w}_2,\mathsf{b}} = 0$ (and similarly for 2-valent white vertices, see [BCdT20, Proposition 50]). The prime form being anti-symmetric, this holds in particular for Fock's Kasteleyn operator.

With this condition satisfied, it can be assumed via reduction of 2-valent white vertices and expansion of 2-valent black vertices that all the white vertices of G are trivalent. For such graphs, Fock's weights take a particularly simple form: it is precisely given by the function $F_s(\alpha,\beta) = \theta(\alpha+\beta-s)E(\alpha,\beta)$ of Section 2.5.3, with $s = s(\mathsf{w})$ constant on the four white vertices appearing in any spider move with black boundary vertices.

Finally, let us consider a dimer model on a bipartite, planar graph G, with Kasteleyn coefficients defined by some function F_s of train-track parameters, as above. Then, the corresponding partition function is invariant under spider moves with black boundary vertices if and only if these coefficients satisfy Equation (10). In particular, Fay's identity directly implies that the dimer models given by Fock's weights are invariant under spider moves, a fact first proved (for urban renewal) by Fock [Foc15, Proposition 1].

As a concluding remark, let us mention that any (finite) bipartite graph whose train-tracks do not self-intersect can be reduced to a minimal one via shrinking/expanding of a 2-valent vertices, spider moves, and *merging parallel edges* as in [KLRR18, Figure 1]: this can be checked using the theory developed by Postnikov [Pos06] as in the proof of [KLRR18, Lemma 3], or the work of Thurston [Thu17] as in [BCdT21, Lemma 33]. Since our models are invariant under the first two transformations, one could hope that the whole theory applies to any (possibly non-minimal) bipartite graphs whose train-tracks do not self-intersect. This is not the case, for the simple reason that dimer models with Fock's weights are clearly *not* invariant by the third move. Another (similar) way to show that minimal graphs form the biggest class on which our work directly applies can be found in [BCdT21, Theorem 31].

Note however that it is in theory possible to study the dimer model on a periodic bipartite weighted graph (G',ν') with no self-intersecting train-track, as follows: first use the three local moves to reduce (G',ν') to a periodic minimal weighted graph (G,ν), then harness Theorem 49 to compute the parameters $\Sigma, t, \boldsymbol{\alpha}$ so that the corresponding Fock weights on G are gauge-equivalent to ν, and finally apply our results.

5.3 Relation to known models, and perspectives

The dimer models studied in the present work are very general, as they are defined on arbitrary minimal graphs and cover all dimer models in the periodic case (recall Theorem 49). As it turns out, particular types of minimal graphs yield interesting classes of dimer models, recovering and extending known models. This study was performed in Section 8.2 of [BCdT20] in the genus 1 case, showing that the elliptic models of [BdTR17, BdTR19, dT21] could be recovered by Fock's elliptic dimer model.

The extension of these results to higher genus is beyond the scope of this article and will be the subject of subsequent work [BCdT]. Let us sketch these constructions very briefly.

Consider a planar graph G, not necessarily bipartite. To this graph, one can associate two natural bipartite graphs: the *double graph* $\mathsf{G} = G^{\mathrm{D}}$, see e.g. [Ken02], and the graph $\mathsf{G} = G^{\mathrm{Q}}$, see e.g. [WL75], both illustrated in Figure 9. One easily checks that if G is an isoradial graph, then the associated planar, bipartite graphs G^{D} and G^{Q} are minimal, so Fock's dimer models can be defined and studied on these graphs.

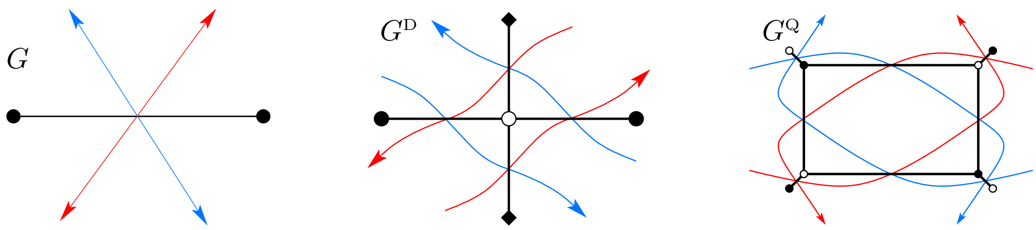

Figure 9: An edge of G with its two adjacent train-tracks (left), and the corresponding parts of $\mathsf{G} = G^{\mathrm{D}}$ (center) and G^{Q} (right) in black lines and white/black vertices, with the four adjacent train-tracks in red and blue lines.

If the M-curve Σ is endowed with a holomorphic involution, any isoradial embedding of G naturally defines a minimal immersion of $\mathsf{G} = G^{\mathrm{D}}$, *i.e.*, an element of X_{G}. The study of the corresponding model can be undertaken using the theory of double (possibly ramified) coverings of Riemann surfaces, as developed for example in [Fay73]. In the ramified case (*i.e.*, when the genus of Σ is even), it can be shown that Fock's Kasteleyn operator on G^{D} is gauge-equivalent to the direct sum of the discrete Laplacian of [Geo19] on G and G^*. In the unramified case, it is a higher odd-genus generalisation of the massive Laplacian of [BdTR17] that appears. These results, together with the study of the resulting Laplace operators and associated Green functions, will be the subject of the upcoming article [BCdT].

As for the dimer models with Fock's weights on minimal graphs of the form $\mathsf{G} = G^{\mathrm{Q}}$, they yield higher genus extensions of the Z-invariant elliptic Ising model of [BdTR19], whose precise nature are yet to be understood and studied.

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