

Dimers on Surface Graphs and Spin Structures. I

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Received: 28 September 2006 / Accepted: 9 January 2007
Published online: 25 July 2007 – © Springer-Verlag 2007

Abstract: Partition functions for dimers on closed oriented surfaces are known to be alternating sums of Pfaffians of Kasteleyn matrices. In this paper, we obtain the formula for the coefficients in terms of discrete spin structures.

1. Introduction

Dimer models on graphs have a long history in statistical mechanics [6, 15]. States in dimer models are perfect matchings between vertices of the graph where only adjacent vertices are matched. The probability of a state is determined by assigning weights to edges.

Dimer models also have many interesting mathematical aspects involving combinatorics, probability theory [11, 3], real algebraic geometry [10, 9], etc.... One of the remarkable facts about dimer models is that the partition function can be written as a linear combination of 2^{2g} Pfaffians of $N \times N$ matrices, where N is the number of vertices in the graph and g the genus of a surface where the graph can be embedded.

The matrices in the Pfaffian formula for the dimer partition function are called Kasteleyn matrices. They involve certain orientations of edges of the graph known as Kasteleyn orientations. Two Kasteleyn orientations are called equivalent if one can be obtained from the other by a sequence of moves reversing orientations of all edges adjacent to a vertex.

The number of non-equivalent Kasteleyn orientations of a surface graph of genus g is 2^{2g} and is equal to the number of non-equivalent spin structures on the surface. The Pfaffian formula expresses the partition function of the dimer model as an alternating sum of Pfaffians of Kasteleyn operators, one for each equivalence class of Kasteleyn orientations. This formula was proved in [6] for the torus, and it was stated in [7] that for other surfaces, the partition function of the dimer model is equal to the sum of 2^{2g} Pfaffians. The formal combinatorial proof of this fact and the exact description of coefficients for all oriented surfaces first appeared in [12] and [16] (see also [4]). A

combinatorial proof of such formula for non-orientable surfaces can also be found in [16].

The partition function of free fermions on a Riemann surface of genus g is also a linear combination of 2^{2g} Pfaffians of Dirac operators. Each term in this sum corresponds to a spin structure [1]. Assuming that dimer models are discretizations of free fermions on Riemann surfaces, one should expect a relation between Kasteleyn orientations and spin structures and between the Kasteleyn operator for a given Kasteleyn orientation and the Dirac operator in the corresponding spinor bundle. Numerical evidence relating the critical dimer model on a square and triangular lattices in the thermodynamical limit with Dirac operators can be found in [2] for $g = 2$.

An explicit construction relating a spin structure on a surface with a Kasteleyn orientation on a graph with dimer configuration was suggested in [11]. Furthermore, for bipartite graphs with critical weights, the Kasteleyn operator can be naturally identified with a discrete version of the Dirac operator [8]. This gives an interesting relation between dimer models and the theory of discrete meromorphic functions [14].

In this paper we investigate further the relation between Kasteleyn orientations and spin structures, and use this relation to give a geometric proof of the Pfaffian formula for closed surfaces. Below is a brief summary of our main results.

Recall some basic notions. A dimer configuration on a graph Γ is a perfect matching on vertices where matched vertices are connected by edges. Given two such configurations D and D' , set $\Delta(D, D') = (D \cup D') \setminus (D \cap D')$. A surface graph is a graph Γ embedded into a surface Σ as the 1-skeleton of a CW-decomposition of Σ . A Kasteleyn orientation of a surface graph is an orientation of edges of the graph, such that the product or relative orientations of boundary edges of each face is negative (see Sect. 4).

One of our results is that any dimer configuration D on a surface graph $\Gamma \subset \Sigma$ induces an isomorphism of affine $H^1(\Sigma; \mathbb{Z}_2)$ -spaces

$$\psi_D: (\mathcal{K}(\Gamma)/\sim) \longrightarrow \mathcal{Q}(H_1(\Sigma; \mathbb{Z}_2), \cdot), \quad [K] \longmapsto q_D^K \tag{1}$$

from the set of equivalence classes of Kasteleyn orientations on $\Gamma \subset \Sigma$ onto the set of quadratic forms on $(H_1(\Sigma; \mathbb{Z}_2), \cdot)$, where \cdot denotes the intersection form on Σ . Furthermore, $\psi_D = \psi_{D'}$ if and only if D and D' are equivalent dimer configurations (that is, $\Delta(D, D')$ is zero in $H_1(\Sigma; \mathbb{Z}_2)$).

Since the affine space of spin structures on Σ is canonically isomorphic to the affine space of such quadratic forms, this establishes an isomorphism between equivalence classes of Kasteleyn orientations and spin structures.

This correspondence implies easily the following identity. Let D_0 be a fixed dimer configuration on a graph Γ . Realize Γ as a surface graph $\Gamma \subset \Sigma$ of genus g . Let K be any Kasteleyn orientation on $\Gamma \subset \Sigma$, and let A^K be the associated Kasteleyn matrix. Then,

$$\text{Pf}(A^K) = \varepsilon^K(D_0) \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} (-1)^{q_{D_0}^K(\alpha)} Z_\alpha(D_0), \tag{2}$$

where $q_{D_0}^K$ is the quadratic form associated to K and D_0 via (1), $\varepsilon^K(D_0)$ is some sign depending on K and D_0 , and

$$Z_\alpha(D_0) = \sum_D w(D),$$

the sum being on all dimer configurations D such that $\Delta(D_0, D)$ is equal to α in $H_1(\Sigma; \mathbb{Z}_2)$.

It follows that the partition function of a dimer model on Γ is given by

$$Z = \frac{1}{2^g} \sum_{[K]} \text{Arf}(q_{D_0}^K) \varepsilon^K(D_0) \text{Pf}(A^K), \tag{3}$$

where the sum is taken over the 2^{2g} equivalence classes of Kasteleyn orientations on $\Gamma \subset \Sigma$, and $\text{Arf}(q_{D_0}^K) = \pm 1$ denotes the Arf invariant of the quadratic form $q_{D_0}^K$. Note that the sign $\text{Arf}(q_{D_0}^K) \varepsilon^K(D_0)$ does not depend on D_0 .

The paper is organized as follows. In Sect. 2, we introduce the dimer model on a graph Γ and define composition cycles. Section 3 deals with dimers on surface graphs $\Gamma \subset \Sigma$ and the definition of an equivalence relation for dimer configurations on surface graphs. In Sect. 4, we recall the definition of a Kasteleyn orientation. We then show that a surface graph $\Gamma \subset \Sigma$ admits such an orientation if and only if the number of vertices of Γ is even. We prove that, in such a case, the set of equivalence classes of Kasteleyn orientations on $\Gamma \subset \Sigma$ is an affine $H^1(\Sigma; \mathbb{Z}_2)$ -space. Finally, we give an algorithmic procedure for the construction of the 2^{2g} non-equivalent Kasteleyn orientations on a given surface graph $\Gamma \subset \Sigma$ of genus g . The core of the paper lies in Sect. 5, where we establish the correspondence (1) stated above. This result is used in Sect. 6 to obtain equations (2) and (3). We also give a formula for the local correlation functions of a dimer model. In the appendix, we collect formulae expressing dimer models in terms of Grassman integrals.

2. The Dimer Model

2.1. Dimer configurations and composition cycles on graphs. Let Γ be a finite connected graph. A *perfect matching* on Γ is a choice of edges of Γ such that each vertex of Γ is adjacent to exactly one of these edges. In statistical mechanics, a perfect matching on Γ is also known as a *dimer configuration* on Γ . The edges of the perfect matching are called *dimers*. An example of a dimer configuration on a graph is given in Fig. 1.

In order to have a perfect matching, a graph Γ clearly needs to have an even number of vertices. However, there are connected graphs with an even number of vertices but no perfect matching. We refer to [13] for combinatorial aspects of matchings. Throughout the paper and unless otherwise stated, we will only consider finite graphs which admit perfect matchings. In particular, all the graphs will have an even number of vertices.

Given two dimer configurations D and D' on a graph Γ , consider the subgraph of Γ given by the symmetric difference $(D \cup D') \setminus (D \cap D')$. The connected components of this subgraph are called (D, D') -*composition cycles* or simply *composition cycles*. Clearly,

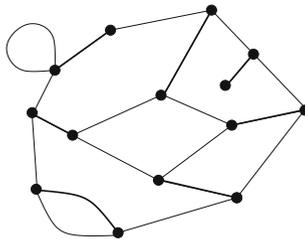


Fig. 1. A dimer configuration on a graph

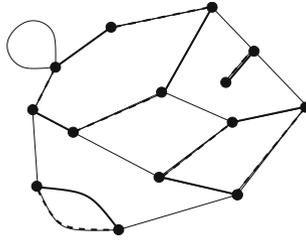


Fig. 2. An example of composition cycles: dimers from D are in solid, and dimers from D' are in traced lines. On this example, there is one (D, D') -composition cycle of length 2, one of length 4, and one of length 6

each composition cycle is a simple closed curve of even length. This is illustrated in Fig. 2, where the two dimer configurations are shown in black and traced lines.

2.2. Edge weight system. Let $\mathcal{D}(\Gamma)$ denote the set of dimer configurations on a graph Γ . A *weight system* on $\mathcal{D}(\Gamma)$ is a positive real-valued function on this set. A weight system w defines a probability distribution on all dimer configurations:

$$\text{Prob}(D) = \frac{w(D)}{Z(\Gamma; w)},$$

where

$$Z(\Gamma; w) = \sum_D w(D)$$

is the partition function. This probability measure is the *Gibbs measure* for the dimer model on the graph Γ with the weight system w .

We shall focus on a particular type of weight system called *edge weight system*. Assign to each edge e of Γ a positive real number $w(e)$, called the *weight* of the edge e . The associated edge weight system on $\mathcal{D}(\Gamma)$ is given by

$$w(D) = \prod_{e \in D} w(e),$$

where the product is over all edges occupied by dimers of D . In statistical mechanics, these weights are called *Boltzmann weights*. Their physical meaning is

$$w(e) = \exp\left(-\frac{E(e)}{T}\right),$$

where $E(e)$ is the energy of dimer occupying the edge e and T is the temperature.

2.3. Local correlation functions. Let e be an edge of Γ . The characteristic function of e is the function σ_e on $\mathcal{D}(\Gamma)$ given by

$$\sigma_e(D) = \begin{cases} 1 & \text{if } e \in D; \\ 0 & \text{otherwise.} \end{cases}$$

The expectation values of products of characteristic functions are called *local correlation functions*, or *dimer-dimer correlation functions*:

$$\langle \sigma_{e_1} \cdots \sigma_{e_k} \rangle = \frac{Z(e_1, \dots, e_k; \Gamma; w)}{Z(\Gamma; w)},$$

where

$$Z(e_1, \dots, e_k; \Gamma; w) = \sum_D \prod_{e \in D} w(e) \prod_{i=1}^k \sigma_{e_i}(D) = \sum_{D \ni e_1, \dots, e_k} w(D).$$

Note that $\langle \sigma_{e_1} \cdots \sigma_{e_k} \rangle = 0$ if some edges $e_i \neq e_j$ share a common vertex. Note also that $\sigma_e \cdot \sigma_e = \sigma_e$. Therefore, it may be assumed that $e_i \neq e_j$ for $i \neq j$.

For any dimer configuration D ,

$$\langle \prod_{e \in D} \sigma_e \rangle = \frac{w(D)}{Z(\Gamma; w)},$$

so we can reconstruct the weight system if we know all local correlation functions. In this sense, local correlation functions carry all the information about the Gibbs measure.

3. Dimers on Surface Graphs

3.1. Surface graphs. Let Σ be a connected oriented closed surface. By a *surface graph*, we mean a graph Γ embedded in Σ as the 1-skeleton of a cellular decomposition X of Σ . We shall assume throughout the paper that the surface Σ is endowed with the counter-clockwise orientation.

Any finite connected graph can be realized as a surface graph. Indeed, such a graph Γ always embeds in a closed oriented surface of genus g , for g sufficiently large. If the genus is minimal, one easily checks that Γ induces a cellular decomposition of the surface Σ .

In this paper we will focus on graphs embedded into a surface of fixed genus.

3.2. Equivalent dimer configurations. A dimer configuration on a surface graph $\Gamma \subset \Sigma$ is simply a dimer configuration on the graph Γ . Such a dimer configuration can be regarded as a 1-chain in the cellular chain complex of X with \mathbb{Z}_2 -coefficients:

$$c_D = \sum_{e \in D} e \in C_1(X; \mathbb{Z}_2).$$

By definition, $\partial c_D = \sum_{v \in \Gamma} v \in C_0(X; \mathbb{Z}_2)$, the sum being on all vertices v of Γ . Therefore, given any pair of dimer configurations D and D' on $\Gamma \subset \Sigma$, $c_D + c_{D'}$ is a 1-cycle:

$$\partial(c_D + c_{D'}) = \partial c_D + \partial c_{D'} = \sum_{v \in \Gamma} (v + v) = 0 \in C_0(X; \mathbb{Z}_2).$$

This 1-cycle is nothing but the union of all (D, D') -composition cycles. Let $\Delta(D, D')$ denote its homology class in $H_1(X; \mathbb{Z}_2) = H_1(\Sigma; \mathbb{Z}_2)$. We shall say that two dimer configurations D and D' are *equivalent* if $\Delta(D, D') = 0$ in $H_1(\Sigma; \mathbb{Z}_2)$.

Note that these concepts make perfect sense when Γ is the 1-skeleton of any CW-complex, not necessarily the cellular decomposition of an oriented closed surface.

4. Kasteleyn Orientations on Surface Graphs

Let $\Gamma \subset \Sigma$ be a surface graph. The counter-clockwise orientation of Σ induces an orientation on each 2-cell, or *face* of X . An orientation K of the edges of Γ is called a *Kasteleyn orientation* if for each face f of X ,

$$\prod_{e \in \partial f} \varepsilon_f^K(e) = -1, \tag{4}$$

where the product is taken over all boundary edges of f , and

$$\varepsilon_f^K(e) = \begin{cases} 1 & \text{if } e \text{ is oriented by } K \text{ as the oriented boundary of the face } f; \\ -1 & \text{otherwise.} \end{cases}$$

This is illustrated in Fig. 3.

Define the operation of *orientation changing at a vertex* as the one which flips the orientation of all the edges adjacent to this vertex, as illustrated in Fig. 4. It is clear that such an operation brings a Kasteleyn orientation to a Kasteleyn orientation. Let us say that two Kasteleyn orientations are *equivalent* if they are obtained one from the other by a sequence of orientation changes at vertices.

4.1. Existence of a Kasteleyn orientation.

Theorem 1. *There exists a Kasteleyn orientation on a surface graph $\Gamma \subset \Sigma$ if and only if the number of vertices of Γ is even.*

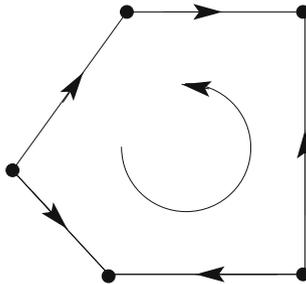


Fig. 3. A Kasteleyn orientation on the boundary edges of a face

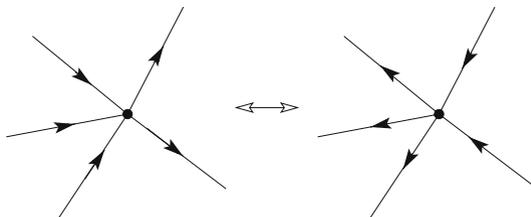


Fig. 4. Orientation change at a vertex

Proof. Let ω be any orientation of the edges of Γ , and let $c_\omega \in C^2(X; \mathbb{Z}_2)$ be defined by the equation

$$(-1)^{c_\omega(f)} = - \prod_{e \in \partial f} \varepsilon_f^\omega(e).$$

Note that ω is Kasteleyn if and only if $c_\omega = 0$. Let V , E and F denote the number of vertices, edges and faces in X , respectively. Since Σ is closed, its Euler characteristic is even, and we get the equality mod 2,

$$0 = \chi(\Sigma) = V + E + F = V + \sum_{f \in F} c_\omega(f).$$

Here, each edge e contributes to the number $c_\omega(f)$, where f is the face whose oriented boundary contains e with the orientation opposite to ω . Therefore, the number of faces f such that $c_\omega(f) = 1$ has the parity of V . Hence, if V is even, then $c_\omega(f) = 1$ for an even number of faces, so c_ω is a 2-coboundary. In other words, $c_\omega = \delta\sigma$ for some $\sigma \in C^1(X; \mathbb{Z}_2)$. Let K be the orientation of the edges of Γ which agrees with ω on e if and only if $\sigma(e) = 0$. Clearly, $c_K = 0$, so K is a Kasteleyn orientation. Conversely, let us assume that there is a Kasteleyn orientation K . This means that $c_K(f) = 1$ for none of the faces. By the argument above, V is even. \square

A more constructive proof of this result will be given in Sect. 4.3.

4.2. Uniqueness of Kasteleyn orientations. Let V be a vector space. Recall that an *affine V -space* is a set S endowed with a map $S \times S \rightarrow V$, $(a, b) \mapsto a - b$ such that:

- i. for every a, b and c in S , we have $(a - b) + (b - c) = a - c$;
- ii. for every b in S , the map $S \rightarrow V$ given by $a \mapsto a - b$ is a bijection.

In other words, an affine V -space is a *V -torsor*: it is a set endowed with a freely transitive action of the abelian group V .

Theorem 2. *Let $\Gamma \subset \Sigma$ be a surface graph with an even number of vertices. Then, the set of equivalence classes of Kasteleyn orientations on $\Gamma \subset \Sigma$ is an affine $H^1(\Sigma; \mathbb{Z}_2)$ -space.*

Corollary 1. *There are exactly 2^{2g} equivalence classes of Kasteleyn orientations on $\Gamma \subset \Sigma$, where g denotes the genus of Σ .*

Proof. Let $\mathcal{K}(\Gamma)$ denote the set of Kasteleyn orientations on $\Gamma \subset \Sigma$. By Theorem 1, it is non-empty. Consider the map $\vartheta : \mathcal{K}(\Gamma) \times \mathcal{K}(\Gamma) \rightarrow C^1(X; \mathbb{Z}_2)$ given by $\vartheta_{K, K'}(e) = 0$ if K and K' agree on the edge e , and $\vartheta_{K, K'}(e) = 1$ otherwise. Since K and K' are Kasteleyn orientations,

$$(-1)^{\vartheta_{K, K'}(\partial f)} = \prod_{e \in \partial f} (-1)^{\vartheta_{K, K'}(e)} = \prod_{e \in \partial f} \varepsilon_f^K(e) \cdot \prod_{e \in \partial f} \varepsilon_f^{K'}(e) = (-1)(-1) = 1$$

for any face f . Therefore, $\delta\vartheta_{K, K'}(f) = \vartheta_{K, K'}(\partial f) = 0$, that is, $\vartheta_{K, K'}$ is a 1-cocycle. Thus, we get a map $\mathcal{K}(\Gamma) \times \mathcal{K}(\Gamma) \xrightarrow{\vartheta} H^1(\Sigma; \mathbb{Z}_2)$. Note that $\vartheta_{K, K'} + \vartheta_{K', K''} = \vartheta_{K, K''}$ for any $K, K', K'' \in \mathcal{K}(\Gamma)$. Also, one easily checks that $\vartheta_{K, K'} = 0$ in $H^1(\Sigma; \mathbb{Z}_2)$ if

and only if there is a sequence of vertices such that K is obtained from K' by reversing the orientation around all these vertices, i.e., if and only if $K \sim K'$. It follows that we have a map

$$(\mathcal{K}(\Gamma)/\sim) \times (\mathcal{K}(\Gamma)/\sim) \longrightarrow H^1(\Sigma; \mathbb{Z}_2), \quad ([K], [K']) \mapsto [K] - [K'] := [\vartheta_{K,K'}]$$

such that for any K' in $\mathcal{K}(\Gamma)$, the map $(\mathcal{K}(\Gamma)/\sim) \rightarrow H^1(\Sigma; \mathbb{Z}_2)$ given by $[K] \mapsto [K] - [K']$ is injective. Finally, let us check that this map is onto. Fix a class in $H^1(\Sigma; \mathbb{Z}_2)$, and represent it by some 1-cycle $\sigma \in Z^1(X; \mathbb{Z}_2)$. Let K be the same orientation as K' whenever $\sigma(e) = 0$, and the opposite when $\sigma(e) = 1$. Obviously, $\vartheta_{K,K'} = \sigma$. Furthermore, K is Kasteleyn since K' is and $\delta\sigma = 0$. Indeed, given a face f ,

$$0 = (\delta\sigma)(f) = \sigma(\partial f) = \vartheta_{K,K'}(\partial f),$$

so $1 = \prod_{e \in \partial f} (-1)^{\vartheta_{K,K'}(e)} = (-1) \prod_{e \in \partial f} \varepsilon_f^K(e)$. This concludes the proof. \square

4.3. How to construct Kasteleyn orientations. Given a surface graph $\Gamma \subset \Sigma$ with an even number of vertices, we know that there are exactly 2^{2g} non-equivalent Kasteleyn orientations on Γ . However, the proof given above is not really constructive. For this reason, we now give an algorithm for the construction of these Kasteleyn orientations. The successive steps of the algorithm are illustrated in Fig. 5.

0. Let $\Gamma \subset \Sigma$ be a surface graph with an even number of vertices, and let g denote the genus of Σ .
1. Consider a system $\alpha = \alpha_1 \cup \dots \cup \alpha_{2g}$ of simple closed curves on Γ such that Σ cut along α is a 2-disc Σ' . (Note that such curves exist since Γ induces a cellular decomposition of Σ .)
2. The surface graph $\Gamma \subset \Sigma$ induces a graph $\Gamma' \subset \Sigma'$ which is the 1-skeleton of a cellular decomposition of the 2-sphere S^2 . Fix a spanning tree T of the graph dual to the surface graph $\Gamma' \subset S^2$, rooted at the vertex corresponding to the face $S^2 \setminus \Sigma'$.

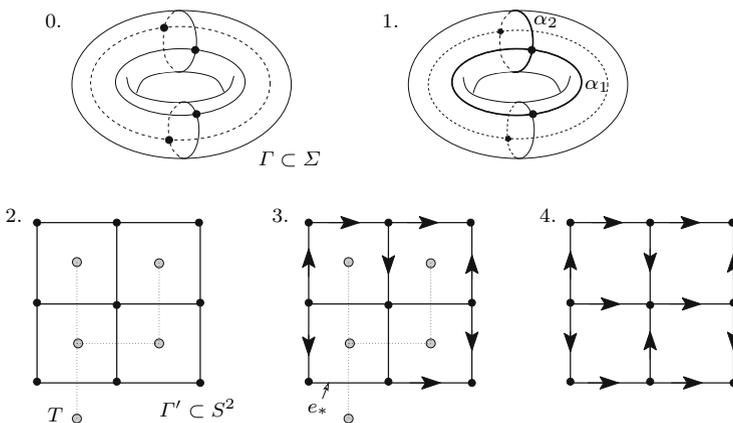


Fig. 5. An example of the explicit construction of a Kasteleyn orientation

3. Orient the edges of Γ' which do not intersect T respecting the following condition: whenever two edges of Γ' in $\partial \Sigma'$ are identified in Σ , their orientations agree. Now, every edge in $\partial \Sigma'$ is oriented, except one. Let us denote it by e_* .
4. Orient edges of Γ' such that the Kasteleyn condition (4) holds for the faces corresponding to the leaves of T . Moving down the tree (from the leaves to the root), orient each crossing edge such that the Kasteleyn condition holds for all faces left behind.

This gives a Kasteleyn orientation of $\Gamma' \subset \Sigma'$. By the condition in step 3, it induces a Kasteleyn orientation on $\Gamma \subset \Sigma$, provided that the orientation of the last edge e_* satisfies this condition. It turns out to be the case if and only if the number of vertices of Γ is even. (This is an easy consequence of Theorem 1.) Therefore, we have constructed a Kasteleyn orientation K on $\Gamma \subset \Sigma$. To obtain the 2^{2g} non-equivalent ones, proceed as follows.

5. Consider a family of simple closed curves $\beta_1, \dots, \beta_{2g}$ on Σ avoiding the vertices of Γ , and forming a basis of $H_1(\Sigma; \mathbb{Z}_2)$.
6. Consider the Kasteleyn orientation K , and some subset $I \subset \{1, \dots, 2g\}$. For all $i \in I$, change the orientation of all the edges in Γ that intersect β_i .

The resulting orientation K_I is clearly Kasteleyn, as $\partial f \cdot \beta_i$ is even for every face f and index i . Furthermore, one easily checks that K_I and K_J are non-equivalent if $I \neq J$. Hence, we have constructed the 2^{2g} non-equivalent Kasteleyn orientations on $\Gamma \subset \Sigma$.

5. Kasteleyn Orientations as Discrete Spin Structures

We saw in Corollary 1 that there are exactly 2^{2g} non-equivalent Kasteleyn orientations on a surface graph $\Gamma \subset \Sigma$, where g denotes the genus of Σ . It is known that this is also the number of non-equivalent spin structures on Σ . This relation between the number of Kasteleyn orientations and the number of spin structures is not accidental. Kasteleyn orientations of surface graphs can be regarded as discrete versions of spin structures. This statement will be made precise in the present section. (See in particular Corollary 3.)

5.1. The quadratic form associated to a Kasteleyn orientation. Let V be a finite dimensional vector space over the field \mathbb{Z}_2 , and let $\varphi: V \times V \rightarrow \mathbb{Z}_2$ be a fixed bilinear form. Recall that a function $q: V \rightarrow \mathbb{Z}_2$ is a *quadratic form* on (V, φ) if

$$q(x + y) = q(x) + q(y) + \varphi(x, y)$$

for all $x, y \in V$. Note that the difference (that is, the sum) of two quadratic forms on (V, φ) is a linear form on V . Therefore, one easily checks that the set $\mathcal{Q}(V, \varphi)$ of quadratic forms on (V, φ) is an affine V^* -space, where V^* denotes the dual of V .

Fix a Kasteleyn orientation K on a surface graph $\Gamma \subset \Sigma$. Given an oriented simple closed curve C on Γ , set

$$\varepsilon_C^K(C) = \prod_{e \in C} \varepsilon_C^K(e),$$

where $\varepsilon_C^K(e)$ is equal to $+1$ (resp. -1) if the orientations on the edge e given by C and K agree (resp. do not agree). For a fixed dimer configuration D on Γ , let $\ell_D(C)$ denote the number of vertices v in C whose adjacent dimer of D sticks out to the left of C in Σ .

Theorem 3. *Given a class $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$, represent it by oriented simple closed curves C_1, \dots, C_m in Γ . If K is a Kasteleyn orientation on $\Gamma \subset \Sigma$, then the function $q_D^K: H_1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ given by*

$$(-1)^{q_D^K(\alpha)} = (-1)^{\sum_{i < j} C_i \cdot C_j} \prod_{i=1}^m (-\varepsilon^K(C_i)) (-1)^{\ell_D(C_i)}$$

is a well-defined quadratic form on $(H_1(\Sigma; \mathbb{Z}_2), \cdot)$, where \cdot denotes the intersection form.

We postpone the proof of this result to the next subsections. Let us first investigate some of its consequences.

Proposition 1. (i) *Let D be a fixed dimer configuration on Γ . If K and K' are two Kasteleyn orientations on $\Gamma \subset \Sigma$, then $q_D^K - q_D^{K'}$ maps to $[K] - [K']$ via the canonical isomorphism $\text{Hom}(H_1(\Sigma; \mathbb{Z}_2); \mathbb{Z}_2) = H^1(\Sigma; \mathbb{Z}_2)$.*

(ii) *Let K be a fixed Kasteleyn orientation on $\Gamma \subset \Sigma$. If D and D' are two dimer configurations on Γ , then $q_D^K - q_{D'}^K \in \text{Hom}(H_1(\Sigma; \mathbb{Z}_2); \mathbb{Z}_2)$ is given by $\alpha \mapsto \alpha \cdot \Delta(D, D')$.*

Proof. Let C be a simple closed curve in Γ representing a class α in $H_1(\Sigma; \mathbb{Z}_2)$. By definition,

$$(-1)^{(q_D^K - q_{D'}^K)(\alpha)} = \prod_{e \in C} \varepsilon_C^K(e) \prod_{e \in C} \varepsilon_C^{K'}(e) = \prod_{e \in C} (-1)^{\vartheta_{K, K'}(e)} = (-1)^{\vartheta_{K, K'}(C)}.$$

This proves the first point. To check the second one, observe that

$$(-1)^{(q_D^K - q_{D'}^K)(\alpha)} = (-1)^{\ell_D(C) + \ell_{D'}(C)}.$$

Clearly, $\ell_D(C) + \ell_{D'}(C) \equiv C \cdot \Delta(D, D') \pmod{2}$, giving the proposition. \square

Corollary 2. *Any dimer configuration D on a surface graph $\Gamma \subset \Sigma$ induces an isomorphism of affine $H^1(\Sigma; \mathbb{Z}_2)$ -spaces*

$$\psi_D: (\mathcal{K}(\Gamma) / \sim) \longrightarrow \mathcal{Q}(H_1(\Sigma; \mathbb{Z}_2), \cdot), \quad [K] \longmapsto q_D^K$$

from the set of equivalence classes of Kasteleyn orientations on $\Gamma \subset \Sigma$ onto the set of quadratic forms on $(H_1(\Sigma; \mathbb{Z}_2), \cdot)$. Furthermore, $\psi_D = \psi_{D'}$ if and only if D and D' are equivalent dimer configurations.

Proof. The first part of Proposition 1 exactly states that ψ_D is an isomorphism of affine $H^1(\Sigma; \mathbb{Z}_2)$ -spaces. By the second part, $\psi_D = \psi_{D'}$ if and only if the homomorphism $H_1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ given by the intersection with $\Delta(D, D')$ is zero. By Poincaré duality, this is the case if and only if $\Delta(D, D') = 0$. \square

5.2. *Spin structures on surfaces.* Recall that the fundamental group of $SO(n)$ is infinite cyclic if $n = 2$ and cyclic of order 2 if $n \geq 3$. Hence, $SO(n)$ admits a canonical 2-fold cover, denoted by $Spin(n) \rightarrow SO(n)$.

Let M be an oriented n -dimensional Riemannian manifold, and let $P_{SO} \rightarrow M$ be the principal $SO(n)$ -bundle associated to its tangent bundle. A *spin structure on M* is a principal $Spin(n)$ -bundle $P \rightarrow M$ together with a 2-fold covering map $P \rightarrow P_{SO}$ which restricts to the covering map $Spin(n) \rightarrow SO(n)$ on each fiber. Equivalently, a spin structure on M is a cohomology class $\xi \in H^1(P_{SO}; \mathbb{Z}_2)$ whose restriction to each fiber F gives the generator of the cyclic group $H^1(F; \mathbb{Z}_2)$.

It is well-known that such a spin structure exists if and only if the second Stiefel-Whitney class of M vanishes. In such a case, the set $\mathcal{S}(M)$ of spin structures on M is endowed with a natural structure of affine $H^1(M; \mathbb{Z}_2)$ -space.

The 2-dimensional case is particularly easy to deal with for several reasons. First of all, any compact orientable surface Σ admits a spin structure, as its second Stiefel-Whitney class is always zero. Furthermore, spin structures can be constructed using a certain class a vector fields, that we now describe.

Let f be a non-vanishing vector field on $\Sigma \setminus \sigma$, where σ is some finite subset of Σ . Recall that the *index of the singularity $x \in \sigma$* of f is defined as the degree of the circle map $t \mapsto f(\gamma_x(t))/|f(\gamma_x(t))|$, where $\gamma_x: S^1 \rightarrow \Sigma \setminus \sigma$ is a (counter-clockwise) parametrization of a simple closed curve separating x from the other singularities of f . Let us denote by $\mathcal{V}_{ev}(\Sigma)$ the set of vector fields on Σ with only even index singularities. We claim that any such vector field f defines a spin structure ξ_f on Σ . Indeed, consider a 1-cycle c in P_{SO} (that is, a closed framed curve in Σ) and let us assume that c avoids the singularities of f . Then, let $\xi_f(c) \in \mathbb{Z}_2$ be the winding number modulo 2 of f along c with respect to the framing of c . Since all the singularities of f have even index, $\xi_f(c) = 0$ if c is a 1-boundary, and $\xi_f(c) = 1$ if c is a small simple closed curve with tangential framing. Therefore, it induces a well-defined cohomology class $\xi_f \in \text{Hom}(H_1(P_{SO}; \mathbb{Z}_2); \mathbb{Z}_2) = H^1(P_{SO}; \mathbb{Z}_2)$ which restricts to the generator of the cohomology of the fibers. So ξ_f is a spin structure on Σ , and we have a map

$$\mathcal{V}_{ev}(\Sigma) \longrightarrow \mathcal{S}(\Sigma), \quad f \longmapsto \xi_f.$$

We shall need one last result about spin structures on surfaces, due to D. Johnson [5]. Given a spin structure $\xi \in \mathcal{S}(\Sigma)$, let $q_\xi: H_1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ be the function defined as follows. Represent $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$ by a collection of disjoint regular simple closed curves $\gamma_1, \dots, \gamma_m: S^1 \hookrightarrow \Sigma$. For all i and all $t \in S^1$, complete the unit tangent vector $\dot{\gamma}_i(t)/|\dot{\gamma}_i(t)|$ to a positive orthonormal basis of $T_{\gamma_i(t)}\Sigma$. This gives disjoint framed closed curves in Σ , that is, a 1-cycle c in P_{SO} . Set $q_\xi(\alpha) = \xi(c) + m$. Johnson's theorem asserts that q_ξ is a well-defined quadratic form on $(H_1(\Sigma; \mathbb{Z}_2), \cdot)$, where \cdot denotes the intersection form. Furthermore, the map

$$\mathcal{S}(\Sigma) \longrightarrow \mathcal{Q}(H_1(\Sigma; \mathbb{Z}_2), \cdot), \quad \xi \longmapsto q_\xi$$

is an isomorphism of affine $H^1(\Sigma; \mathbb{Z}_2)$ -spaces.

5.3. *Proof of Theorem 3.* Let $\Gamma \subset \Sigma$ be a surface graph, with Σ counter-clockwise oriented. Given a Kasteleyn orientation K and a dimer configuration D on $\Gamma \subset \Sigma$, Kuperberg [11] constructs a vector field $f(K, D) \in \mathcal{V}_{ev}(\Sigma)$ as follows. Around each vertex of Γ , make the vectors point to the vertex. At the middle of each edge, make the

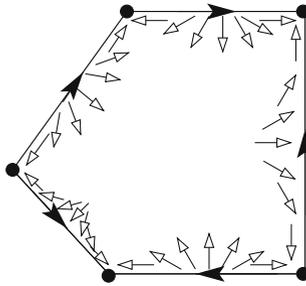


Fig. 6. Kuperberg's construction

vector point 90 degrees clockwise relative to the orientation K of the edge. Extend this continuously to the whole edges, as described in Fig. 6.

The Kasteleyn condition (4) ensures that the vector field extends to the faces with one singularity of even index in the interior of each face. However, this vector field $\tilde{f}(K)$ has an odd index singularity at each vertex of Γ . This is where the dimer configuration D enters the game: contract the odd index singularities in pairs along the dimers of D . The resulting vector field $f(K, D)$ has even index singularities: one in the interior of each face of Σ , and one in the middle of each dimer of D .

Gathering the results of the previous section and Kuperberg's construction, we get the following composition of maps:

$$\mathcal{K}(\Gamma) \times \mathcal{D}(\Gamma) \longrightarrow \mathcal{V}_{ev}(\Sigma) \longrightarrow \mathcal{S}(\Sigma) \longrightarrow \mathcal{Q}(H_1(\Sigma; \mathbb{Z}_2), \cdot), \quad (K, D) \longmapsto q_{\xi_{f(K,D)}}.$$

We are left with the proof that, given any Kasteleyn orientation K and dimer configuration D , the resulting quadratic form $q = q_{\xi_{f(K,D)}}$ coincides with the function q_D^K defined in the statement of Theorem 3. So, given $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$, represent it by a collection of simple closed curves C_1, \dots, C_m in Γ . (This is always possible as Γ induces a cellular decomposition of Σ .) Since q is a quadratic form and $\alpha = \sum_{i=1}^m [C_i]$,

$$(-1)^{q(\alpha)} = (-1)^{\sum_{i < j} C_i \cdot C_j} \prod_{i=1}^m (-1)^{q([C_i])}.$$

Therefore, we just need to check that if C is an oriented simple closed curve in Γ , then $(-1)^{q([C])+1} = \varepsilon^K(C)(-1)^{\ell_D(C)}$. Consider the oriented regular curve γ in Σ which follows C slightly on its left, and goes around the middle of each dimer it meets, except if it meets the same dimer twice. In this case, γ stays close to C , as illustrated in Fig. 7. Clearly, γ is a regular oriented simple closed curve in Σ . Furthermore, it is homologous to C and it avoids all the singularities of $f(K, D)$. We now have to check that the winding number ω of $f(K, D)$ along γ with respect to its tangential framing satisfies $(-1)^\omega = \varepsilon^K(C)(-1)^{\ell_D(C)}$. One easily checks that ω is equal (mod 2) to the winding number ω_0 of $f(K, D)$ along γ_0 , where γ_0 is the regular curve which goes around the middle of each dimer it meets, including the ones it meets twice. By construction of $f(K, D)$, ω_0 is equal to the winding number $\tilde{\omega}$ of $\tilde{f}(K)$ along $\tilde{\gamma}$, where $\tilde{\gamma}$ is the regular curve which avoids all the dimers of D and all the vertices of Γ , as described in Fig. 7.

The latter winding number $\tilde{\omega}$ can be computed locally by cutting $\tilde{\gamma}$ into pieces: one piece $\tilde{\gamma}_e$ for each edge e of C , and one piece $\tilde{\gamma}_v$ for each vertex v of Γ adjacent to a

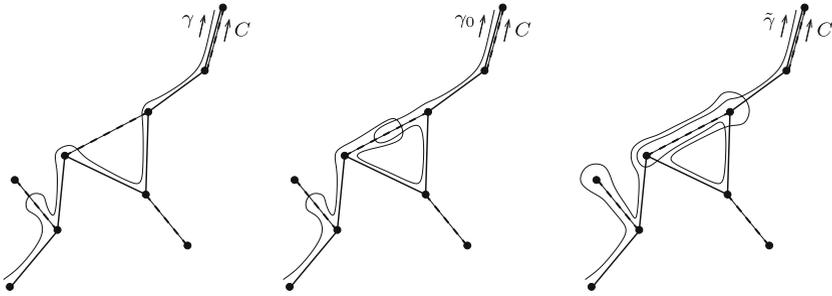


Fig. 7. The oriented simple closed curve $C \subset \Gamma$ and its associated regular curves γ , γ_0 and $\tilde{\gamma}$. The curve C is in solid, the dimers are in traced lines

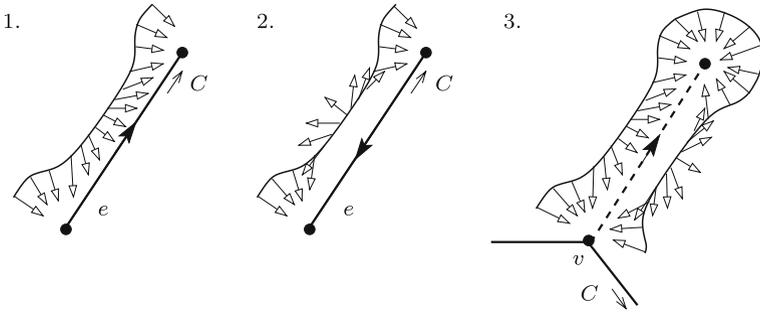


Fig. 8. Computation of the local winding numbers

dimer of D sticking out to the left of C . The theorem now follows from the following case study.

1. Let us assume that e is an edge of C such that $\varepsilon_C^K(e) = +1$. In this case, the vector field $\tilde{f}(K)$ along $\tilde{\gamma}_e$ in the tangential framing of $\tilde{\gamma}_e$ defines a curve which is homotopically trivial, as illustrated in Fig. 8. Hence, its contribution to $\tilde{\omega}$ is null.

2. Consider now the case of an edge e of C such that $\varepsilon_C^K(e) = -1$. This time, $\tilde{f}(K)$ along $\tilde{\gamma}_e$ defines a simple close curve around the origin (see Fig. 8). Its contribution to $\tilde{\omega}$ is equal to 1 (mod 2).

3. Let v be a vertex of C with a dimer of D sticking out of v to the left of C . Then, the vector field $\tilde{f}(K)$ along $\tilde{\gamma}_v$ induces a simple closed curve around the origin, so its contribution to $\tilde{\omega}$ is equal to 1 (mod 2). The case illustrated in Fig. 8 is when K orients the dimer from v to its other boundary vertex. The other case is similar.

Gathering all the pieces, the winding number of $\tilde{f}(K)$ along $\tilde{\gamma}$ is equal to $\prod_{e \in C} \varepsilon_C^K(e) (-1)^{\ell_D(C)}$. This concludes the proof of Theorem 3. \square

Using Johnson’s theorem, we have the following immediate consequence of Corollary 2.

Corollary 3. Any dimer configuration D on a surface graph $\Gamma \subset \Sigma$ induces an isomorphism of affine $H^1(\Sigma; \mathbb{Z}_2)$ -spaces

$$\psi_D: (\mathcal{K}(\Gamma)/ \sim) \longrightarrow \mathcal{S}(\Sigma)$$

from the set of equivalence classes of Kasteleyn orientations on $\Gamma \subset \Sigma$ onto the set of spin structures on Σ . Furthermore, $\psi_D = \psi_{D'}$ if and only if D and D' are equivalent dimer configurations.

6. Pfaffian Formulae for the Partition Function and Correlation Functions

6.1. *The Kasteleyn matrix and its Pfaffian.* Let K be a Kasteleyn orientation on a surface graph $\Gamma \subset \Sigma$ with an even number of vertices. Enumerate these vertices by $1, 2, \dots, N = 2n$. The *Kasteleyn matrix* is the $2n \times 2n$ skew-symmetric matrix $A^K(\Gamma; w) = A^K$ whose entry a_{ij}^K is the total weight of all edges from i to j minus the total weight of all edges from j to i . More formally,

$$a_{ij}^K = \sum_e \varepsilon_{ij}^K(e)w(e),$$

where the sum is on all edges e in Γ between the vertices i and j , $\varepsilon_{ij}^K(e) = 0$ if $i = j$, and

$$\varepsilon_{ij}^K(e) = \begin{cases} 1 & \text{if } e \text{ is oriented by } K \text{ from } i \text{ to } j; \\ -1 & \text{otherwise,} \end{cases}$$

if $i \neq j$.

Consider a dimer configuration D on Γ given by edges e_1, \dots, e_n matching vertices i_ℓ and j_ℓ for $\ell = 1, \dots, n$. It determines an equivalence class of permutations $\sigma: (1, \dots, 2n) \mapsto (i_1, j_1, \dots, i_n, j_n)$ with respect to permutations of pairs (i_ℓ, j_ℓ) and transpositions $(i_\ell, j_\ell) \mapsto (j_\ell, i_\ell)$. We will write this as $\sigma \in D$. Given such a permutation σ , define

$$\varepsilon^K(D) = (-1)^\sigma \prod_{\ell=1}^n \varepsilon_{i_\ell j_\ell}^K(e_\ell),$$

where $(-1)^\sigma$ denotes the sign of the permutation σ . Note that this expression does not depend on the choice of $\sigma \in D$, but only on the dimer configuration D .

Theorem 4. *Let $\Gamma \subset \Sigma$ be a surface graph with an even number of vertices. For any dimer configuration D_0 on Γ and any Kasteleyn orientation K on $\Gamma \subset \Sigma$,*

$$\varepsilon^K(D_0)Pf(A^K) = \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} (-1)^{q_{D_0}^K(\alpha)} Z_\alpha(D_0), \tag{5}$$

where $q_{D_0}^K$ is the quadratic form on $H_1(\Sigma; \mathbb{Z}_2)$ associated to K and D_0 , and

$$Z_\alpha(D_0) = \sum_D w(D),$$

the sum being on all dimer configurations D such that $\Delta(D_0, D) = \alpha$.

Proof. Recall that the Pfaffian of a skew-symmetric matrix $A = (a_{ij})$ of size $2n$ is given by

$$\text{Pf}(A) = \sum_{[\sigma] \in \Pi} (-1)^\sigma a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)},$$

where the sum is on the set Π of matchings of $\{1, \dots, 2n\}$. Therefore,

$$\begin{aligned} \text{Pf}(A^K) &= \sum_{[\sigma] \in \Pi} (-1)^\sigma a_{\sigma(1)\sigma(2)}^K \cdots a_{\sigma(2n-1)\sigma(2n)}^K \\ &= \sum_{[\sigma] \in \Pi} (-1)^\sigma \sum_{e_1} \varepsilon_{\sigma(1)\sigma(2)}^K(e_1) w(e_1) \cdots \sum_{e_n} \varepsilon_{\sigma(2n-1)\sigma(2n)}^K(e_n) w(e_n) \\ &= \sum_D (-1)^\sigma \prod_{\ell=1}^n \varepsilon_{\sigma(2\ell-1)\sigma(2\ell)}^K(e_\ell) w(e_\ell) \\ &= \sum_D \varepsilon^K(D) w(D), \end{aligned}$$

where the sum is on all dimer configurations D on Γ . Hence,

$$\varepsilon^K(D_0) \text{Pf}(A^K) = \sum_D \varepsilon^K(D_0) \varepsilon^K(D) w(D).$$

Let us denote by e_1, \dots, e_n the edges of Γ occupied by dimers of D , and by e_1^0, \dots, e_n^0 the edges occupied by dimers of D_0 . Fix permutations σ and τ representing the dimer configurations D and D_0 , respectively, and set $\nu = \tau \circ \sigma^{-1}$. By definition,

$$\begin{aligned} \varepsilon^K(D_0) \varepsilon^K(D) &= (-1)^\tau \prod_{\ell=1}^n \varepsilon_{\tau(2\ell-1)\tau(2\ell)}^K(e_\ell^0) \cdot (-1)^\sigma \prod_{\ell=1}^n \varepsilon_{\sigma(2\ell-1)\sigma(2\ell)}^K(e_\ell) \\ &= (-1)^\nu \prod_{\ell=1}^n \varepsilon_{\nu(\sigma(2\ell-1))\nu(\sigma(2\ell))}^K(e_\ell^0) \varepsilon_{\sigma(2\ell-1)\sigma(2\ell)}^K(e_\ell). \end{aligned}$$

Note that the permutation ν depends on the choice of $\sigma \in D$ and $\tau \in D_0$, but it always brings the perfect matching D to D_0 . Moreover, one can choose representatives $\sigma \in D$ and $\tau \in D_0$ such that ν is the counter-clockwise rotation by one edge of every (D_0, D) -composition cycle C_1, \dots, C_m . For this particular choice of representatives, we have

$$\varepsilon^K(D_0) \varepsilon^K(D) = (-1)^{\sum_{i=1}^m (\text{length}(C_i)+1)} \prod_{i=1}^m \varepsilon^K(C_i) = \prod_{i=1}^m (-\varepsilon^K(C_i)).$$

Here, we use the fact that the length of a permutation cycle is the length of the corresponding composition cycle, and that the length of each composition cycle is even. Recall the quadratic form $q_{D_0}^K$ of Theorem 3. Since the C_i 's are disjoint (D_0, D) -composition cycles, $C_i \cdot C_j = 0$ and $\ell_{D_0}(C_i) = 0$ for all i, j . Therefore,

$$\prod_{i=1}^m (-\varepsilon^K(C_i)) = (-1)^{q_{D_0}^K(\Delta(D_0, D))}.$$

The theorem follows. \square

6.2. *The partition function.* Given a dimer configuration D_0 , Theorem 4 provides 2^{2g} linear equations (one for each equivalence class of Kasteleyn orientation) with 2^{2g} unknowns (the functions $Z_\alpha(D_0)$). We want to use these equations to express the dimer partition function $Z = \sum_\alpha Z_\alpha(D_0)$ in terms of Pfaffians.

Let $V \times V \rightarrow \mathbb{Z}_2$, $(\alpha, \beta) \mapsto \alpha \cdot \beta$ be a non-degenerate bilinear form on a \mathbb{Z}_2 -vector space V . Recall that the Arf invariant of a quadratic form $q: V \rightarrow \mathbb{Z}_2$ on (V, \cdot) is given by

$$\text{Arf}(q) = \frac{1}{|V|} \sum_{\alpha \in V} (-1)^{q(\alpha)}.$$

Lemma 1. *Let q, q' be two quadratic forms on (V, \cdot) . Then,*

$$\text{Arf}(q)\text{Arf}(q') = (-1)^{q(\Delta)} = (-1)^{q'(\Delta)},$$

where $\Delta \in V$ satisfies $(q + q')(\alpha) = \Delta \cdot \alpha$ for all $\alpha \in V$.

Proof. First note that $q + q'$ is a linear form on V . Since the bilinear form \cdot is non-degenerate, there exists $\Delta \in V$ such that $(q + q')(\alpha) = \alpha \cdot \Delta$ for all $\alpha \in V$. Furthermore, $(q + q')(\Delta) = \Delta \cdot \Delta = 0$, so $(-1)^{q(\Delta)} = (-1)^{q'(\Delta)}$. Let us now compute the product of the Arf invariants:

$$\text{Arf}(q)\text{Arf}(q') = \frac{1}{|V|} \sum_{\alpha, \beta \in V} (-1)^{q(\alpha) + q'(\beta)} = \frac{1}{|V|} \sum_{\alpha, \beta \in V} (-1)^{q(\alpha) + q'(\beta + \Delta)}.$$

Using the equality

$$\begin{aligned} q(\alpha) + q'(\beta + \Delta) &= q(\alpha) + q'(\beta) + q'(\Delta) + \beta \cdot \Delta \\ &= q(\alpha) + q(\beta) + q'(\Delta) = q(\alpha + \beta) + \alpha \cdot \beta + q(\Delta), \end{aligned}$$

we obtain

$$\begin{aligned} \text{Arf}(q)\text{Arf}(q') &= \frac{(-1)^{q(\Delta)}}{|V|} \sum_{\alpha, \beta} (-1)^{q(\alpha + \beta) + \alpha \cdot \beta} \\ &= \frac{(-1)^{q(\Delta)}}{|V|} \left(\sum_{\alpha} 1 + \sum_{\alpha \neq \beta} (-1)^{q(\alpha + \beta) + \alpha \cdot \beta} \right). \end{aligned}$$

We are left with the proof that the latter sum is zero:

$$\sum_{\alpha \neq \beta} (-1)^{q(\alpha + \beta) + \alpha \cdot \beta} = \sum_{\gamma \neq 0} (-1)^{q(\gamma)} \sum_{\alpha + \beta = \gamma} (-1)^{\alpha \cdot \beta} = \sum_{\gamma \neq 0} (-1)^{q(\gamma)} (n_\gamma^0 - n_\gamma^1),$$

where n_γ^i is the cardinality of the set

$$N_\gamma^i = \{(\alpha, \beta) \in V \times V \mid \alpha + \beta = \gamma \text{ and } \alpha \cdot \beta = i\}$$

for $i = 0, 1$. Since $\gamma \neq 0$ and (V, \cdot) is non-degenerate, there exists $x \in V$ such that $x \cdot \gamma = 1$. Then, the map $(\alpha, \beta) \mapsto (\alpha + x, \beta + x)$ induces a bijection $N_\gamma^0 \rightarrow N_\gamma^1$. Hence $n_\gamma^0 = n_\gamma^1$ for all $\gamma \neq 0$, and the lemma is proved. \square

Using this lemma, it is easy to check that if $\dim(V) = 2n$, then there are exactly $2^{2n-1} + 2^{n-1}$ quadratic forms on (V, \cdot) with Arf invariant 1 and $2^{2n-1} - 2^{n-1}$ forms with Arf invariant -1 .

Theorem 5. *The partition function of a dimer model on a surface graph $\Gamma \subset \Sigma$ for a closed surface of genus g is given by the formula*

$$Z = \frac{1}{2^g} \sum_{[K]} \text{Arf}(q_{D_0}^K) \varepsilon^K(D_0) \text{Pf}(A^K), \tag{6}$$

where the sum is taken over all equivalence classes of Kasteleyn orientations. Each summand is defined for a Kasteleyn orientation but it depends only on its equivalence class. Furthermore, the sign $\text{Arf}(q_{D_0}^K) \varepsilon^K(D_0)$ does not depend on D_0 .

Proof. Recall that there is a free, transitive action of $H^1(\Sigma; \mathbb{Z}_2)$ on the set of equivalence classes of Kasteleyn orientations on $\Gamma \subset \Sigma$ (Theorem 2). Let us denote by K_ϕ the result of the action of $\phi \in H^1(\Sigma; \mathbb{Z}_2)$ on a Kasteleyn orientation K . Then, by the first part of Proposition 1,

$$(-1)^{(q_{D_0}^K + q_{D_0}^{K_\phi})(\alpha)} = (-1)^{\phi(\alpha)} =: \chi_\alpha(\phi).$$

Here χ_α 's are characters of irreducible representations of $H^1(\Sigma; \mathbb{Z}_2)$.

By Eq. (5),

$$\varepsilon^{K_\phi}(D_0) \text{Pf}(A^{K_\phi}) = \sum_{\alpha \in H_1(\Sigma; \mathbb{Z})} (-1)^{q_{D_0}^{K_\phi}(\alpha)} Z_\alpha(D_0)$$

for all $\phi \in H^1(\Sigma; \mathbb{Z})$. Multiplying these equations by $(-1)^{q_{D_0}^{K_\phi}(\beta)}$ and taking the sum over all $\phi \in H^1(\Sigma; \mathbb{Z})$, we get

$$\begin{aligned} \sum_{\phi} (-1)^{q_{D_0}^{K_\phi}(\beta)} \varepsilon^{K_\phi}(D_0) \text{Pf}(A^{K_\phi}) &= \sum_{\phi} \sum_{\alpha} (-1)^{q_{D_0}^{K_\phi}(\alpha) + q_{D_0}^{K_\phi}(\beta)} Z_\alpha(D_0) \\ &= \sum_{\alpha} (-1)^{q_{D_0}^K(\alpha) + q_{D_0}^K(\beta)} \sum_{\phi} \chi_\alpha(\phi) \chi_\beta(\phi) Z_\alpha(D_0). \end{aligned}$$

Using the orthogonality formula $\sum_{\phi} \chi_\alpha(\phi) \chi_\beta(\phi) = 2^{2g} \delta_{\alpha\beta}$, we obtain

$$Z_\alpha(D_0) = \frac{1}{2^{2g}} \sum_{\phi} (-1)^{q_{D_0}^{K_\phi}(\alpha)} \varepsilon^{K_\phi}(D_0) \text{Pf}(A^{K_\phi}).$$

Therefore, the partition function $Z = \sum_{\alpha} Z_\alpha(D_0)$ is given by

$$Z = \frac{1}{2^g} \sum_{\phi \in H^1(\Sigma; \mathbb{Z}_2)} \sigma^{K_\phi} \text{Pf}(A^{K_\phi}),$$

where

$$\sigma^{K_\phi} = \varepsilon^{K_\phi}(D_0) \frac{1}{2^g} \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}_2)} (-1)^{q_{D_0}^{K_\phi}(\alpha)} = \varepsilon^{K_\phi}(D_0) \text{Arf}(q_{D_0}^{K_\phi}).$$

This gives the formula

$$Z = \frac{1}{2^g} \sum_{\phi \in H^1(\Sigma; \mathbb{Z}_2)} \text{Arf}(q_{D_0}^{K_\phi}) \varepsilon^{K_\phi}(D_0) \text{Pf}(A^{K_\phi}).$$

Since $H^1(\Sigma, \mathbb{Z}_2)$ acts transitively and freely on the equivalence classes of Kasteleyn orientations, equality (6) follows.

If K and K' are equivalent Kasteleyn orientations, then $q_{D_0}^K = q_{D_0}^{K'}$. Therefore, $\text{Arf}(q_{D_0}^K) = \text{Arf}(q_{D_0}^{K'})$. On the other hand, $\varepsilon^K(D_0) = (-1)^\mu \varepsilon^{K'}(D_0)$ and $\text{Pf}(A^K) = (-1)^\mu \text{Pf}(A^{K'})$, where μ is the number of vertices of Γ around which the orientation was flipped. Therefore, the summand $\text{Arf}(q_{D_0}^K) \varepsilon^K(D_0) \text{Pf}(A^K)$ does not depend on the choice of the representative in the equivalence class $[K]$.

Let us finally check that the sign $\text{Arf}(q_{D_0}^K) \varepsilon^K(D_0)$ does not depend on D_0 . Let D be another dimer configuration on Γ . By Proposition 1, Lemma 1 and the proof of Theorem 4,

$$A(q_{D_0}^K) \varepsilon^K(D_0) A(q_D^K) \varepsilon^K(D) = (-1)^{q_D^K(\Delta(D_0, D))} \text{Arf}(q_{D_0}^K) A(q_D^K) = 1.$$

This concludes the proof of the theorem. \square

6.3. Local correlation functions. In order to express local correlation functions $\langle \sigma_{e_1} \cdots \sigma_{e_k} \rangle$ as combinations of Pfaffians, let us recall several facts of linear algebra.

Let $A = (a_{ij})$ be a matrix of size $2n$. Given an ordered subset I of the ordered set $\alpha = (1, \dots, 2n)$, let A_I denote the matrix obtained from A by removing the i^{th} row and the i^{th} column for all $i \in I$. Also, let $(-1)^{\sigma(I)}$ denote the signature of the permutation which sends α to the ordered set $I(\alpha \setminus I)$. If A is skew-symmetric, then for all ordered sets of indices $I = (i_1, j_1, \dots, i_k, j_k)$,

$$\frac{\partial^k \text{Pf}(A)}{\partial a_{i_1 j_1} \cdots \partial a_{i_k j_k}} = (-1)^{\sigma(I)} \text{Pf}(A_I).$$

Furthermore, if A is invertible, then $\text{Pf}(A) \neq 0$ and

$$(-1)^{\sigma(I)} \text{Pf}(A_I) = (-1)^k \text{Pf}(A) \text{Pf}((A^{-1})_{\alpha \setminus I}).$$

So, let e_1, \dots, e_k be edges of the graph Γ , and let i_ℓ, j_ℓ be the two boundary vertices of e_ℓ for $\ell = 1, \dots, k$. For simplicity, we shall assume that Γ has no multiple edges. Finally, set $I = (i_1, j_1, \dots, i_k, j_k)$. Applying the identities above to the Kasteleyn matrices $A^\phi = A^{K_\phi}$, Theorem 5 gives

$$\begin{aligned} \prod_{\ell=1}^k w(e_\ell) \frac{\partial^k Z}{\partial w(e_1) \cdots \partial w(e_k)} &= \frac{1}{2^g} \sum_{\phi} \sigma^{K_\phi} \prod_{\ell=1}^k a_{i_\ell j_\ell}^\phi \frac{\partial^k}{\partial a_{i_1 j_1}^\phi \cdots \partial a_{i_k j_k}^\phi} \text{Pf}(A^\phi) \\ &= \frac{1}{2^g} \sum_{\phi} \sigma^{K_\phi} \prod_{\ell=1}^k a_{i_\ell j_\ell}^\phi (-1)^{\sigma(I)} \text{Pf}(A_I^\phi). \end{aligned}$$

If the Kasteleyn matrix is invertible for any Kasteleyn orientation of Γ , this expression is equal to

$$\frac{(-1)^k}{2^g} \sum_{\phi} \sigma^{K_{\phi}} \prod_{\ell=1}^k a_{i_{\ell} j_{\ell}}^{\phi} \text{Pf}(A^{\phi}) \text{Pf}((A^{\phi})_{\alpha \setminus I}^{-1}).$$

Since

$$\langle \sigma_{e_1} \cdots \sigma_{e_k} \rangle = \frac{w(e_1) \cdots w(e_k)}{Z} \frac{\partial^k Z}{\partial w(e_1) \cdots \partial w(e_k)},$$

the correlation functions are given by

$$\langle \sigma_{e_1} \cdots \sigma_{e_k} \rangle = (-1)^k \frac{\sum_{\phi} \sigma^{K_{\phi}} \text{Pf}(A^{\phi}) \prod_{\ell} a_{i_{\ell} j_{\ell}}^{\phi} \text{Pf}((A^{\phi})_{\alpha \setminus I}^{-1})}{\sum_{\phi} \sigma^{K_{\phi}} \text{Pf}(A^{\phi})}.$$

if the Kasteleyn matrix is invertible for all possible Kasteleyn orientations of Γ .

If Γ is a planar graph, then the Kasteleyn matrix is always invertible since its Pfaffian is equal to the partition function. Therefore,

$$\langle \sigma_{e_1} \cdots \sigma_{e_k} \rangle = (-1)^k a_{i_1 j_1}^K \cdots a_{i_k j_k}^K \text{Pf}((A^K)_{\alpha \setminus I}^{-1}),$$

where K is any Kasteleyn orientation on $\Gamma \subset S^2$.

On the other hand, there are graphs where some Kasteleyn matrix is not invertible. For example, consider a square lattice on a torus. Then, the Kasteleyn matrix corresponding to the spin structure with Arf invariant -1 is not invertible (see [15]).

A. Dimers and Grassman Integrals

A.1 Grassman integrals and Pfaffians. Let V be an n -dimensional vector space. Its exterior algebra $\wedge V = \bigoplus_{k=0}^n \wedge^k V$ is called the *Grassman algebra of V* . The choice of a linear basis in V induces an isomorphism between $\wedge V$ and the algebra generated by elements ϕ_1, \dots, ϕ_n with defining relations $\phi_i \phi_j = -\phi_j \phi_i$. The isomorphism identifies the linear basis in V with the generators ϕ_i .

Choose an orientation on V . Together with the basis in V , this defines a basis in the top exterior power of V . The *integral over the Grassman algebra of V* of an element $a \in \wedge V$ is the coordinate of a in the top exterior power of V with respect to this basis. It is denoted by

$$\int a \, d\phi.$$

In physics, elements ϕ are called *Fermionic fields*. More precisely, they are called neutral fermionic fields (or neutral fermions). They are called charged fermions if there is an action of $U(1)$ on V .

Recall that the Pfaffian of a skew symmetric matrix A of even size n is given by

$$\text{Pf}(A) = \frac{1}{2^{\frac{n}{2}} (\frac{n}{2})!} \sum_{\sigma \in S_n} (-1)^{\sigma} a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(n-1)\sigma(n)},$$

where the sum is over all permutations $\sigma \in S_n$. (This formula is easily seen to be equivalent to the one stated in Sect. 6.) Expanding the exponent into a power series, one gets the following identity in $\wedge V$:

$$\pi \left(\exp \left(\frac{1}{2} \sum_{i,j=1}^n a_{ij} \phi_i \wedge \phi_j \right) \right) = \text{Pf}(A) \phi_1 \wedge \cdots \wedge \phi_n,$$

where $\pi : \wedge V \rightarrow \wedge^n V$ is the projection to the top exterior power. In terms of Grassmann integral, this can be expressed as

$$\int \exp \left(\frac{1}{2} \sum_{i,j=1}^n \phi_i a_{ij} \phi_j \right) d\phi = \text{Pf}(A). \tag{7}$$

Let us now assume that the space V is polarized, i.e. that $V = W \oplus W^*$ with W some vector space and W^* its dual. Then, the Grassman algebra of V is isomorphic to the tensor product of Grassman algebras for W and for W^* in the category of super-vector spaces. That is, if the dimension of W is k , the Grassman algebra of V is isomorphic to the algebra generated by $\psi_i, \psi_i^*, i = 1, \dots, k$ with defining relations $\psi_i \psi_j = -\psi_j \psi_i, \psi_i \psi_j^* = -\psi_j^* \psi_i$, and $\psi_i^* \psi_j^* = -\psi_j^* \psi_i^*$. Such an isomorphism is specified by the choice of a linear basis in W . Note that such a choice induces a basis in V (take the dual basis in W^*) and an orientation on V given by the ordering $\psi_1, \dots, \psi_n, \psi_1^*, \dots, \psi_n^*$. The Grassmann integral of an element $a \in \wedge V$ with respect to this choice of basis in W is denoted by

$$\int a d\psi d\psi^*.$$

Expanding the exponent, we obtain

$$\int \exp \left(\sum_{i,j=1}^k \psi_i a_{ij} \psi_j^* \right) d\psi d\psi^* = (-1)^{\frac{k(k-1)}{2}} \text{Det}(A). \tag{8}$$

Comparing this formula with (7), we obtain the following well known identity

$$\text{Pf} \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} = (-1)^{\frac{k(k-1)}{2}} \text{Det}(A).$$

Now consider the space $U = V \oplus V^*$. Let ϕ_1, \dots, ϕ_n be a basis in V and $\phi_1^*, \dots, \phi_n^*$ be the dual basis in V^* . Using the change of variables $\chi_i = \frac{\phi_i + \sqrt{-1} \phi_i^*}{\sqrt{2}}, \chi_i^* = \frac{\phi_i - \sqrt{-1} \phi_i^*}{\sqrt{2}}$, together with the equalities (7) and (8), we obtain

$$\begin{aligned} \text{Pf}(A)^2 &= \int \exp \left(\frac{1}{2} \sum_{i,j=1}^n a_{ij} \phi_i \phi_j + \frac{1}{2} \sum_{i,j=1}^n a_{ij} \phi_i^* \phi_j^* \right) d\phi d\phi^* \\ &= (-1)^{\frac{n(n-1)}{2}} \int \exp \left(\sum_{i,j=1}^n \chi_i a_{ij} \chi_j^* \right) d\chi d\chi^* \\ &= \text{Det}(A). \end{aligned}$$

One can easily derive other Pfaffian identities in a similar way .

A.2 Dimer models on graphs and Grassmann integrals. Let $\Gamma \subset \Sigma$ be a surface graph, and let $A^K = (a_{ij}^K)$ be the Kasteleyn matrix associated with a Kasteleyn orientation K on $\Gamma \subset \Sigma$. By Theorem 5 and identity (7), the partition function for dimers on Γ is given by

$$Z = \frac{1}{2^g} \sum_{[K]} \text{Arf}(q_{D_0}^K) \varepsilon^K(D_0) \int \exp\left(\frac{1}{2} \sum_{i,j \in V(\Gamma)} \phi_i a_{ij}^K \phi_j\right) d\phi,$$

where $V(\Gamma)$ denotes the set of vertices of Γ .

Recall that $\varepsilon^K(D_0) = (-1)^\sigma \prod_{\ell=1}^n \varepsilon_{i_\ell j_\ell}^K$, where the dimer configuration D_0 matches vertices i_ℓ and j_ℓ for $\ell = 1, \dots, n$, and σ denotes the permutation $(1, \dots, 2n) \mapsto (i_1, j_1, \dots, i_n, j_n)$. Taking into account the identity

$$(-1)^\sigma \prod_{\ell=1}^n \varepsilon_{i_\ell j_\ell}^K d\phi_1 \dots d\phi_{2n} = \prod_{\ell=1}^n \varepsilon_{i_\ell j_\ell}^K d\phi_{i_\ell} d\phi_{j_\ell},$$

the formula for the partition function can be written as

$$Z = \frac{1}{2^g} \sum_{[K]} \int \exp\left(\frac{1}{2} \sum_{i,j \in V(\Gamma)} \phi_i a_{ij}^K \phi_j\right) D^K \phi,$$

where

$$D^K \phi = \text{Arf}(q_{D_0}^K) \varepsilon^K(D_0) d\phi = \text{Arf}(q_{D_0}^K) \prod_{\ell=1}^n \varepsilon_{i_\ell j_\ell}^K d\phi_{i_\ell} d\phi_{j_\ell}.$$

Similarly, local correlation functions can be written as

$$\langle \sigma_{e_1} \dots \sigma_{e_k} \rangle = \frac{1}{2^g Z} \sum_{[K]} \int \exp\left(\frac{1}{2} \sum_{i,j \notin I} \phi_i a_{ij}^K \phi_j\right) \prod_{l=1}^k a_{i_l j_l}^K \phi_{i_l} \phi_{j_l} D^K \phi.$$

Here i_l and j_l are the boundary vertices of the edge e_l , and $I = (i_1, j_1, \dots, i_k, j_k)$.

Acknowledgements. We are grateful to Peter Teichner and Andrei Okounkov for discussions, and to Richard Kenyon and Greg Kuperberg for useful remarks and comments. We also thank Faye Yaeger who was kind to type a large part of this paper. Finally, we thankfully acknowledge the hospitality of the Department of Mathematics of the University of Aarhus, via the Niels Bohr initiative. The work of D.C. was supported by the Swiss National Science Foundation. The work of N.R. was supported by the NSF grant DMS-0307599, by the CRDF grant RUM1-2622, and by the Humboldt Foundation.

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Communicated by L. Takhtajan