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Alexander Invariants of Multilinks

THÈSE

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David CIMASONI
de
Bellinzone (TI)

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Contents

Introduction

A **multilink** is an oriented link $L = L_1 \cup \ldots \cup L_n$ in an oriented homology 3-sphere Σ, together with an integer m_i associated with each component L_i. The following convention is used: $m_i L_i = (-m_i)(-L_i)$, where $-L_i$ represents L_i with the opposite orientation. Of course, a multilink with multiplicities ± 1 is simply an oriented link.

In a more formal way, a multilink can be defined as an oriented link L with a homology class $\underline{m} = (m_1, \ldots, m_n)$ in $\mathrm{H}_1 L$. Classical theorems of JAMES ALEXANDER and HENRY WHITEHEAD imply that this $\mathbb{Z}$-module is isomorphic to the group of homotopy classes of maps $X \xrightarrow{\varphi} S^1$, where X denotes the complement $\Sigma - L$ of L. A multilink is said to be **fibered** if there is a fiber bundle in the homotopy class corresponding to the multiplicities. If the multilink is an oriented link, this is the usual definition of a fibered link. Moreover, standard techniques of algebraic topology show that the homotopy classes of maps $X \xrightarrow{\varphi} S^1$ are in one-to-one correspondence with the infinite cyclic coverings of X; therefore, the set of all multilink structures on a given link L is in bijection with the set of infinite cyclic coverings of the complement X of L. In other words, the choice of multiplicities $\underline{m}$ determines a $\mathbb{Z}$-covering $\widetilde{X}(\underline{m}) \xrightarrow{p} X$. If one denotes by t a generator of the infinite cyclic group of this covering, the homology $\mathrm{H}_1 \widetilde{X}(\underline{m})$ is endowed with a natural structure of module over $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials with integer coefficients. This is the **Alexander module** of the multilink. The determinant of a square presentation matrix of this module is called the **Alexander polynomial** of the multilink. Of course, if all the multiplicities are ± 1, these definitions coincide with the usual definitions of the Alexander module and polynomial of an oriented link.

In this work, we first give a systematic study of the Alexander module of multilinks, and an original method to compute this invariant. Then, we analyze the consequences of these results on the Alexander polynomial. Finally, this technique of computation is implemented on remarkable classes of multilinks: Seifert multilinks, and several graph multilinks.

The concept of multilink was first introduced in 1985 by DAVID EISENBUD and WALTER NEUMANN in their book *Three-dimensional link theory and invariants of plane curve singularities* [11]. Clearly, it is a natural generalization of the notion of oriented link; nevertheless, the reasons of the introduction of multilinks are far from obvious at first sight. Therefore, we will now present, in an informal way, three fields where the development of a coherent theory requires the use of multilinks.

Complex Plane Curve Singularities

Consider a polynomial application $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$, and the algebraic curve V given by

$$V = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}.$$

The object of the study is the topology of V in the neighborhood of a given point $a = (x_0, y_0) \in V$. If $f'(a)$ does not vanish, a is called a **regular point** of f. In this case, V is a smooth surface in the neighborhood of a, so there is nothing to say from the topological point of view. On the other hand, if $\frac{\partial f}{\partial x}(a) = \frac{\partial f}{\partial y}(a) = 0$, a is a **singular point** of f, and the topology of V near a is very interesting.

The fundamental idea of this theory is due to WILHELM WIRTINGER and to his student KARL BRAUNER [5]: for $\epsilon > 0$ sufficiently small, consider the intersection K_ϵ of V and of a 3-sphere $S_\epsilon = \partial D_\epsilon \subset \mathbb{C}^2$ of radius ϵ and center a. It turns out that K_ϵ is a smooth 1-manifold, that is, a link: it is called the **link of the singularity**.[1] Furthermore, the topology of $V \cap D_\epsilon$ is determined by this link; more precisely, JOHN MILNOR [25, Theorem 2.10] shows that for $\epsilon > 0$ sufficiently small, the pair $(D_\epsilon, V \cap D_\epsilon)$ is homeomorphic to the cone on the pair (S_ϵ, K_ϵ). For example, the origin is a singular point of $f(x,y) = xy$, and the link of this singularity is the Hopf link; therefore, the algebraic curve $V = \{(x,y) \in \mathbb{C}^2 \mid xy = 0\}$ near the origin is homeomorphic to the cone on the Hopf link. Another simple example is given by the application $f(x,y) = x^p + y^q$, where p and q are coprime integers greater than one: the origin is a singular point, and the link of this singularity is a torus knot of type (p,q).

Another fundamental result is Milnor's celebrated fibration theorem [25, Theorem 4.8]: for $\epsilon > 0$ sufficiently small, the map $\phi = \frac{f}{|f|}\colon S_\epsilon - K_\epsilon \longrightarrow S^1$ is a smooth fiber bundle. It is called the **Milnor fibration**, and $F = \phi^{-1}(1)$ is the **Milnor fiber**. Considering the examples given above, this theorem shows that the Hopf link and the torus knots are fibered links. The Milnor fiber provides a Seifert surface for the corresponding link.

Now, let a be a singular point of a polynomial application $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$; by translation, it may be assumed that a is the origin. Consider f as an element of $\mathbb{C}\{x,y\}$, the ring of convergent power series with complex coefficients. Since this is a factorial ring, f can be written in a unique way as a product

$$f = f_1^{m_1} \cdot f_2^{m_2} \cdots f_n^{m_n}$$

of irreducible elements of $\mathbb{C}\{x,y\}$, with $m_i \geq 1$. If all the multiplicities m_i are equal to one, f is said to be **reduced**; in fact, $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ is reduced if and only if the singularity is isolated. For all i, the space $L_i = \{(x,y) \in S_\epsilon \mid f_i(x,y) = 0\}$ is connected. Since K_ϵ is the union $L_1 \cup L_2 \cup \ldots \cup L_n$, the number of components of the link of the singularity is equal to the number of irreducible factors of f in $\mathbb{C}\{x,y\}$, or **branches** of f.

As a set, the link K_ϵ does not depend on the integers $m_i \geq 1$. On the other hand, the Milnor fibration does depend on these multiplicities: indeed, the smooth fiber bundle $S_\epsilon - K_\epsilon \xrightarrow{\phi} S^1$ is in the homotopy class given by $\underline{m} = (m_1, \ldots, m_n)$. In other words, Milnor's theorem states that the multilink $K_\epsilon = m_1 L_1 \cup \ldots \cup m_n L_n$ is fibered. Equivalently, the Milnor fiber provides a Seifert surface for the multilink $K_\epsilon = m_1 L_1 \cup \ldots \cup m_n L_n$. Finally, the Alexander module of the singularity is the Alexander module of this multilink.

[1] A link that can be realized as the link of a singularity is called an **algebraic link**.

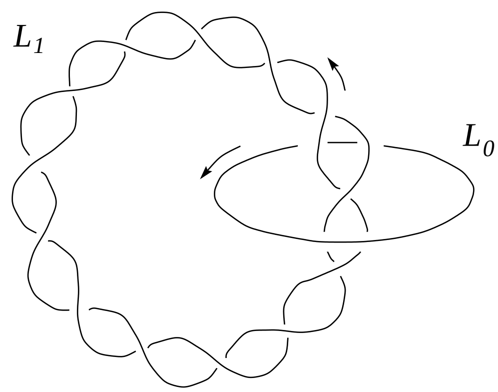
To summarize, one can associate a multilink to any complex plane curve singularity. If the singularity is isolated, this multilink is simply an oriented link.

Additivity of Link Invariants

With the previous paragraph, one might think that multilinks only occur when the singularity in non-isolated. In fact, multilinks are important for the computation of the Alexander polynomial of a singularity, even if this singularity is isolated. Let us explain this fact on an example. Consider the singularity given by the polynomial application

$$f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7.$$

One can show that f is irreducible in $\mathbb{C}\{x,y\}$; in particular, the origin is an isolated singular point of f. Moreover, the knot K of this singularity is obtained in the following way. Let K' be an oriented trefoil knot, and let L be the link illustrated below, consisting of a $(13,2)$-torus knot L_1 and an unknot L_0 linking L_1 twice.



Consider tubular neigborhoods $\mathcal{N}(L_0)$ and $\mathcal{N}(K')$ with standard parallels and meridiens $P, M \subset \partial\mathcal{N}(L_0)$ and $P', M' \subset \partial\mathcal{N}(K')$. Then, K is the image of L_1 under the map

$$S^3 - \mathcal{N}(L_0) \hookrightarrow (S^3 - \mathcal{N}(L_0)) \cup_h (S^3 - \mathcal{N}(K')),$$

where $h: \partial\mathcal{N}(L_0) \longrightarrow \partial\mathcal{N}(K')$ is a homeomorphism mapping P onto M' and M onto P'. We say that K is obtained by **splicing** L and K' along L_0 et K'. In a less formal way, K should be visualized as a $(13,2)$-torus knot on a torus embedded as a trefoil knot.

More generally, every algebraic link is an **iterated torus link**: it is obtained by splicing a certain number of torus links. This number is equal to the number of characteristic Puiseux pairs of the singularity.[2] The idea is to use this canonical decomposition to compute some invariants of the singularity, for example, its Alexander polynomial. But in order to have a multiplicativity formula for the Alexander polynomial via splicing, it is necessary to introduce multiplicities on the components along which the splicing is performed: the multiplicity on one of these components is given by the linking number of the other with the rest of the link

[2] The singularity given above is the simplest with two characteristic Puiseux pairs. This is the reason why you find this example in almost every textbook.

it belongs to. The Alexander polynomial of the multilinks obtained is multiplicative under splicing. In the example above, the Alexander polynomial of K is equal to the product of the Alexander polynomials of the multilinks $L_1 \cup 0L_0$ and $2K'$. Hence, the problem of the computation of the Alexander polynomial of a singularity can be reduced to the (very simple) corresponding problem for torus multilinks.

The necessity of the introduction of multiplicities in the process of splicing also appeared in the topological context of knot theory. Indeed, HERBERT SEIFERT [42] proved in 1950 the following assertion: consider K' a knot in S^3, $\mathcal{N}(K')$ a closed tubular neighborhood of K', and $f: \mathcal{N}(K') \longrightarrow S^1 \times D^2$ an orientation preserving homeomorphism sending K' onto $S^1 \times \{0\}$ and a standard parallel of $\mathcal{N}(K')$ onto $S^1 \times \{1\}$; if K is a knot in the interior of $\mathcal{N}(K')$ with $K \sim m \cdot K'$ in $\mathrm{H}_1(\mathcal{N}(K'))$, their Alexander polynomials satisfy the equality

$$\Delta_K(t) = \Delta_{K'}(t^m) \cdot \Delta_{f(K)}(t).$$

This formula was generalized to the multivariable case by GUILLERMO TORRES [44] in 1953. The result is known as the *Seifert-Torres formula* (see corollary 3.2.11).

In short, if we want to compute the Alexander polynomial of a link L obtained by splicing other links, it is useful to consider them as multilinks: indeed, the Alexander polynomial of L is simply the product of the Alexander polynomials of the other links, considered as multilinks. This technique works particularly well for the algebraic multilinks: these multilinks can be expressed in a canonical way as the splice of torus multilinks.

But splicing is not just a useful trick in the algebraic context. The decomposition theorem of WILLIAM JACO, PETER SHALEN and KLAUS JOHANSSON [17, 18], and WILLIAM THURSTON's hyperbolization theorem imply that any link L in S^3 can be expressed in a canonical way as the splice of torus links and hyperbolic links: these links are called the **splice components** of L. Furthermore, a multilink structure $\underline{m}$ on L (for example: an orientation) determines a structure of multilink on every splice component. It is then possible to prove numerous results connecting properties of the multilink $L(\underline{m})$ with corresponding properties of the multilink splice components (see [11]). For example, a multilink is fibered if and only if it is irreducible and all his multilink splice components are fibered. Of course, this result is false if we don't consider the splice components as multilinks.

Plane Curves and Links at Infinity

Surprisingly, multilinks also appear in the study of the global topology of algebraic curves in $\mathbb{C}^2$, as explained by WALTER NEUMANN and LEE RUDOLPH [28, 29]. Let us summarize this very briefly. Consider $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ a polynomial application; a fiber $f^{-1}(c)$ is called **regular** if there exists a neighborhood D of c in $\mathbb{C}$ such that $f|: f^{-1}(D) \longrightarrow D$ is a locally trivial fibration.[3] One might think that if c is not a singular value of f, then $f^{-1}(c)$ is regular; this is false. For example, Broughton's famous polynomial $g(x,y) = x + x^2y$ has no singular value, but $g^{-1}(0)$ is not regular. In fact, the following additional condition is required: a fiber $f^{-1}(c)$ is **regular at infinity** if there exists a neighborhood D of c in $\mathbb{C}$ and a compact K in $\mathbb{C}^2$ such that f restricted to $f^{-1}(D) - K$ is a locally trivial fibration. HÀ HUY VUI and LÊ

[3] Note that for a fixed f, there are only finite many non-regular fibers.

DUNG TRANG [13] showed that $f^{-1}(c)$ is regular if and only if it is non-singular and regular at infinity.

Given $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ and c in $\mathbb{C}$, the intersection of the fiber $f^{-1}(c)$ with a sufficiently large sphere S^3_R about the origin defines a link called the **link at infinity** of $f^{-1}(c)$. It turns out that two fibers $f^{-1}(c)$ and $f^{-1}(c')$ that are regular at infinity determine two isotopic links at infinity; this link is called the **regular link at infinity** of f, and it is denoted by $\mathcal{L}(f,\infty)$. It is a very special type of link: there exists a fibered multilink $L(\underline{m})$ with multiplicities $m_i \geq 1$, such that $\mathcal{L}(f,\infty)$ is equal to the boundary of the fiber of $L(\underline{m})$.

Let us now mention a first remarkable result: all the fibers $f^{-1}(c)$ are regular at infinity if and only if all the multiplicities of the multilink $L(\underline{m})$ are equal to 1 (that is, $L(\underline{m})$ is equal to $\mathcal{L}(f,\infty)$). This condition is satisfied if and only if $\mathcal{L}(f,\infty)$ is fibered. In short, the defect of regularity at infinity of a polynomial application f is measured by the multiplicities of a multilink.

Let us conclude with the main theorem of [29]: the topology of a regular algebraic plane curve $V \subset \mathbb{C}^2$, as an embedded smooth manifold, is determined by the associated multilink at infinity $L(\underline{m})$. In fact, V is properly isotopic to the embedded surface obtained from F, the fiber of $L(\underline{m})$, by attaching a collar out to infinity in $\mathbb{C}^2$ to the boundary of F.

New Results

We conclude this introduction with a list of the principal new results of our work.
The first chapter is dedicated to a systematic study of the Alexander module of multilinks. In particular, we prove that this module can be computed using Seifert surfaces, as defined by Eisenbud and Neumann [11]. But not any Seifert surface: it must have the least possible number of connected components.

Proposition 1.2.2.
Let $L(\underline{m})$ be a multilink, and let d be the greatest common divisor of $m_1, \ldots, m_n$; then, there exists a Seifert surface for $L(\underline{m})$ with d connected components. Moreover, there is no Seifert surface for $L(\underline{m})$ with less then d connected components.

Such a Seifert surface will be called a **good Seifert surface** for $L(\underline{m})$. Given $F \subset \Sigma - L$ a good Seifert surface for $L(\underline{m})$, let us denote by $\overline{F}$ the union $F \cup L$. Consider the bilinear forms

$$\begin{array}{ccc} \mathrm{H}_1 F \times \mathrm{H}_1 \overline{F} & \longrightarrow & \mathbb{Z} \\ (x,y) & \longmapsto & \ell k(i_{\pm}x, y), \end{array}$$

where $i_+, i_-: \mathrm{H}_1 F \longrightarrow \mathrm{H}_1(\Sigma - \overline{F})$ are the morphisms induced by the push in the positive or negative normal direction off F. Let us denote by A^+_F and A^-_F matrices of these forms, called **Seifert matrices**.

Theorem 1.2.7.
Given F any good Seifert surface for a multilink $L(\underline{m})$, $A^+_F - tA^-_F$ is a square presentation matrix of the Alexander module of $L(\underline{m})$.

Corollary 1.2.8.
Let $L(\underline{m})$ be a multilink; if F is a good Seifert surface for $L(\underline{m})$, a representative up to sign of the Alexander polynomial $\Delta^{L(\underline{m})}(t)$ is given by $\det(A^+_F - tA^-_F)$.

In chapter 2, we study the consequences of this corollary on the Alexander polynomial of a multilink $L(\underline{m})$. Let us use the notations $d = \gcd(\underline{m}) > 0$, $\ell_{ij} = \ell k(L_i, L_j)$ if $i \neq j$, $\ell_{ii} = 0$, and $d_i = \gcd(m_i, \sum_j m_j \ell_{ij}) \geq 0$.

Theorem 2.1.6.
There exists a unique polynomial $\nabla^{L(\underline{m})}(t)$ in $\mathbb{Z}[t, t^{-1}]$ which satisfies:

- $\Delta^{L(\underline{m})}(t) \doteq \frac{1}{t^d - 1} \prod_{i=1}^{n} (t^{d_i} - 1) \cdot \nabla^{L(\underline{m})}(t)$;
- $\nabla^{L(\underline{m})}(t^{-1}) = \nabla^{L(\underline{m})}(t)$, *and the leading coefficient of $\nabla^{L(\underline{m})}(t)$ is positive.*

Furthermore, we have:

- $|\nabla^{L(\underline{m})}(1)| = \frac{d^2\ D}{d_1 \cdots d_n m_1 \cdots m_n}$, *where D is any $(n-1) \times (n-1)$-minor of the matrix*

$$\begin{pmatrix} -\sum_j m_1 m_j \ell_{1j} & m_1 m_2 \ell_{12} & \dots & m_1 m_n \ell_{1n} \\ m_1 m_2 \ell_{12} & -\sum_j m_2 m_j \ell_{2j} & \dots & m_2 m_n \ell_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_1 m_n \ell_{1n} & m_2 m_n \ell_{2n} & \dots & -\sum_j m_n m_j \ell_{nj} \end{pmatrix}.$$

- $\nabla^{L(m_1, \dots, m_{n-1}, 0)}(t) = \nabla^{L(m_1, \dots, m_{n-1})}(t)$.

Corollary 2.1.7.
For any multilink $L(\underline{m})$, $\Delta^{L(\underline{m})}$ is equal to $\Delta^{L(-\underline{m})}$.

Corollary 2.1.9.
Let H be a monodromy matrix for a fibered multilink $L(\underline{m})$; then, the determinant of H is given by $\det H = (-1)^{n+d+1+\sum_i d_i}$.

In section 2.2, we explain the consequences of theorem 2.1.6 on the multivariable Alexander polynomial. Indeed, the Alexander polynomial $\Delta^{L(\underline{m})}$ of a multilink is related to the multivariable Alexander polynomial Δ_L of the underlying oriented ordered link. Therefore, theorem 2.1.6 can be translated into several conditions on Δ_L. It turns out that these conditions are equivalent to the well known *Torres conditions*.

In chapter 4, we give a closed formula for the Alexander module of any Seifert multilink, that is, any multilink whose exterior admits a Seifert fibration. If the multilink is non-fibered, theorem 1.2.7 can be applied, yielding the following surprisingly simple result.

Theorem 4.3.1.
Let $L(\underline{m})$ be the non-fibered Seifert multilink $(\Sigma(\alpha_1, \dots, \alpha_k), m_1 L_1 \cup \dots \cup m_n L_n)$, with $d = \gcd(\underline{m}) \neq 0$. Then, the Alexander module of $L(\underline{m})$ is presented by

$$\begin{pmatrix} \alpha_{n+1} \cdots \alpha_k (t^d - 1) & \overbrace{0 \quad 0 \quad \dots \quad 0}^{n-2} \end{pmatrix}.$$

Corollary 4.3.2.
The Alexander polynomial of a non-fibered Seifert multilink is given by

$$\Delta^{L(\underline{m})}(t) \doteq \begin{cases} \alpha_{n+1}\cdots\alpha_k(t^d-1) & \text{if } n=2; \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if the Seifert multilink is fibered, the method of chapter 1 does not apply very well. Nevertheless, it is possible to compute its Alexander module using techniques of Waldhausen.

Theorem 4.4.6.
Let $(\Sigma, L(\underline{m})) = (\Sigma(\alpha_1,\ldots,\alpha_k), m_1L_1\cup\ldots\cup m_nL_n)$ be a fibered Seifert multilink with $n<k$.[4]
Let $\beta_1,\ldots,\beta_k$ be integers such that $\sum_{j=1}^k \beta_j\alpha_1\cdots\widehat{\alpha_j}\cdots\alpha_k = 1$. Let us note

- $u = |\underline{m}(\gamma)| = |\sum_{i=1}^n m_i\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_k|$;
- $n_0 = -\underline{m}(\gamma)\sum_{i=n+1}^k \frac{\beta_i}{\alpha_i}$, *and* $d_0 = \frac{u}{\alpha_{n+1}\cdots\alpha_k}$;
- $n_i = \frac{m_i-\beta_i\underline{m}(\gamma)}{\alpha_i}$, *and* $d_i = \gcd(n_i,u)$ *for* $i=1,\ldots,n-1$;
- $d^i = \gcd(d_0,d_1,\ldots,d_i)$ *for* $i=0,\ldots,n-1$.

Finally, for $i=1,\ldots,n-2$, let $\gamma_i^0,\ldots,\gamma_i^{i+1}$ be polynomials in $\mathbb{Z}[t]$ satisfying

$$\sum_{j=0}^{i}\gamma_i^j(t)\cdot\frac{t^{n_j}-1}{t^{d^i}-1}+\gamma_i^{i+1}(t)\cdot\frac{t^u-1}{t^{d^i}-1}=1$$

and such that $\gamma_i^0,\ldots,\gamma_i^i$ and $t^{d^i}-1$ don't have any common factor in $\mathbb{Z}[t]$.
Then, the Alexander module of $(\Sigma, L(\underline{m}))$ is given by the direct sum of

$$\bigoplus_{i=n+2}^{k}\mathbb{Z}[t,t^{-1}]\Big/\frac{(t^u-1)(t^{u/\alpha_{n+1}\cdots\alpha_i}-1)}{(t^{u/\alpha_{n+1}\cdots\alpha_{i-1}}-1)(t^{u/\alpha_i}-1)}$$

and of the $\mathbb{Z}[t,t^{-1}]$-module presented by

$$\begin{pmatrix} p_{11} & & & \\ p_{21} & p_{22} & & \\ \vdots & & \ddots & \\ p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,n-1} \end{pmatrix},$$

where the p_{ij} are defined recursively by

- $p_{ii}(t) = \frac{(t^u-1)(t^{d^i}-1)}{(t^{d^{i-1}}-1)}$, *and*
- $p_{ij}(t) = \frac{t^{d^j}-1}{t^{d^{j-1}}-1}\sum_{\ell=j+1}^i p_{i\ell}(t)\cdot\frac{t^{n_\ell}-1}{t^{d^\ell}-1}\cdot\gamma_{\ell-1}^j(t)$ *for* $1\le j<i$.

[4]Since $\Sigma(\alpha_1,\ldots,\alpha_n)=\Sigma(\alpha_1,\ldots,\alpha_n,1)$, we can always assume that $n<k$.

Therefore, the problem of the computation of the Alexander module of Seifert multilinks is settled. In chapter 5, we turn to the corresponding question for graph multilinks: these are the multilinks whose exterior is a graph manifold, that is, a manifold whose splice components are all Seifert fibered. In other words, a graph multilink is a multilink obtained by splicing Seifert multilinks. Graph multilinks where classified by Eisenbud and Neumann with the help of decorated trees named **splice diagrams**. All our results are expressed in terms of these combinatorial objects.

The general case seems out of reach, but we obtain closed formulas for several types of graph multilinks. In particular, we have the following generalization of theorem 4.3.1.

Theorem 5.2.2.
Let $L(m_1, \dots, m_n)$ be a graph multilink (with some $m_i \neq 0$) whose splice components are all non-fibered. Let us consider a splice diagram Γ for $L(\underline{m})$. If $\mathcal{N}$ denotes its set of nodes, every element $v \in \mathcal{N}$ corresponds to some non-fibered Seifert multilink; let us note d_v the greatest common divisor of its multiplicities, γ_v its generic Seifert fiber and α_v the product of the weights on the arrowhead edges of Γ adjacent to v.

Then, an Alexander matrix of $L(\underline{m})$ is given by

$$v\in\mathcal{N}\left\{ \left(\overbrace{\frac{\ell k(\gamma_v,\gamma_w)}{\alpha_w} \cdot (t^{d_v} - 1)}^{w\in\mathcal{N}} \quad \overbrace{\quad 0 \quad}^{n-1-\#\mathcal{N}} \right)\right. .$$

It is possible to simplify the problem by considering rings "bigger" then $\mathbb{Z}[t,t^{-1}]$, in other words, by scalar extension. For example, we give a closed formula for the rank of the Alexander module of any graph multilink $L(\underline{m})$. In order to state this result, let us introduce several notations. Let Γ be a minimal splice diagram of $L(\underline{m})$. Each node of Γ corresponds to a Seifert multilink, fibered or non-fibered; we will speak of **fibered nodes** and **non-fibered nodes**. Also, a node is a **0-node** if all the multiplicities of the corresponding Seifert multilink are zero. Decompose Γ along the edges connecting a fibered node with a non-fibered one, and delete all the connected components that correspond to fibered multilinks. Let us denote the result by Γ^{NF}.

Theorem 5.3.1.
The rank over $\mathbb{Z}[t,t^{-1}]$ of the Alexander module of a graph multilink $L(\underline{m})$ is given by

$$rk\,\mathcal{M}_{\mathbb{Z}}^{L(\underline{m})} = \begin{cases} n & \text{if } \underline{m} = \underline{0}; \\ n - r - k + c - 1 & \text{else,} \end{cases}$$

where c denotes the number of connected components of Γ (that is, the number of irreducible components of L), r the number of connected components of Γ^{NF}, n the number of arrowheads in Γ^{NF} and k the number of non-fibered nodes that are not 0-nodes.

Another natural scalar extension is given by the **Novikov ring** $\widehat{\Lambda} = \mathbb{Z}[\![t]\!][t^{-1}]$. The module

$$\widehat{H}_L(\underline{m}) = \mathrm{H}_1\widetilde{X}(\underline{m}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[\![t]\!][t^{-1}]$$

is called the **Novikov homology** of $L(\underline{m})$. With the notations introduced previously, we have the following result.

Theorem 5.4.2.
Let $L(\underline{m})$ be a graph multilink (with $\underline{m} \neq \underline{0}$) given by a splice diagram Γ. Then, its Novikov homology is equal to

$$\widehat{H}_L(\underline{m}) = \mathcal{T} \oplus \widehat{\Lambda}^{n-r-k+c-1},$$

where $\mathcal{T}$ is the $\widehat{\Lambda}$-module presented by the $(k \times k)$-matrix $\mathcal{P} = (p_{vw})$, where $p_{vw} = \frac{\ell k(\gamma_v, \gamma_w)}{\alpha_w}$ if v and w are in the same connected component of Γ^{NF}, and $p_{vw} = 0$ otherwise.
Furthermore, if Γ is minimal, then $\mathcal{T}$ is the torsion submodule of $\widehat{H}_L(\underline{m})$, and $n-r-k+c-1$ its rank.

Corollary 5.4.3.
Let L be a graph link with n components. Then, there exists a finite number of hyperplanes in $\mathbb{Z}^n$, defined by homogeneous linear equations over the integers, such that the Novikov homology $\widehat{H}_L(\underline{m})$ only depends on which hyperplanes $\underline{m}$ belongs.

Corollary 5.4.4.
Let L be a graph link with c irreducible components and b splice components. Then, the number of non-isomorphic Novikov modules (corresponding to the different multiplicities) is bounded above by $3^b - 2(b-c)$.

Finally, we compute the Alexander module over $\mathbb{C}[t, t^{-1}]$ of a wide class of graph multilinks. Namely, we make the following assumption: if one decomposes Γ, a splice diagram for $L(\underline{m})$, along the set $\mathcal{E}'$ of edges connecting a fibered node and a non-fibered one, then the subdiagram Γ^F gathering all the fibered nodes is connected.

The multilink corresponding to Γ^F is fibered, so its Alexander module can be computed using results of Eisenbud and Neumann. Let us note ℓ the number of components of this multilink, d the greatest common divisor of its multiplicities, $\Delta(t)$ the characteristic polynomial of its monodromy h_*, and $\Delta'(t)$ the polynomial whose roots are the eigenvalues of h_* corresponding to the 2×2 Jordan blocks. Given $E \in \mathcal{E}'$, let d_E be the greatest common divisor of the multiplicities involved in the corresponding splicing. Also, let $\mathcal{N}_1$ be the set of non-fibered nodes that are not 0-nodes, and let r (as above) denote the number of connected components of Γ^{NF}. Finally, let us note

$$\epsilon(t) = \begin{cases} \gcd\left(t^d - 1, \{\frac{t^{d_E}-1}{t^{d_v}-1}\}_{(v,E)} \ ; \ E \in \mathcal{E}' \text{ is adjacent to } v \in \mathcal{N}_1\right) & \text{if } \#\mathcal{E}' = \ell; \\ 1 & \text{if } \#\mathcal{E}' < \ell. \end{cases}$$

Theorem 5.5.2.
The Alexander module $\mathcal{M}_{\mathbb{C}}^{L(\underline{m})}$ is determined by the following properties.

- *The rank of $\mathcal{M}_{\mathbb{C}}^{L(\underline{m})}$ is equal to $n - \#\mathcal{N}_1 - r$.*
- *The order ideal of the torsion submodule $\mathcal{T} = Tor\, \mathcal{M}_{\mathbb{C}}^{L(\underline{m})}$ is generated by*

$$\widetilde{\Delta}(t) = \frac{\Delta(t) \cdot \epsilon(t) \cdot \prod_{v \in \mathcal{N}_1}(t^{d_v} - 1)}{\prod_{E \in \mathcal{E}'}(t^{d_E} - 1)}.$$

- *The Jordan normal form of t restricted to $\mathcal{T}$ has blocks of dimension at most 2; the eigenvalues corresponding to the 2×2 blocks are the roots of $\Delta'(t)$.*

Given $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ a polynomial application, its regular link at infinity $\mathcal{L}(f,\infty)$ is in the class of graph multilinks defined above. Therefore, we can compute its Alexander module over $\mathbb{C}[t,t^{-1}]$.

Theorem 5.6.1.
Let $L(m_1,\ldots,m_n)$ be the multilink associated with $\mathcal{L}(f,\infty)$, and let us note $d = \gcd(\underline{m})$, $d_i = \gcd(m_i, \sum_{j\neq i} m_j \ell k(L_i,L_j))$, $\Delta(t)$ the characteristic polynomial of the monodromy of $L(\underline{m})$, and $\Delta'(t)$ the polynomial corresponding to the 2×2 Jordan blocks. Then, the Alexander module $\mathcal{M}_{\mathbb{C}}^{\mathcal{L}}$ is given by the following properties.

- *rk* $\mathcal{M}_{\mathbb{C}}^{\mathcal{L}} = \sum_{i=1}^{n}(d_i - 1)$.
- *The order ideal of the torsion of $\mathcal{M}_{\mathbb{C}}^{\mathcal{L}}$ is generated by*

$$\widetilde{\Delta}(t) = \frac{\Delta(t)}{\frac{t-1}{t^d-1}\prod_{i=1}^{n}\frac{t^{d_i}-1}{t-1}} .$$

- *The Jordan normal form of the monodromy restricted to the torsion of $\mathcal{M}_{\mathbb{C}}^{\mathcal{L}}$ has blocks of dimension at most two; the 2×2 blocks correspond to the roots of $\Delta'(t)$.*

Finally, chapter 6 is dedicated to the computation of the Conway potential function of graph links.

Theorem 6.2.7.
Let L be a graph link with n components given by a splice diagram Γ. Then, its Conway potential function is equal to

$$\nabla_L(t_1,\ldots,t_n) = (-1)^{k_-}\prod_v (t_1^{\ell_{1v}}\cdots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}}\cdots t_n^{-\ell_{nv}})^{\delta_v - 2},$$

where the product is over all non-arrowhead vertices v of Γ, δ_v is the valency of the vertex v, ℓ_{iv} denotes the linking number of L_i with the component corresponding to v, and k_- is equal to the number of (-1)-weighted arrowheads.

Chapter 1

The Alexander Module of Multilinks

Multilinks appear naturally in the study of complex plane curve singularities. They can be considered as a generalization of oriented links. Many classical invariants of oriented links, such as the Alexander module, the Alexander polynomial and the torsion numbers can be extended to multilinks. Also, the notion of "fibered multilink" can be defined, that generalizes the usual concept of fibered link.

In this chapter, we give an algorithm to compute the Alexander module of any multilink in a homology sphere. Generalized Seifert surfaces and Seifert forms are introduced for this purpose. Then, we apply Fox free differential calculus to obtain a presentation of the Alexander module of a multilink in S^3 from a given projection. Finally, the computation of torsion numbers for multilinks is discussed.

1.1 Basic Facts about Multilinks

A **multilink** is an oriented link $L = L_1 \cup \ldots \cup L_n$ in an oriented homology 3-sphere Σ, together with an integer m_i associated with each component L_i. The following notations will be used for multilinks:

$$m_1 L_1 \cup \ldots \cup m_n L_n = L(m_1, \ldots, m_n) = L(\underline{m}) \ .$$

We also need the convention that $m_i L_i = (-m_i)(-L_i)$, where $-L_i$ represents L_i with the opposite orientation.

Of course, a set of multiplicities $\underline{m} = (m_1, \ldots, m_n)$ can be interpreted as an element of $\mathrm{H}_1 L = \bigoplus_{i=1}^n \mathbb{Z} L_i$. If X denotes the complement $\Sigma - L$ of L, the Alexander duality and the universal coefficient theorem give the isomorphisms

$$\mathrm{H}_1 L \simeq \mathrm{H}^1 X \simeq \mathrm{Hom}(\mathrm{H}_1 X, \mathbb{Z}).$$

Hence, $\underline{m}$ also represents the morphism $\mathrm{H}_1 X \longrightarrow \mathbb{Z}$ given by

$$\gamma \longmapsto \sum_{i=1}^{n} m_i \cdot \ell k(\gamma, L_i).$$

Furthermore, using Hurewicz theorem and a theorem of Whitehead, we get

$$\mathrm{Hom}(\mathrm{H}_1 X, \mathbb{Z}) \simeq \mathrm{Hom}(\pi_1 X, \mathbb{Z}) \simeq [X, S^1].$$

So a set of multiplicities defines a homotopy class $\underline{m} \in [X, S^1]$; this class is determined by the following criterion:

$$\phi \in \underline{m} \Leftrightarrow \deg(\phi|_{M_i}) = m_i \text{ for all } i = 1, \dots, n,$$

where M_i is an oriented meridian of L_i such that $\ell k(M_i, L_i) = +1$.

As a consequence, assigning a set of multiplicities to an oriented link is a way to specify a preferred infinite cyclic covering $\widetilde{X}(\underline{m}) \xrightarrow{p} X$: it is the unique regular covering of X given by the morphism $\pi_1 X \xrightarrow{\underline{m}} \mathbb{Z}$. Equivalently, it can be defined as the pullback $\mathbb{Z}$-bundle $\phi^* \exp$, where $\mathbb{R} \xrightarrow{\exp} S^1$ is the universal $\mathbb{Z}$-bundle and $X \xrightarrow{\phi} S^1$ any map in the homotopy class $\underline{m}$:

$$\begin{array}{ccc} \widetilde{X}(\underline{m}) & \longrightarrow & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \exp} \\ X & \xrightarrow{\phi} & S^1 . \end{array}$$

Note that $\widetilde{X}(\underline{m})$ is not connected in general.

Choosing a generator t of the infinite cyclic group of the covering endows $\mathrm{H}_*\widetilde{X}(\underline{m})$ with a structure of module over $\mathbb{Z} < t >= \mathbb{Z}[t, t^{-1}]$. Most of these modules are not interesting: we will prove that $\mathrm{H}_0\widetilde{X}(\underline{m}) \simeq \mathbb{Z}[t, t^{-1}]/(t^d - 1)$, where d is the greatest common divisor of the multiplicities m_i, and that $\mathrm{H}_2\widetilde{X}(\underline{m})$ is a free module with the same rank as $\mathrm{H}_1\widetilde{X}(\underline{m})$ (see corollary 1.1.2 and proposition 1.1.4). Of course, $\mathrm{H}_i\widetilde{X}(\underline{m}) = 0$ for all $i \geq 3$. Therefore, the only interesting module is $\mathrm{H}_1\widetilde{X}(\underline{m})$: it is called the **Alexander module** of the multilink $L(\underline{m})$. This is a (cumbersome) invariant of the multilink.

To extract handy information from a given module, the standard trick is to consider its elementary ideals. Although these objects are widely used, their definitions vary according to the authors (see e.g. [7, 10]). Therefore, let us now clarify the meaning of these concepts in the present work. Let M be a module over a ring R. A **finite presentation** of M is an exact sequence $F \xrightarrow{\phi} E \longrightarrow M \longrightarrow 0$, where E and F are free R-modules with finite basis. If A is a matrix of ϕ, then its transposed A^T is a **presentation matrix** of M. Each column of A^T corresponds to a generator of M, and each line to a relation.[1] If M has an $m \times n$ presentation matrix P, the $\mathbf{r^{th}}$ **elementary ideal** of M, denoted by $E_r M$, is the ideal of R generated by the $(n-r) \times (n-r)$ minors of P. It is easy to check that these ideals do not depend on the presentation of M. Note that if $n = m$, P is a square matrix and $E_0 M$ is the principal ideal of R generated by the determinant of P. Now, let us denote by $\Delta_r M$ a generator of the smallest principal ideal that contains $E_r M$; if R is factorial, $\Delta_r M$ is simply the greatest common divisor of the $(n-r) \times (n-r)$ minors of a presentation matrix. Note that this element $\Delta_r M$ of R is only defined up to multiplication by a unit of R. Given Δ and Δ' in a ring R, let us note $\Delta \doteq \Delta'$ if $\Delta = u \cdot \Delta'$ for some unit u of R.

Let us now turn back to multilinks. Given a multilink $L(\underline{m})$, we just defined the Alexander module of $L(\underline{m})$ as some module $\mathrm{H}_1\widetilde{X}(\underline{m})$ over the ring $\mathbb{Z}[t, t^{-1}]$. A presentation matrix of

[1] Here, we follow the convention of Rolfsen [35]. Many authors consider A as the presentation matrix.

this module is an **Alexander matrix** of $L(\underline{m})$. The r$^{\text{th}}$ elementary ideal $E_r(\mathrm{H}_1\widetilde{X}(\underline{m}))$ is the **r$^{\text{th}}$ Alexander ideal** of $L(\underline{m})$. The Laurent polynomial $\Delta_r(\mathrm{H}_1\widetilde{X}(\underline{m}))$ is called the **r$^{\text{th}}$ Alexander polynomial** of $L(\underline{m})$; $\Delta_0(\mathrm{H}_1\widetilde{X}(\underline{m}))$ is called the **Alexander polynomial** of $L(\underline{m})$, and is denoted by $\Delta^{L(\underline{m})}$. Again, note that $\Delta^{L(\underline{m})}$ is only defined up to multiplication by a unit of $\mathbb{Z}[t,t^{-1}]$, that is, by $\pm t^i$.

Of course, if a multilink has weights $m_1 = \ldots = m_n = \pm 1$, all these Alexander invariants correspond to the usual Alexander invariants of the corresponding oriented link. We will speak of this special case as the **usual case**.

Let us start with several basic properties of the Alexander invariants of multilinks.

1.1.1 Proposition.
Let d be a positive integer; then, there is an isomorphism

$$H_*\widetilde{X}(d\cdot\underline{m}) \;=\; H_*\widetilde{X}(\underline{m}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[T,T^{-1}]\,,$$

where $\mathbb{Z}[T,T^{-1}]$ is endowed with the structure of $\mathbb{Z}[t,t^{-1}]$-algebra given by $t \longmapsto T^d$.

Proof. Let $X \xrightarrow{\phi} S^1$ be a map in the homotopy class $\underline{m}$. Then, ϕ^d is in the homotopy class $d\cdot\underline{m}$. By definition of the pullback,

$$\begin{aligned}\widetilde{X}(d\cdot\underline{m}) &= \left\{ (x,\lambda) \in X\times\mathbb{R} \mid (\phi(x))^d = e^{2i\pi\lambda} \right\} \\ &= \bigsqcup_{k=0}^{d-1} \left\{ (x,\lambda) \in X\times\mathbb{R} \mid \phi(x) = e^{2i\pi(\lambda+k)/d} \right\} \\ &\simeq \bigsqcup_{k=0}^{d-1} \widetilde{X}(\underline{m}).\end{aligned}$$

Therefore, $\widetilde{X}(d\cdot\underline{m})$ is given by d copies of $\widetilde{X}(\underline{m})$. Furthermore, a generator T of the group of the covering $\widetilde{X}(d\cdot\underline{m}) \longrightarrow X$ permutes these copies cyclically, and T^d acts on each copy as a generator t of the group of the covering $\widetilde{X}(\underline{m}) \longrightarrow X$. This gives the isomorphism stated in the proposition. □

1.1.2 Corollary.
$H_0\widetilde{X}(\underline{m}) \simeq \mathbb{Z}[t,t^{-1}]/(t^d-1)$, *where* $d = \gcd(m_1,\ldots,m_n)$.

Proof. If $d=1$, then $\pi_1 X \xrightarrow{\underline{m}} \mathbb{Z}$ is onto, so $\widetilde{X}(\underline{m})$ is connected and the result holds. The general case in settled by proposition 1.1.1. □

1.1.3 Corollary.
Let $\mathcal{P}(t)$ be an Alexander matrix of $L(\underline{m})$, and let d be a positive integer. Then, $\mathcal{P}(t^d)$ is an Alexander matrix of $L(d\cdot\underline{m})$. In particular, for the computation of the Alexander module of a multilink, it may be assumed that $\gcd(m_1,\ldots,m_n)=1$.

Proof. Let $\bigoplus_{i=1}^m \mathbb{Z}[t,t^{-1}]\,e_i \xrightarrow{\phi} \bigoplus_{j=1}^n \mathbb{Z}[t,t^{-1}]\,f_j \longrightarrow \mathrm{H}_1\widetilde{X}(\underline{m}) \longrightarrow 0$ be the presentation of $\mathrm{H}_1\widetilde{X}(\underline{m})$ corresponding to $\mathcal{P}(t)$; this means that the coefficients $p_{ij}(t)$ of $\mathcal{P}(t)$ are defined by $\phi(e_i) = \sum_{j=1}^{n} p_{ij}(t)\, f_j$. Tensoring this exact sequence by $\otimes_{\mathbb{Z}[t,t^{-1}]}\mathbb{Z}[T,T^{-1}]$, and using proposition 1.1.1, it follows that

$$\bigoplus_{i=1}^{m} \mathbb{Z}[T,T^{-1}]\,(e_i\otimes 1) \xrightarrow{\phi\otimes I} \bigoplus_{j=1}^{n} \mathbb{Z}[T,T^{-1}]\,(f_j\otimes 1) \longrightarrow \mathrm{H}_1\widetilde{X}(d\cdot \underline{m}) \longrightarrow 0$$

is a presentation of $\mathrm{H}_1\widetilde{X}(d\cdot\underline{m})$. Finally, the equalities

$$(\phi\otimes I)(e_i\otimes 1) = \phi(e_i)\otimes 1 = (\sum_j p_{ij}(t)\, f_j)\otimes 1 = \sum_j p_{ij}(T^d)\,(f_j\otimes 1)$$

show that this presentation corresponds to the matrix $\mathcal{P}(T^d)$. □

1.1.4 Proposition.
$H_2\widetilde{X}(\underline{m})$ is a free $\mathbb{Z}[t,t^{-1}]$-module. If the multiplicities are not all zero, then $rk\, H_2\widetilde{X}(\underline{m}) = rk\, H_1\widetilde{X}(\underline{m})$. In particular, if $H_2\widetilde{X}(\underline{m})$ is non trivial, then the Alexander polynomial $\Delta^{L(\underline{m})}$ vanishes.

Proof. Let us choose an equivariant cellular decomposition of $\widetilde{X} = \widetilde{X}(\underline{m})$; this gives free chain modules $C_i\widetilde{X}$. The exterior X of the link is a 3-manifold with boundary, so it has the homotopy type of a finite connected complex of dimension 2. Therefore, $\mathrm{H}_2\widetilde{X} = Z_2\widetilde{X}$. Hence, there is an exact sequence of $\mathbb{Z}[t,t^{-1}]$-modules

$$0 \longrightarrow \mathrm{H}_2\widetilde{X} \longrightarrow C_2\widetilde{X} \xrightarrow{\partial} B_1\widetilde{X} \longrightarrow 0\,,$$

which means that $\mathrm{H}_2\widetilde{X}$ is the first syzygy module of $B_1\widetilde{X}$ (with respect to the set of generators given by the cellular chains). Now, $B_1\widetilde{X}$ is a submodule of the free module $C_1\widetilde{X}$. The global dimension of the ring $\mathbb{Z}[t,t^{-1}]$ being equal to 2 (use [48, Proposition 3.3.10 and Theorem 4.3.7]), $B_1\widetilde{X}$ has projective dimension less or equal to 1. This means that in any projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} B_1\widetilde{X} \longrightarrow 0\,,$$

the image of $P_1 \longrightarrow P_0$ is projective. In particular, this holds for any first syzygy module of $B_1\widetilde{X}$, so $\mathrm{H}_2\widetilde{X}$ is projective. Since $\widetilde{K}_0(\mathbb{Z}[t,t^{-1}])$ is trivial (see [36, Chapter 3.3]), $\mathrm{H}_2\widetilde{X}$ is free.[2]

It remains to check that $\mathrm{rk}\,\mathrm{H}_2\widetilde{X} = \mathrm{rk}\,\mathrm{H}_1\widetilde{X}$. Since X has the homotopy type of a finite 2-complex, and $\chi(X) = 1-n+(n-1) = 0$, there exists a cellular decomposition of this complex (also denoted by X) with a single vertex X^0, μ 1-cells, $(\mu-1)$ 2-cells and no cell of higher dimension (for some $\mu \geq 1$). Choosing an equivariant cellular decomposition of $\widetilde{X}$, we obtain chain modules $C_i\widetilde{X}$ that are free $\mathbb{Z}[t,t^{-1}]$-modules of rank b_i, with $b_0 = 1$, $b_1 = \mu$, $b_2 = \mu - 1$ and $b_i = 0$ if $i \geq 3$. Hence,

$$\mathrm{rk}\,\mathrm{H}_0\widetilde{X} - \mathrm{rk}\,\mathrm{H}_1\widetilde{X} + \mathrm{rk}\,\mathrm{H}_2\widetilde{X} = b_0 - b_1 + b_2 = 0.$$

By corollary 1.1.2, $\mathrm{H}_0\widetilde{X}$ is a torsion module unless $d = 0$. So if the multiplicities are not all zero, the equality $\mathrm{rk}\,\mathrm{H}_0\widetilde{X} = 0$ follows, as well as the proposition. □

[2] I wish to thank Professor Eisenbud for suggesting this argument.

1.1.5 Proposition.
For any multilink, there exists a square Alexander matrix. In particular, $E_0(H_1\widetilde{X}(\underline{m}))$ is a principal ideal, and the Alexander polynomial of a multilink can be defined as the determinant of a square Alexander matrix.

Proof. As described in the proof of proposition 1.1.4 above, there exists an equivariant cellular decomposition of the 2-complex $\widetilde{X}$, so that the chain modules $C_i\widetilde{X}$ are free $\mathbb{Z}[t,t^{-1}]$-modules of rank b_i, with $b_0 = 1$, $b_1 = \mu$, $b_2 = \mu - 1$ and $b_i = 0$ if $i \geq 3$. Let $\widetilde{X}^r$ denotes the r-skeleton of $\widetilde{X}$; the exact sequence of the triple $\widetilde{X}^0 \subset \widetilde{X}^1 \subset \widetilde{X}^2 = \widetilde{X}$ gives

$$\begin{array}{ccccccc}
\mathrm{H}_2(\widetilde{X}^2,\widetilde{X}^1) & \longrightarrow & \mathrm{H}_1(\widetilde{X}^1,\widetilde{X}^0) & \longrightarrow & \mathrm{H}_1(\widetilde{X}^2,\widetilde{X}^0) & \longrightarrow & \mathrm{H}_1(\widetilde{X}^2,\widetilde{X}^1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_2\widetilde{X} & \longrightarrow & C_1\widetilde{X} & \longrightarrow & \mathrm{H}_1(\widetilde{X},\widetilde{X}^0) & \longrightarrow & 0
\end{array}$$

where all the vertical arrows are canonical isomorphisms. This is a presentation of $\mathrm{H}_1(\widetilde{X},\widetilde{X}^0)$ with μ generators and $\mu - 1$ relations. The exact sequence of the pair $(\widetilde{X},\widetilde{X}^0)$ gives

$$0 \to \mathrm{H}_1\widetilde{X} \to \mathrm{H}_1(\widetilde{X},\widetilde{X}^0) \to \mathrm{H}_0(\widetilde{X}^0) \xrightarrow{\epsilon} \mathrm{H}_0\widetilde{X} \to 0.$$

It it is easy to see that Ker (ϵ) is free of rank one. Thus, the sequence splits, and a presentation matrix of $\mathrm{H}_1\widetilde{X}$ can be obtained from a presentation matrix of $\mathrm{H}_1(\widetilde{X},\widetilde{X}^0)$ by adding one relation. This gives a square Alexander matrix of the multilink. □

We will give two other proofs of this fact in the next chapter, one via Seifert surfaces, the other using free differential calculus.

1.1.6 Proposition.
Let $L(\underline{m})$ be a multilink with n components and some non-zero multiplicity. Then, the rank of $H_1\widetilde{X}(\underline{m})$ is at most $n-1$.

Proof. It follows from the presentation

$$C_2\widetilde{X} \to C_1\widetilde{X} \to \mathrm{H}_1(\widetilde{X},\widetilde{X}^0) \to 0$$

that $\mathrm{H}_1(\widetilde{X},\widetilde{X}^0) \otimes \mathbb{Z} \simeq \mathrm{H}_1(X,X^0) = \mathrm{H}_1X$, which by Alexander duality is free of rank n. Therefore, the rank of $\mathrm{H}_1(\widetilde{X},\widetilde{X}^0)$ as a $\mathbb{Z}[t,t^{-1}]$-module is at most n. The exact sequence of the pair $(\widetilde{X},\widetilde{X}^0)$ implies that $\mathrm{rk}\,\mathrm{H}_1(\widetilde{X},\widetilde{X}^0) = \mathrm{rk}\,\mathrm{H}_1\widetilde{X} + 1$, and the proposition is proved. □

Let us now define a very interesting class of multilinks, that generalizes the concept of fibered link: a **fibered multilink** is a multilink $L(\underline{m})$ such that there exists a locally trivial fibration $X \xrightarrow{\varphi} S^1$ in the homotopy class $\underline{m} \in [X, S^1]$. Let us note $F := \varphi^{-1}(1)$, the **fiber** of $L(\underline{m})$. The diagram

$$\begin{array}{ccc}
\widetilde{X}(\underline{m}) & \xrightarrow{\Phi} & \mathbb{R} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \exp} \\
X & \xrightarrow{\varphi} & S^1
\end{array}$$

can now be understood as defining the pullback fibration $\Phi = \exp^* \varphi$. Since $\mathbb{R}$ is contractible, there exists a homeomorphism $F \times \mathbb{R} \longrightarrow \widetilde{X}(\underline{m})$ such that the following diagram commutes:

$$\begin{array}{ccc} F \times \mathbb{R} & \longrightarrow & \widetilde{X}(\underline{m}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathbb{R} & = & \mathbb{R}. \end{array}$$

Hence, the generator $\widetilde{X}(\underline{m}) \xrightarrow{t} \widetilde{X}(\underline{m})$ of the infinite cyclic group of the covering p can be seen as the transformation

$$\begin{array}{ccc} F \times \mathbb{R} & \longrightarrow & F \times \mathbb{R} \\ (x, z) & \longmapsto & (h(x), z+1), \end{array}$$

where $h\colon F \longrightarrow F$ is some homeomorphism, unique up to isotopy, called the **geometric monodromy**. In our context, it is called the **monodromy** of the multilink $L(\underline{m})$. The induced homomorphism $h_*\colon \mathrm{H}_1 F \longrightarrow \mathrm{H}_1 F$ is the **algebraic monodromy** of the multilink.

Let us give several easy properties of fibered multilinks with fiber $F \subset \Sigma - L$ and monodromy $h\colon F \longrightarrow F$.

1.1.7 Proposition.
The fiber of a multilink has $d = \gcd(m_1, \dots, m_n)$ connected components.

Proof. The exact homotopy sequence of the fibration takes the form

$$0 \longrightarrow \pi_1 F \longrightarrow \pi_1 X \xrightarrow{\varphi_\#} \mathbb{Z} \longrightarrow \pi_0 F \longrightarrow 0 ,$$

where $\varphi_\#(\gamma) = \sum_i m_i \,\ell k(\gamma, L_i)$. Hence, $\pi_0 F = Coker\ \varphi_\# \simeq \mathbb{Z}/d\mathbb{Z}$. □

1.1.8 Proposition.
Let us note $\overline{F} = F \cup L$. Then, the Betti numbers $b_i = rk\,\widetilde{H}_i F$ and $\overline{b}_i = rk\,\widetilde{H}_i \overline{F}$ satisfy the relation $b_i = \overline{b}_{2-i}$ for all i.

Proof. The fibration $\Sigma - L \longrightarrow S^1$ yields a fibration $\Sigma - \overline{F} \longrightarrow (0,1)$. Therefore, $\Sigma - \overline{F}$ is homeomorphic to $F \times (0,1)$. Using this homeomorphism and Alexander duality, it follows that

$$\widetilde{\mathrm{H}}_i F = \widetilde{\mathrm{H}}_i(F \times (0,1)) \simeq \widetilde{\mathrm{H}}_i(\Sigma - \overline{F}) \simeq \widetilde{\mathrm{H}}^{2-i} \overline{F} \simeq \widetilde{\mathrm{H}}_{2-i} \overline{F} ,$$

which gives the result. □

1.1.9 Proposition.
An Alexander matrix of a fibered multilink is given by $H^T - tI$, where H is any matrix of the algebraic monodromy $h_*\colon H_1 F \longrightarrow H_1 F$. In particular, the Alexander polynomial of a fibered multilink is the characteristic polynomial of the monodromy.

Proof. By the above discussion, there is an isomorphism of $\mathbb{Z}$-modules $f\colon \mathrm{H}_1 F \longrightarrow \mathrm{H}_1 \widetilde{X}(\underline{m})$ such that $t \cdot f(x) = f(h_*(x))$. Choosing a $\mathbb{Z}$-basis $e_1, \dots, e_\mu$ of $\mathrm{H}_1 F$, there is an exact sequence of $\mathbb{Z}[t, t^{-1}]$-modules

$$\bigoplus_{i=1}^{\mu} \mathbb{Z}[t, t^{-1}]\, e_i \xrightarrow{h_* - t} \bigoplus_{i=1}^{\mu} \mathbb{Z}[t, t^{-1}]\, e_i \xrightarrow{f_*} \mathrm{H}_1 \widetilde{X}(\underline{m}) \longrightarrow 0,$$

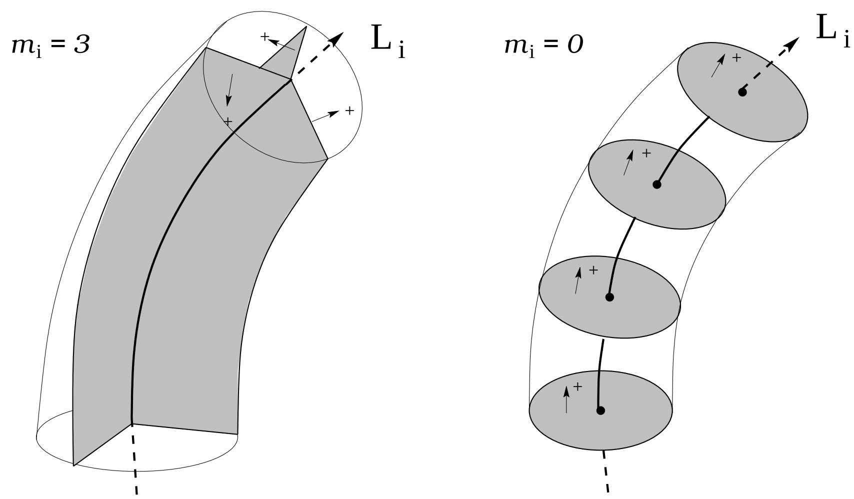


Figure 1.1: A Seifert surface near the multilink.

where f_* denotes the $\mathbb{Z}[t, t^{-1}]$-linear extension of f. This is a finite presentation of $\mathrm{H}_1\widetilde{X}(\underline{m})$; therefore, $(H - tI)^T$ is a presentation matrix of this module. □

1.2 Seifert Surfaces and Seifert Forms

The aim of this paragraph is to give an algorithm for the computation of the Alexander module of a multilink in a homology sphere. Our method will be to mimic as much as possible the fibered case. In particular, the fiber will be replaced by a "good Seifert surface", that we define now.

The concept of Seifert surface for multilinks was introduced by Eisenbud and Neumann [11]; it is the obvious generalization of Seifert surfaces for oriented links. More precisely: a **Seifert surface** for a multilink $L(\underline{m}) = m_1 L_1 \cup \ldots \cup m_n L_n$ is an open embedded oriented surface $F \subset \Sigma - L$, such that, if $\mathcal{N}(L_i)$ denotes a closed tubular neighborhood of L_i and F_0 the closure of $F - (F \cap \bigcup_i \mathcal{N}(L_i))$, then for all i:

- If $m_i \neq 0$, $\overline{F} \cap \mathcal{N}(L_i)$ consists of $|m_i|$ sheets meeting along L_i ; F is oriented such that $\partial F_0 = m_i L_i$ in $\mathrm{H}_1(\mathcal{N}(L_i))$.
- If $m_i = 0$, $\overline{F} \cap \mathcal{N}(L_i)$ consists of discs transverse to L_i; F is oriented such that the intersection number of L_i with each of these discs is the same (either always $+1$ or always -1).

This is illustrated by figure 1.1.

Note that $F \subset \Sigma - L$ and $F_0 \subset \Sigma - \mathcal{N}(L)$ determine each other up to isotopy; to simplify the notation, we will consider both of them as Seifert surfaces, and denote both by F. We will denote by $\overline{F}$ the union of $F \subset \Sigma - L$ and L; if no multiplicity is zero, this is simply the closure of F in $\Sigma - L$.

The following proposition can be considered as an alternative definition of a Seifert surface.

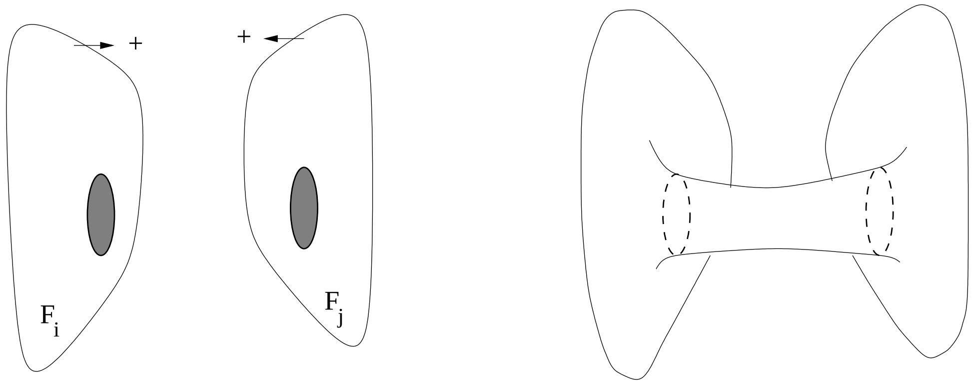


Figure 1.2: Surgery.

1.2.1 Proposition. (Eisenbud-Neumann [11])
An oriented surface F properly embedded in $\Sigma - \mathcal{N}(L)$ is a Seifert surface for $L(\underline{m}) = m_1L_1 \cup \ldots \cup m_nL_n$ if and only if, for all i,

$$F \cap \partial\mathcal{N}(L_i) = d_iL_i(p_i, q_i),$$

a $d_i(p_i, q_i)$-cable on L_i, where d_i is the greatest common divisor of $m'_i := -\sum_{j\neq i} m_j\ell k(L_i, L_j)$ and m_i, sgn(d_i)=sgn(m_i)[3], $p_i = \frac{m_i}{d_i}$ and $q_i = \frac{m'_i}{d_i}$. □

If $L(\underline{m})$ is a fibered multilink, its fiber F is a Seifert surface for $L(\underline{m})$ with $d = \gcd(m_1, \ldots, m_n)$ connected components (recall proposition 1.1.7). The following statement generalizes this fact to any multilink.

1.2.2 Proposition.
Let $L(\underline{m})$ be a multilink, and let d be the greatest common divisor of $m_1, \ldots, m_n$; then, there exists a Seifert surface for $L(\underline{m})$ with d connected components. Moreover, there is no Seifert surface for $L(\underline{m})$ with less then d connected components.

Such a Seifert surface will be called a **good Seifert surface** for $L(\underline{m})$.

Proof. Let $F \subset \Sigma - L$ be any Seifert surface for $L(\underline{m})$ without closed component, and let us note $\overline{F} = F \cup L$. Also, let $i_+, i_-\colon \mathrm{H}_0F \longrightarrow \mathrm{H}_0(\Sigma - \overline{F})$ be the epimorphisms induced by the push in the positive or negative normal direction off F.

If i_+ or i_- is not an isomorphism, there are two distinct connected components F_i and F_j of F that can be connected by a tube as in figure 1.2 to give a new Seifert surface F' with less connected components than F. This operation is called surgery. Let us suppose that all the possible surgeries have been made, yielding $F = F_1 \cup \ldots \cup F_r$ with $\mathrm{H}_0F \simeq \mathrm{H}_0(\Sigma - \overline{F})$. We have to see that $r = d$.

Let us consider $\overline{F}$ as the union $F \cup M$, where $M = \mathcal{N}(L) \cap \overline{F}$; the Mayer-Vietoris exact sequence gives

$$0 \longrightarrow \mathrm{H}_2\overline{F} \longrightarrow \mathrm{H}_1(\partial F) \longrightarrow \mathrm{H}_1F \oplus \mathrm{H}_1M.$$

[3] If $m_i = 0$, set $d_i = m'_i$, that can be chosen positive.

In particular, the natural homomorphism $\mathrm{H}_2\overline{F} \xrightarrow{\overline{\partial}} \mathrm{H}_1 M$ is zero. Since F is good, it follows that $\mathrm{H}_0 F \simeq \mathrm{H}_0(\Sigma - \overline{F})$, giving an isomorphism $\psi\colon \mathrm{H}_2\overline{F} \longrightarrow \widetilde{\mathrm{H}}_0 F$ equal to the composition

$$\mathrm{H}_2\overline{F} \simeq \mathrm{H}^2\overline{F} \simeq \widetilde{\mathrm{H}}_0(\Sigma - \overline{F}) \simeq \widetilde{\mathrm{H}}_0 F.$$

Furthermore, M has the homotopy type of $\mathcal{N}(L)$, giving a natural isomorphism $\phi\colon \mathrm{H}_1 M \longrightarrow \mathrm{H}_1(\mathcal{N}(L))$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}_2\overline{F} & \xrightarrow{\overline{\partial}} & \mathrm{H}_1 M \\ \psi\downarrow & & \downarrow\phi \\ \widetilde{\mathrm{H}}_0 F & \xrightarrow{\partial} & \mathrm{H}_1(\mathcal{N}(L)), \end{array}$$

where ∂ is induced by the morphism that sends a connected component F_i of $F \subset \Sigma - \mathcal{N}(L)$ to the homology class of its boundary ∂F_i in $\mathrm{H}_1(\mathcal{N}(L))$. Since $\overline{\partial}$ is zero and ϕ, ψ are isomorphisms, ∂ is also zero. This means that for all $i, j = 1, \dots, r$, $\partial F_i = \partial F_j \in \mathrm{H}_1(\mathcal{N}(L))$; it gives the equality

$$\sum_{i=1}^{n} m_i L_i = \partial F = \sum_{j=1}^{r} \partial F_j = r \cdot \partial F_1 \in \mathrm{H}_1(\mathcal{N}(L)) = \bigoplus_{i=1}^{n} \mathrm{H}_1(\mathcal{N}(L_i)).$$

Hence, r divides m_i for all i, that is, r divides d. This concludes the proof for $d = 1$; if $d > 1$, a good Seifert surface is obtained by d copies of a connected surface for $L(\frac{m_1}{d}, \dots, \frac{m_n}{d})$.

It remains to show that if $d > 1$, there is no surface with less then d connected components. Consider a Seifert surface F with $r < d$ components; by the argument above, one component of F would provide a connected Seifert surface for $L(\frac{\underline{m}}{r})$, with $\gcd(\frac{\underline{m}}{r}) = \frac{d}{r} > 1$. Therefore, it is sufficient to check that there is no connected surface for a multilink with $d > 1$. So, let F be a connected Seifert surface for $L(\underline{m})$ with $d > 1$. Since $i_+\colon \mathrm{H}_0 F \longrightarrow \mathrm{H}_0(\Sigma - \overline{F})$ is surjective, $\Sigma - \overline{F}$ is connected. Take P in F, and γ a path in $\Sigma - \overline{F}$ joining i_+P and i_-P. This gives a cycle γ such that $\underline{m}(\gamma) = \ell k(L(\underline{m}), \gamma) = \gamma \cdot F = 1$. But $\mathrm{Im}\,(\underline{m}) = d\mathbb{Z}$, with $d > 1$: a contradiction. □

1.2.3 Corollary.
F is good $\iff$ $H_0 F \simeq H_0(\Sigma - \overline{F})$, where $\overline{F} = F \cup L$. □

If $L(\underline{m})$ is a fibered multilink with fiber F, the (reduced) Betti numbers b_i of F and $\overline{b}_i$ of $\overline{F}$ are related by $b_i = \overline{b}_{2-i}$ for all i (proposition 1.1.8). The following corollary generalizes this fact to any multilink with good Seifert surface F.

1.2.4 Corollary.
Let F be a Seifert surface for $L(m_1, \dots, m_n)$; let b_i be the rank of $\widetilde{H}_i F$ and $\overline{b}_i$ the rank of $\widetilde{H}_i\overline{F}$. Then F is good $\iff$ $\overline{b}_2 = b_0$, $\overline{b}_1 = b_1 + r$ and $\overline{b}_0 = b_2 + r$, where r is the number of indices i such that $m_i = m'_i = 0$.

Proof. It is clear that all the $\mathbb{Z}$-modules considered here are free. In general, there is an epimorphism

$$\widetilde{H}_0 F \twoheadrightarrow \widetilde{H}_0(\Sigma - \overline{F}) \simeq \mathrm{H}^2\overline{F} \simeq \mathrm{H}_2\overline{F};$$

by corollary 1.2.3, the first arrow is an isomorphism if and only if F is good. In that case, it is easy to see that $\overline{F}$ has $r+1$ connected components. Furthermore, $\overline{F}$ is the union $F \cup M$, with $M = \mathcal{N}(L) \cap \overline{F} \searrow L$ and $F \cap M \searrow \partial F$. Therefore, $\chi(\overline{F}) = \chi(F) + \chi(M) - \chi(F \cap M) = \chi(F)$. This equality and the fact that $\mathrm{H}_2 F = 0$ give the result. □

This concludes our study of the Seifert surfaces of multilinks. We now turn to the natural generalization of the Seifert form to multilinks. Let F be a good Seifert surface for $L(\underline{m})$; the **Seifert forms** associated with F are the bilinear forms

$$\begin{array}{rccc} \alpha_F^+, \alpha_F^- \colon & \mathrm{H}_1 F \times \mathrm{H}_1 \overline{F} & \longrightarrow & \mathbb{Z} \\ & (x, y) & \longmapsto & \ell k(i_\pm x, y), \end{array}$$

where $i_+, i_- \colon \mathrm{H}_1 F \longrightarrow \mathrm{H}_1(\Sigma - \overline{F})$ are the morphisms induced by the push in the positive or negative normal direction off F. We will use the notation $A_F^\pm$ for matrices of these forms, called **Seifert matrices**.

In the usual case, the inclusion of F in $\overline{F}$ is a homotopy equivalence, giving a canonical isomorphism $\mathrm{H}_1 F = \mathrm{H}_1 \overline{F}$. The bilinear form $\alpha_F^+ \colon \mathrm{H}_1 F \times \mathrm{H}_1 F \longrightarrow \mathbb{Z}$ is the usual Seifert form, and A_F^+ is a usual Seifert matrix. Furthermore, $A_F^- = \left(A_F^+\right)^T$ since $\ell k(i_- x, y) = \ell k(x, i_+ y) = \ell k(i_+ y, x)$. In the general case of a multilink however, there is no canonical isomorphism between $\mathrm{H}_1 F$ and $\mathrm{H}_1 \overline{F}$. Therefore, both Seifert matrices A_F^+ and A_F^- need to be considered.

1.2.5 Proposition.
If the multilink is fibered with fiber F, the matrices $A_F^\pm$ are unimodular. Furthermore, a matrix of the monodromy is given by $(A_F^+ \cdot (A_F^-)^{-1})^T$.

Proof. The bilinear form $\alpha_F^+ \colon \mathrm{H}_1 \overline{F} \times \mathrm{H}_1 F \longrightarrow \mathbb{Z}$ can be understood as a homomorphism $\alpha_F^+ \colon \mathrm{H}_1 F \longrightarrow \mathrm{Hom}(\mathrm{H}_1 \overline{F}, \mathbb{Z}) \simeq \mathrm{H}^1 \overline{F}$. The composition of this morphism with the Alexander duality isomorphism $\mathrm{H}^1 \overline{F} \simeq \mathrm{H}_1(\Sigma - \overline{F})$ is nothing but the natural morphism $i_+ \colon \mathrm{H}_1 F \longrightarrow \mathrm{H}_1(\Sigma - \overline{F})$. The same holds for α_F^- and i_-. As a consequence, the Seifert matrices $A_F^\pm$ with respect to basis $\mathcal{A}$ of $\mathrm{H}_1 F$ and $\overline{\mathcal{A}}$ of $\mathrm{H}_1 \overline{F}$ are equal to the transposed matrices of $i_+, i_- \colon \mathrm{H}_1 F \longrightarrow \mathrm{H}_1(\Sigma - \overline{F})$ with respect to the basis $\mathcal{A}$ and $\overline{\mathcal{A}}^*$, where $\overline{\mathcal{A}}^*$ is the dual basis of $\overline{\mathcal{A}}$ via Alexander duality $\mathrm{H}_1 \overline{F} \simeq \mathrm{H}^1(\Sigma - \overline{F})$. If the multilink is fibered with fiber F, $i_+, i_- \colon F \longrightarrow \Sigma - \overline{F}$ are homotopy equivalences, so $i_+, i_- \colon \mathrm{H}_1 F \longrightarrow \mathrm{H}_1(\Sigma - \overline{F})$ are isomorphisms. Hence the associated matrices are unimodular.

The monodromy of a fibered multilink can be defined as the composition $(i_-)^{-1} \circ (i_+)$. Hence, a matrix of the monodromy is given by $H = ((A_F^-)^T)^{-1} \cdot (A_F^+)^T = (A_F^+ \cdot (A_F^-)^{-1})^T$. □

1.2.6 Proposition.
If the multilink is fibered with fiber F, a monodromy matrix and the Seifert matrices $A_F^\pm$ yield the same information.

Proof. We have already shown that a monodromy matrix can be recovered from Seifert matrices. Now, let us fix a basis $\mathcal{A}$ of $\mathrm{H}_1 F$, and let H be the matrix of the monodromy with respect to $\mathcal{A}$. Let us note $\overline{\mathcal{A}}$ the basis of $\mathrm{H}_1 \overline{F}$ obtained from $\mathcal{A}$ via the isomorphisms

$$\mathrm{H}_1 F \xrightarrow{i_-} \mathrm{H}_1(\Sigma - \overline{F}) \simeq \mathrm{H}^1 \overline{F} \simeq \mathrm{Hom}(\mathrm{H}_1 \overline{F}, \mathbb{Z}) \simeq \mathrm{H}_1 \overline{F}.$$

The Seifert matrices with respect to $\mathcal{A}$ and $\overline{\mathcal{A}}$ are given by $A_F^- = I$ and $A_F^+ = H^T$. □

This proposition is very important to understand that the matrices $A_F^\pm$ are no proper generalization of the usual Seifert matrix (for oriented links). In the usual case, the Seifert matrix of a fibered link yield much more information than the monodromy. This is lost in the case of multilinks because two basis (for $\mathrm{H}_1 F$ and $\mathrm{H}_1 \overline{F}$) have to be chosen. These two $\mathbb{Z}$-modules are most of the time isomorphic (that is, when r=0); nevertheless there is no canonical isomorphism between them that gives the identity on $\mathrm{H}_1 F$ when all the multiplicities are ± 1. In particular, $A_F^\pm$ cannot be considered as authentic bilinear forms, nor as a generalization of the usual Seifert form.

Still, there is one essential property of the usual Seifert matrix that is preserved: it is very helpful to compute the Alexander module. Recall that in the fibered case, an Alexander matrix is given by $H^T - tI$ (proposition 1.1.9); by proposition 1.2.5, this matrix is $A_F^+ \cdot (A_F^-)^{-1} - tI$ which is equivalent to $A_F^+ - tA_F^-$. It turns out that this is always an Alexander matrix.

1.2.7 Theorem.
An Alexander matrix of $L(\underline{m})$ is given by $A_F^+ - tA_F^-$, where F is any good Seifert surface for $L(\underline{m})$.

Proof. The demonstration relies on the possibility of constructing $\widetilde{X} = \widetilde{X}(\underline{m})$ with a good Seifert surface, and the study of the associated Mayer-Vietoris exact sequence.

Let F be a good Seifert surface for $L(\underline{m})$, and let us note $Y = \Sigma - \overline{F}$; by proposition 1.2.2, F exists and it is possible to number the connected components $F = F_1 \cup \ldots \cup F_d$ and $Y = Y_1 \cup \ldots \cup Y_d$ such that $i_+ F_k = Y_k$ and $i_- F_k = Y_{k-1}$ (with the indices modulo d). Let us set $N = F \times (-1;1)$ an open bicollar of F, $N_+ = F \times (0;1)$, $N_- = F \times (-1;0)$ and $\{Y^i\}_{i\in\mathbb{Z}}$ (resp. $\{N^i\}_{i\in\mathbb{Z}}$) copies of Y (resp. N). Define

$$E = \bigsqcup_{i\in\mathbb{Z}} Y^i \sqcup \bigsqcup_{i\in\mathbb{Z}} N^i \Big/ \sim ,$$

where $Y^i \supset N_+ \sim N_+ \subset N^i$ and $Y^i \supset N_- \sim N_- \subset N^{i+1}$. The obvious projection $E \xrightarrow{p} X$ is the infinite cyclic covering $\widetilde{X}(\underline{m}) \longrightarrow X$ determined by $\underline{m}$. Indeed, a loop γ in X lifts to a loop in E if and only if the intersection number of γ with F is zero, that is, if

$$0 = \gamma \cdot F = \ell k(L(\underline{m}), \gamma) = \underline{m}(\gamma).$$

Therefore, $E \xrightarrow{p} X$ is the $\mathbb{Z}$-regular covering determined by $\underline{m}$. Of course, a generator t of the group of the covering is given by the obvious homeomorphism sending Y^i to Y^{i+1} and N^i to N^{i+1}.

Consider the Mayer-Vietoris exact sequence of $\mathbb{Z}[t,t^{-1}]$-modules associated with the decomposition $\widetilde{X} = (\bigcup_i Y^i) \cup (\bigcup_i N^i)$; it gives

$$\begin{array}{cl} 0 \longrightarrow & \mathrm{H}_2(\Sigma - \overline{F}) \otimes \mathbb{Z}[t,t^{-1}] \longrightarrow \mathrm{H}_2\widetilde{X} \longrightarrow (\mathrm{H}_1 F \oplus \mathrm{H}_1 F) \otimes \mathbb{Z}[t,t^{-1}] \\ \xrightarrow{\phi_1} & (\mathrm{H}_1(\Sigma - \overline{F}) \oplus \mathrm{H}_1 F) \otimes \mathbb{Z}[t,t^{-1}] \xrightarrow{\psi} \mathrm{H}_1\widetilde{X} \longrightarrow (\mathrm{H}_0 F \oplus \mathrm{H}_0 F) \otimes \mathbb{Z}[t,t^{-1}] \\ \xrightarrow{\phi_0} & (\mathrm{H}_0(\Sigma - \overline{F}) \oplus \mathrm{H}_0 F) \otimes \mathbb{Z}[t,t^{-1}] \longrightarrow \mathrm{H}_0\widetilde{X} \longrightarrow 0. \end{array}$$

The homomorphism ϕ_0 is given by $\phi_0(\alpha,\beta) = (i_+\alpha + t\cdot i_-\beta, \alpha+\beta)$. Since F is good, the homomorphisms $i_\pm\colon \mathrm{H}_0F \longrightarrow \mathrm{H}_0(\Sigma - \overline{F})$ are injective, and so is ϕ_0. Therefore, ψ is surjective and there is an exact sequence

$$(\mathrm{H}_1F \oplus \mathrm{H}_1F)\otimes \mathbb{Z}[t,t^{-1}] \xrightarrow{\phi_1} (\mathrm{H}_1(\Sigma-\overline{F})\oplus \mathrm{H}_1F)\otimes \mathbb{Z}[t,t^{-1}] \longrightarrow \mathrm{H}_1\widetilde{X} \longrightarrow 0,$$

with $\phi_1(\alpha,\beta) = (i_+\alpha + t\cdot i_-\beta, \alpha+\beta)$. This can be transformed into

$$\mathrm{H}_1F\otimes \mathbb{Z}[t,t^{-1}] \xrightarrow{\tilde{\phi}} \mathrm{H}_1(\Sigma-\overline{F})\otimes \mathbb{Z}[t,t^{-1}] \longrightarrow \mathrm{H}_1\widetilde{X}(\underline{m}) \longrightarrow 0,$$

where $\tilde{\phi}(\alpha) = i_+\alpha - t\cdot i_-\alpha$. Since A_F^+ (resp. A_F^-) is the transposed matrix of i_+ (resp. i_-), this concludes the proof. □

In the usual case, it is Seifert's famous result: given any Seifert matrix V for an oriented link L, $V - tV^T$ is a presentation matrix of the Alexander module of L.

1.2.8 Corollary.
Let $L(\underline{m})$ be a multilink with no index i such that $m_i = m_i' = 0$; if F is good, a representative up to sign of the Alexander polynomial $\Delta^{L(\underline{m})}(t)$ is given by

$$\Delta^F(t) = \det(A_F^+ - tA_F^-).$$

If there is an index i such that $m_i = m_i' = 0$, then $\Delta^{L(\underline{m})}(t)$ is equal to zero. □

Let us now give several consequences of this theorem. Note that the main application is to be found in section 4.3, where the Alexander module of non-fibered Seifert multilinks is computed.

1.2.9 Proposition.
Let $\mathcal{M}'$ (resp. $\mathcal{M}''$) be the Alexander module of $L'(\underline{m}')$ (resp. $L''(\underline{m}'')$); if $\underline{m}' \neq \underline{0}$ and $\underline{m}'' \neq \underline{0}$, the Alexander module $\mathcal{M}$ of the disjoint sum $L'(\underline{m}') + L''(\underline{m}'')$ is equal to $\mathcal{M}' \oplus \mathcal{M}'' \oplus \mathbb{Z}[t,t^{-1}]$. If $\underline{m}' = \underline{0}$ or $\underline{m}'' = \underline{0}$, then $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$.

Proof. Let F' (resp. F'') be a good Seifert surface for $L(\underline{m}')$ (resp. $L''(\underline{m}'')$); then, a good Seifert surface F for $L(\underline{m}) = L'(\underline{m}') + L''(\underline{m}'')$ is obtained from $F_0 = F' \sqcup F''$ via $d' + d'' - d$ surgeries, where $d' = \gcd(\underline{m}')$, $d'' = \gcd(\underline{m}'')$ and $d = \gcd(d', d'')$. The effect of a surgery on H_1F_0 is very simple; if the result of the operation is F_1, its homology is given by

$$\mathrm{H}_1F_1 \simeq \mathrm{H}_1F_0 \oplus \mathbb{Z}c_1\,,$$

with c_1 the 1-cycle represented in figure 1.3.

On the other hand, the effect of the surgery on $\mathrm{H}_1\overline{F}_0$ depends on whether it reduces $\overline{b}_0 = \mathrm{rk}\,\widetilde{H}_0\overline{F}_0$ and/or $\overline{b}_2 = \mathrm{rk}\,\mathrm{H}_2\overline{F}_0$. Using the notations σ_1 and c_1 as in figure 1.3 it is not hard to prove the following assertions.

(i) If the surgery reduces $\overline{b}_0$ and $\overline{b}_2$, then $\mathrm{H}_1\overline{F}_1 \simeq \mathrm{H}_1\overline{F}_0$.

(ii) If the surgery reduces $\overline{b}_2$, not $\overline{b}_0$, then $\mathrm{H}_1\overline{F}_1 \simeq \mathrm{H}_1\overline{F}_0 \oplus \mathbb{Z}\sigma_1$.

(iii) If the surgery reduces $\overline{b}_0$, not $\overline{b}_2$, then $\mathrm{H}_1\overline{F}_1 \simeq \mathrm{H}_1\overline{F}_0 \oplus \mathbb{Z}c_1$.

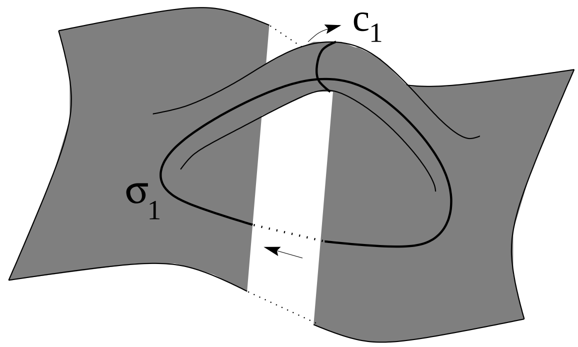


Figure 1.3: The new cycles appearing via surgery.

(iv) If it does not reduce any rank, then $\mathrm{H}_1\overline{F}_1 \simeq \mathrm{H}_1\overline{F}_0 \oplus \mathbb{Z}c_1 \oplus \mathbb{Z}\sigma_1$.

If $d' = 0$ or $d'' = 0$, there is no surgery to do: $F = F_0 = F' \sqcup F''$ is good. The proposition then follows trivially from theorem 1.2.7. Let us now suppose that $d' > 0$ and $d'' > 0$; there will be one surgery of the first type, $d' + d'' - d - 2$ of the second type, and a last one of the fourth type (unless $d' = d'' = 1$, in which case there will be a unique surgery of the third type). The Seifert forms of $L(\underline{m})$ relatively to the good Seifert surface F are given by

$$A_F^+ = \begin{pmatrix} A_{F'}^+ & 0 & & \overbrace{\ast \quad\quad}^{d'+d''-d-1} & & 0 \\ 0 & A_{F''}^+ & & \ast & & 0 \\ 0 & 0 & & \ast & & 0 \\ & & 1 & \ast & \ast & \\ 0 & 0 & & \ddots & \ast & 0 \\ & & & & 1 & \end{pmatrix} \qquad A_F^- = \begin{pmatrix} A_{F'}^- & 0 & & \overbrace{\ast \quad\quad}^{d'+d''-d-1} & & 0 \\ 0 & A_{F''}^- & & \ast & & 0 \\ 0 & 0 & & \ast & & 0 \\ & & 0 & \ast & \ast & \\ 0 & 0 & & \ddots & \ast & 0 \\ & & & & 0 & \end{pmatrix}.$$

By theorem 1.2.7, we have the Alexander matrix

$$\mathcal{P} = \begin{pmatrix} A_{F'}^+ - tA_{F'}^+ & 0 & & \ast & & 0 \\ 0 & A_{F''}^+ - tA_{F''}^- & & \ast & & 0 \\ 0 & 0 & & \ast & & 0 \\ & & 1 & \ast & \ast & \\ 0 & 0 & & \ddots & \ast & 0 \\ & & & & 1 & \end{pmatrix} \sim \begin{pmatrix} \mathcal{P}' & 0 & 0 \\ 0 & \mathcal{P}'' & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $\mathcal{P}'$ (resp. $\mathcal{P}''$) Alexander matrices of $L'(\underline{m}')$ (resp. $L'(\underline{m}')$). This concludes the proof. □

1.2.10 Proposition.
Let F be a good Seifert surface for $L(\underline{m})$ with $\underline{m} \neq \underline{0}$. Then, the matrix $A_F^+ - A_F^-$ presents the free $\mathbb{Z}$-module of rank $n-1$.

Proof. Similarly to the proof of theorem 1.2.7, consider the Mayer-Vietoris exact sequence of abelian groups associated with $\Sigma - L = F \times (-1;1) \cup (\Sigma - \overline{F})$; it it given by

$$\mathrm{H}_1F \oplus \mathrm{H}_1F \xrightarrow{\phi} \mathrm{H}_1F \oplus \mathrm{H}_1(\Sigma - \overline{F}) \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z} \longrightarrow 0,$$

with $\phi(\alpha,\beta) = (\alpha+\beta, i_+\alpha + i_-\beta)$. This leads to the exact sequence

$$\mathrm{H}_1 F \xrightarrow{\tilde{\phi}} \mathrm{H}_1(\Sigma - \overline{F}) \longrightarrow \mathbb{Z}^{n-1} \longrightarrow 0,$$

where $\tilde{\phi}(\alpha) = i_+\alpha - i_-\alpha$. Since A_F^+ (resp. A_F^-) is the transposed matrix of i_+ (resp. i_-), this concludes the proof. □

1.2.11 Corollary.
Let $\mathcal{P}(t)$ be any Alexander matrix for a multilink with $\underline{m} \neq \underline{0}$; then, $\mathcal{P}(1)$ presents the abelian group $\mathbb{Z}^{n-1}$. □

1.3 Fox Free Differential Calculus

In this paragraph, we intend to apply Fox differential calculus to multilinks. This will enable us to compute an Alexander matrix of a multilink in S^3 from a given projection.

Let $L(\underline{m}) = m_1L_1 \cup \ldots \cup m_nL_n$ be a multilink with $\gcd(m_1,\ldots,m_n) = d$. Let us note X^0 a base point in $X = \Sigma - \mathcal{N}(L)$, $\pi := \pi_1(X, X^0)$, $\varphi\colon \pi \longrightarrow < t >$ the morphism defined by the multiplicities (that is: $\gamma \longmapsto t^{\underline{m}(\gamma)}$, where $\underline{m}(\gamma) = \sum_{i=1}^n m_i \ell k(\gamma, L_i)$), and $\widetilde{X} \xrightarrow{p} X$ the associated infinite cyclic covering of X. Let us also note φ the linear extension $\mathbb{Z}\pi \longrightarrow \mathbb{Z}[t,t^{-1}]$; finally, let $\widetilde{X}^0 = p^{-1}(X^0)$ be the fiber of the covering.

If a presentation of π is given by $\pi = \langle\ G_1,\ldots,G_r \mid R_1,\ldots,R_m\ \rangle$, let us note $\mathcal{F} := \langle\ G_1,\ldots,G_r\ \rangle$, $\psi\colon \mathcal{F} \longrightarrow \pi$ the natural projection and also $\psi\colon \mathbb{Z}\mathcal{F} \longrightarrow \mathbb{Z}\pi$ its extension. Recall that the Fox derivatives $\frac{\partial}{\partial G_j}\colon \mathbb{Z}\mathcal{F} \longrightarrow \mathbb{Z}\mathcal{F}$ are characterized by the following properties:

- $\mathbb{Z}$-linearity;
- $\frac{\partial}{\partial G_j}(G_i) = \delta_{ij}$;
- $\frac{\partial}{\partial G_j}(\xi\cdot\eta) = \frac{\partial}{\partial G_j}(\xi)\cdot\epsilon(\eta) + \xi\cdot\frac{\partial}{\partial G_j}(\eta)$, where $\epsilon\colon \mathbb{Z}\mathcal{F} \longrightarrow \mathbb{Z}$ is the augmentation morphism.

The following proposition is due to R. Fox. For the proof, see [8, Proposition 9.9].

1.3.1 Proposition. (Fox)
The matrix $\left(\varphi\psi(\frac{\partial R_j}{\partial G_i})\right)_{i,j}$ presents $H_1(\widetilde{X}, \widetilde{X}^0)$. □

Let us consider the exact sequence of the pair $(\widetilde{X}, \widetilde{X}^0)$:

$$0 \longrightarrow \mathrm{H}_1\widetilde{X} \longrightarrow \mathrm{H}_1(\widetilde{X}, \widetilde{X}^0) \xrightarrow{\partial} \mathrm{H}_0(\widetilde{X}^0) \xrightarrow{i} \mathrm{H}_0\widetilde{X} \longrightarrow 0.$$

Clearly, $\mathrm{H}_0(\widetilde{X}^0) \simeq \mathbb{Z}[t,t^{-1}]\cdot P$, and $\mathrm{H}_0\widetilde{X} \simeq \mathbb{Z}[t,t^{-1}]/(t^d-1)\cdot P$, where P is any element of $\widetilde{X}^0$. Thus, $\operatorname{Ker} i \simeq \mathbb{Z}[t,t^{-1}]\,(t^d-1)\cdot P$ and the short exact sequence

$$0 \longrightarrow \mathrm{H}_1\widetilde{X} \longrightarrow \mathrm{H}_1(\widetilde{X}, \widetilde{X}^0) \xrightarrow{\partial} \operatorname{Ker} i \longrightarrow 0$$

splits, giving the isomorphism

$$\mathrm{H}_1(\widetilde{X}, \widetilde{X}^0) \ \simeq\ \mathrm{H}_1\widetilde{X} \ \oplus\ \mathbb{Z}[t,t^{-1}]\cdot w$$

where w is any element of $\mathrm{H}_1(\widetilde{X}, \widetilde{X}^0)$ such that $\partial(w) = (t^d - 1) \cdot P$. Hence, a presentation matrix of $\mathrm{H}_1\widetilde{X}$ is obtained via one of $\mathrm{H}_1(\widetilde{X}, \widetilde{X}^0)$ by adding the relation $w = 0$, where w satisfies $\partial(w) = (t^d - 1) \cdot P$.

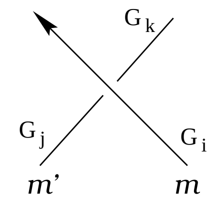
Let us now suppose that $L(\underline{m})$ is a multilink in S^3 given by a diagram with multiplicities. Proposition 1.3.1 applied to the Wirtinger presentation of the group π leads to the following algorithm.

1.3.2 Proposition.
Let $L(\underline{m})$ be a multilink in S^3 given by a diagram with multiplicities.

Number the bridges of the projection $G_1, \ldots, G_r$. To each double point (except one), assign the relation

$$(t^{m'} - 1) \cdot G_i \; - \; t^m \cdot G_j \; + \; G_k \; = \; 0 \, .$$

The $(r-1) \times r$ matrix obtained presents the module $H_1(\widetilde{X}, \widetilde{X}^0)$. □



In the usual case, there is a well known trick to compute an Alexander matrix from this presentation matrix of $\mathrm{H}_1(\widetilde{X}, \widetilde{X}^0)$: just delete any column. It is a little more complicated in the case of multilinks.

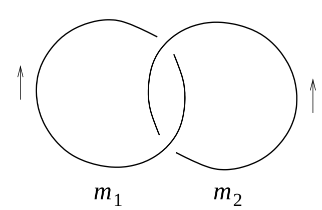
1.3.3 Proposition.
Let us number the bridges of the projection in such a way that $\varphi\psi(G_i) = t^{m_i}$ for $i = 1, \ldots, n$ and let us choose integers $\mu_1, \ldots, \mu_n \in \mathbb{Z}$ such that $\sum_{i=1}^n \mu_i m_i = d$. Then, an Alexander matrix is given by the matrix of proposition 1.3.2 together with the relation $\sum_{i=1}^n p_i G_i = 0$, where

$$p_i = \begin{cases} t^{\mu_1 m_1 + \ldots + \mu_{i-1} m_{i-1}} \cdot \frac{t^{\mu_i m_i} - 1}{t^{m_i} - 1} & \text{if } m_i \neq 0; \\ 0 & \text{if } m_i = 0. \end{cases}$$

Proof. By the previous discussion, we just need to check that $\partial(\sum_{i=1}^n p_i G_i) = t^d - 1$;

$$\begin{aligned} \partial(\sum_{i=1}^n p_i G_i) &= \sum_{i=1}^n p_i(t^{m_i} - 1) = \sum_{i=1}^n t^{\mu_1 m_1 + \ldots + \mu_{i-1} m_{i-1}} \cdot (t^{\mu_i m_i} - 1) \\ &= t^{\mu_1 m_1 + \ldots + \mu_n m_n} - 1 = t^d - 1. \end{aligned}$$ □

1.3.4 Example. Consider the Hopf link with multiplicities m_1, m_2 and $\gcd(m_1, m_2) = d$.



If $\mu_1 m_1 + \mu_2 m_2 = d$, proposition 1.3.3 gives the Alexander matrix

$$\mathcal{P} = \begin{pmatrix} t^{m_2} - 1 & 1 - t^{m_1} \\ \frac{t^{\mu_1 m_1} - 1}{t^{m_1} - 1} & t^{\mu_1 m_1} \frac{t^{\mu_2 m_2} - 1}{t^{m_2} - 1} \end{pmatrix}.$$

By Euclid's algorithm, the presentation matrix $(t^{m_2}-1 \quad 1-t^{m_1})$ is equivalent to the matrix $(t^d-1 \quad 0)$. Transforming $\mathcal{P}$ with the same moves, we get

$$\mathcal{P} \sim \mathcal{P}' = \begin{pmatrix} t^d-1 & 0 \\ r(t) & s(t) \end{pmatrix}.$$

But since $t^d-1 = \det \mathcal{P} \doteq \det \mathcal{P}' = s(t)(t^d-1)$, $s(t)$ is a unit. Therefore, $\mathcal{P}$ is equivalent to (t^d-1), and the Alexander module of this Hopf multilink is given by $\mathbb{Z}[t,t^{-1}]/(t^d-1)$.

1.4 Torsion Numbers

Let $L(\underline{m})$ be a multilink in Σ with exterior X, and let us fix an integer $k \geq 1$. To avoid unnecessary complications, let us assume for the rest of the section that $\gcd(\underline{m}) = 1$. The **$\mathbf{k^{th}}$ cyclic covering** of the multilink $L(\underline{m})$ is the regular covering $\widetilde{X}_k \longrightarrow X$ determined by the epiomorphism φ_k defined as the composition

$$\pi_1 X \xrightarrow{\underline{m}} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/k\,,$$

where π is the canonical projection $\pi(\alpha) = [\alpha]_k$.

1.4.1 Proposition.
$H_2\widetilde{X}_k$ is a free $\mathbb{Z}$-module of rank equal to rk $H_1\widetilde{X}_k - 1$, and $H_q\widetilde{X}_k = 0$ for all $q \geq 3$.

Proof. Since $\widetilde{X}_k$ is a 3-manifold with boundary, it has the homotopy type of a 2-complex. Therefore, $\mathrm{H}_q\widetilde{X}_k = 0$ for all $q \geq 3$ and $\mathrm{H}_2\widetilde{X}_k$ is free. Finally, the equality $\chi(\widetilde{X}_k) = k\cdot\chi(X) = 0$ concludes the proof. □

By this proposition, all the homological information of $\widetilde{X}_k$ is contained in the first homology group. It is a finitely generated abelian group, so it can be written uniquely

$$\mathrm{H}_1\widetilde{X}_k = \mathbb{Z}^{\mathrm{rk}} \oplus \mathbb{Z}/n_1 \oplus \mathbb{Z}/n_2 \oplus \ldots \oplus \mathbb{Z}/n_s$$

with n_j dividing n_{j+1} for $j = 1,\ldots,s-1$. These n_i are the **$\mathbf{k^{th}}$ torsion numbers** of the multilink $L(\underline{m})$. In the usual case, this definition coincides with the definition of the torsion numbers of an oriented link.

1.4.2 Lemma.
Two sets of multiplicities $m_1,\ldots,m_n$ and $m'_1,\ldots,m'_n$ on a given link L yield multilinks with the same k^{th} torsion numbers if $m'_i \equiv m_i \pmod{k}$ for all $i = 1,\ldots,n$. In particular, one can choose multiplicities m'_i such that $0 \leq m'_i \leq k-1$ for all $i = 1,\ldots,n$.

Proof. The homomorphism φ_k can be written as

$$\varphi_k([\gamma]) = \left[\sum_{i=1}^{n} m_i \ell k(L_i,\gamma)\right]_k = \left[\sum_{i=1}^{n} m'_i \ell k(L_i,\gamma)\right]_k$$

as soon as $m'_i \equiv m_i \pmod{k}$ for all i. Since φ_k determines the covering, we are done. □

The next theorem asserts that the torsion numbers can be computed using Seifert matrices.

1.4.3 Theorem.
Let $L(m_1, \dots, m_n)$ be a multilink with $d = 1$ and k^{th} cyclic covering space $\widetilde{X}_k$; let F be a connected surface for $L(m'_1, \dots, m'_n)$, where m'_i is any integer such that $m'_i \equiv m_i \pmod k$ and let $A^\pm$ be associated Seifert matrices. Then, a presentation matrix of $H_1\widetilde{X}_k$ is given by

$$k{-}1 \text{ blocs}\left\{\left(\begin{array}{c|ccccc} \overbrace{}^{n} & \multicolumn{5}{c}{\overbrace{\hspace{6cm}}^{k-1 \text{ blocs}}} \\ & A^+ & & & & A^- \\ & -A^- & A^+ & & & A^- \\ 0 & & \ddots & \ddots & & \vdots \\ & & & & -A^- & A^+ + A^- \end{array}\right)\right..$$

Proof. By the above lemma, $H_1\widetilde{X}_k$ isomorphic to $H_1\widetilde{X}'_k$, where $\widetilde{X}'_k$ is the k$^{\text{th}}$ cyclic covering space of the multilink $L(m'_1, \dots, m'_n)$. As in the proof of theorem 1.2.7, $\widetilde{X}'_k$ can be constructed using a connected Seifert surface F for $L(\underline{m}')$. If Y^i denotes copies of $\Sigma - \overline{F}$ and N^i copies of an open bicollar of F, the Mayer-Vietoris sequence associated with $\widetilde{X}'_k = (Y^0 \cup \dots \cup Y^{k-1}) \cup (N^0 \cup \dots \cup N^{k-1})$ easily gives the fact that $H_1\widetilde{X}'_k$ is presented as a $\mathbb{Z}$-module by the following matrix made of $k \times k$ blocs

$$\begin{pmatrix} 0 & A^+ & & & -A^- \\ 0 & -A^- & A^+ & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & -A^- & A^+ \end{pmatrix}.$$

It is equivalent to

$$\begin{pmatrix} 0 & A^+ - A^- & & & & -A^- \\ 0 & 0 & A^+ & & & A^- \\ 0 & 0 & -A^- & A^+ & & A^- \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & & -A^- & A^+ + A^- \end{pmatrix}.$$

By proposition 1.2.10, $A^+ - A^-$ presents the free $\mathbb{Z}$-module of rank $n - 1$. Therefore, this matrix is equivalent to the one stated in the theorem. □

1.4.4 Example. The second torsion numbers of a multilink $L(m_1, \dots, m_n)$ are given by the matrix $A^+_F + A^-_F$, were F is a Seifert surface for $L(m'_1, \dots, m'_n)$ with $m'_i \in \{0; 1\}$ such that $m_i \equiv m'_i \pmod 2$.

1.4.5 Remark. (Branched cyclic coverings) In the usual case, it is interesting to extend $\widetilde{X}_k \longrightarrow X$ to $\widehat{X}_k \longrightarrow \Sigma$, the k$^{\text{th}}$ cyclic covering of Σ branched over L. Indeed, the n border tori of X lift to n border tori in $\widetilde{X}_k$ whose meridiens form a basis of the free submodule in $H_1\widetilde{X}_k$. Therefore, $\widehat{X}_k$ is a 3-manifold (without boundary) whose first homology is presented as a $\mathbb{Z}$-module by

$$\begin{pmatrix} V & & & V^T \\ -V^T & V & & V^T \\ & \ddots & \ddots & V^T \\ & & -V^T & V + V^T \end{pmatrix},$$

where V is a Seifert matrix. In particular, one can prove that the order ideal of $\mathrm{H}_1\widetilde{X}_k$ is generated by $\left|\prod_{i=1}^{k-1} \Delta^L(\zeta^i)\right|$, where $\zeta = \mathrm{e}^{\frac{2i\pi}{k}}$.

In the case of multilinks, $\widehat{X}_k$ has $\beta := \sum_{i=1}^n \gcd(d_i, k)$ boundary components. We can construct a branched cyclic covering $\widehat{X}_k \xrightarrow{p_k} \Sigma$ by pasting β solid tori along $\partial\widetilde{X}_k$. The resulting space $\widehat{X}_k$ is a 3-manifold, and the projection p_k satisfies

$$\#p_k^{-1}(x) = \begin{cases} k & \text{if } x \in \Sigma - L; \\ \gcd(d_i, k) & \text{if } x \in L_i\,,\ i = 1, \ldots, n. \end{cases}$$

However, this process kills $\beta \geq n$ cycles in the space $\widetilde{X}_k$, that is, more than just the free submodule of rank n (unless $\beta = n$). Therefore, the homology of this manifold can not be described using Seifert matrices.

Another possibility is to paste n solid tori along $\partial\widetilde{X}_k$, identifying the $\gcd(d_i, k)$ boundary components above $\partial\mathcal{N}(L_i)$. But the space $\widehat{X}_k$ obtained is not a manifold in general.

1.4.6 Remark. (Signature) In the case of an oriented link L, an invariant is given by the signature of the symmetric matrix $V + V^T$, where V is a Seifert matrix for L. This invariant, called the signature of L, is denoted by $\sigma(L)$. Equivalently, $\sigma(L)$ can be defined as the signature of the intersection form $\mathrm{H}_2\widetilde{D^4} \otimes \mathrm{H}_2\widetilde{D^4} \longrightarrow \mathbb{Z}$, where $\widetilde{D^4} \longrightarrow D^4$ is the double covering of D^4 branched along a surface $F \subset D^4$ such that $\partial F = L \subset S^3 = \partial D^4$ (see [20]).

Unfortunately, none of these definitions can be generalized to multilinks. On one hand, the matrix $A_F^+ + A_F^-$ is not symmetric in general. On the other hand, the double covering $\widetilde{D^4} \longrightarrow D^4$ branched along $\overline{F}$ is not a manifold in general; indeed its boundary is $\partial\widetilde{D^4} = \widehat{X}_2$, which is not a 3-manifold as noted in remark 1.4.5.

Chapter 2

The Alexander Polynomial

It was in his 1934 paper [41] that HERBERT SEIFERT introduced the notions of Seifert sufaces and Seifert forms associated to an oriented link L. His aim was not really to give an algorithm for the computation of the Alexander polynomial Δ_L: JAMES ALEXANDER himself had suggested a satisfactory method in 1928 (see [2]). The true goal of Seifert was to derive general properties of Δ_L from the study of the associated surfaces and forms. In particular, he was able to characterize among polynomials in $\mathbb{Z}[t, t^{-1}]$, those which are Alexander polynomials of knots. This same method was used by FUJITSUGU HOSOKAWA [16] to characterize the one-variable Alexander polynomials of oriented links.

In this chapter, we try to walk along the same path. Since the Alexander polynomial of a multilink is given by

$$\Delta^{L(\underline{m})}(t) \doteq \det(A_F^+ - tA_F^-),$$

general properties of $\Delta^{L(\underline{m})}$ will follow from a carefull study of $\mathrm{H}_1 F$ and $\mathrm{H}_1 \overline{F}$. A complete characterization of the Alexander polynomials of multilinks seems out of reach. Nevertheless, the necessary conditions found in this chapter are very interesting. There is a dictionary between the Alexander polynomial of a given multilink and the multivariable Alexander of the underlying oriented ordered link. And it turns out that these conditions translate exactly into the celebrated "Torres conditions", that generalize Seifert's conditions to the multivariable case. The hope is, of course, to find new conditions by this mean.

2.1 The Alexander Polynomial of a Multilink

Let us fix the notations once and for all: given a multilink $L(m_1, \ldots, m_n)$, let us note $\Delta^{L(\underline{m})}(t)$ its Alexander polynomial, $d = \gcd(m_1, \ldots, m_n) > 0$, $\ell_{ij} = \ell k(L_i, L_j)$ if $i \neq j$, $\ell_{ii} = 0$, $m'_i = -\sum_j m_j \ell_{ij}$ and $d_i = \gcd(m_i, m'_i) \geq 0$. A Seifert surface is always assumed to be good.

2.1.1 Proposition.
We have the following Torres formula for multilinks:

$$\Delta^{L(m_1,\ldots,m_{n-1},0)}(t) \doteq (t^{\sum_{i=1}^{n-1} m_i \ell_{in}} - 1) \cdot \Delta^{L(m_1,\ldots,m_{n-1})}(t).$$

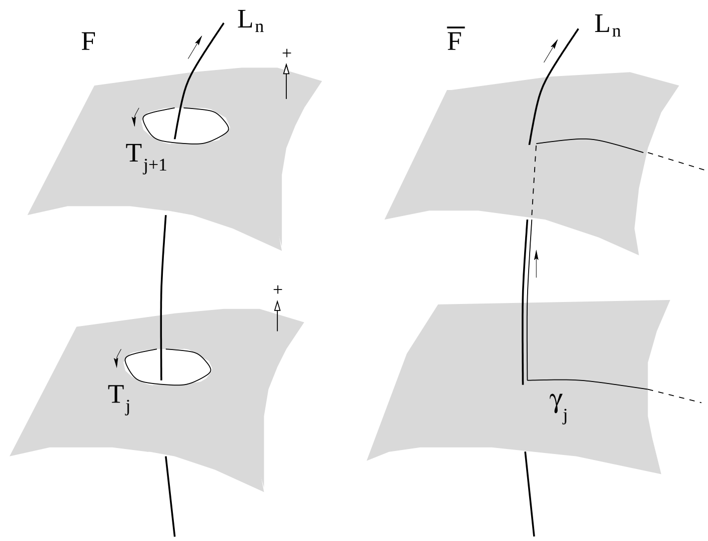


Figure 2.1: Cycles T_j and γ_j in proposition 2.1.1.

Proof. Since $\Delta^{L(\underline{m})}(t) \doteq \Delta^{L(\frac{\underline{m}}{d})}(t^d)$, it may be assumed that $d = 1$. Let us note $L(\underline{m}) = L(m_1, \ldots, m_{n-1}, 0)$ and $L_0(\underline{m}) = L(m_1, \ldots, m_{n-1})$. A Seifert surface F_0 for $L_0(\underline{m})$ is obtained from a Seifert surface F for $L(\underline{m})$ by filling up the holes in F near L_n. By proposition 1.2.1, there are $d_n = \gcd(0, m_n') = \sum_{i=1}^{n-1} m_i \ell_{in}$ such holes. Furthermore, $\overline{F} = \overline{F}_0 \cup L_n$. Hence, we get the isomorphisms

$$\mathrm{H}_1 F = \mathrm{H}_1 F_0 \oplus \bigoplus_{j=1}^{d_n} \mathbb{Z} T_j \quad , \quad \mathrm{H}_1 \overline{F} = \mathrm{H}_1 \overline{F}_0 \oplus \bigoplus_{j=1}^{d_n} \mathbb{Z} \gamma_j,$$

where the T_j are the border of the holes, and the γ_j are the transverse cycles depicted in figure 2.1. Computing the Seifert matrices, an Alexander matrix of $L(\underline{m})$ is given by

$$A_F^+ - tA_F^- = \begin{pmatrix} A_{F_0}^+ - tA_{F_0}^- & & 0 & & \\ & 1 & & & -t \\ & -t & 1 & & \\ * & & \ddots & \ddots & \\ & & & -t & 1 \end{pmatrix} \sim \begin{pmatrix} A_{F_0}^+ - tA_{F_0}^- & 0 \\ * & t^{d_n} - 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} \Delta^{L(m_1, \ldots, m_{n-1}, 0)}(t) &\doteq \det(A_F^+ - tA_F^-) = (t^{d_n} - 1) \cdot \det(A_{F_0}^+ - tA_{F_0}^-) \\ &\doteq (t^{d_n} - 1) \cdot \Delta^{L(m_1, \ldots, m_{n-1})}(t), \end{aligned}$$

and we are done. □

To simplify the exposition of the next lemmas, we will only consider the case where $d = \gcd(m_1, \ldots, m_n) = 1$. Let F be a connected Seifert surface for a multilink $L(m_1, \ldots, m_n)$. For $i = 1, \ldots, n$, let us note $M_i = \overline{F} \cap \mathcal{N}(L_i)$ and $T_1^i, \ldots, T_{d_i}^i$ the components of ∂M_i. If F is of genus g, its homology is clearly given as follows.

2.1.2 Lemma.

$$H_1 F \;\simeq\; \bigoplus_{j=1}^{2g} \mathbb{Z}x_j \,\oplus\, \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{d_i} \mathbb{Z}T_j^i \Big/ \sum_{i,j} T_j^i \,. \qquad \square$$

The computation of $\mathrm{H}_1\overline{F}$ is a little more tricky. Since M_i has an "open book" structure with d_i sheets, $d_i - 1$ independent new cycles appear in $\overline{F}$, as illustrated in figure 2.2. We will denote them by $\gamma_1^i, \ldots, \gamma_{d_i-1}^i$.

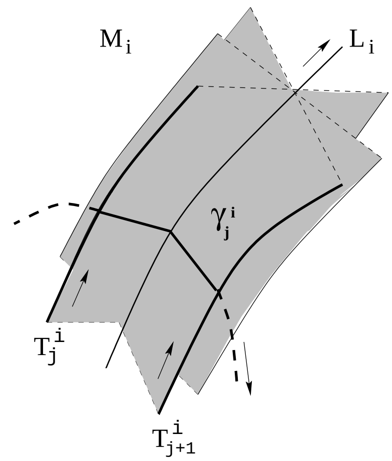


Figure 2.2: New cycles in $\overline{F}$.

2.1.3 Lemma.
If $d_1 = \ldots = d_r = 0$ and $d_i \neq 0$ for $i > r$, then

$$\mathrm{H}_1\overline{F} \;\simeq\; \bigoplus_{j=1}^{2g} \mathbb{Z}x_j \,\oplus\, \bigoplus_{i=r+1}^{n}\bigoplus_{j=1}^{d_i-1} \mathbb{Z}\gamma_j^i \,\oplus\, \bigoplus_{i=1}^{n} \mathbb{Z}L_i \Big/ \sum_i m_i L_i \,.$$

Proof. Consider the Mayer-Vietoris exact sequence associated with $\overline{F} = F \cup M$:

$$0 \longrightarrow \mathrm{H}_1(F\cap M) \stackrel{\phi}{\longrightarrow} \mathrm{H}_1 F \oplus \mathrm{H}_1 M \longrightarrow \mathrm{H}_1\overline{F} \stackrel{\partial}{\longrightarrow} \widetilde{\mathrm{H}}_0(F\cap M) \stackrel{i}{\longrightarrow} \widetilde{\mathrm{H}}_0 M.$$

Clearly, $F\cap M$ retracts by deformation on $\partial F = \bigsqcup_{i=1}^n \bigsqcup_{j=1}^{d_i} T_j^i$ and M retracts by deformation on $L = \bigsqcup_{i=1}^n L_i$. Since $\mathrm{Ker}\, i$ is free, we get the isomorphism

$$\mathrm{H}_1\overline{F} \;\simeq\; s(\mathrm{Ker}\, i) \oplus \left(\mathrm{H}_1 F \oplus \bigoplus_{i=1}^{n} \mathbb{Z}L_i \right) \Big/ \mathrm{Im}\, \phi \,,$$

where $s\colon \mathrm{Ker}\, i \longrightarrow \mathrm{H}_1\overline{F}$ is any section of ∂. Clearly, a basis of $s(\mathrm{Ker}\, i)$ is given by the $\sum_{i=r+1}^n (d_i - 1)$ cycles γ_j^i introduced above. Now, the homomorphism

$$\bigoplus_{i=r+1}^{n}\bigoplus_{j=1}^{d_i} \mathbb{Z}T_j^i \stackrel{\phi}{\longrightarrow} \bigoplus_{j=1}^{2g} \mathbb{Z}x_j \,\oplus\, \bigoplus_{i=r+1}^{n}\bigoplus_{j=1}^{d_i-1} \mathbb{Z}T_j^i \Big/ \sum_{i,j} T_j^i \,\oplus\, \bigoplus_{i=1}^{n} \mathbb{Z}L_i$$

is given by $T^i_j \longmapsto (0, T^i_j, p_i L_i)$, where $p_i = \frac{m_i}{d_i}$. Therefore,

$$\left(\mathrm{H}_1 F \oplus \bigoplus_{i=1}^{n} \mathbb{Z} L_i\right) \Big/ \mathrm{Im}\ \phi \ \simeq \ \bigoplus_{j=1}^{2g} \mathbb{Z} x_j \ \oplus \ \bigoplus_{i=1}^{n} \mathbb{Z} L_i \Big/ \sum_i d_i p_i L_i \ ,$$

proving the lemma. □

Since $\gcd(m_1, \ldots, m_n) = 1$, $\mathrm{H}_1 \overline{F}$ is free of rank

$$\mathrm{rk}\, \mathrm{H}_1 \overline{F} = 2g + \sum_{i=r+1}^{n} (d_i - 1) + n - 1 = 2g + \sum_{i=1}^{n} d_i + r - 1 = \mathrm{rk}\, \mathrm{H}_1 F + r.$$

We check that the equality of corollary 1.2.4 is satisfied. Nevertheless, there is no way to exhibit a canonical base of $\mathrm{H}_1 \overline{F}$. We can only use the following algebraic lemma.

2.1.4 Lemma.
Given integers $m_1, \ldots, m_n$ with $\gcd(m_1, \ldots, m_n) = 1$, there is an isomorphism

$$\bigoplus_{i=1}^{n} \mathbb{Z} L_i \Big/ (m_1 L_1 + \ldots + m_n L_n) \ \simeq \ \bigoplus_{j=1}^{n-1} \mathbb{Z} e_j,$$

where $e_j = \sum_{i=1}^{n} e_{ij} L_i$ with any integer coefficients e_{ij} such that the matrix

$$\begin{pmatrix} e_{11} & e_{12} & \ldots & e_{1\,n-1} & m_1 \\ e_{21} & e_{22} & \ldots & e_{2\,n-1} & m_2 \\ \vdots & \vdots & & \vdots & \vdots \\ e_{n1} & e_{n2} & \ldots & e_{n\,n-1} & m_n \end{pmatrix}$$

is unimodular.

Proof. Using Euclid's algorithm, we see that the presentation matrix $(\, m_1 \quad \cdots \quad m_n \,)$ can be transformed into $(\, d \quad 0 \quad \cdots \quad 0 \,)$, where $d = \gcd(m_1, \ldots, m_n)$, by a base change. Therefore, there exists a unimodular matrix Q such that $(\, m_1 \quad \cdots \quad m_n \,) \cdot Q = (\, d \quad 0 \quad \cdots \quad 0 \,)$. Since we assumed that $d = 1$, the matrix stated in the lemma is simply $(Q^{-1})^T$. □

Given M a $(m \times n)$-matrix (with coefficients in a factorial ring) with $m \geq n$ and $0 \leq k \leq n$ an integer, let us denote by $\Delta_k M$ the greatest common divisor of the $(n-k) \times (n-k)$-minors of M. This is defined up to multiplication by a unit of the ring.
In the next lemma, the notations are the ones used in the previous lemmas.

2.1.5 Lemma.
Let F be a connected Seifert surface for a multilink $L(\underline{m})$ with $r = 0$. Let V be the free $\mathbb{Z}$-module with base $\left\langle \{x_i\}_{i=1}^{2g}, \{T^i_1, \ldots, T^i_{d_i}\}_{i=1}^{n} \right\rangle$, and let $\overline{V}$ be the free $\mathbb{Z}$-module with base $\left\langle \{x_i\}_{i=1}^{2g}, \{\gamma^i_1, \ldots, \gamma^i_{d_i - 1}\}_{i=1}^{n}, \{L_i\}_{i=1}^{n} \right\rangle$. Finally, let $\widetilde{A}^+_F$ (resp. $\widetilde{A}^-_F$) be the matrix of the bilinear form $V \times \overline{V} \longrightarrow \mathbb{Z}$ given by $(x, y) \longmapsto \ell k(i_+ x, y)$ (resp. $(x, y) \longmapsto \ell k(i_- x, y)$). Then,

$$\Delta^{L(\underline{m})}(t) \doteq \Delta_1(\widetilde{A}^+_F - t\widetilde{A}^-_F).$$

Proof. By corollary 1.2.8, we just need to check that $\det(A_F^+ - tA_F^-) \doteq \Delta_1(\widetilde{A}_F^+ - t\widetilde{A}_F^-)$. Since $\sum_{i,j} T_j^i$ vanishes in $\mathrm{H}_1 F$, we clearly have $\det(A_F^+ - tA_F^-) \doteq \Delta_0 M$, where M is a matrix of the bilinear form $V \times \mathrm{H}_1\overline{F} \longrightarrow \mathbb{Z}[t,t^{-1}]$ given by $(x,y) \longmapsto \ell k(i_+x, y) - t \cdot \ell k(i_-x, y)$. Trivially, $\Delta_0 M \doteq \Delta_1 \left(M \mid 0 \right)$, where $\left(M \mid 0 \right)$ denotes the matrix M with an additional column of zeros. Let us denote by Q the $(n \times n)$-matrix given in lemma 2.1.4. Since Q is unimodular, it follows that

$$\Delta_1 \left(M \mid 0 \right) \doteq \Delta_1 \left(\left(M \mid 0 \right) \cdot \begin{pmatrix} I & 0 \\ 0 & Q^{-1} \end{pmatrix} \right).$$

By construction, and using the fact that $\sum_i m_i L_i$ vanishes in $\mathrm{H}_1\overline{F}$, we have

$$(\widetilde{A}_F^+ - t\widetilde{A}_F^-) \cdot \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} = \left(M \mid 0 \right).$$

Therefore, $\Delta_1 \left(M \mid 0 \right) \doteq \Delta_1(\widetilde{A}_F^+ - t\widetilde{A}_F^-)$, and the lemma is proved. □

We are now ready to state the main result of this section.
Let $L(\underline{m}) = L(m_1, \ldots, m_n)$ be a multilink with Alexander polynomial $\Delta^{L(\underline{m})}(t)$. If there is an index i such that $d_i = 0$, we know (by corollary 1.2.8) that $\Delta^{L(\underline{m})}(t)$ vanishes. So let us suppose that $d_i > 0$ for all i (that is: $r = 0$). In the fibered case, $\Delta^{L(\underline{m})}(t)$ is just the characteristic polynomial of the monodromy $h\colon F \longrightarrow F$. Clearly, h permutes cyclically the d_i boundary components of F near L_i; hence $\Delta^{L(\underline{m})}(t)$ will have a factor of the form $\frac{1}{t^d-1}\prod_{i=1}^n (t^{d_i} - 1)$. Once again, this is true for any multilink.

2.1.6 Theorem.
There exists a unique polynomial $\nabla^{L(\underline{m})}(t)$ in $\mathbb{Z}[t,t^{-1}]$ which satisfies the following properties:

- $\Delta^{L(\underline{m})}(t) \doteq \frac{1}{t^d-1}\prod_{i=1}^n (t^{d_i} - 1) \cdot \nabla^{L(\underline{m})}(t)$;
- $\nabla^{L(\underline{m})}(t^{-1}) = \nabla^{L(\underline{m})}(t)$, *and the leading coefficient of $\nabla^{L(\underline{m})}(t)$ is positive.*[1]

Furthermore, we have:

- $|\nabla^{L(\underline{m})}(1)| = \frac{d^2\, D}{d_1 \cdots d_n m_1 \cdots m_n}$, *where D is any $(n-1) \times (n-1)$-minor of the matrix*

$$\begin{pmatrix} -\sum_j m_1 m_j \ell_{1j} & m_1 m_2 \ell_{12} & \ldots & m_1 m_n \ell_{1n} \\ m_1 m_2 \ell_{12} & -\sum_j m_2 m_j \ell_{2j} & \ldots & m_2 m_n \ell_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_1 m_n \ell_{1n} & m_2 m_n \ell_{2n} & \ldots & -\sum_j m_n m_j \ell_{nj} \end{pmatrix}.^2$$

- $\nabla^{L(m_1,\ldots,m_{n-1},0)}(t) = \nabla^{L(m_1,\ldots,m_{n-1})}(t)$.

Proof. By corollary 1.1.3, $\Delta^{L(\underline{m})}(t) \doteq \Delta^{L(\frac{\underline{m}}{d})}(t^d)$; therefore, it may be assumed for the whole proof that $d = 1$. Let F be a connected Seifert surface for $L(\underline{m})$. By lemma 2.1.5, the

[1] There is no way to define an intrinsic sign for $\nabla^{L(\underline{m})}(t)$; see remark 2.2.10.
[2] Here, an expression of the form $\frac{m_i}{m_i}$ should be formally simplified before replacing m_i by its value.

Alexander polynomial is given by $\Delta^{L(\underline{m})}(t) \doteq \Delta_1(\widetilde{A}_F^+ - t\widetilde{A}_F^-)$. Using the notations of this lemma, we see with the help of figure 2.2 that for all $i = 1, \dots, n$ and $j = 1, \dots, d_i$

$$\begin{aligned}
\ell k(i_-T_j^i, \gamma_j^i) &= \ell k(i_+T_j^i, \gamma_j^i) + 1 \\
\ell k(i_+T_j^i, \gamma_j^i) &= \ell k(i_-T_{j+1}^i, \gamma_j^i) \\
\ell k(i_-T_j^i, \gamma_j^i) &= \ell k(i_+T_{j+1}^i, \gamma_j^i) = \ell k(i_\pm T_k^i, \gamma_j^i) \quad \text{for all } k \neq j,\ j+1.
\end{aligned}$$

Furthermore, $\ell k(i_\pm T_j^i, y)$ does not depend on j for all element y of the basis of $\overline{V}$, $y \neq \gamma_1^i, \dots, \gamma_{d_i-1}^i$. Let us fix an index i, and focus on the $d_i \times (d_i - 1)$-submatrix of $\widetilde{A}_F^+ - t\widetilde{A}_F^-$ given by $T_1^i, \dots, T_{d_i}^i$ and $\gamma_1^i, \dots, \gamma_{d_i-1}^i$. Using the facts just mentioned, this matrix is of the form

$$\widetilde{A}_F^+ - t\widetilde{A}_F^- = \begin{array}{c} \\ \\ T_1^i \\ T_2^i \\ T_3^i \\ \vdots \\ T_{d_i}^i \end{array} \begin{array}{c} \\ \left(\begin{array}{cccccc} & \gamma_1^i & \gamma_2^i & \gamma_3^i & \cdots & \gamma_{d_i-1}^i \\ R & & & * & & \\ v_i(1-t) & a_1 - 1 & a_2 & a_3 & \cdots & a_{d_i-1} \\ v_i(1-t) & a_1 + t & a_2 - 1 & a_3 & \cdots & a_{d_i-1} \\ v_i(1-t) & a_1 & a_2 + t & a_3 - 1 & \cdots & a_{d_i-1} \\ \vdots & & & \ddots & & \\ v_i(1-t) & a_1 & a_2 & a_3 & \cdots & a_{d_i-1} + t \end{array}\right) \end{array},$$

which is equivalent to

$$\begin{pmatrix} R & * \\ v_i(1-t) & -1 + \sum_{j=1}^{d_i-1} a_j(1-t^j) \\ 0 & \frac{1-t^{d_i}}{1-t} \end{pmatrix}.$$

Hence, $\Delta^{L(\underline{m})}(t) \doteq \Delta_1 M$, where M is the matrix

$$\begin{array}{r} x_1,\dots,x_{2g}\left\{ \right. \\ T^1,\dots,T^n\left\{ \right. \\ \\ \end{array} \begin{pmatrix} \overbrace{G - tG^T}^{x_1,\dots,x_{2g}} & \overbrace{W(1-t)}^{L_1,\dots,L_n} & & * & \\ V(1-t) & L(1-t) & & * & \\ & & \frac{1-t^{d_1}}{1-t} & & \\ 0 & 0 & & \ddots & \\ & & & & \frac{1-t^{d_n}}{1-t} \end{pmatrix}.$$

Since $\sum_i m_i L_i$ vanishes in $\mathrm{H}_1\overline{F}$, as well as $\sum_i d_i T^i$ in $\mathrm{H}_1 F$, all the minors of dimension $2g + 2n - 1$ of M vanish, except the ones corresponding to a row T^i and a column L_j. Since every such minor is a multiple of

$$(1-t)^{n-1} \prod_{i=1}^{n} \frac{t^{d_i} - 1}{t - 1} = \frac{1}{1-t} \prod_{i=1}^{n} (t^{d_i} - 1),$$

so is $\Delta_1 M$. We get

$$\Delta^{L(\underline{m})}(t) \doteq \frac{1}{1-t} \prod_{i=1}^{n} (t^{d_i} - 1) \cdot \Delta_1 M', \quad \text{where} \quad M' = \begin{pmatrix} G - tG^T & W \\ V(1-t) & L \end{pmatrix}.$$

Let us note $\Delta^F(t) = \det(A_F^+ - tA_F^-)$ and $\nabla^F(t) = \frac{(t-1)\cdot\Delta^F(t)}{\prod_{i=1}^n (t^{d_i}-1)}$.

We will now prove that $\nabla^F(t^{-1}) = t^\nu \cdot \nabla^F(t)$ for some even integer ν. Renumbering the components, it may be assumed that $m_1 \neq 0$. Working over $\mathbb{Q}$, it is easy to see that the polynomial $\nabla^F_{\mathbb{Q}}(t)$ is given by the determinant of the matrix

$$\begin{matrix} & \overbrace{\quad x_1,\dots,x_{2g} \quad} & \overbrace{\quad T^2,\dots,T^n \quad} \\ x_1,\dots,x_{2g}\left\{\right. & G - tG^T & V^T \\ T^2,\dots,T^n\left\{\right. & V(1-t) & \ell = \ell^T \end{matrix}.$$

Hence, we have the equalities

$$\begin{aligned} \nabla^F_{\mathbb{Q}}(t^{-1}) &= \det\begin{pmatrix} G - t^{-1}G^T & V^T \\ V(1-t^{-1}) & \ell \end{pmatrix} \\ &= t^{-2g}\cdot\det\begin{pmatrix} G^T - tG & V^T \\ V(1-t) & \ell \end{pmatrix} \\ &= t^{-2g}\cdot\det\begin{pmatrix} G - tG^T & V^T(1-t) \\ V & \ell^T \end{pmatrix} \\ &= t^{-2g}\cdot\det\begin{pmatrix} G - tG^T & V^T \\ V(1-t) & \ell \end{pmatrix} \\ &= t^{-2g}\cdot\nabla^F_{\mathbb{Q}}(t). \end{aligned}$$

Of course, $\nabla^F(t) = \alpha\cdot\nabla^F_{\mathbb{Q}}(t)$ for some integer α; hence,

$$\nabla^F(t^{-1}) = \alpha\cdot\nabla^F_{\mathbb{Q}}(t^{-1}) = t^{-2g}\cdot\alpha\cdot\nabla^F_{\mathbb{Q}}(t) = t^{-2g}\cdot\nabla^F(t).$$

As a consequence, we can define

$$\nabla^{L(\underline{m})}(t) = \epsilon\cdot t^{-g}\cdot\nabla^F(t),$$

where $\epsilon = \pm 1$ is chosen such that the leading coefficient of $\nabla^{L(\underline{m})}$ is positive. This gives the unique polynomial satisfying the first points of the theorem.

Let us now compute $|\nabla^{L(\underline{m})}(1)|$. By the beginning of the proof, $|\nabla^{L(\underline{m})}(1)|$ is equal to $\Delta_1 M''$, where M'' is the following matrix:

$$\begin{matrix} & \overbrace{\quad x_1,\dots,x_{2g} \quad} & \overbrace{\quad L_1,\dots,L_n \quad} \\ x_1,\dots,x_{2g}\left\{\right. & G - G^T & * \\ T^1,\dots,T^n\left\{\right. & 0 & L \end{matrix}.$$

But $G - G^T$ is the intersection matrix of a closed surface of genus g, so it is unimodular. Therefore, $|\nabla^{L(\underline{m})}(1)| = \Delta_1 L$. Recall that L is the $(n\times n)$-matrix given by $L_{i,j} = \ell k(T^i, L_j)$;

by proposition 1.2.1, it is equal to

$$\begin{pmatrix} -\frac{1}{d_1}\sum m_j\ell_{1j} & \frac{m_1}{d_1}\ell_{12} & \dots & \frac{m_1}{d_1}\ell_{1n} \\ \frac{m_2}{d_2}\ell_{12} & -\frac{1}{d_2}\sum m_j\ell_{2j} & \dots & \frac{m_2}{d_2}\ell_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m_n}{d_n}\ell_{1n} & \frac{m_n}{d_n}\ell_{2n} & \dots & -\frac{1}{d_n}\sum m_j\ell_{nj} \end{pmatrix}.$$

Let us now consider the $(n \times n)$-matrix (m_{ij}) given by

$$m_{ij} = \begin{cases} m_i m_j \ell_{ij} & \text{if } i \neq j \ ; \\ -\sum_k m_i m_k \ell_{ik} & \text{if } i = j \ , \end{cases}$$

and let D be any $(n-1)\times(n-1)$-minor of this matrix. If L^{ij} denotes the (i,j)−minor of L, then

$$\begin{aligned} |\nabla^{L(\underline{m})}(1)| &= \gcd\left(L^{ij} \ ; \ i,j = 1,\dots,n\ \right) \\ &= \gcd\left(\frac{d_i m_j D}{d_1\cdots d_n m_1\cdots m_n} \ ; \ i,j=1,\dots,n\ \right) \\ &= \gcd\left(d_i m_j \ ; \ i,j = 1,\dots,n\ \right)\frac{D}{d_1\cdots d_n m_1\cdots m_n}. \end{aligned}$$

Since $\gcd(d_i \ ; \ i = 1,\dots,n\ ) = \gcd(m_j \ ; \ j = 1,\dots,n\ ) = 1$, it follows that $|\nabla^{L(\underline{m})}(1)| = \frac{D}{d_1\cdots d_n m_1\cdots m_n}$.

Finally, let us note $L(\underline{m}) = L(m_1,\dots,m_{n-1},0)$ and $L_0(\underline{m}) = L(m_1,\dots,m_{n-1})$. By the proof of proposition 2.1.1, $\Delta^F(t) = (t^{d_n}-1)\cdot\Delta^{F_0}(t)$ with F and F_0 of the same genus g. Therefore,

$$\begin{aligned} \nabla^{L(\underline{m})}(t) &= \epsilon\cdot t^{-g}\cdot\nabla^F(t) = \epsilon\cdot t^{-g}\cdot\frac{(t-1)\cdot\Delta^F(t)}{\prod_{i=1}^n(t^{d_i}-1)} \\ &= \epsilon\cdot t^{-g}\cdot\frac{(t-1)\cdot\Delta^{F_0}(t)}{\prod_{i=1}^{n-1}(t^{d_i}-1)} = \epsilon\cdot t^{-g}\cdot\nabla^{F_0}(t) \\ &= \nabla^{L_0(\underline{m})}(t). \end{aligned}$$

This concludes the proof of the theorem. □

This theorem has an important consequence: for any multilink $L(\underline{m})$ in a homology sphere, there exists a unique preferred representative of the Alexander polynomial. If $d = 1$, it is given by

$$\Delta^{L(\underline{m})}(t) = \epsilon\cdot t^{-g}\cdot\det(A_F^+ - tA_F^-),$$

where F is any connected Seifert surface of genus g, and $\epsilon = \pm1$ is chosen such that the leading coefficient of the polynomial is positive. For $d > 1$, $\Delta^{L(\underline{m})}(t)$ is given by $\Delta^{L(\frac{\underline{m}}{d})}(t^d)$. This polynomial is a multiple of $\frac{1}{t^d-1}\prod_{i=1}^n(t^{d_i}-1)$, and the quotient

$$\nabla^{L(\underline{m})}(t) = \frac{(t^d-1)\Delta^{L(\underline{m})}(t)}{\prod_{i=1}^n(t^{d_i}-1)}$$

satisfies $\nabla^{L(\underline{m})}(t^{-1}) = \nabla^{L(\underline{m})}(t)$. From now on, $\Delta^{L(\underline{m})}(t)$ will denote this preferred polynomial. Of course, the sign of $\Delta^{L(\underline{m})}$ is not intrinsic: the polynomial $\Delta^{L(\underline{m})} = t^{-g}\det(A_F^+ - tA_F^-)$ is only defined up to sign. Indeed, replacing any element of the basis of $\mathrm{H}_1 F$ by its opposite (without changing the basis of $\mathrm{H}_1\overline{F}$) turns $\Delta^{L(\underline{m})}$ into $-\Delta^{L(\underline{m})}$. On the question of the impossibility to define an intrinsic sign for the Alexander polynomial of a multilink, see remark 2.2.10.

2.1.7 Corollary.
For any multilink $L(\underline{m})$, $\Delta^{L(\underline{m})}$ is equal to $\Delta^{L(-\underline{m})}$.

Proof. It may be assumed that $d = 1$. If F is a good Seifert surface for $L(\underline{m})$, then $-F$ is a good Seifert surface for $L(-\underline{m})$. Hence,

$$\begin{aligned}\Delta^{L(-\underline{m})}(t) &\doteq \det(A^+_{-F} - tA^-_{-F}) = \det(A^-_F - tA^+_F)\\ &\doteq \det(A^+_F - t^{-1}A^-_F) = \Delta^{L(\underline{m})}(t^{-1}) \doteq \Delta^{L(\underline{m})}(t).\end{aligned}$$

Since $\frac{(t-1)\Delta^{L(\underline{m})}(t)}{\prod_i (t^{d_i}-1)}$ and $\frac{(t-1)\Delta^{L(-\underline{m})}(t)}{\prod_i (t^{d_i}-1)}$ are symmetric with positive leading coefficient, $\Delta^{L(\underline{m})}$ and $\Delta^{L(-\underline{m})}$ are equal. □

2.1.8 Corollary.
For any connected Seifert surface F, $\det A^+_F = (-1)^{\mu} \cdot \det A^-_F$, where $\mu = n + \sum_{i=1}^n d_i$.

Proof. Let us fix a connected Seifert surface F of genus g. Since the polynomial $\nabla^{L(\underline{m})}(t) = t^{-g}(t-1)\frac{\det(A^+_F - tA^-_F)}{\prod_i (t^{d_i}-1)}$ satisfies $\nabla^{L(\underline{m})}(t^{-1}) = \nabla^{L(\underline{m})}(t)$, it follows

$$\begin{aligned}\det(A^-_F - tA^+_F) &= (-t)^{2g+\sum_i d_i - 1} \cdot \det(A^+_F - t^{-1}A^-_F)\\ &= (-1)^{1+\sum_i d_i} \cdot t^{g+\sum_i d_i - 1} \cdot \frac{1}{t^{-1}-1}\prod_{i=1}^n (t^{-d_i}-1) \cdot \nabla^{L(\underline{m})}(t^{-1})\\ &= (-1)^{n+\sum_i d_i} \cdot t^g \cdot \frac{1}{t-1}\prod_{i=1}^n (t^{d_i}-1) \cdot \nabla^{L(\underline{m})}(t)\\ &= (-1)^{n+\sum_i d_i} \cdot \det(A^+_F - tA^-_F).\end{aligned}$$

Hence, the following equality holds:

$$\det A^-_F + a_1 t + \ldots + a_\nu t^\nu = (-1)^{n+\sum_i d_i} \cdot \det A^+_F + b_1 t + \ldots + b_\nu t^\nu .$$

In particular, $\det A^+_F = (-1)^{n+\sum_i d_i} \cdot \det A^-_F$. □

2.1.9 Corollary.
Let H be a monodromy matrix for a fibered multilink $L(\underline{m})$; then, the determinant of H is given by $\det H = (-1)^{n+d+1+\sum_i d_i}$.

Proof. If F denotes the fiber, we know by proposition 1.2.5 that $H^T = A^+_F \cdot (A^-_F)^{-1}$; if $d = 1$, corollary 2.1.8 leads to

$$\det H = \det H^T = \det A^+_F \cdot (\det A^-_F)^{-1} = (-1)^{n+\sum_{i=1}^n d_i}.$$

Now, if d is greater then 1, a monodromy matrix H for $L(\underline{m})$ is given by the $(d \times d)$-blocks matrix

$$H = \begin{pmatrix} & I & & \\ & & \ddots & \\ & & & I \\ \widetilde{H} & & & \end{pmatrix},$$

where $\widetilde{H}$ is any monodromy matrix for $L(\frac{\underline{m}}{d})$. Each block being of size $2g + \sum_i \frac{d_i}{d} - 1$, it follows

$$\begin{aligned} \det H &= (-1)^{(d-1)(2g+\sum_i \frac{d_i}{d}-1)} \cdot \det \begin{pmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & \widetilde{H} \end{pmatrix} \\ &= (-1)^{(d+1)(\sum_i \frac{d_i}{d}+1)} \cdot (-1)^{n+\sum_i \frac{d_i}{d}} = (-1)^{n+d+1+\sum_i d_i}. \end{aligned}$$

□

2.2 The Multivariable Alexander Polynomial

Let $L = L_1 \cup \ldots \cup L_n$ be an oriented link in a homology 3-sphere Σ, and let X be its complement. The Hurewicz homomorphism $\pi_1 X \longrightarrow \mathrm{H}_1 X$ is given by

$$\begin{array}{rcl} \pi_1 X & \longrightarrow & \mathbb{Z}^n \\ \gamma & \longmapsto & (\ell k(L_1, \gamma), \ldots, \ell k(L_n, \gamma)) \ . \end{array}$$

This homomorphism defines a regular $\mathbb{Z}^n$-covering of X called the **universal abelian covering** of X; let us denote this covering by $\widehat{X} \xrightarrow{\hat{p}} X$, and its fiber by $\widehat{X}^0$. The $\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$-module $\mathrm{H}_1(\widehat{X}, \widehat{X}^0)$ is the **Alexander module** of the link L. The **multivariable Alexander polynomial** of L is equal to

$$\Delta_L(t_1, \ldots, t_n) = \Delta_1 \mathrm{H}_1(\widehat{X}, \widehat{X}^0)\,,$$

the greatest common divisor of $E_1 \mathrm{H}_1(\widehat{X}, \widehat{X}^0)$, the first elementary ideal of the Alexander module.[3] Note that $\Delta_L(t_1, \ldots, t_n)$ is only defined up to multiplication by $\pm t_1^{\nu_1} \cdots t_n^{\nu_n}$.

Let us recall the relation between the Alexander polynomial $\Delta^{L(\underline{m})} \in \mathbb{Z}[t, t^{-1}]$ of a multilink and the Alexander polynomial $\Delta_L \in \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ of the underlying oriented link. For a proof, see [11, Proposition 5.1] or [46, Theorem 1.1.3].

2.2.1 Proposition.

$$\Delta^{L(\underline{m})}(t) \doteq \begin{cases} (t^d - 1) \cdot \Delta_L(t^{m_1}, \ldots, t^{m_n}) & \text{if } n > 1; \\ \Delta_L(t^{m_1}) & \text{if } n = 1. \end{cases}$$

□

Using theorem 2.1.6 and proposition 2.2.1, it is now very easy to prove several classical results.

2.2.2 Corollary. (Hosokawa [16])
Given an oriented link $L = L_1 \cup \ldots \cup L_n$ with $\ell_{ij} = \ell k(L_i, L_j)$, the polynomial ∇_L defined by

$$\nabla_L(t) \doteq \begin{cases} (\frac{1}{t-1})^{n-2} \Delta_L(t, \ldots, t) & \text{if } n > 1; \\ \Delta_L(t) & \text{if } n = 1 \end{cases}$$

[3] Levine [23] showed that $\Delta_L(t_1, \ldots, t_n)$ is also equal to $\Delta_0 \mathrm{H}_1 \widehat{X}$.

satisfies $\nabla_L(t^{-1}) = t^\nu \cdot \nabla_L(t)$ *with* ν *even. Furthermore,* $|\nabla_L(1)|$ *is equal to any* $(n-1)$*-minor of the matrix*

$$\begin{pmatrix} -\sum \ell_{1j} & \ell_{12} & \dots & \ell_{1n} \\ \ell_{12} & -\sum \ell_{2j} & \dots & \ell_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{1n} & \ell_{2n} & \dots & -\sum \ell_{nj} \end{pmatrix}.$$

Proof. By proposition 2.2.1, the Hosakawa polynomial ∇_L is equal to $\nabla^{L(1,\dots,1)}$. The second and third points of theorem 2.1.6 give the formulas. □

2.2.3 Corollary. (Kidwell [21])
Let $L = L_1 \cup \dots \cup L_n$ *be an oriented link with* $n \geq 2$ *and* $\ell k(L_i, L_j) = 0$ *for all* $i \neq j$.

- *If* n *is even,* $(t-1)^{2n-2}$ *divides* $\Delta_L(t,\dots,t)$.
- *If* n *is odd,* $(t-1)^{2n-3}$ *divides* $\Delta_L(t,\dots,t)$.

Proof. By proposition 2.2.1, $\Delta^{L(1,\dots,1)}(t) \doteq (t-1)\cdot\Delta_L(t,\dots,t)$. We also know by the proof of theorem 2.1.6 that $\nabla^{L(1,\dots,1)}(t) \doteq (\frac{1}{t-1})^{n-1}\cdot\Delta^{L(1,\dots,1)}(t)$ is given by the determinant of

$$\begin{matrix} 2g\left\{ \vphantom{\begin{matrix} a \\ b \end{matrix}} \right. \\ n-1\left\{ \vphantom{\begin{matrix} a \\ b \end{matrix}} \right. \end{matrix} \begin{pmatrix} \overbrace{G - tG^T}^{2g} & \overbrace{V^T}^{n-1} \\ V(1-t) & \widetilde{L} \end{pmatrix}.$$

By hypothesis, $\widetilde{L} = 0$, so

$$\nabla^{L(1,\dots,1)}(t) = (t-1)^{n-1}\cdot\det\begin{pmatrix} G - tG^T & V^T \\ V & 0 \end{pmatrix} =: (t-1)^{n-1}\cdot\det P(t).$$

Hence, $\Delta_L(t,\dots,t) = (t-1)^{2n-3}\cdot\det P(t)$. Furthermore,

$$\det P(1) = (-1)^{n-1}\cdot\det(-P(1)) = (-1)^{n-1}\cdot\det P(1)^T = (-1)^{n-1}\cdot\det P(1).$$

So if n is even, $\det P(1) = 0$ and we have one more factor $(t-1)$. □

To derive the classical results of Torres on the multivariable Alexander polynomial, we also need the following lemma.

2.2.4 Lemma.
Let Δ *and* Δ' *be two elements of* $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$.

(*i*) *If* $\Delta(t^{m_1},\dots,t^{m_n}) = \Delta'(t^{m_1},\dots,t^{m_n})$ *in* $\mathbb{Z}[t,t^{-1}]$ *for all integers* $m_1,\dots,m_n$, *then* $\Delta = \Delta'$ *in* $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$.

(*ii*) *If* $\Delta(t^{m_1},\dots,t^{m_n}) \doteq \Delta'(t^{m_1},\dots,t^{m_n})$ *in* $\mathbb{Z}[t,t^{-1}]$ *for all integers* $m_1,\dots,m_n$, *then* $\Delta \doteq \Delta'$ *in* $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$.

(iii) If $\Delta(t^{m_1},\ldots,t^{m_n}) = t^{\nu(m_1,\ldots,m_n)}\Delta'(t^{m_1},\ldots,t^{m_n})$ *in* $\mathbb{Z}[t,t^{-1}]$ *for all* $(m_1,\ldots,m_n)\in\mathbb{Z}^n$ *(where* ν *denotes a function* $\mathbb{Z}^n \xrightarrow{\nu} \mathbb{Z}$*), then* $\Delta(t_1,\ldots,t_n) = t_1^{\nu_1}\cdots t_n^{\nu_n}\Delta'(t_1,\ldots,t_n)$ *in* $\mathbb{Z}[t_1,t_1^{-1},\ldots,t_n,t_n^{-1}]$ *for some integers* $\nu_1,\ldots,\nu_n$*.*

Proof. Let us note $\Delta = \sum a_{i_1\cdots i_n} t_1^{i_1}\cdots t_n^{i_n}$ and $\Delta' = \sum b_{j_1\cdots j_n} t_1^{j_1}\cdots t_n^{j_n}$. Choose a positive integer N such that

$$a_{i_1\cdots i_n}\neq 0 \Rightarrow |i_k| < N \ \ \forall k \text{ , and } \quad b_{j_1\cdots j_n}\neq 0 \Rightarrow |j_k| < N \ \ \forall k.$$

Taking $m_1 = 1$, $m_2 = 2N$, $\ldots, m_n = (2N)^{n-1}$, it follows

$$\sum a_{i_1\cdots i_n} t^{i_1+(2N)i_2+\ldots+(2N)^{n-1}i_n} = \sum b_{j_1\cdots j_n} t^{j_1+(2N)j_2+\ldots+(2N)^{n-1}j_n} \qquad (*)$$

with $|i_k| < N$ and $|j_k| < N$ for all k. But $i_1+(2N)i_2+\ldots+(2N)^{n-1}i_n = j_1+(2N)j_2+\ldots+(2N)^{n-1}j_n$ with $|i_k|<N$ and $|j_k|<N$ for all k implies that $(i_1,\cdots,i_n) = (j_1,\cdots,j_n)$. So for all multi-index $I = (i_1,\cdots,i_n)$, $a_I = b_I$. Hence, $\Delta = \Delta'$, and the first point is proved.

If Δ is zero, the second point holds; so let us suppose that $\Delta \neq 0$. Without loss of generality, it may be assumed that Δ has the form $\Delta = \sum a_{i_1\cdots i_n} t_1^{i_1}\cdots t_n^{i_n}$ with $a_{0\ldots 0} > 0$ and $a_{i_1\cdots i_n} = 0$ as soon as some index i_k is negative. Similarly, it may be assumed that $\Delta' = \sum b_{j_1\cdots j_n} t_1^{j_1}\cdots t_n^{j_n}$ with $b_{0\ldots 0} > 0$ and $b_{j_1\cdots j_n} = 0$ if $j_k < 0$ for some k. By hypothesis, the equality

$$\Delta(t^{m_1},\ldots,t^{m_n}) = \sum a_{i_1\cdots i_n} t^{\sum m_k i_k} \doteq \sum b_{j_1\cdots j_n} t^{\sum m_k j_k} = \Delta'(t^{m_1},\ldots,t^{m_n})$$

holds for all integers $(m_1,\ldots,m_n)\in\mathbb{Z}^n$. This means that there exists functions $\mathbb{Z}^n \xrightarrow{\epsilon} \{\pm 1\}$ and $\mathbb{Z}^n \xrightarrow{\nu} \mathbb{Z}$ such that the equality

$$\sum a_{i_1\cdots i_n} t^{\sum m_k i_k} = \epsilon(m_1,\ldots,m_n)\cdot t^{\nu(m_1,\ldots,m_n)}\cdot \sum b_{j_1\cdots j_n} t^{\sum m_k j_k}$$

holds for all integers $(m_1,\ldots,m_n)\in\mathbb{Z}^n$. Let us choose $m_1 = 1$, $m_2 = 2N$, $\ldots, m_n = (2N)^{n-1}$ as above. Since these integers are positive, as well as $a_{0\ldots 0}$ and $b_{0\ldots 0}$, it follows that $\epsilon(1,\ldots,(2N)^{n-1}) = +1$ and $\nu(1,\ldots,(2N)^{n-1}) = 0$. So we get back the equation $(*)$, and we can conclude as for the first point.

For the third point, it may be assumed that Δ and Δ' have a non-zero constant term and only positive powers of t_i. The argument then follows the proof of the second point. □

2.2.5 Corollary. (The Torres formula [44])
Let $L = L_1\cup\ldots\cup L_n$ *be an oriented link with* $n\geq 2$*,* L_0 *be the sublink* $L_1\cup\ldots\cup L_{n-1}$*, and* $\ell_i = \ell k(L_i, L_n)$*; then,*

$$\Delta_L(t_1,\ldots,t_{n-1},1) \doteq \begin{cases} (t_1^{\ell_1}\cdots t_{n-1}^{\ell_{n-1}} - 1)\cdot \Delta_{L_0}(t_1,\ldots,t_{n-1}) & \text{if } n>2; \\ \frac{t_1^{\ell_1}-1}{t_1-1}\cdot \Delta_{L_0}(t_1) & \text{if } n=2. \end{cases}$$

Proof. Let us denote by Δ' the right-hand side of Torres formula. By the second point of lemma 2.2.4, we just need to show that $\Delta'(t^{m_1},\ldots,t^{m_{n-1}}) \doteq \Delta_L(t^{m_1},\ldots,t^{m_{n-1}},1)$ for all integers $m_1, \ldots, m_{n-1}$.

$$\begin{aligned}
\Delta'(t^{m_1}, \ldots, t^{m_{n-1}}) &= \begin{cases} (t^{\sum m_i \ell_i} - 1) \cdot \Delta_{L_0}(t^{m_1}, \ldots, t^{m_{n-1}}) & \text{if } n > 2; \\ \frac{t^{m_1 \ell_1} - 1}{t^{m_1} - 1} \cdot \Delta_{L_0}(t^{m_1}) & \text{if } n = 2, \end{cases} \\
&= \begin{cases} (t^{d_n} - 1) \cdot \Delta_{L_0}(t^{m_1}, \ldots, t^{m_{n-1}}) & \text{if } n > 2; \\ \frac{t^{d_2} - 1}{t^{m_1} - 1} \cdot \Delta_{L_0}(t^{m_1}) & \text{if } n = 2, \end{cases} \\
\text{(by proposition 2.2.1)} &\doteq \frac{1}{t^d - 1} \cdot (t^{d_n} - 1) \cdot \Delta^{L_0(m_1, \ldots, m_{n-1})}(t) \\
\text{(by proposition 2.1.1)} &\doteq \frac{1}{t^d - 1} \cdot \Delta^{L_0(m_1, \ldots, m_{n-1}, 0)}(t) \\
&\doteq \Delta_L(t^{m_1}, \ldots, t^{m_{n-1}}, 1).
\end{aligned}$$

□

2.2.6 Corollary. (Seifert [41], Torres-Fox [45])
Let $L = L_1 \cup \ldots \cup L_n$ be a link with Alexander polynomial Δ_L.

- *If $n = 1$, Δ_L satisfies $\Delta_L(t^{-1}) = t^\nu \cdot \Delta_L(t)$ with $\nu \equiv 0 \pmod 2$ and $\Delta_L(1) = \pm 1$.*
- *If $n > 1$, Δ_L satisfies*

$$\Delta_L(t_1^{-1}, \ldots, t_n^{-1}) = (-1)^n \cdot t_1^{\nu_1} \cdots t_n^{\nu_n} \cdot \Delta_L(t_1, \ldots, t_n)$$

with integers ν_i such that $\nu_i \equiv 1 + \sum_j \ell_{ij} \pmod 2$ if $\Delta_L \neq 0$.

Proof. In the case of a knot, $\nabla^{L(m_1)}(t) \doteq \Delta^{L(m_1)}(t) \doteq \Delta_L(t^{m_1})$; in particular, $\Delta_L(t) \doteq \nabla^{L(1)}(t)$ and Seifert's result is given by the second and third points of theorem 2.1.6.

Let us now suppose that $n \geq 2$, and denote by $\Delta_L \neq 0$ some fixed representative of the multivariable Alexander polynomial of L. Let us fixe some integers $m_1, \ldots, m_n$, and define $\nabla(t) = \frac{(t^d - 1)^2 \Delta_L(t^{m_1}, \ldots, t^{m_n})}{\prod_{i=1}^n (t^{d_i} - 1)}$. By proposition 2.2.1, $\nabla \doteq \nabla^{L(\underline{m})}$. By the second point of theorem 2.1.6, it follows that $\nabla(t^{-1}) = t^\nu \cdot \nabla(t)$ for some even integer ν. Therefore,

$$\begin{aligned}
\Delta_L(t^{-m_1}, \ldots, t^{-m_n}) &= \frac{\prod_{i=1}^n (t^{-d_i} - 1)}{(t^{-d} - 1)^2} \cdot \nabla(t^{-1}) \\
&= (-1)^n \cdot t^{2d + \nu - \sum d_i} \cdot \frac{\prod_{i=1}^n (t^{d_i} - 1)}{(t^d - 1)^2} \cdot \nabla(t) \\
&= (-1)^n \cdot t^{\nu + 2d - \sum d_i} \cdot \Delta_L(t^{m_1}, \ldots, t^{m_n}).
\end{aligned}$$

Hence, for all integers $m_1, \ldots, m_n$, the equality

$$\Delta_L(t^{-m_1}, \ldots, t^{-m_n}) = (-1)^n \cdot t^{\nu(m_1, \ldots, m_n)} \cdot \Delta_L(t^{m_1}, \ldots, t^{m_n}),$$

holds, for some function $\mathbb{Z}^n \xrightarrow{\nu} \mathbb{Z}$. By the third point of lemma 2.2.4, it follows that

$$\Delta_L(t_1^{-1}, \ldots, t_n^{-1}) = (-1)^n \cdot t_1^{\nu_1} \cdots t_n^{\nu_n} \cdot \Delta_L(t_1, \ldots, t_n). \qquad (\star)$$

It is a well known (and easy) fact that the Torres formula, together with $(\star)$, imply that $\nu_i \equiv 1 + \sum_j \ell_{ij} \pmod 2$; see Hillman [15, pp.85–86] or Torres-Fox [45] for the proof. □

These two last results provide very strong necessary conditions for a polynomial $\Delta \in \Lambda_n = \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ to be the Alexander polynomial of a link. They are known as the **Torres conditions**. The obvious question is now to ask which of the polynomials satisfying these conditions can be realized as an Alexander polynomial. Let us state this problem in a more precise way.

Let $n \geq 1$ be an integer; according to Levine, an **n-form** is an integral symmetric $(n \times n)$-matrix with zero diagonal entries. Let $\ell = (\ell_{ij})$ be any n-form; we define a subset $\mathcal{T}_\ell$ of Λ_n by induction on n.

- If $n = 1$, let $\mathcal{T}_\ell$ be the set of polynomials Δ in Λ_1 satisfying

$$\Delta(t^{-1}) = \Delta(t) \quad \text{and} \quad \Delta(1) = 1.$$

- If $n \geq 2$, let ℓ^i be the $(n-1)$-form obtained from ℓ by deleting the i-th row and column; then, $\mathcal{T}_\ell$ is the set of polynomials satisfying

$$\Delta(t_1^{-1}, \ldots, t_n^{-1}) = (-1)^n \cdot t_1^{\nu_1} \cdots t_n^{\nu_n} \cdot \Delta(t_1, \ldots, t_n)$$

  with $\nu_i = \sum_j \ell_{ij} - 1$, and for all i,

$$\Delta(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) = \begin{cases} (\prod_j t_j^{\ell_{ij}} - 1) \cdot \Delta_i(t_1, \ldots, \widehat{t_i}, \ldots, t_n) & \text{if } n > 2; \\ \frac{t_j^{\ell_{12}} - 1}{t_j - 1} \cdot \Delta_i(t_j) \qquad (i \neq j) & \text{if } n = 2, \end{cases}$$

  with Δ_i some element of $\mathcal{T}_{\ell^i}$.

If $\Delta \in \Lambda_n$ is the Alexander polynomial of an ℓ-link, there exists some representative of Δ in $\mathcal{T}_\ell$. In other words,

$$\{\, \Delta \in \Lambda_n \,;\, \Delta \doteq \Delta_L \text{ for some } \ell\text{-link } L \,\} \subset \Lambda_n^* \cdot \mathcal{T}_\ell.$$

It is known since Seifert [41] that the converse is true when $n = 1$: every element of $\mathcal{T}_{(0)}$ is the Alexander polynomial of a knot. For $n = 2$, let us note b the 2-form given by $\ell_{12} = b$. Levine [23] proved that the Torres conditions are also sufficiant when $b = 0$ or ± 1. But Hillman [15] gave an example of a polynomial in $\mathcal{T}_6$ that cannot be realized as an Alexander polynomial.

Theorem 2.1.6 and proposition 2.2.1 provide a new set of necessary conditions for Δ in Λ_n to be the Alexander polynomial of a link.

2.2.7 Corollary.
Let $L = L_1 \cup \ldots \cup L_n$ be a link, $n \geq 2$, with $\ell k(L_i, L_j) = \ell_{ij}$ and $\ell_{ii} = 0$; then, its Alexander polynomial Δ_L satisfies the following conditions.
For all integers $\underline{m} = (m_1, \ldots, m_n)$ with $d = gcd(m_1, \ldots, m_n)$ and $d_i = gcd(m_i, \sum_j m_j \ell_{ij}) > 0$, there exists some polynomial $\nabla^{L(\underline{m})}(t)$ in $\mathbb{Z}[t^d, t^{-d}]$ which satisfies:

- $\nabla^{L(\underline{m})}(t) \doteq \frac{(t^d-1)^2 \Delta_L(t^{m_1}, \ldots, t^{m_n})}{\prod_i (t^{d_i} - 1)}$;
- $\nabla^{L(\underline{m})}(t^{-1}) = \pm \nabla^{L(\underline{m})}(t)$;

- $|\nabla^{L(\underline{m})}(1)| = \frac{d^2\, D}{d_1 \cdots d_n m_1 \cdots m_n}$, *where D is any minor of the matrix given in theorem 2.1.6;*
- $\nabla^{L(m_1,\dots,m_{n-1},0)}(t) = \pm\nabla^{L(m_1,\dots,m_{n-1})}(t)$. □

Given an n-form ℓ, let us denote by $\mathcal{C}_\ell$ the set of polynomials in Λ_n which satisfie the above conditions. The proof of corollaries 2.2.5 and 2.2.6 show that these conditions are as strong as the Torres conditions, that is,

$$\{\,\Delta \in \Lambda_n\,;\; \Delta \doteq \Delta_L \text{ for some } \ell\text{-link } L\,\} \subset \mathcal{C}_\ell \subset \Lambda_n^* \cdot \mathcal{T}_\ell.$$

Unfortunately, it turns out that these conditions on the multivariable Alexander polynomial are equivalent to the Torres conditions. In other "words":

2.2.8 Proposition.
$\mathcal{C}_\ell = \Lambda_n^* \cdot \mathcal{T}_\ell$.

Proof. Let Δ in $\mathcal{T}_\ell$, and let us fix integers $m_1,\dots,m_n$. Without loss of generality, it may be assumed that $d = 1$. Let us note $d_i = \gcd(m_i, m_i')$, where $m_i' = \sum_j m_j \ell_{ij}$.

• We have to show that $\prod_i (t^{d_i} - 1)$ divides $(t-1)^2\Delta(t^{m_1},\dots,t^{m_n})$; since the greatest common divisor of $t^{d_1}-1,\dots,t^{d_n}-1$ is equal to $t-1$, we just need to prove the two following facts:

(i) $(t-1)^n$ divides $(t-1)^2\Delta(t^{m_1},\dots,t^{m_n})$, that is, $(t-1)^{n-2}$ divides $\Delta(t^{m_1},\dots,t^{m_n})$;
(ii) $(t^{d_i}-1)$ divides $\Delta(t^{m_1},\dots,t^{m_n})$ for all i.

It is a well known fact that $\mathcal{T}_\ell \subset I^{n-2}$, where $I = (t_1 - 1,\dots,t_n - 1)$ is the augmentation ideal (see [15, p. 88]). Hence, $\Delta \in \mathcal{T}_\ell$ implies $\Delta(t^{m_1},\dots,t^{m_n}) \in J^{n-2}$, where $J = (t^{m_1} - 1,\dots,t^{m_n}-1) = (t-1)$ and the first point is proved.
To show the second point, we just need to check that $\Delta(\zeta^{m_1},\dots,\zeta^{m_n}) = 0$ for all ζ d_i-th root of the unity. Indeed, this would show that $t^{d_i}-1$ divides $\Delta(t^{m_1},\dots,t^{m_n})$ in $\mathbb{C}[t,t^{-1}]$, which implies that it divides it in $\mathbb{Z}[t,t^{-1}]$. Now, since d_i divides m_i, we have $\zeta^{m_i} = 1$. Using Torres formula, this gives

$$\Delta(\zeta^{m_1},\dots,1,\dots,\zeta^{m_n}) = \begin{cases} \frac{(\zeta^{m_i'}-1)}{(\zeta^{m_j}-1)} \cdot \Delta_i(\zeta) \quad (i \neq j) & \text{if } n = 2 \\ (\zeta^{m_i'}-1)\cdot\Delta_i(\zeta) & \text{if } n > 2\,, \end{cases}$$

which is equal to zero because d_i divides m_i', and the second point is proved. So, there exists a polynomial $\nabla(t)$ which satisfies the first point of corollary 2.2.7.

• Using the definition of ∇ and Torres condition on the symmetry of Δ, we find that $\nabla(t^{-1}) = t^\nu \cdot \nabla(t)$, with $\nu = \sum_i d_i - 2 + \sum_i m_i\nu_i$ and $\nu_i \equiv 1 - \sum_j \ell_{ij} \pmod 2$. So computing modulo 2, we get

$$\begin{aligned} \nu &\equiv \sum_i d_i + \sum_i m_i(1 + \sum_j \ell ij) = \sum_i (d_i + m_i) + \sum_i m_i \sum_j \ell_{ij} \\ &= \sum_i (d_i + m_i) + \sum_j\sum_i m_i\ell_{ij} = \sum_i (d_i + m_i) + \sum_j m_j' \\ &= \sum_i (gcd(m_i, m_i') + m_i + m_i') \equiv \sum_i m_i m_i' = \sum_i m_i \sum_j m_j \ell_{ij} \\ &= \sum_i\sum_j m_i m_j \ell_{ij} = 2\sum_{i<j} m_i m_j \ell_{ij} \equiv 0. \end{aligned}$$

Hence, we can choose $\nabla(t)$ such that $\nabla(t^{-1}) = \pm\nabla(t)$.

• The fact that $\nabla^{L(m_1,\dots,m_{n-1},0)}(t) = \pm\nabla^{L(m_1,\dots,m_{n-1})}(t)$ is the exact translation for $\nabla(t)$ of the Torres formula for $\Delta(t_1,\dots,t_n)$.

• Finally, the expression of $|\nabla^{L(\underline{m})}(1)|$ is a direct consequence of the following lemma.

2.2.9 Lemma.
Let $\ell = (\ell_{ij})$ be an n-form with $n \geq 2$, and let us define φ_ℓ, $\psi_\ell\colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$ by

$$\varphi_\ell(m_1,\dots,m_n) = \frac{\Delta(t^{m_1},\dots,t^{m_n})}{(t-1)^{n-2}}\Big|_{t=1}, \quad \psi_\ell(m_1,\dots,m_n) = \frac{D}{m_1\cdots m_n},$$

where $\Delta \in \mathcal{T}_\ell$ and D is the determinant of

$$\begin{pmatrix} \sum m_1 m_j \ell_{1j} & -m_1 m_2 \ell_{12} & \dots & -m_1 m_n \ell_{1\,n-1} \\ -m_1 m_2 \ell_{12} & \sum m_2 m_j \ell_{2j} & \dots & -m_2 m_n \ell_{2\,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -m_1 m_{n-1} \ell_{1\,n-1} & -m_2 m_{n-1} \ell_{2\,n-1} & \dots & \sum m_{n-1} m_j \ell_{n-1\,j} \end{pmatrix}.$$

Then, $\varphi_\ell = \psi_\ell$. In particular, φ_ℓ does not depend on the choice of $\Delta \in \mathcal{T}_\ell$.

Proof. Let us prove the lemma by induction on n, starting with the case $n = 2$:

$$\begin{aligned} \psi_\ell(m_1,m_2) &= \frac{\det(m_1 m_2 \ell_{12})}{m_1 m_2} = \ell_{12}, \quad \text{and} \\ \varphi_\ell(m_1,m_2) &= \Delta(t^{m_1},t^{m_2})\,|_{t=1} = \Delta(1,1) = \ell_{12}; \end{aligned}$$

indeed, $\Delta \in \mathcal{T}_\ell$ implies that $\Delta(t,1) = \frac{t^{\ell_{12}}-1}{t-1}\Delta'(t)$ with $\Delta'(1) = 1$.

Let us now suppose that $\varphi_{\ell'} = \psi_{\ell'}$ for all k-form ℓ' with $k \leq n-1$; consider an n-form ℓ with $n \geq 3$. We have

$$\varphi_\ell(0,m_2,\dots,m_n) = \frac{\Delta(1,t^{m_2},\dots,t^{m_n})}{(t-1)^{n-2}}\Big|_{t=1} = \frac{t^{\sum_i m_i \ell_{1i}}-1}{t-1}\frac{\Delta'(t^{m_2},\dots,t^{m_n})}{(t-1)^{n-3}}\Big|_{t=1},$$

with $\Delta' \in \mathcal{T}_{\ell'}$, ℓ' being the $(n-1)$-form obtained from ℓ by deleting the first line and the first column. By induction, it follows

$$\varphi_\ell(0,m_2,\dots,m_n) = \sum_i m_i \ell_{1i} \cdot \psi_{\ell'}(m_2,\dots,m_n).$$

Using the definition of ψ_ℓ, it is very easy to see that

$$\psi_\ell(0,m_2,\dots,m_n) = \sum_i m_i \ell_{1i} \cdot \psi_{\ell'}(m_2,\dots,m_n).$$

Hence, the equality $\varphi_\ell(0,m_2,\dots,m_n) = \psi_\ell(0,m_2,\dots,m_n)$ holds. Similarly, φ_ℓ and ψ_ℓ coincide on every hyperplane of the form $m_i = 0$, for $i = 1,\dots,n-1$.

Note that both φ_ℓ and ψ_ℓ are either homogeneous polynomial applications of degree $n-2$, or identically zero. (This is clear for ψ_ℓ, and easy for φ_ℓ once you recall that $\mathcal{T}_\ell \subset I^{n-2}$, where $I = (t_1-1,\dots,t_n-1)$.) Therefore, their difference $\Upsilon_\ell = \varphi_\ell - \psi_\ell$ is a polynomial of degree at most $n-2$. Since it vanishes on every hyperplan $m_i = 0$ for $i = 1,\dots,n-1$, $t_1 t_2 \cdots t_{n-1}$ divides $\Upsilon_\ell(t_1,\dots,t_n)$. As $\deg\Upsilon_\ell \leq n-2$, Υ_ℓ is identically zero. This proves the lemma, as well as the proposition. □

2.2.10 Remark. (Conway multivariable potential function) Following ideas of Conway [9], Hartley [14] constructed an isotopy invariant ∇_L for oriented links with n components. This invariant is characterized up to sign by the properties

$$\begin{aligned} \nabla_L(t_1, \ldots, t_n) &\doteq \begin{cases} \frac{1}{t_1 - t_1^{-1}} \cdot \Delta_L(t_1^2) & \text{if } n = 1; \\ \Delta_L(t_1^2, \ldots, t_n^2) & \text{if } n \geq 2, \end{cases} \\ \nabla_L(t_1^{-1}, \ldots, t_n^{-1}) &= (-1)^n \cdot \nabla_L(t_1, \ldots, t_n). \end{aligned}$$

Furthermore, Murakami [27] gave an axiomatic characterization of this invariant via a certain number of skein-type formulas.

The obvious question for us is: does the polynomial $\nabla_L(t^{m_1}, \ldots, t^{m_n})$ provide an invariant for multilinks? Such a result would have two interesting offshoots:

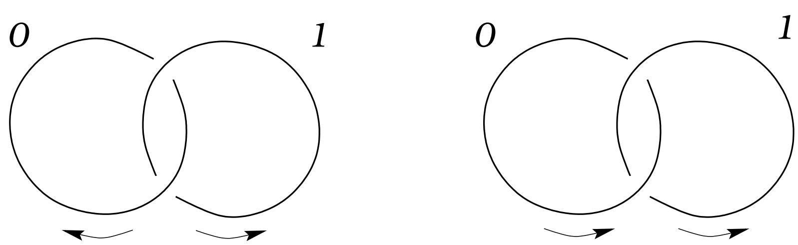
(i) Using proposition 2.2.1 and the proof of corollary 2.1.8, it is easy to show that the polynomial $\widetilde{\nabla}^{L(\underline{m})}(t) := \frac{1}{t^d - t^{-d}} \det(t^{-1}A_F^+ - tA_F^-)$, which is only defined up to sign, satisfies the equalities

$$\begin{aligned} \widetilde{\nabla}^{L(\underline{m})}(t) &\doteq \begin{cases} \Delta_L(t^{2m_1}, \ldots, t^{2m_n}) & \text{if } n > 1; \\ \frac{1}{t^{m_1} - t^{-m_1}} \cdot \Delta_L(t^{2m_1}) & \text{if } n = 1. \end{cases} \\ \widetilde{\nabla}^{L(\underline{m})}(t^{-1}) &= (-1)^n \cdot \widetilde{\nabla}^{L(\underline{m})}(t). \end{aligned}$$

Therefore, we have the equality $\nabla_L(t^{m_1}, \ldots, t^{m_n}) = \pm \widetilde{\nabla}^{L(\underline{m})}(t)$. In other words, the existence of an invariant of multilinks $\nabla_L(t^{m_1}, \ldots, t^{m_n})$ would provide an intrinsic sign for the polynomial $\frac{1}{t^d - t^{-d}} \det(t^{-1}A_F^+ - tA_F^-)$.

(ii) This polynomial invariant would be easy to compute from a given projection of the multilink using skein-type relations.

Unfortunately, this is not the case. Indeed, let us consider the following multilinks.



As multilinks, they are equivalent: recall that $m_i L_i = (-m_i)(-L_i)$. Nevertheless, the underlying oriented links are not equivalent: one is a positive clasp, with $\nabla_L = 1$, the other a negative clasp, with $\nabla_{L'} = -1$. This trivial observation shows that there is no way to define an intrinsic sign for the Alexander polynomial of a multilink.

Chapter 3

The Alexander Module via Splicing

In this chapter, we begin by introducing the notion of splicing, as defined by Larry Siebenmann [43], David Eisenbud and Walter Neumann [11]. From a classical link theoretical point of view, it is an interesting construction as it generalizes the operations of connected sum, disjoint sum, and cabling. But the main point of this operation is to be found in three-dimensional topology. Indeed, we will see in the next chapter that any link (resp. multilink) can be expressed in a canonical way as the splicing of "elementary" links (resp. multilinks) which are endowed with some geometric structure. Moreover, the Alexander polynomial of multilinks turns out to be multiplicative under splicing.

Our aim in this chapter is to understand the behaviour of the Alexander module under splicing. It turns out to be very tricky: no general closed formula is derived. Nevertheless, several partial results are obtained.

3.1 Definitions and Fundamental Results

In the sequel, we will denote by a bold letter $\mathbf{L}$ the topological pair (Σ, L) consisting of a link L in a homology 3-sphere Σ. The term "link" can designate L itself, or the pair $\mathbf{L} = (\Sigma, L)$.

Consider two links $\mathbf{L}' = (\Sigma', L'_0 \cup L'_1 \cup \ldots \cup L'_{n'})$ and $\mathbf{L}'' = (\Sigma'', L''_0 \cup L''_1 \cup \ldots \cup L''_{n''})$. Choose tubular neighborhoods $\mathcal{N}(L'_0)$ and $\mathcal{N}(L'_0)$ together with standard oriented parallels and meridians $P',\ M' \subset \partial\mathcal{N}(L'_0)$, $P'',\ M'' \subset \partial\mathcal{N}(L''_0)$, and set

$$\Sigma = (\Sigma' - \mathcal{N}(L'_0)) \cup_h (\Sigma'' - \mathcal{N}(L''_0)) \ ,$$

where $h\colon \partial\mathcal{N}(L'_0) \longrightarrow \partial\mathcal{N}(L''_0)$ is a homeomorphism sending P' onto M'' and M' onto P''. It is easy to check that Σ is a homology sphere. The link $\mathbf{L} = (\Sigma, L'_1 \cup \ldots \cup L'_{n'} \cup L''_1 \cup \ldots \cup L''_{n''})$ is called the **splice** of $\mathbf{L}'$ and $\mathbf{L}''$ along L'_0 and L''_0 and is denoted by

$$\mathbf{L} = \mathbf{L}' \underset{L'_0 \quad L''_0}{\overline{\qquad\qquad}} \mathbf{L}'' \ .$$

Conversely, let us consider $\mathbf{L} = (\Sigma, L_1 \cup \ldots \cup L_n)$ a link and $T^2 \subset \Sigma - (L_1 \cup \ldots \cup L_n)$ an embedded torus; then $\mathbf{L}$ is the result of a splicing operation $\mathbf{L} = \mathbf{L}' \underset{L'_0 \quad L''_0}{\overline{\qquad\qquad}} \mathbf{L}''$ along this torus, uniquely determined up to reversing the orientations of both L'_0 and L''_0 [11, Proposition 2.1]. We will say that $\mathbf{L}$ is **despliced** along T^2.

Now, suppose that an unoriented link $\mathbf{L}$ is the result of the splicing of two unoriented links $\mathbf{L}'$ and $\mathbf{L}''$, and let us denote by X, X' and X'' the respective link exteriors. By definition of splicing, X is the union of X' and X'' pasted along some torus. Hence, any multilink structure $\underline{m}$ on L (for example, an orientation of L) induces multilink structures $\underline{m}'$ on L' and $\underline{m}''$ on L''; indeed, a multilink structure on L is nothing but a cohomology class $\underline{m} \in \mathrm{H}^1 X = \mathrm{H}^1(X' \cup X'')$ which restricts to classes $\underline{m}' \in \mathrm{H}^1 X'$ and $\underline{m}'' \in \mathrm{H}^1 X''$. The important thing is that even if you start with an oriented link with multiplicities ± 1, wou may well end up with multilink splice components having multiplicities greater than one.

Conversely, consider two multilinks $(\Sigma', L'(\underline{m}'))$ and $(\Sigma'', L''(\underline{m}''))$ with exteriors X' and X''. When is the splice $(\Sigma, L) = (\Sigma', L') \frac{}{L_0' \quad L_0''} (\Sigma'', L'')$ a multilink with the same multiplicities on the remaining components? We have homomorphisms $\underline{m}' \colon \mathrm{H}_1 X' \longrightarrow \mathbb{Z}$ and $\underline{m}'' \colon \mathrm{H}_1 X'' \longrightarrow \mathbb{Z}$, and we are looking for a homomorphism $\underline{m} \colon \mathrm{H}_1 X \longrightarrow \mathbb{Z}$, where $X = \Sigma - \mathcal{N}(L) = X' \cup_{T^2} X''$. Consider the diagram

$$\begin{array}{ccccccc}
\mathrm{H}_1 T^2 & \longrightarrow & \mathrm{H}_1 X' \oplus \mathrm{H}_1 X'' & \longrightarrow & \mathrm{H}_1 X & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \underline{m}' + \underline{m}'} & & \downarrow{\scriptstyle \underline{m}} & & \\
\mathbb{Z} & = & \mathbb{Z} & = & \mathbb{Z}\,, & &
\end{array}$$

where the first line is an exact sequence; $\underline{m}$ is defined if and only if $\underline{m}'$ and $\underline{m}''$ agree on $\mathrm{H}_1 T^2 = \mathbb{Z}P' \oplus \mathbb{Z}M' = \mathbb{Z}M'' \oplus \mathbb{Z}P''$. Since $\underline{m}'(M') = m_0'$ and $\underline{m}''(M'') = m_0''$, the condition is given by the equations

$$\begin{cases} m_0' &= \underline{m}''(P'') = \sum_{j>0} m_j''\, \ell k(L_0'', L_j'') \\ m_0'' &= \underline{m}'(P') \;\;= \sum_{i>0} m_i'\, \ell k(L_0', L_i')\,. \end{cases}$$

Note that, by proposition 1.2.1, this is equivalent to asking that any Seifert surfaces F' for $L'(\underline{m}')$ and F'' for $L''(\underline{m}'')$ can be pasted along $T^2 = \partial\mathcal{N}(L_0') = \partial\mathcal{N}(L_0'')$, giving a Seifert surface for the multilink $L(\underline{m})$.

3.2 Mayer-Vietoris for Splicing

Let us consider the splicing $(\Sigma, L(\underline{m}))$ of two multilinks $(\Sigma', L'(\underline{m}'))$ and $(\Sigma'', L''(\underline{m}''))$ along $m_0' L_0'$ and $m_0'' L_0''$. Let us note $d' = gcd(\underline{m}')$, $d'' = gcd(\underline{m}'')$, $d = gcd(\underline{m}) = gcd(d', d'')$ and $m = gcd(m_0', m_0'')$. The case $d = 0$ being trivial, we will assume that $d > 0$. Also, let us denote by $\widetilde{X}$ (resp. $\widetilde{X}', \widetilde{X}''$) the total space of the infinite cyclic covering of X (resp. X', X'') given by the multiplicities $\underline{m}$ (resp. $\underline{m}', \underline{m}''$).

Of course, the main tool is the Mayer-Vietoris exact sequence of the union $\widetilde{X} = \widetilde{X}' \cup \widetilde{X}''$. Here, two cases need to be considered.

Case I: m=0

In this case, the meridien M' and the parallel P' lift to an infinite number of loops $\{t^i \cdot \widetilde{M}'\}_{i \in \mathbb{Z}}$ and $\{t^i \cdot \widetilde{P}'\}_{i \in \mathbb{Z}}$. The same holds for M'' and P''. Therefore, $\widetilde{X}' \cap \widetilde{X}''$ is the disjoint union of an infinite number of tori, and

$$\begin{aligned} \mathrm{H}_1(\widetilde{X}' \cap \widetilde{X}'') &= \mathbb{Z}[t, t^{-1}] \cdot \widetilde{M}' \,\oplus\, \mathbb{Z}[t, t^{-1}] \cdot \widetilde{P}' \\ &= \mathbb{Z}[t, t^{-1}] \cdot \widetilde{P}'' \,\oplus\, \mathbb{Z}[t, t^{-1}] \cdot \widetilde{M}''. \end{aligned}$$

Hence, the Mayer-Vietoris exact sequence takes the form

$$\begin{aligned}
0 &\longrightarrow \mathbb{Z}[t,t^{-1}] \longrightarrow \mathrm{H}_2\widetilde{X}' \oplus \mathrm{H}_2\widetilde{X}'' \longrightarrow \mathrm{H}_2\widetilde{X} \longrightarrow \mathbb{Z}[t,t^{-1}] \oplus \mathbb{Z}[t,t^{-1}] \\
&\xrightarrow{j_1} \mathrm{H}_1\widetilde{X}' \oplus \mathrm{H}_1\widetilde{X}'' \longrightarrow \mathrm{H}_1\widetilde{X} \xrightarrow{\partial} \mathbb{Z}[t,t^{-1}] \xrightarrow{j_0} \mathbb{Z}[t,t^{-1}]/(t^{d'}-1) \oplus \mathbb{Z}[t,t^{-1}]/(t^{d''}-1) \\
&\longrightarrow \mathbb{Z}[t,t^{-1}]/(t^{d}-1) \longrightarrow 0,
\end{aligned}$$

where $j_1(1,0) = (\widetilde{M}', -\widetilde{P}'')$ and $j_1(0,1) = (\widetilde{P}', -\widetilde{M}'')$. Since j_0 is the obvious projection, the image of ∂ is given by

$$\operatorname{Im}\partial = \operatorname{Ker} j_0 = \left(t^{d'}-1\right) \cap \left(t^{d''}-1\right) = \left(\frac{(t^{d'}-1)(t^{d''}-1)}{(t^d-1)}\right).$$

If d' and d'' are non-zero, then $\operatorname{Im}\partial$ is free of rank one, leading to the following result.

3.2.1 Proposition.
If $(\Sigma, L(\underline{m}))$ is the splice of $(\Sigma', L'(\underline{m}'))$ and $(\Sigma'', L''(\underline{m}''))$ along $0 \cdot L'_0$ and $0 \cdot L''_0$, with $d' \neq 0$ and $d'' \neq 0$, the Alexander modules are related by

$$H_1\widetilde{X} \;\simeq\; \mathbb{Z}[t,t^{-1}] \;\oplus\; \left(H_1\widetilde{X}' \oplus H_1\widetilde{X}''\right) \Big/ \widetilde{M}' \sim \widetilde{P}'',\ \widetilde{P}' \sim \widetilde{M}''. \qquad \square$$

On the other hand, let us suppose that d'' (for example) is equal to 0, that is, the n weights of the multilink $L''(\underline{m}'')$ are null. Here, $\mathrm{H}_1\widetilde{X}''$ is simply a free $\mathbb{Z}[t,t^{-1}]$-module of rank n with a natural basis given by the lifts $\widetilde{M}''_i$ of the meridians of the link. In this case, $\operatorname{Im}\partial$ is equal to zero, leading to the following formula.

3.2.2 Proposition.
If $(\Sigma, L(\underline{m}))$ is the splice of $(\Sigma', L'(\underline{m}'))$ and $(\Sigma'', L''(\overbrace{0,\dots,0}^{n}))$ along $0 \cdot L'_0$ and $0 \cdot L''_0$, the Alexander modules are related by

$$H_1\widetilde{X} \;\simeq\; \left(H_1\widetilde{X}' \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}[t,t^{-1}] \cdot \widetilde{M}''_i\right) \Big/ \widetilde{M}' \sim \sum_{i=1}^{n-1} \ell k(L''_0, L''_i) \cdot \widetilde{M}''_i. \qquad \square$$

3.2.3 Remark. If $\mathrm{H}_1\widetilde{X}'$ is interpretated as a quotient of $\mathrm{H}_1(\Sigma' - \overline{F}') \otimes \mathbb{Z}[t,t^{-1}]$, where F' is a good Seifert surface for $L'(\underline{m}')$ (recall theorem 1.2.7), then the element $\widetilde{M}'$ (resp. $\widetilde{P}'$) of $\mathrm{H}_1\widetilde{X}'$ corresponds to the class of the meridien $M' \subset \Sigma' - \overline{F}'$ (resp. of the parallel $P' \subset \Sigma' - \overline{F}'$). The same holds for $\widetilde{M}''$ and $\widetilde{P}''$ in $\mathrm{H}_1\widetilde{X}''$.

Case II: m>0

This time, the intersection $\widetilde{X}' \cap \widetilde{X}''$ consists of m disjoint copies of $S^1 \times \mathbb{R}$. This is fairly easy to see using the construction of $\widetilde{X}'$ (for example) by mean of a Seifert surface for $L'(\underline{m}')$. So the Mayer-Vietoris sequence takes the form

$$\begin{aligned}
0 &\longrightarrow \mathrm{H}_2\widetilde{X}' \oplus \mathrm{H}_2\widetilde{X}'' \longrightarrow \mathrm{H}_2\widetilde{X} \longrightarrow \mathbb{Z}[t,t^{-1}]/(t^m-1) \xrightarrow{j_1} \mathrm{H}_1\widetilde{X}' \oplus \mathrm{H}_1\widetilde{X}'' \\
&\longrightarrow \mathrm{H}_1\widetilde{X} \xrightarrow{\partial} \mathbb{Z}[t,t^{-1}]/(t^m-1) \xrightarrow{j_0} \mathbb{Z}[t,t^{-1}]/(t^{d'}-1) \oplus \mathbb{Z}[t,t^{-1}]/(t^{d''}-1) \\
&\longrightarrow \mathbb{Z}[t,t^{-1}]/(t^d-1) \longrightarrow 0.
\end{aligned}$$

3.2.4 Proposition.
If $L(\underline{m})$ is the splice of $L'(\underline{m}')$ and $L''(\underline{m}'')$ along $m_0'L_0'$ and $m_0''L_0''$ with $(m_0', m_0'') \neq (0,0)$, then $rk\, H_1\widetilde{X} = rk\, H_1\widetilde{X}' + rk\, H_1\widetilde{X}''$.

Proof. The exact sequence above gives $0 \longrightarrow \mathrm{H}_2\widetilde{X}' \oplus \mathrm{H}_2\widetilde{X}'' \longrightarrow \mathrm{H}_2\widetilde{X} \longrightarrow \mathcal{M} \longrightarrow 0$, with $\mathcal{M}$ a torsion $\mathbb{Z}[t,t^{-1}]$-module. Tensoring with the quotient field $Q(\mathbb{Z}[t,t^{-1}])$, it follows $\mathrm{rk}\,\mathrm{H}_2\widetilde{X} = \mathrm{rk}\,\mathrm{H}_2\widetilde{X}' + \mathrm{rk}\,\mathrm{H}_2\widetilde{X}''$. The proof is completed via proposition 1.1.4. □

Consider the natural projections

$$\begin{array}{rcl} \mathbb{Z}[t,t^{-1}] & \xrightarrow{\pi'} & \mathbb{Z}[t,t^{-1}]/(t^{d'}-1) \\ \mathbb{Z}[t,t^{-1}] & \xrightarrow{\pi''} & \mathbb{Z}[t,t^{-1}]/(t^{d''}-1) \\ \mathbb{Z}[t,t^{-1}] & \xrightarrow{\pi} & \mathbb{Z}[t,t^{-1}]/(t^{m}-1). \end{array}$$

Since d' and d'' divide m, there is a unique homomorphism ψ such that the composition

$$\mathbb{Z}[t,t^{-1}] \xrightarrow{\pi} \mathbb{Z}[t,t^{-1}]/(t^m-1) \xrightarrow{\psi} \mathbb{Z}[t,t^{-1}]/(t^{d'}-1) \oplus \mathbb{Z}[t,t^{-1}]/(t^{d''}-1)$$

is equal to $(\pi', -\pi'')$. This unique morphism is the morphism j_0. Therefore, $\mathrm{Im}\,\partial$ is equal to

$$\mathrm{Ker}\, j_0 = \pi\left(\mathrm{Ker}\,(\pi', -\pi'')\right) = \pi\left(\frac{(t^{d'}-1)(t^{d''}-1)}{(t^d-1)}\right) = \mathbb{Z}[t,t^{-1}]\Big/\left(\frac{(t^m-1)(t^d-1)}{(t^{d'}-1)(t^{d''}-1)}\right).$$

On the other hand, interpretating $\mathrm{H}_1\widetilde{X}'$ and $\mathrm{H}_1\widetilde{X}''$ as in remark 3.2.3, j_1 sends a generator x of $\mathbb{Z}[t,t^{-1}]/(t^m-1)$ to

$$j_1(x) = (i_+T', i_+T''),$$

where T' (resp. T'') is (the class of) any component of $F' \cap \mathcal{N}(L_0')$ (resp. $F'' \cap \mathcal{N}(L_0'')$). Note that we could have chosen (i_-T', i_-T''), since $t \cdot i_-T' = i_+T'$.
This gives the following result.

3.2.5 Proposition.
If $L(\underline{m}) = L'(\underline{m}')\underset{m_0'L_0' \quad m_0''L_0''}{\rule{3em}{0.4pt}}L''(\underline{m}'')$ with $gcd(m_0', m_0'') = m > 0$, there is a short exact sequence

$$0 \longrightarrow (H_1\widetilde{X}' \oplus H_1\widetilde{X}'')/(i_+T' + i_+T'') \xrightarrow{i} H_1\widetilde{X} \xrightarrow{\partial} \mathbb{Z}[t,t^{-1}]/(p(t)) \longrightarrow 0,$$

where $p(t) = \frac{(t^m-1)(t^d-1)}{(t^{d'}-1)(t^{d''}-1)}$. □

Since d' and d'' divide m, the case $m = 1$ is settled.

3.2.6 Corollary.
If $L(\underline{m}) = L'(\underline{m}')\underset{m_0'L_0' \quad m_0''L_0''}{\rule{3em}{0.4pt}}L''(\underline{m}'')$ with $gcd(m_0', m_0'') = 1$, there is an isomorphism

$$H_1\widetilde{X} \simeq (H_1\widetilde{X}' \oplus H_1\widetilde{X}'')/(i_+T' + i_+T'')\ .$$ □

Unfortunately, the sequence given in proposition 3.2.5 does not split in general. Therefore, we don't have any general formula giving the module $\mathrm{H}_1\widetilde{X}$ from $\mathrm{H}_1\widetilde{X}'$ and $\mathrm{H}_1\widetilde{X}''$.

Nevertheless, it is possible to restate proposition 3.2.5 in a handy way using the interpretation of $\mathrm{H}_1\widetilde{X}$ as a quotient of $\mathrm{H}_1(\Sigma - \overline{F}) \otimes \mathbb{Z}[t, t^{-1}]$. If $\mathcal{P}'$ (resp. $\mathcal{P}''$) is a presentation matrix of $\mathrm{H}_1\widetilde{X}'$ (resp. $\mathrm{H}_1\widetilde{X}''$) with respect to a basis of $\mathrm{H}_1\overline{F}'$ (resp. $\mathrm{H}_1\overline{F}''$), then a presentation matrix of $(\mathrm{H}_1\widetilde{X}' \oplus \mathrm{H}_1\widetilde{X}'')/(i_+T' + i_+T'')$ is given by

$$\begin{pmatrix} \mathcal{P}' & 0 \\ 0 & \mathcal{P}'' \\ \ell k(i_+T',-) & \ell k(i_+T'',-) \end{pmatrix}.$$

To continue this study, we need an easy algebraic lemma.

3.2.7 Lemma.
Let $0 \longrightarrow \mathcal{N} \stackrel{i}{\longrightarrow} \mathcal{M} \stackrel{\partial}{\longrightarrow} R/(r) \longrightarrow 0$ *be an exact sequence of R-modules, where $\mathcal{N}$ is presented by $\langle\, x_1, \ldots, x_g \mid r_1, \ldots, r_s \,\rangle$. Choose $m \in \mathcal{M}$ such that $\partial(m)$ is a generator of $R/(r)$, and $n \in \mathcal{N}$ such that $i(n) = r \cdot m$. If $n = \sum_j \lambda_j \cdot x_j$, then $\mathcal{M}$ is presented by*

$$\mathcal{M} = \langle\, x_1, \ldots, x_g, m \mid r_1, \ldots, r_s, r \cdot m - \textstyle\sum_j \lambda_j \cdot x_j \,\rangle.$$

□

Before applying this lemma to our problem, let us try to understand geometrically the elements involved.
A good Seifert surface F for $L(\underline{m})$ is obtained by pasting good Seifert surfaces F' for $L'(\underline{m}')$ and F'' for $L''(\underline{m}'')$ (and making some surgeries). On F, several new cycles, called **big cycles**, can appear. Since $F' \cap F''$ has m connected components, there is no such cycle if $m = 1$, or if $m = d'$ or $m = d''$. In fact, these big cycles correspond exactly to the torsion module $\mathbb{Z}[t, t^{-1}]/(p(t))$ where $p(t) = \frac{(t^m-1)(t^d-1)}{(t^{d'}-1)(t^{d''}-1)}$. More precisely: let λ be a big cycle in H_1F; consider $i_+\lambda \in \mathrm{H}_1(\Sigma - \overline{F}) \otimes \mathbb{Z}[t, t^{-1}]$ and its class $[i_+\lambda] \in \mathrm{H}_1\widetilde{X}$. Then, $\partial([i_+\lambda])$ is a generator of $\mathbb{Z}[t, t^{-1}]/(p(t))$. By exactness, there exists a unique element in $(\mathrm{H}_1\widetilde{X}' \oplus \mathrm{H}_1\widetilde{X}'')/(i_+T' + i_+T'')$ sent by i on $p(t) \cdot \lambda$. Hence, there exists λ' in $\mathrm{H}_1(\Sigma' - \overline{F}')$ and λ'' in $\mathrm{H}_1(\Sigma'' - \overline{F}'')$, not uniquely defined, that satisfy $i([[\lambda'], [\lambda']]) = p(t) \cdot \lambda$. Putting all this together, we get the following formula.

3.2.8 Proposition.
If $L(\underline{m}) = L'(\underline{m}')\overline{{}_{m_0'L_0'}\quad{}_{m_0''L_0''}}L''(\underline{m}'')$ with $gcd(m_0', m_0'') = m > 0$, and if $\mathcal{P}'$ (resp. $\mathcal{P}''$) is a presentation matrix of $\mathrm{H}_1\widetilde{X}'$ (resp. $\mathrm{H}_1\widetilde{X}''$) with respect to a basis of $\mathrm{H}_1\overline{F}'$ (resp. $\mathrm{H}_1\overline{F}''$), then a presentation matrix of $\mathrm{H}_1\widetilde{X}$ is given by

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}' & 0 & 0 \\ 0 & \mathcal{P}'' & 0 \\ \ell k(i_+T',-) & \ell k(i_+T'',-) & 0 \\ -\ell k(\lambda',-) & -\ell k(\lambda'',-) & \frac{(t^m-1)(t^d-1)}{(t^{d'}-1)(t^{d''}-1)} \end{pmatrix},$$

where λ' (resp. λ'') is some 1-cycle in $\Sigma' - \overline{F}'$ (resp. in $\Sigma'' - \overline{F}''$). □

Of course, this cannot be considered as a closed formula unless we understand λ' and λ''.

In conclusion, the behaviour of the Alexander module via splicing is very tricky, especially compared with the behaviour of the Alexander polynomial. Indeed, Eisenbud and Neumann proved the following theorem, which is one of the motivations of the introduction of multi-links.[1]

3.2.9 Theorem. (Eisenbud-Neumann [11])
Suppose that $\mathbf{L}(\underline{m}) = (\Sigma, m_1 L_1 \cup \ldots \cup m_n L_n)$ *is obtained by splicing*

$$\mathbf{L}'(\underline{m}') = (\Sigma', m_0' L_0' \cup m_1 L_1 \cup \ldots \cup m_k L_k)$$

and

$$\mathbf{L}''(\underline{m}'') = (\Sigma'', m_0'' L_0'' \cup m_{k+1} L_{k+1} \cup \ldots \cup m_n L_n)$$

along L_0' *and* L_0'', *with* $0 \leq k < n$. *Let us also note* $d' = gcd(m_0', m_1, \ldots, m_k)$, $d'' = gcd(m_0'', m_{k+1}, \ldots, m_n)$ *and* $d = gcd(m_1, \ldots, m_n) = gcd(d', d'')$. *Then, the Alexander polynomials are related by the formula*

$$\frac{\Delta^{L(\underline{m})}(t)}{t^d - 1} \doteq \frac{\Delta^{L'(\underline{m}')}(t)}{t^{d'} - 1} \cdot \frac{\Delta^{L''(\underline{m}'')}(t)}{t^{d''} - 1},$$

unless $m_0' = m_0'' = 0$ *and* $k = 0$, *in which case*

$$\Delta^{L(\underline{m})}(t) \doteq \Delta^{(L'' - L_0'')(\underline{m})}(t).$$

Proof. Let us first suppose that $m_0' = m_0'' = 0$. In this case, $\Delta^{L'(\underline{m}')} = \Delta^{L''(\underline{m}'')} = 0$ by corollary 1.2.8. If $d' \neq 0$ and $d'' \neq 0$, then $\Delta^{L(\underline{m})}$ also vanishes by proposition 3.2.1. If $d' = 0$ or $d'' = 0$, then $\Delta^{L(\underline{m})} = 0$ by corollary 1.2.8, unless $k = 0$. In this case, proposition 3.2.2 gives the isomorphism $\mathrm{H}_1 \widetilde{X} \simeq \mathrm{H}_1 \widetilde{X}'' / \widetilde{M}''$, that is, the Alexander module of $L(\underline{m})$ is isomorphic to the Alexander module of $(L'' - L_0'')(\underline{m})$. In particular, their Alexander polynomials are equal.

Let us now suppose that $m > 0$. If $\mathrm{H}_2 \widetilde{X}$ is not trivial, then $\Delta^{L(\underline{m})}$ vanishes (by proposition 1.1.4). Furthermore, since $\mathrm{H}_2 \widetilde{X}$, $\mathrm{H}_2 \widetilde{X}'$ and $\mathrm{H}_2 \widetilde{X}''$ are free (proposition 1.1.4) and $\mathrm{rk}\, \mathrm{H}_2 \widetilde{X} = \mathrm{rk}\, \mathrm{H}_2 \widetilde{X}' + \mathrm{rk}\, \mathrm{H}_2 \widetilde{X}''$ (proposition 3.2.4), it follows that $\mathrm{H}_2 \widetilde{X}'$ or $\mathrm{H}_2 \widetilde{X}''$ is non trivial. Therefore, the product $\Delta^{L'(\underline{m}')} \cdot \Delta^{L''(\underline{m}'')}$ vanishes. Hence, we can assume that $\mathrm{H}_2 \widetilde{X} = 0$, giving the exact sequence

$$\begin{aligned} 0 \longrightarrow\ & \mathbb{Z}[t, t^{-1}]/(t^m - 1) \longrightarrow \mathrm{H}_1 \widetilde{X}' \oplus \mathrm{H}_1 \widetilde{X}'' \longrightarrow \mathrm{H}_1 \widetilde{X} \longrightarrow \mathbb{Z}[t, t^{-1}]/(t^m - 1) \\ \longrightarrow\ & \mathbb{Z}[t, t^{-1}]/(t^{d'} - 1) \oplus \mathbb{Z}[t, t^{-1}]/(t^{d''} - 1) \longrightarrow \mathbb{Z}[t, t^{-1}]/(t^d - 1) \longrightarrow 0. \end{aligned}$$

The formula then follows from an algebraic lemma of Levine [23, Lemma 5] on the order ideals of modules in an exact sequence. □

Here is an analogous result for the multivariable Alexander polynomial of oriented links.

[1] Our demonstration of this theorem is by no mean original: it is basically Eisenbud and Neumann's argument. Nevertheless, it is worth giving the proof in some details, to show that this important theorem is an easy consequence of several results of the present work.

3.2.10 Theorem.
Suppose that $L = L_1 \cup \ldots \cup L_n$ is obtained by splicing $L' = L_0' \cup L_1 \cup \ldots \cup L_k$ and $L'' = L_0'' \cup L_{k+1} \cup \ldots \cup L_n$ along L_0' and L_0'', with $0 \leq k < n$. Then the Alexander polynomials are related by the formula

$$\Delta_L(t_1, \ldots, t_n) \doteq \begin{cases} \Delta_{L'}(t_{k+1}^{\ell''_{k+1}} \cdots t_n^{\ell''_n}, t_1, \ldots, t_k) \cdot \Delta_{L''}(t_1^{\ell'_1} \cdots t_k^{\ell'_k}, t_{k+1}, \ldots, t_n) & \text{if } k > 0; \\ \Delta_{L'}(t_1^{\ell''_1} \cdots t_n^{\ell''_n}) \cdot \Delta_{L''-L_0''}(t_1, \ldots, t_n) & \text{if } k = 0, \end{cases}$$

where $\ell_i' = \ell k(L_0', L_i)$ and $\ell_j'' = \ell k(L_0'', L_j)$.

Proof. Let us denote by Δ' the right hand side of the formula. By the second point of lemma 2.2.4, we just need to check that $\Delta'(t^{m_1}, \ldots, t^{m_n}) \doteq \Delta_L(t^{m_1}, \ldots, t^{m_n})$ for all integers $m_1, \ldots, m_n$. Let us first assume that $m_0' := \sum_{i=k+1}^n m_i \ell_i''$, $m_0'' := \sum_{i=1}^k m_i \ell_i'$ and k are not all zero. In this case, we get

$$\begin{aligned} \Delta'(t^{m_1}, \ldots, t^{m_n}) &= \begin{cases} \Delta_{L'}(t^{m_0'}, t^{m_1}, \ldots, t^{m_k}) \cdot \Delta_{L''}(t^{m_0''}, t^{m_{k+1}}, \ldots, t^{m_n}) & \text{if } k > 0; \\ \Delta_{L'}(t^{m_0'}) \cdot \Delta_{L''-L_0''}(t^{m_1}, \ldots, t^{m_n}) & \text{if } k = 0, \end{cases} \\ \text{(by proposition 2.1.1)} &\doteq \begin{cases} \Delta_{L'}(t^{m_0'}, t^{m_1}, \ldots, t^{m_k}) \cdot \Delta_{L''}(t^{m_0''}, t^{m_{k+1}}, \ldots, t^{m_n}) & \text{if } k > 0; \\ \Delta_{L'}(t^{m_0'}) \cdot \frac{\Delta_{L''}(t^0, t^{m_1}, \ldots, t^{m_n})}{t^{m_0'} - 1} & \text{if } k = 0, \end{cases} \\ \text{(by proposition 2.2.1)} &\doteq \begin{cases} \frac{\Delta^{L'(\underline{m}')}(t)}{t^{d'} - 1} \cdot \frac{\Delta^{L''(\underline{m}'')}(t)}{t^{d''} - 1} & \text{if } k > 0; \\ \frac{\Delta^{L'(m_0')}(t)}{t^{m_0'} - 1} \cdot \frac{\Delta^{L''(\underline{m}'')}(t)}{t^{d''} - 1} & \text{if } k = 0, \end{cases} \\ \text{(by theorem 3.2.9)} &\doteq \frac{\Delta^{L(\underline{m})}(t)}{t^d - 1} \\ \text{(by proposition 2.2.1)} &\doteq \Delta_L(t^{m_1}, \ldots, t^{m_n}). \end{aligned}$$

On the other hand, if $m_0' = m_0'' = k = 0$, then

$$\begin{aligned} \Delta'(t^{m_1}, \ldots, t^{m_n}) &= \overbrace{\Delta_{L'}(1)}^{\doteq 1} \cdot \Delta_{L''-L_0''}(t^{m_1}, \ldots, t^{m_n}) \\ \text{(by proposition 2.2.1)} &\doteq \begin{cases} \frac{\Delta^{(L''-L_0'')(\underline{m})}(t)}{t^d - 1} & \text{if } n > 1; \\ \Delta^{(L''-L_0'')(\underline{m})}(t) & \text{if } n = 1, \end{cases} \\ \text{(by theorem 3.2.9)} &\doteq \begin{cases} \frac{\Delta^{L(\underline{m})}(t)}{t^d - 1} & \text{if } n > 1; \\ \Delta^{L(\underline{m})}(t) & \text{if } n = 1, \end{cases} \\ \text{(by proposition 2.2.1)} &\doteq \Delta_L(t^{m_1}, \ldots, t^{m_n}). \end{aligned}$$

□

Of course, Eisenbud and Neumann already had a result of this kind (see [11, theorem 5.3]). But they did not state it under this form, since they did not use Torres formula.

3.2.11 Corollary. (The Seifert-Torres formula [42], [44])
Let K be a knot in S^3, and let $\mathcal{N}(K)$ be a closed tubular neiborhood of K. Consider an orientation preserving homeomorphism $\mathcal{N}(K) \xrightarrow{f} S^1 \times D^2$ sending K to the core of the solid

torus and a standard parallel of K to $S^1 \times \{1\}$. If $L = L_1 \cup \ldots \cup L_n$ is a link in the interior of $\mathcal{N}(K)$ with $L_i \sim \mu_i \cdot K$ in $H_1(\mathcal{N}(K))$, then

$$\Delta_L(t_1, \ldots, t_n) \doteq \Delta_K(t_1^{\mu_1} \cdots t_n^{\mu_n}) \cdot \Delta_{f(L)}(t_1, \ldots, t_n).$$

Proof. The link L is obtained by splicing $f(L) \cup M$ and K along M and K, where M is a meridian of $f(\mathcal{N}(K))$. The case $k = 0$ of theorem 3.2.10 gives the formula. □

3.3 Splicing a Knot with a Multilink

As an immediate application of the previous section, let us mention the following result.

3.3.1 Theorem.
Let $\mathbf{L}'(\underline{m}') = (\Sigma', m'L')$ be a "multiknot", and $\mathbf{L}''(\underline{m}'') = (\Sigma'', m_0''L_0'' \cup m_1 L_1 \cup \ldots \cup m_n L_n)$ a multilink with $n \geq 1$; the splicing $\mathbf{L}(\underline{m}) = \mathbf{L}'(m') \underset{L' \quad L_0''}{\overline{\qquad}} \mathbf{L}''(\underline{m}'')$ is defined if and only if $m' = \sum_{j=1}^{n} m_j \ell k(L_0'', L_j)$ and $m_0'' = 0$; let us suppose so. Then, the Alexander module of $L(\underline{m})$ is given by

$$\mathcal{M}^{L(\underline{m})} = \begin{cases} \mathcal{M}^{L'(m')} \oplus \mathcal{M}^{(L-L_0'')(\underline{m})} & \text{if } m' \neq 0; \\ \mathcal{M}^{(L-L_0'')(\underline{m})} & \text{if } m' = 0. \end{cases}$$

Proof. Let us suppose that $m' \neq 0$. With the notations of proposition 3.2.5, we have $m = d'$ and $d = d''$; hence, $p(t) = \frac{(t^m-1)(t^d-1)}{(t^{d'}-1)(t^{d''}-1)} = 1$ and there is an isomorphism

$$\mathrm{H}_1\widetilde{X} \simeq (\mathrm{H}_1\widetilde{X}' \oplus \mathrm{H}_1\widetilde{X}'')/(i_+T' + i_+T'') .$$

Now, T' is the unique border component of a compact surface, so $T' = 0$ in $\mathrm{H}_1 F'$ and $i_+T' = 0$ in $\mathrm{H}_1\widetilde{X}'$. Hence,

$$\mathrm{H}_1\widetilde{X} \simeq \mathrm{H}_1\widetilde{X}' \oplus \mathrm{H}_1\widetilde{X}''/(i_+T'') .$$

To kill i_+T'' in $\mathrm{H}_1\widetilde{X}''$ means to fill up the $|m'|$ holes in a Seifert surface F'' near L_0'' (recall figure 2.1). The result is a Seifert surface F_0'' for $(L - L_0'')(\underline{m})$; therefore,

$$\mathrm{H}_1\widetilde{X} \simeq \mathrm{H}_1\widetilde{X}' \oplus \mathrm{H}_1\widetilde{X}_0'' ,$$

where $\mathrm{H}_1\widetilde{X}_0''$ denotes the Alexander module of $(L - L_0'')(\underline{m})$. On the other hand, if $m' = 0$, proposition 3.2.2 gives the result. □

3.3.2 Remark. In fact, we have the following stronger result:

$$A_F^{\pm} \equiv \begin{cases} A_{F'}^{\pm} \oplus A_{F_0''}^{\pm} & \text{if } m' \neq 0; \\ A_{F_0''}^{\pm} & \text{if } m' = 0. \end{cases}$$

Eisenbud and Neumann stated and proved the isomorphism $A = A' \oplus A_0''$ between the usual Seifert forms associated with a knot and a multilink. Since these forms do not give an Alexander matrix in general, they could not deduce theorem 3.3.1.

Chapter 4

The Alexander Module of Seifert Multilinks

The celebrated splitting theorem of WILLIAM JACO, PETER SHALEN and KLAUS JOHANSSON implies that the exterior of an irreducible link can be canonically expressed as the splicing of elementary links, each of which is either Seifert fibered, or simple (see section 4.1 below for precise definitions and statements). Furthermore, all the multilinks appearing in algebraic geometry turn out to have only Seifert fibered components: they are *graph multilinks*. This remarkable class of multilinks was studied extensively by DAVID EISENBUD and WALTER NEUMANN. We recall some of their fundamental results in section 4.2.

An ideal achievement would be to give a closed formula for the Alexander module of any graph multilink. This seems a little too ambitious, but the computation of the Alexander module of the elementary pieces, that is: of Seifert multilinks, is completely settled in this chapter. The methods are completely different, according to if the Seifert multilink is fibered or not. The main tool in the study of the fibered case is a theorem of FRIEDHELM WALDHAUSEN. On the other hand, the non-fibered case is treated with the methods of chapter 1. Therefore, the discussion is divided into two distinct sections.

4.1 A Review of Three-Dimensional Topology

In order to state the Jaco-Shalen-Johansson splitting theorem, let us recall several basic definitions. In three-dimensional topology, the terminology has evolved a lot along these last 25 years (see e.g [17, 22, 33, 3]). We will mainly follow M. Boileau's definitions [3]. Throughout this section, M denotes a connected, compact, orientable 3-manifold and F a compact, orientable surface properly embedded in M.

M is **irreducible** if every embedding $S^3 = \partial B^3 \hookrightarrow M$ extends to an embedding $B^3 \hookrightarrow M$. An important class of irreducible manifolds is given by the exterior of knots in S^3. More generally, given L a link in a homology 3-sphere, its exterior is irreducible if and only if L cannot be expressed as a non-trivial disjoint sum of links.

F is **incompressible** in M if, for all connected component F_i of F, the inclusion $F_i \subset M$ induces a monomorphism $\pi_1 F \to \pi_1 M$, and F_i does not bound a 3-ball in M. F is **boundary-parallel** in M if there exists an embedding $h\colon F \times I \hookrightarrow M$ with $h(x, 0) = x$, $h(x, 1) \in \partial M$

for all x in F, and $h(\partial F \times I) \subset \partial M$. Finally, F is **essential** in M if it is incompressible and not boundary-parallel in M.

M is **simple** if it contains no essential torus. For example, the exterior of a torus knot is simple. More generally, it follows from works of H. Shubert [39] that the exterior of a prime knot that is not a satellite knot is simple.

We now turn to the definition of Seifert fibered manifolds. Let (α, β) be a pair of relatively prime integers, with $\alpha > 0$, and let $D^2 = \{\, (r, \vartheta) \ : \ 0 \leq r \leq 1\,,\ 0 \leq \vartheta < 2\pi \,\}$ be the unit disc in $\mathbb{R}^2$. A **fibered solid torus of type** (α, β) is the quotient of the cylinder $D^2 \times I$ via the identification $((r, \vartheta), 1) = ((r, \vartheta + 2\pi\beta/\alpha), 0)$. A **fiber** is the image of an arc $\{x\} \times I$. It is easy to see that up to fiber preserving homeomorphism, it may be assumed that $\frac{\alpha}{2} > \beta \geq 0$. Note that the core of the fibered solid torus (that is, the image of $\{0\} \times I$) meets the disc $D^2 \times \{0\}$ once, but every other fiber meets it exactly α times. The number α is called the **index** of the fiber $\{0\} \times I$.

A **Seifert fibered** 3-manifold is a 3-manifold M which is the union of circles, called **Seifert fibers**, such that each circle has a closed neighborhood in M which is homeomorphic to a fibered solid torus via a fiber-preserving homeomorphism. A fiber is called a **regular fiber** (or **generic fiber**) if it has a fibered solid torus neighborhood of type $(1, 0)$. Otherwise, it is a **singular fiber** (or **exceptional fiber**). By compactness of M, the number of singular fibers is finite.

The quotient space of M obtained by identifying each fiber to a point is a 2-manifold B, called the **orbit manifold**. The natural projection $M \longrightarrow B$ is the **Seifert fibration**. Note that a Seifert fibration is not a circle bundle in general, not even a fibration is the homotopy theoretical sense (unless there is no singular fiber).

We are finally ready to present the Jaco-Shalen-Johansson decomposition theorem. We state it here for 3-manifolds with boundary, but the theorem is valid for a wider class a 3-manifolds, namely "Haken manifolds" or "sufficiently large manifolds".

4.1.1 Theorem. (Jaco-Shalen [17], Johansson [18])
Let M be an irreducible oriented compact 3-manifold with boundary. Then, there exists a finite family of disjoint embedded essential tori $\{T_i\}$, unique up to ambient isotopy, such that each component of M cut along $\{T_i\}$ is Seifert fibered or simple, and such that $\{T_i\}$ is minimal with this property. □

Note that the statement "Seifert fibered or simple" is not exclusiv. For example, the exterior of a torus knot is Seifert fibered and simple. It is known that if M is an irreducible simple 3-manifold with ∂M consisting of incompressible tori, and if M is not Seifert fibered, then it is atoroidal (that is: every subgroup $\mathbb{Z} + \mathbb{Z}$ of $\pi_1 M$ is conjugate to the fundamental group of a component of ∂M). On the other hand, since M is simple and not Seifert fibered, the Jaco-Shalen mapping theorem [17, Chapter III] implies that M does not contain any essential annulus. Therefore, we can use Thurston's hyperbolization theorem to conclude that the interior of M has a complete hyperbolic structure of finite volume.

Hence, we have the following geometrization theorem for irreducible 3-manifolds with boundary (again, this theorem is valid for Haken manifolds).

4.1.2 Theorem. (Thurston)
Let M be an irreducible oriented compact 3-manifold with boundary. Then, there exists a finite family of disjoint embedded essential tori $\{T_i\}$, such that each component of M cut along $\{T_i\}$ is either Seifert fibered or hyperbolic. Moreover, there exists a minimal such family, which is unique up to ambient isotopy. □

Of course, one of the most interesting class of oriented compact 3-manifolds with boundary is given by the exterior of a link. We say that a link L in a homology sphere Σ is an **irreducible link** (resp. a **simple link**, a **Seifert link**, a **hyperbolic link**) if its exterior $M_L = \Sigma - \mathcal{N}(L)$ is irreducible (resp. simple, Seifert fibered, admits a complete hyperbolic structure). Note that a link in S^3 is a Seifert link if and only if it is a torus link.

The core of the work of Eisenbud and Neumann lies in the following reformulation of theorem 4.1.2.

4.1.3 Theorem. (Eisenbud-Neumann)
Let L be an irreducible link in a homology 3-sphere Σ. Then, there exists a finite family of disjoint essential tori $\{T_i\}$ embedded in the link exterior, such that if one desplices L along all these tori, each resulting link is either a Seifert link or a hyperbolic link. Moreover, there exists a minimal collection with this property, unique up to ambient isotopy. □

These "elementary pieces" of a link are called its **splice components**; they form a **splice decomposition** of the link. Furthermore, the choice of multiplicities $\underline{m}$ for L endows each splice component with a natural structure of multilink (even if L had only multiplicities ± 1). Therefore, we can restate the previous theorem as follows.

4.1.4 Corollary. (Eisenbud-Neumann)
Let $L(\underline{m})$ be an irreducible multilink. Then there exists a unique minimal splice decomposition of $L(\underline{m})$ into hyperbolic and Seifert multilinks.

Moreover, the problem of telling whether a multilink is fibered can be reduced to the same question concerning its splice components. Indeed, Eisenbud and Neumann showed that a theorem of Roussarie [37] implies the following striking result.

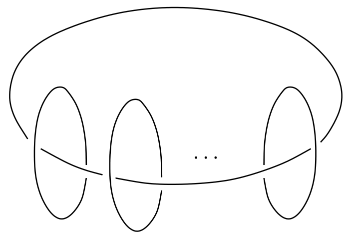
4.1.5 Theorem. (Eisenbud-Neumann)
A multilink is fibered if and only if it is irreducible and each of its multilink splice component is fibered. □

Note that this result would not be true if we replace multilinks by links.

4.2 Eisenbud and Neumann's Work on Graph Multilinks

Eisenbud and Neumann are particularly interested in multilinks arising in algebraic geometry. Of course, they form a very special class of multilinks: every multilink cannot be realized as the multilink of a complex plane curve singularity, or as the multilink at infinity of a polynomial application. Indeed, such multilinks have the property (*inter alia*) that each of their splice component is Seifert fibered. This motivates the following definition: a **graph multilink** is a multilink whose splice components are all Seifert fibered. A link in the standard sphere S^3 is a graph link if and only if it is an iterated torus link. Graph links (and more generally, graph multilinks) were studied extensively by Eisenbud and Neumann. Let us mention, among their numerous results, those that we will need in the sequel.

Recall that a Seifert link is a link (Σ, L) whose exterior is a Seifert fibered manifold. It turns out that the whole of Σ is Seifert fibered, and L is a collection of fibers, unless L is the following link in S^3.



Seifert fibered homology spheres were classified by Seifert himself in his 1933 founding article [40]. Let us recall his construction. Let $((\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k))$ be integers with $\gcd(\alpha_i,\beta_i)=1$. Set $F = S^2 - \left(\mathrm{int}D^2_1 \cup \ldots \cup \mathrm{int}D^2_k\right)$. Note that

$$\pi_1(F\times S^1) = \pi_1 F \times \pi_1 S^1 = \langle\, x_1,\dots,x_k \mid x_1\cdots x_k = 1\,\rangle \times \langle\, h\,\rangle\,,$$

where x_i is a loop representing $\partial D^2_i \times \{1\}$, oriented as the boundary of $F\times\{1\}$.

Set

$$M\left((\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k)\right) = \left(S^1\times D^2_1 \cup\ldots\cup S^1\times D^2_k\right) \cup_f \left(F\times S^1\right)\,,$$

where $f\colon S^1\times\partial D^2_1\cup\ldots\cup S^1\times\partial D^2_k \longrightarrow \partial D^2_1\times S^1\cup\ldots\cup\partial D^2_k\times S^1$ is a homeomorphism such that

$$f_*(\{1\}\times\partial D^2_i) = \alpha_i\cdot x_i + \beta_i\cdot h \quad \text{in} \quad \mathrm{H}_1(\partial D^2_i\times S^1)$$

for all $i=1,\dots,k$.

4.2.1 Example. If $k\leq 2$, we obtain lens spaces. Indeed, it is clear that $M((\alpha,\beta)) = L(\beta,\alpha)$ and it can be shown that $M\left((\alpha_1,\beta_1),(\alpha_2,\beta_2)\right) = L(\alpha_1\beta_2+\alpha_2\beta_1,\alpha_1\alpha_2)$ (see e.g. [38, Chap. 6]).

Using the van Kampen theorem, we get the following presentation of the fundamental group of $M\left((\alpha_1,\beta_1),\dots,(\alpha_k,\beta_k)\right)$:

$$\pi_1 M = \left\langle x_1,\dots,x_k,h \;\middle|\; x_ih = hx_i\,,\ x_i^{\alpha_i}h^{\beta_i}=1\,,\ x_1\cdots x_k = 1\right\rangle.$$ [1]

Hence, a presentation matrix of $\mathrm{H}_1 M = (\pi_1 M)_{ab}$ is given by

$$\mathcal{P} = \begin{pmatrix} \alpha_1 & & & \beta_1 \\ & \ddots & & \vdots \\ & & \alpha_k & \beta_k \\ 1 & \dots & 1 & 0 \end{pmatrix}.$$

[1] Note that the center of this group is non-trivial. In fact, this is a characteristic property of irreducible Seifert fibered manifold with infinite fundamental group.

Therefore, M is a homology sphere if and only if $\det \mathcal{P} = \pm 1$, that is:

$$\sum_{i=1}^{k} \beta_i \, \alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_k = \pm 1. \tag{$*$}$$

In particular, the α_i must be pairwise coprime. It can be shown that the manifold M does not depend on the choice of the β_i, as long as they satisfy equality $(*)$. Let us denote this Seifert fibered homology sphere by $\Sigma(\alpha_1, \ldots, \alpha_k)$. Moreover, the foliation being orientable, there is a canonical orientation on this manifold: the orientation such that the linking number of two generic fibers (oriented in a coherent way) is equal to $\alpha_1 \cdots \alpha_k$. $\Sigma(\alpha_1, \ldots, \alpha_k)$ will denote the manifold with the canonical orientation, and $-\Sigma(\alpha_1, \ldots, \alpha_k)$ the manifold with the opposite orientation.

This notation is not unique; for example, the manifolds $\Sigma(\alpha_1, \ldots, \alpha_k, 1)$ and $\Sigma(\alpha_1, \ldots, \alpha_k)$ are clearly homeomorphic. Also, $\Sigma(\alpha_1, \alpha_2)$ is simply S^3 foliated by (α_1, α_2)-torus knots, for any choice of coprime (α_1, α_2).

Here is Seifert's classification theorem.

4.2.2 Theorem. (Seifert [40])
Let Σ be a Seifert fibered homology 3-sphere, $\Sigma \neq S^3$. Then, there exists a unique unordered k-uple $\{\alpha_1, \ldots, \alpha_k\}$ of pairwise coprime integers, with $k > 2$ and $\alpha_i \geq 2$ for all i, such that $\Sigma = \pm\Sigma(\alpha_1, \ldots, \alpha_k)$ via an orientation preserving homeomorphism. □

In particular, the "simplest" non-trivial Seifert fibered homology sphere is $\Sigma(2, 3, 5)$. This is nothing but the famous Poincaré manifold.

As a consequence of this theorem, every Seifert link can be written

$$(\pm\Sigma(\alpha_1, \ldots, \alpha_k), \pm L_1 \cup \ldots \cup \pm L_n),$$

with $\alpha_i \geq 0$ pairwise coprime and $k \geq n$, where the component L_i is the Seifert fiber corresponding to the index α_i (with the same orientation). Here, we need to consider indices equal to 1 in case a component of the Seifert link is a generic fiber. Also, some index can be zero: the exotic Seifert link illustrated above is denoted by

$$(\Sigma(0, 1, \ldots, 1), \pm L_1 \cup \ldots \cup \pm L_n).$$

Another important example of Seifert link in S^3 is $(\Sigma(\overbrace{1, \ldots, 1}^{n}, p, q), L_1 \cup \ldots \cup L_n)$; this is simply n parallel copies of a (p, q)-torus knot.

It is easy to check that

$$(-\Sigma(\alpha_1, \alpha_2, \ldots, \alpha_k), L_1 \cup L_2 \cup \ldots \cup L_n) = (\Sigma(-\alpha_1, \alpha_2, \ldots, \alpha_k), -L_1 \cup L_2 \cup \ldots \cup L_n).$$

Therefore, every Seifert link can be written

$$(\Sigma(\alpha_1, \ldots, \alpha_k), \pm L_1 \cup \ldots \cup \pm L_n),$$

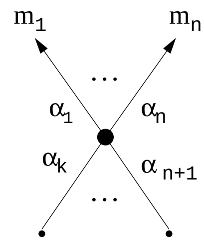


Figure 4.1: Splice diagram of a Seifert multilink.

with α_i pairwise coprime integers and $k \geq n$.[2] Of course, the classification can be extended to multilinks: every Seifert multilink can be written

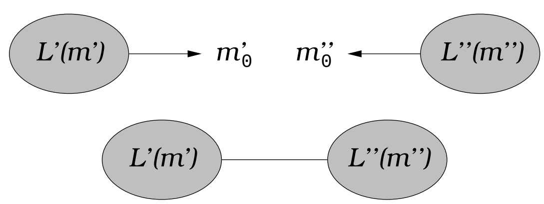
$$(\Sigma(\alpha_1, \ldots, \alpha_k), m_1 L_1 \cup \ldots \cup m_n L_n),$$

with α_i pairwise coprime integers and $k \geq n$. It is symbolized by the diagram of figure 4.1; in this diagram, the arrowhead vertices correspond to the components of the link, while the node corresponds to the generic Seifert fiber.

Now, to represent every graph multilink by a diagram, we just need to define the splicing operation for diagrams: the splice of two multilinks

$$L'(\underline{m}')\underset{m_0'L_0' \qquad m_0''L_0''}{\overline{\hspace{4cm}}}L''(\underline{m}'')$$

is represented by the diagrams of $L'(\underline{m}')$ and $L''(\underline{m}'')$ linked along their arrowhead edges corresponding to L_0' and L_0'', as illustrated below.



The disjoint union of diagrams represents the disjoint sum of multilinks.

Therefore, every graph multilink can be represented by such a diagram, called the **splice diagram** of the multilink. Of course, different splice diagrams may represent the same graph multilink; we speak of **equivalent splice diagrams**. For example, Eisenbud and Neumann show the following result.

4.2.3 Lemma. (Eisenbud-Neumann [11])
The transformation (i) of figure 4.2 always yield equivalent splice diagrams. The transformation (ii) yield equivalent splice diagrams if and only if $\alpha_0\alpha_0' = \alpha_1 \cdots \alpha_r \alpha_1' \cdots \alpha_k'$. □

[2] This has the advantage of simplifying a little the notation, at the expense of allowing negativ indices.

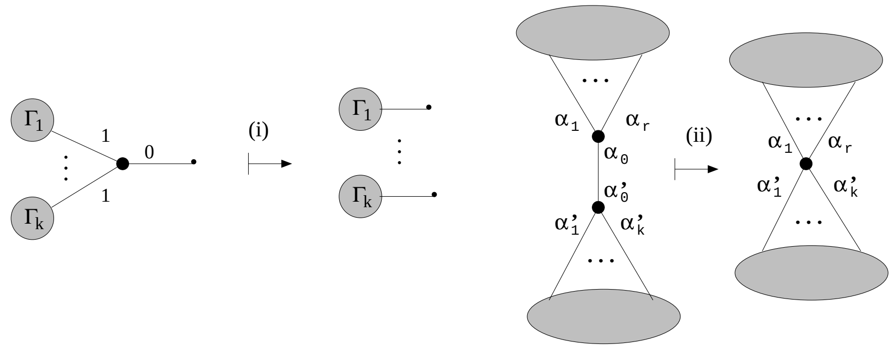


Figure 4.2: Transformations of a splice diagram.

The integer $\alpha_0\alpha'_0 - \alpha_1 \cdots \alpha_r \alpha'_1 \cdots \alpha'_k$ is called the **determinant** of the splicing. A splice diagram will be called **minimal** if it cannot be reduced via transformations (i) and (ii).[3]

Splice diagrams are very convenient for many purposes. For example, to compute linking numbers.

4.2.4 Proposition. (Eisenbud-Neumann [11])
Given two vertices v and w, the linking number $\ell(v, w)$ of their corresponding components (generic fiber or component of the link) is given by the following formula: let $\sigma(v, w)$ be the simple path in the diagram joining v and w, including v and w; then, $\ell(v, w)$ is the product of the weights on the edges adjacent to $\sigma(v, w)$ but not in $\sigma(v, w)$.

4.2.5 Example. Consider the Seifert multilink $(\Sigma(\alpha_1, \ldots, \alpha_k), m_1 L_1 \cup \ldots \cup m_n L_n)$ with generic Seifert fiber γ. Then,

$$\begin{aligned} \ell k(L_i, L_j) &= \alpha_1 \cdots \widehat{\alpha}_i \cdots \widehat{\alpha}_j \cdots \alpha_k \quad \text{for all } i \neq j \ ; \\ \ell k(\gamma, L_i) &= \alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_k \quad \text{for all } i \ . \end{aligned}$$

Also, it is very easy to read on a splice diagram whether a graph multilink is fibered or not. We saw that an irreducible graph multilink is fibered if and only if each of its (Seifert) splice component is fibered. Furthermore:

4.2.6 Proposition. (Eisenbud-Neumann [11])
Suppose that $L(\underline{m})$ is a Seifert multilink (different from the Hopf multilinks) given by a minimal splice diagram as in figure 4.1; then, $L(\underline{m})$ is fibered if and only if

$$\underline{m}(\gamma) = \sum_{i=1}^{n} m_i \, \alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_k \neq 0. \qquad \square$$

In other words, a Seifert multilink is fibered if and only if its linking number with a generic Seifert fiber does not vanish. It is worth mentioning that this result follows very easily from the *vertical-horizontal lemma* of Waldhausen (see [47]).

[3] This is not the definition of Eisenbud and Neumann; indeed, there are other possible ways to minimise a splice diagram. Nevertheless, they are not important for our computations.

4.3 The Non-Fibered Case

In this section, the algorithm described in chapter 1 is used to give a closed formula for the Alexander module of any non-fibered Seifert multilink, that is, of any multilink as in figure 4.1 with $\sum_{i=1}^{n} m_i \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_k = 0$. In all this section, d denotes the greatest common divisor of the multiplicities of the multilink.

4.3.1 Theorem.
Let $\mathbf{L}(\underline{m})$ be the non-fibered Seifert multilink $(\Sigma(\alpha_1, \dots, \alpha_k), m_1 L_1 \cup \dots \cup m_n L_n)$, with $d \neq 0$. An Alexander matrix for $\mathbf{L}(\underline{m})$ is given by

$$\mathcal{P} = \left(\alpha_{n+1} \cdots \alpha_k (t^d - 1) \quad \overbrace{0 \quad 0 \quad \dots \quad 0}^{n-2} \right).$$

Before starting the proof, let us give several consequences of this result .

4.3.2 Corollary.
The Alexander polynomial of a non-fibered Seifert multilink is given by

$$\Delta^{L(\underline{m})}(t) \doteq \begin{cases} \alpha_{n+1} \cdots \alpha_k (t^d - 1) & \text{if } n = 2; \\ 0 & \text{otherwise.} \end{cases}$$

□

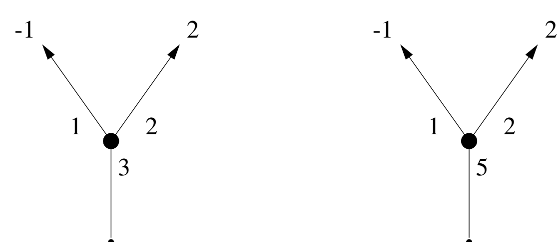
It is well known that the Alexander polynomial of a fibered graph multilink is a product of cyclotomic polynomials. This is false in general.

4.3.3 Corollary.
The Alexander polynomial of a graph multilink is not *always a product of cyclotomic polynomials.*

Proof. The Alexander polynomial of $(\Sigma(1,3,2), -L_1 \cup 3L_2)$ is equal to $2(t-1)$. □

4.3.4 Corollary.
The Alexander module over $\mathbb{Q}$ $(\mathcal{M}_{\mathbb{Q}})$ of a Seifert multilink does not determine the Alexander module over $\mathbb{Z}$ $(\mathcal{M}_{\mathbb{Z}})$.

Proof. Consider the following Seifert multilinks in S^3.

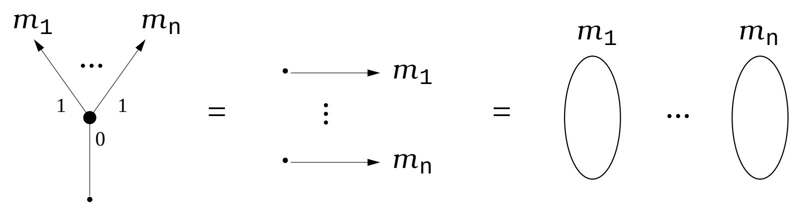


Both have $\mathcal{M}_{\mathbb{Q}} \simeq \mathbb{Q}[t,t^{-1}]/(t-1) \simeq \mathbb{Q}$, but $\mathcal{M}_{\mathbb{Z}} \simeq \mathbb{Z}[t,t^{-1}]/3(t-1)$ for the first one and $\mathcal{M}_{\mathbb{Z}} \simeq \mathbb{Z}[t,t^{-1}]/5(t-1)$ for the second. □

4.3.5 Corollary.
A Seifert multilink $L(\underline{m})$ is fibered if and only if there exists a good Seifert surface F for $L(\underline{m})$ such that the Seifert matrices $A_F^{\pm}$ are unimodular.[4]

Proof. The Seifert matrices of a fibered multilink with respect to a fiber F are always unimodular (proposition 1.2.5). Let us now suppose that $L(\underline{m})$ is a non-fibered Seifert multilink. Using corollary 4.3.2 together with the fact that the lowest coefficient of $\Delta^{L(\underline{m})}$ is equal to $\det A_F^{\pm}$ (up to sign), we see that its Seifert forms are unimodular if and only if $n = 2$, $\underline{m} \neq \underline{0}$ and $\alpha_{n+1} = \ldots = \alpha_k = 1$. This means that $L(\underline{m})$ is a Hopf multilink with $\underline{m} \neq \underline{0}$, which is always fibered. □

Let us begin the proof of the theorem. Using Fox calculus or proposition 1.2.9, it is easy to show that the theorem holds for the case of a trivial n-component multilink, illustrated below.



From now on, we will assume that we are not in this case.

Let $\mathbf{L}(\underline{m}) = (\Sigma(\alpha_1, \ldots, \alpha_k), m_1 L_1 \cup \ldots \cup m_n L_n)$ be a non-fibered Seifert multilink. For $i = 1, \ldots, n$, let us note $d_i = \gcd(m_i, m_i')$, where $m_i' = -\sum_{j \neq i} m_j \ell k(L_i, L_j)$, with $\operatorname{sgn}(d_i) = \operatorname{sgn}(m_i) \cdot \operatorname{sgn}(\alpha_i)$ if $m_i \neq 0$, and $\operatorname{sgn}(d_i) = \operatorname{sgn}(m_i')$ if $m_i = 0$. Note that if $\alpha_i \geq 0$, this corresponds to the definition of d_i given in proposition 1.2.1; in any case $|d_i|$ is equal to the number of components of $F \cap \partial \mathcal{N}(L_i)$, where F denotes a Seifert surface for $L(\underline{m})$.

4.3.6 Lemma.

(*i*) $\sum_{i=1}^n d_i = 0$

(*ii*) *If* $\alpha_i \neq 0$, *then* $d_i = \frac{m_i}{\alpha_i}$.

(*iii*) *If* $\alpha_i = 0$ *for some* i, *then* $m_i = 0$.

Proof. Let us first assume that $\alpha_i \neq 0$ for all i. We have

$$m_i' = -\sum_{\substack{j=1 \\ j \neq i}}^n m_j \ell k(L_i, L_j) = -\sum_{\substack{j=1 \\ j \neq i}}^n m_j \alpha_1 \cdots \widehat{\alpha_i} \cdots \widehat{\alpha_j} \cdots \alpha_k.$$

By proposition 4.2.6, it follows

$$0 = \underline{m}(\gamma) = \sum_{i=1}^n m_i \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_k = m_i \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_k - \alpha_i m_i'.$$

Since $\alpha_i \neq 0$ and the α_j are coprime, α_i divides m_i, and

$$d_i = \gcd(m_i, \frac{m_i}{\alpha_i} \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_k) = \frac{m_i}{\alpha_i}.$$

[4]We will see in example 5.4.7 that this is false for graph multilinks.

Dividing the equation $\underline{m}(\gamma) = 0$ by $\alpha_1 \cdots \alpha_k \neq 0$, it follows $\sum_{i=1}^{n} d_i = 0$.

We are left with the case of $(\Sigma(0,1,\ldots,1), m_1L_1 \cup m_2L_2 \cup \ldots \cup m_nL_n)$. Since $\underline{m}(\gamma) = m_1$, $L(\underline{m})$ is non-fibered if and only if $m_1 = 0$. In this case, we have $d_1 = -(m_2 + \ldots + m_n)$ and $d_i = m_i$ for all $i > 1$. Therefore, the equality $\sum_{i=1}^{n} d_i = 0$ still holds, as well as $d_i = \frac{m_i}{\alpha_i}$ for $i > 1$. □

Theorem 4.3.1 is a direct consequence of the following result.

4.3.7 Proposition.
Let $(\Sigma(\alpha_1,\ldots,\alpha_k), m_1L_1 \cup \ldots \cup m_nL_n)$ be a non-fibered multilink. Suppose that

$$\sum_{i=1}^{n_2} d_i = \sum_{i=n_2+1}^{n_3} d_i = \ldots = \sum_{i=n_b+1}^{n} d_i = 0\,,$$

each sum being minimal with this property. Choose integers $\beta_1,\ldots,\beta_n$ such that:

$$\sum_{i=1}^{n_2} \beta_i\, \alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_{n_2} = \ldots = \sum_{i=n_b+1}^{n} \beta_i\, \alpha_{n_b+1} \cdots \widehat{\alpha}_i \cdots \alpha_n = 1.$$

Then, the Alexander module is given by

$$\left\langle \delta_1^*,\ldots,\delta_b^*, c_{b+1},\ldots,c_{n-1} \;\middle|\; \tfrac{\alpha_1\cdots\alpha_k}{\alpha_1\cdots\alpha_{n_2}}(t^d-1)\cdot\delta_1^* \;+\; \ldots \;+\; \tfrac{\alpha_1\cdots\alpha_k}{\alpha_{n_b+1}\cdots\alpha_n}(t^d-1)\cdot\delta_b^* = 0 \right\rangle\,,$$

where

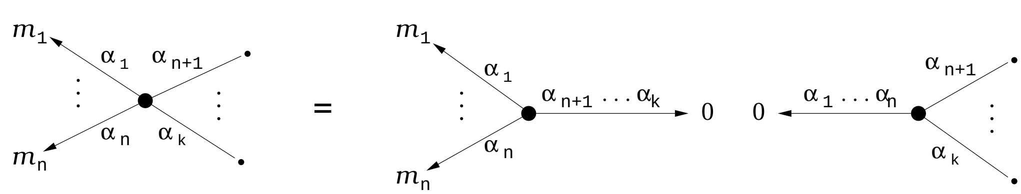
$$\delta_1 = \sum_{i=1}^{n_2} \beta_i L_i,\ ,\ldots,\ \delta_b = \sum_{i=n_b+1}^{n} \beta_i L_i\,,$$

$d = \gcd(m_1,\ldots,m_n) = \gcd(d_1,\ldots,d_n)$ and * denotes the image via $H_1\overline{F} \simeq H^1(\Sigma - \overline{F}) \simeq H_1(\Sigma - \overline{F})$.

It is not so hard to prove the general case directly; nevertheless, we will use several results obtained earlier to restrict ourselves to a special case. The proof will then be a little less technical.

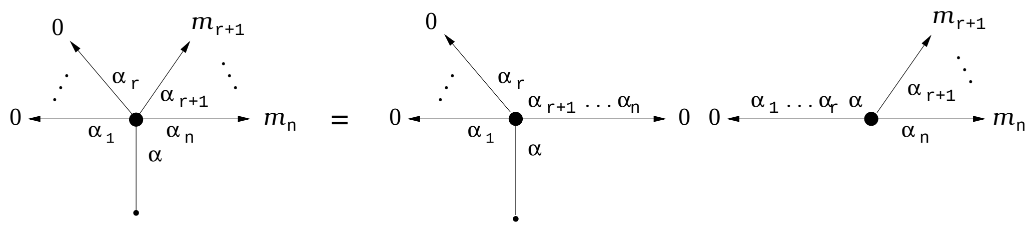
• Reduction 1: Since an Alexander matrix of $L(d \cdot \underline{m})$ is given by an Alexander matrix of $L(\underline{m})$ with $t \longmapsto t^d$ (corollary 1.1.3), it is sufficient to give the proof when $d = 1$.

• Reduction 2: The multilink $(\Sigma(\alpha_1,\ldots,\alpha_k), m_1L_1 \cup \ldots \cup m_nL_n)$ can be seen as the splice illustrated below.



By theorem 3.3.1, the Alexander module of this multilink is equal to the Alexander module of $(\Sigma(\alpha_1,\ldots,\alpha_n,\alpha_{n+1}\cdots\alpha_k), m_1L_1 \cup \ldots \cup m_nL_n)$. Hence, it may be assumed that $k = n+1$.

• Reduction 3: Consider a non fibered Seifert multilink with $r > 1$ weights equal to zero. It can be seen as a (non-minimal) splice of two Seifert multilinks, one having $r + 1$ components with all weights zero, the other having one single zero weight. This is illustrated below. Using proposition 3.2.2, it is easy to check that if proposition 4.3.7 holds for the second splice



component, it also holds for the splice. So it may be assumed that the multilink has at most one zero weight.

• Reduction 4: Let us consider

$$\begin{aligned} \mathbf{L} &= (\Sigma(\alpha_1, \ldots, \alpha_n, \alpha), m_1 L_1 \cup \ldots \cup m_n L_n) \\ \mathbf{L}' &= (\Sigma(1, \alpha_1, \ldots, \alpha_n, \alpha), 0L_0 \cup m_1 L_1 \cup \ldots \cup m_n L_n), \end{aligned}$$

with $m_i \neq 0$ for all i. Clearly, a good Seifert surface F' for $\mathbf{L}'$ is given by a good Seifert surface F for $\mathbf{L}$. Furthermore, $\overline{F}' = \overline{F} \sqcup L_0$ giving $\mathrm{H}_1\overline{F}' = \mathrm{H}_1\overline{F} \oplus \mathbb{Z}L_0$. Using theorem 1.2.7, it follows

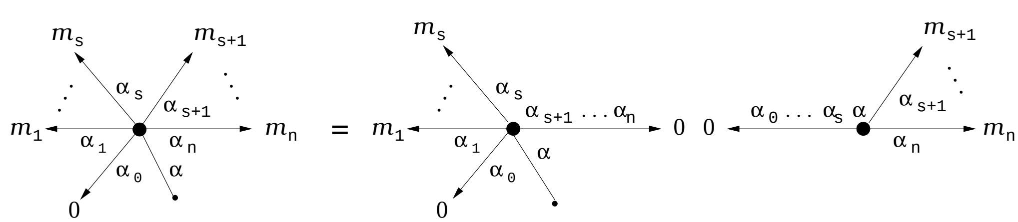
$$\begin{aligned} \mathrm{H}_1\widetilde{X} &\simeq \mathrm{H}_1(\Sigma - \overline{F}) \otimes \mathbb{Z}[t, t^{-1}]/(i_+(\mathrm{H}_1 F) = t \cdot i_-(\mathrm{H}_1 F)) \\ &\simeq \mathrm{H}_1(\Sigma - \overline{F}') \otimes \mathbb{Z}[t, t^{-1}]/(L_0^* \ , \ i_+(\mathrm{H}_1 F') = t \cdot i_-(\mathrm{H}_1 F')) \\ &\simeq \mathrm{H}_1\widetilde{X'}/L_0^* \ . \end{aligned}$$

Using this equality, we see that if proposition 4.3.7 holds for $\mathbf{L}'$, then it also holds for $\mathbf{L}$. Hence, it may be assumed that there is exactly one weight equal to zero.

• Reduction 5: Finally, suppose that

$$(\Sigma(\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha), 0L_0 \cup m_1 L_1 \cup \ldots \cup m_n L_n)$$

satisfies $m_i \neq 0$ and $\sum_{i=1}^{s} d_i = \sum_{i=s+1}^{n} d_i = 0$, with $0 < s < n$. Then, it can be splice-decomposed as follows.



Using proposition 3.2.1, it is easy to show that if proposition 4.3.7 holds for the splice components, it also holds for the splice. Using this argument inductively, it can be assumed that

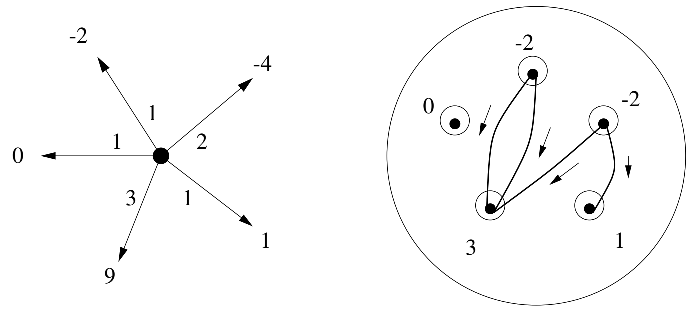


Figure 4.3: Construction of a vertical Seifert surface.

$b = 2$, that is, that there is no way to write $\sum_{i\in I} \frac{m_i}{\alpha_i} = 0$ with I a proper subset of $\{1, \ldots, n\}$.

So, we are left with the proof of the following proposition.

4.3.8 Proposition.
Let $(\Sigma(\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha), 0L_0 \cup m_1L_1 \cup \ldots \cup m_nL_n)$ be a non-fibered multilink, with $m_i \neq 0$ and $\gcd(m_1, \ldots, m_n) = 1$. Furthermore, suppose that there is no way to write $\sum_{i\in I} \frac{m_i}{\alpha_i} = 0$ with I a proper subset of $\{1, \ldots, n\}$. Then, the Alexander module is given by

$$\langle\ M_0, \delta^*, c_1, \ldots, c_{n-2} \mid \alpha_1 \cdots \alpha_n \alpha(t-1) \cdot M_0 \ +\ \alpha_0\alpha(t-1) \cdot \delta^* = 0\ \rangle\ ,$$

where M_0 is a meridian of L_0, $\delta = \sum_{i=1}^{n} \beta_i L_i$ and the β_i satisfy $\sum_{i=1}^{n} \beta_i \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_n = 1$.

Proof. The method of the proof is the following. We start with a Seifert surface F' for $L(\underline{m})$ saturated with Seifert fibers (a **vertical Seifert surface**). Such a Seifert surface exists because $L(\underline{m})$ is non-fibered. Since F' in the union of Seifert fibers, the linking numbers are easy to compute and we can determine the $A^{\pm}_{F'}$ explicitly. But F' is not good in general, so these matrices don't present the Alexander module. Nevertheless, it is always possible to obtain a good Seifert surface F from F' via surgeries (proposition 1.2.2); hence, we just have to understand how the $A^{\pm}_{F'}$ are transformed via surgeries. The final matrices $A^{\pm}_{F}$ give the requested presentation by theorem 1.2.7.

Seifert forms associated with a vertical Seifert surface

Consider the orbit space S^2 of the Seifert fibration $\Sigma(\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha) \xrightarrow{\pi} S^2$. Each component L_i of the link is a Seifert fiber, so $\pi(L_i)$ is a point v_i in S^3. Let us note $P = S^2 - (D_0^2 \cup \ldots \cup D_n^2)$, where D_i^2 is an open neighborhood of v_i in S^3. Since $\sum_{i=0}^{n} d_i = 0$ (lemma 4.3.6), it is possible to construct an oriented graph $\Gamma \subset S^2$ with vertices $\{v_0, v_1, \ldots, v_n\}$, such that the degree of v_i (computed with respect to the orientation of the edges) is equal to d_i for all i. Furthermore, since $d_0 = 0$ and since there is no way to write $\sum_{i\in I} d_i = 0$ other than $I = \{1, \ldots, n\}$, Γ will have two connected components: one consists of the vertex v_0, the other gathers all the other vertices $v_1, \ldots, v_n$ (see figure 4.3 for an example). A vertical Seifert surface F' is given by

$$F' = \pi^{-1}(\Gamma \cap P).$$

It is oriented via the orientation of Γ. Let us also note $\overline{F}' = \pi^{-1}(\Gamma)$.

Our aim is now to compute the Seifert matrices $A^{\pm}_{F'}$ associated with this vertical Seifert surface F'. The first step is to compute the homology of F' and $\overline{F}'$. Let $\overline{b}_i$ be the rank of $\mathrm{H}_i\overline{F}'$ and $b_i(\Gamma)$ the rank of $\mathrm{H}_i\Gamma$. The homology of F' is very simple: F' is a union $F'_1 \cup \ldots \cup F'_\mu$, with $\mu = \frac{1}{2}\sum_i |d_i|$, where each F'_i is a ring saturated by generic Seifert fibers. Therefore,

$$\mathrm{H}_1 F' = \mathbb{Z}\gamma_1 \oplus \ldots \oplus \mathbb{Z}\gamma_\mu\,, \qquad (\star)$$

where γ_i is a generic fiber of the Seifert fibration.

To compute $\mathrm{H}_1\overline{F}'$, let us consider the Mayer-Vietoris exact sequence associated with $\overline{F}' = F' \cup M$ (where $M = \mathcal{N}(L) \cap \overline{F}'$):

$$0 \longrightarrow \mathrm{H}_2\overline{F}' \longrightarrow \mathrm{H}_1(\partial F') \xrightarrow{\varphi_1} \mathrm{H}_1 F' \oplus \mathrm{H}_1 M \longrightarrow \mathrm{H}_1\overline{F}' \xrightarrow{\partial} \mathrm{H}_0(\partial F') \xrightarrow{\varphi_0} \mathrm{H}_0 F' \oplus \mathrm{H}_0 M\,.$$

Since L is a deformation retract of M, there is an isomorphism

$$\mathrm{H}_1\overline{F}' \simeq \Big((\mathrm{H}_1 F' \oplus \mathrm{H}_1 L)/\mathrm{Im}\,\varphi_1\Big) \;\oplus\; s(\mathrm{Ker}\,\varphi_0)\,,$$

where $s\colon \mathrm{Ker}\,\varphi_0 \longrightarrow \mathrm{H}_1\overline{F}'$ is any section of ∂.

4.3.9 Lemma.

(i) $\mu = n + b_1(\Gamma) - 1\,;$

(ii) $s(\mathit{Ker}\,\varphi_0)$ *is naturally isomorphic to* $H_1\Gamma$*;*

(iii) $(H_1F' \oplus H_1 M)/\mathit{Im}\,\varphi_1 \;\simeq\; \mathbb{Z}L_0 \oplus \mathbb{Z}\delta$*, where* $\delta = \sum_{i=1}^n \beta_i L_i$ *and the integers* β_i *satisfy* $\sum_{i=1}^n \beta_i\,\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_n = 1$*. (Such* β_i *exist because the* α_i *are coprime.)*

Proof. By the Euler-Poincaré formula, $b_0(\Gamma) - b_1(\Gamma) = (n+1) - \mu$; since $b_0(\Gamma) = 2$, the first point is proved. The Mayer-Vietoris exact sequence given above implies the equality

$$\begin{aligned} 0 &= \overline{b}_2 - 2\mu + (\mu + n + 1) - \overline{b}_1 + \mathrm{rk}\,\mathrm{Ker}\,\varphi_0 \\ &= \overbrace{\chi(\overline{F}')}^{=0} - 2 - \mu + n + 1 + \mathrm{rk}\,\mathrm{Ker}\,\varphi_0 \\ &= \mathrm{rk}\,\mathrm{Ker}\,\varphi_0 - b_1(\Gamma), \end{aligned}$$

by the equality (i). Therefore, $s(\mathrm{Ker}\,\varphi_0)$ and $\mathrm{H}_1\Gamma$ have the same rank. Furthermore, the choice of a basis of $\mathrm{H}_1\Gamma$ gives a basis of $s(\mathrm{Ker}\,\varphi_0) \subset \mathrm{H}_1\overline{F}'$ in the following way: a cycle in $\mathrm{H}_1\Gamma$ corresponds to a sequence of F_i that form an abstract torus in $\overline{F}'$. A meridian of this torus will give the corresponding cycle in $\mathrm{H}_1\overline{F}'$.

To prove the last point, let us note $M_i = M \cap \mathcal{N}(L_i)$ for all $i = 1, \ldots, n$. The inclusion $\partial F' \cap M_i \hookrightarrow M_i$ induces the morphism $\bigoplus_{j=1}^{|d_i|} \mathbb{Z}T^i_j \longrightarrow \mathbb{Z}L_i$ with $T^i_j \longmapsto \frac{m_i}{d_i} L_i = \alpha_i L_i$ (lemma 4.3.6). This gives

$$(\mathrm{H}_1 F' \oplus \mathrm{H}_1 L)/\mathrm{Im}\,\varphi_1 \;\simeq\; \mathbb{Z}L_0 \oplus \bigoplus_{i=1}^n \mathbb{Z}L_i/(\alpha_j L_j = \alpha_k L_k)\,.$$

The result then follows from the fact that the α_i are coprime (and a little bit of algebra). □

As a consequence, the homology of $\overline{F}'$ is given by

$$\mathrm{H}_1\overline{F}' \;\simeq\; \mathbb{Z}L_0 \oplus \mathbb{Z}\delta \oplus \mathrm{H}_1\Gamma\,, \qquad (\star\star)$$

where $\delta = \sum_{i=1}^n \beta_i L_i$ and the integers β_i satisfy $\sum_{i=1}^n \beta_i\, \alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_n = 1$.

With $(\star)$ and $(\star\star)$, we are now in position to describe the $A^{\pm}_{F'}$: we just need to compute $\ell k(i_\pm x, y)$, with x element of a basis of H_1F', y of $\mathrm{H}_1\overline{F}'$. Proposition 4.2.4 gives

$$\begin{aligned}
\ell k(i_\pm\gamma_i, L_0) &= \ell k(\gamma, L_0) \;=\; \alpha_1\cdots\alpha_n\alpha \\
\ell k(i_\pm\gamma_i, \delta) &= \ell k(\gamma, \delta) \;=\; \sum_{i=1}^n \beta_i\, \ell k(\gamma, L_i) \\
&= \sum_{i=1}^n \beta_i\, \alpha_0\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_n\alpha \\
&= \alpha_0\alpha \underbrace{\sum_{i=1}^n \beta_i\, \alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_n}_{1} \;=\; \alpha_0\alpha.
\end{aligned}$$

Using the proof of lemma 4.3.9, it is easy to compute $\ell k(i_\pm\gamma_i, y)$, for all y in a basis of $\mathrm{H}_1\Gamma \simeq s(\mathrm{Ker}\,\varphi_0) \subset \mathrm{H}_1\overline{F}'$. Of course, this will depend on Γ and on the choice of a basis of $\mathrm{H}_1\Gamma$.

Before proceeding with the next step, let us recall the ranks of the $\mathbb{Z}$-modules involved; note that these numbers are completely determined by the multilink. In particular, they don't depend on the choice of Γ. Let us denote by a the rank $b_1(\Gamma)$; we have

- $\mathrm{rk}\,\mathrm{H}_0F' = \mathrm{rk}\,\mathrm{H}_1F' = \frac{1}{2}\sum_{i=1}^n |\frac{m_i}{\alpha_i}| = n + a - 1$.
- $\mathrm{rk}\,\mathrm{H}_2F' = 0$.
- $\mathrm{rk}\,\mathrm{H}_0\overline{F}' = \mathrm{rk}\,\mathrm{H}_0\Gamma = 2$.
- $\mathrm{rk}\,\mathrm{H}_1\overline{F}' = 2 + a$.
- $\mathrm{rk}\,\mathrm{H}_2\overline{F}' = a$, since $\chi(\overline{F}') = 0$.

Homology and Surgery

Recall that in our context, a surgery on $F' =: F_0$ consists of taking away two open disks on two distinct connected components of F_0 and joining their boundaries with a cylinder (as described in figure 1.2). The effect of a surgery on H_1F_0 and $\mathrm{H}_1\overline{F}_0$ has already been studied in the proof of proposition 1.2.9. Using the notations of figure 1.3, and denoting by F_1 the result of the surgery [5], we have the following results.

- $\mathrm{H}_1F_1 \;\simeq\; \mathrm{H}_1F_0 \oplus \mathbb{Z}c_1$.
- If the surgery reduces $\overline{b}_2$, then $\mathrm{H}_1\overline{F}_1 \simeq \mathrm{H}_1\overline{F}_0 \oplus \mathbb{Z}\sigma_1$: let us speak of surgery of type (I).

[5] Note that if F_0 is planar (as is our vertical surface F'), then F_1 remains planar.

- If it does not reduce $\overline{b}_2$, then $\mathrm{H}_1\overline{F}_1 \simeq \mathrm{H}_1\overline{F}_0 \oplus \mathbb{Z}c_1 \oplus \mathbb{Z}\sigma_1$: surgery of type (II).

Since $d = 1$, a Seifert surface is good if and only if it is connected. Hence, we need to make $\mu - 1$ surgeries to obtain $F := F_{\mu-1}$ good.

4.3.10 Lemma.
It is possible to make the surgeries in the following order: at first, a surgeries of type (I), so that $\mathrm{H}_2\overline{F}_a = 0$; then, $n-2$ of type (II), giving $F := F_{\mu-1}$ connected.

Proof. Once $\mathrm{H}_2\overline{F} = 0$, the Alexander duality implies that $\Sigma - \overline{F}$ is connected; then, every surgery is possible. Hence, we only need to show that it is always possible to reduce $\overline{b}_2$ (if $\overline{b}_2 > 0$) with a surgery. Let us suppose that it is impossible to diminish $\overline{b}_2 > 0$. By the proof of proposition 1.2.2, it is possible to obtain a good surface F; thus, we must be able to "unlock the situation" with a surgery. "Unlock the situation" means: allow two points in $\Sigma - \overline{F}$ to be linked, that were not in the same connected component of $\Sigma - \overline{F}$ before; that is: diminish $\operatorname{rk} \widetilde{H}_0(\Sigma - \overline{F}) = \operatorname{rk} \mathrm{H}_2\overline{F} = \overline{b}_2$. □

Before proceeding, let us check that we get the right dimensions for $\mathrm{H}_1 F$ and $\mathrm{H}_1\overline{F}$:

$$\begin{aligned}
\operatorname{rk} \mathrm{H}_1 F &= \mu \,+\, \mu - 1 \,=\, 2\mu - 1\\
\operatorname{rk} \mathrm{H}_1 \overline{F} &= a + 2 \,+\, 1 \cdot a \,+\, 2 \cdot (n-2)\\
&= 2 \cdot (n + a - 1) \,=\, 2\mu \,.
\end{aligned}$$

Since F is a connected planar surface with 2μ boundary components, we had to find $\operatorname{rk} \mathrm{H}_1 F = 2\mu - 1$. Furthermore, F being good implies that $\operatorname{rk} \mathrm{H}_1\overline{F} = \operatorname{rk} \mathrm{H}_1 F + r$ by corollary 1.2.4; since $r = 1$ in our case, we get the right result.

Computing the Alexander Matrix

Now, we know that the matrices $A^{\pm}_{F'} = A^{\pm}_{F_0}$ are of the form

$$A^{\pm}_{F_0} = \left(\begin{array}{c|c} \mathcal{Q} & \mathcal{L}^{\pm} \end{array} \right) \Big\}\mu \,,$$

where

$$\mathcal{Q} = \begin{pmatrix} \alpha_1 \cdots \alpha_n \alpha & \alpha_0 \alpha \\ \alpha_1 \cdots \alpha_n \alpha & \alpha_0 \alpha \\ \vdots & \vdots \\ \alpha_1 \cdots \alpha_n \alpha & \alpha_0 \alpha \end{pmatrix}$$

and $\mathcal{L}^{\pm}$ is a $(\mu \times a)$-matrix that only depends on Γ and on the choice of a basis $\{y_1, \ldots, y_a\}$ of $\mathrm{H}_1\Gamma$. Also, remember that the first column represents L_0 and the second δ in $\mathrm{H}_1\overline{F}'$.

Performing the surgeries, we can choose the cycles c_i and σ_i such that

$$\begin{aligned}
\ell k(i_{\pm}c_i, L_j) &= \ell k(i_{\pm}c_i, y_\ell) \,=\, 0 \;\text{ for all }\; j, \ell\,,\\
\ell k(i_{\pm}c_i, \sigma_j) &= 0 \;\text{ for all }\; j < i,\\
\ell k(i_{+}c_i, \sigma_i) &= +1,\\
\ell k(i_{-}c_i, \sigma_i) &= 0.
\end{aligned}$$

In general, we can say nothing about $\ell k(i_{\pm}\gamma_\ell, \sigma_i)$. This gives the following Seifert matrices.

$$
A_F^+ = \begin{array}{r}\mu\{\\ a\{\\ n-2\{\end{array}
\left(\begin{array}{ccccc}
\overbrace{\mathcal{Q}}^{2} & \overbrace{\mathcal{L}^+}^{a} & \overbrace{* \; * \; *}^{a} & \overbrace{0}^{n-2} & \overbrace{* \; * \; *}^{n-2} \\
0 & 0 & \begin{smallmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{smallmatrix} & 0 & * \; * \; * \\
0 & 0 & 0 & 0 & \begin{smallmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{smallmatrix}
\end{array}\right)
$$

$$
A_F^- = \begin{array}{r}\mu\{\\ a\{\\ n-2\{\end{array}
\left(\begin{array}{ccccc}
\overbrace{\mathcal{Q}}^{2} & \overbrace{\mathcal{L}^-}^{a} & \overbrace{* \; * \; *}^{a} & \overbrace{0}^{n-2} & \overbrace{* \; * \; *}^{n-2} \\
0 & 0 & \begin{smallmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{smallmatrix} & 0 & * \; * \; * \\
0 & 0 & 0 & 0 & \begin{smallmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{smallmatrix}
\end{array}\right)
$$

Computing $\mathcal{P} = A_F^+ - tA_F^-$, we get

$$
\mathcal{P} = \left(\begin{array}{ccccc}
\mathcal{Q}(1-t) & \mathcal{L}^+ - t\mathcal{L}^- & * \; * \; * & 0 & * \; * \; * \\
0 & 0 & \begin{smallmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{smallmatrix} & 0 & * \; * \; * \\
0 & 0 & 0 & 0 & \begin{smallmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{smallmatrix}
\end{array}\right)
$$

which, as a presentation matrix, is equivalent to

$$
\left(\begin{array}{cccc}
\alpha_1 \cdots \alpha_n \alpha(1-t) & \alpha_0 \alpha(1-t) & & 0 \\
\vdots & \vdots & \overbrace{\mathcal{L}^+ - t\mathcal{L}^-}^{a} & \overbrace{\vdots}^{n-2} \\
\alpha_1 \cdots \alpha_n \alpha(1-t) & \alpha_0 \alpha(1-t) & & 0
\end{array}\right).
$$

Proposition 4.3.8 now follows from the next lemma.

4.3.11 Lemma.
For all $\alpha', \alpha'' \in \mathbb{Z}$, for all $d_1, \dots, d_n \in \mathbb{Z}$ with $\gcd(d_1, \dots, d_n) = 1$ and $\sum_{i=1}^n d_i = 0$, and for all Γ realizing these integers $d_1, \dots, d_n$, the matrix

$$
\left(\begin{array}{ccc}
\alpha'(1-t) & \alpha''(1-t) & \\
\vdots & \vdots & \mathcal{L}^+ - t\mathcal{L}^- \\
\alpha'(1-t) & \alpha''(1-t) &
\end{array}\right)
$$

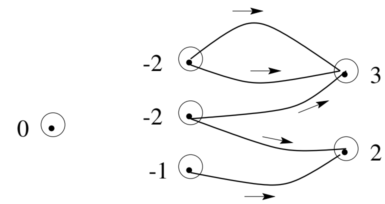


Figure 4.4: Construction of Γ for $d_i = (0, 2, -2, 3, -1, -2)$.

presents the $\mathbb{Z}[t, t^{-1}]$*-module* $\langle\, x, y \mid \alpha'(t-1) \cdot x + \alpha''(t-1) \cdot y = 0 \,\rangle$.

Proof. Since the module presented does not depend on the choice of Γ, we just need to prove the lemma for one Γ. The only problem now is to find a systematic way of constructing Γ; the rest will be simple computation. Here is one way (perhaps not the best): order the d_i so that $d_0 = 0$, $d_1, \dots, d_s < 0$ and $d_{s+1}, \dots, d_n > 0$; then, place $d_1, \dots, d_s$ on one column, $d_{s+1}, \dots, d_n$ on another, and link them with edges oriented from left to right (see figure 4.4 for an example).

The graph obtained is a sequence of subgraphs Γ_k consisting of two vertices joined by k edges oriented in the same direction. With this convention, our sequence $d_1, \dots, d_n$ determines a sequence of integers $k_1, \dots, k_{n-1}$. The example illustrated above gives: $(k_1, k_2, k_3, k_4) = (2, 1, 1, 1)$. It is easy to prove by induction that $\gcd(k_1, \dots, k_{n-1}) = \gcd(d_1, \dots, d_n)$, which is assumed to be equal to one.

Using the proof of lemma 4.3.9, we obtain the matrix

$$\mathcal{P} = \begin{pmatrix} \alpha'(1-t) & \alpha''(1-t) & \mathcal{P}_{k_1} & & \\ \vdots & \vdots & 0 & \ddots & 0 \\ \alpha'(1-t) & \alpha''(1-t) & & & \mathcal{P}_{k_{n-1}} \end{pmatrix},$$

where $\mathcal{P}_k$ is empty if $k = 0$ and

$$\mathcal{P}_k = \begin{pmatrix} -t & & & \\ 1 & -t & & \\ & 1 & & \\ & & \ddots & \\ & & & -t \\ & & & 1 \end{pmatrix},$$

a $k \times (k-1)$-matrix, if $k \geq 1$. With the appropriate transformations (keeping the basis of $\mathrm{H}_1\overline{F}$ fixed), we can see that

$$\begin{aligned} \mathcal{P} \;&\sim\; \begin{pmatrix} \alpha'(1-t^{k_1}) & \alpha''(1-t^{k_1}) \\ \vdots & \\ \alpha'(1-t^{k_{n-1}}) & \alpha''(1-t^{k_{n-1}}) \end{pmatrix} \\ &\sim\; \begin{pmatrix} \alpha'(1-t^{\gcd(k_1,\dots,k_{n-1})}) & \alpha''(1-t^{\gcd(k_1,\dots,k_{n-1})}) \end{pmatrix} \\ &=\; \begin{pmatrix} \alpha'(1-t) & \alpha''(1-t) \end{pmatrix}. \end{aligned}$$

This concludes the proof of this lemma, of propositions 4.3.8, 4.3.7 and of theorem 4.3.1. $\square$

4.4 The Fibered Case

The aim of the present section is to give an Alexander matrix over $\mathbb{Z}[t,t^{-1}]$ of any fibered Seifert multilink. The method is not original: we use techniques developped by C. Weber and F. Michel in [26]. Therefore, we will skip the tedious proofs.

Let

$$(\Sigma, L(\underline{m})) = (\Sigma(\alpha_1,\ldots,\alpha_k), m_1L_1\cup\ldots\cup m_nL_n)$$

be a Seifert multilink with $d=\gcd(m_1,\ldots,m_n)$, let $X=\Sigma-\mathcal{N}(L)$ be its exterior, $\Sigma \xrightarrow{\pi} S^2$ the associated Seifert fibration and γ a generic fiber. Also, let us denote by b_i the element of the orbit space S^2 corresponding to the Seifert fiber of index α_i (for $i=1,\ldots,k$). Clearly, the restriction to X of the Seifert fibration yields $X \xrightarrow{\pi|} P_n$, where P_n is the n-punctured sphere $S^2-(D_1^2\cup\ldots\cup D_n^2)$, D_i^2 being an open neighborhood of b_i.

We know by proposition 4.2.6 that if the number

$$\underline{m}(\gamma)=\sum_{i=1}^{n} m_i\,\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_k$$

is not equal to zero, there exists a locally trivial fibration $X \xrightarrow{\phi} S^1$ with fiber F in the homotopy class of $\underline{m}$. In this context, a theorem of Waldhausen [47] can be stated in the following way. (See also [11, Lemma 11.4].)

4.4.1 Theorem. (Waldhausen)
It is possible to choose a topological monodromy $F \xrightarrow{h} F$ and a homeomorphism $T(h) \xrightarrow{H} X$, where $T(h)=F\times[0,1]/(x,0)\sim(h(x),1)$ is the mapping torus of h, satisfying the following conditions:

- *the diagram* $\begin{array}{ccc} T(h) & \xrightarrow{H} & X \\ & \searrow \quad \swarrow \phi & \\ & S^1 & \end{array}$ *commutes up to homotopy;*
- *h is of finite order $u=|\underline{m}(\gamma)|$;*
- *the diagram* $\begin{array}{ccc} T(h) & \xrightarrow{H} & X \\ & \searrow \quad \swarrow \pi| & \\ & P_n & \end{array}$ *commutes, where the arrow $T(h)\longrightarrow P_n$ is the natural projection corresponding to the equivalence relation: $x\sim y \iff x,y$ are in the same trajectory parallel to $[0,1]$;*
- *the action of $\mathbb{Z}/u=\langle h \mid h^u=1\rangle$ on F gives an u-fold branched cyclic covering $F \xrightarrow{p} P_n$, branched over $b_{n+1},\ldots,b_k$ with branch indices $\alpha_{n+1},\ldots,\alpha_k$.* □

As a consequence, we obviously have an unbranched u-fold cyclic covering $F_0 \xrightarrow{p_0} P_k$, where $F_0=F-p^{-1}(D_{n+1}^2\cup\ldots\cup D_k^2)$, $p_0=p|_{F_0}$ and $P_k=S^2-(D_1^2\cup\ldots\cup D_k^2)$. The group of p_0 is cyclic of order u, generated by $h_0=h|_{F_0}$. Clearly, $F_0 \xrightarrow{h_0} F_0$ is the monodromy of the multilink

$$(\Sigma, L_0(\underline{m_0}))=(\Sigma(\alpha_1,\ldots,\alpha_k), m_1L_1\cup\ldots\cup m_nL_n\cup m_{n+1}L_{n+1}\cup\ldots\cup m_kL_k),$$

with $m_{n+1} = \ldots = m_k = 0$. This covering $F_0 \xrightarrow{p_0} P_k$ is determined by a morphism $\mathrm{H}_1 P_k \xrightarrow{\varphi} \mathbb{Z}/u$, that is, by $\varphi(v_1), \ldots, \varphi(v_k) \in \mathbb{Z}/u$ where v_i is the boundary component of P_k corresponding to D_i^2.

The first step is to compute these $\varphi(v_i)$. Using proposition 1.2.1 and proposition 4.2.4 together with the third point of Waldhausen's theorem, we get the following result.

4.4.2 Lemma.
Let $(\Sigma, L(\underline{m})) = (\Sigma(\alpha_1, \ldots, \alpha_k), m_1 L_1 \cup \ldots \cup m_k L_k)$ be a Seifert multilink with $\underline{m}(\gamma) \neq 0$, $v_1, \ldots, v_k$ the boundary components of P_k corresponding to $\alpha_1, \ldots, \alpha_k$, and $\beta_1, \ldots, \beta_k$ integers such that $\sum_{j=1}^{k} \beta_j \alpha_1 \cdots \widehat{\alpha_j} \cdots \alpha_k = 1$. Then, the morphism $\mathrm{H}_1 P_k \xrightarrow{\varphi} \mathbb{Z}/u$ is given by

$$\varphi(v_i) = \frac{m_i - \beta_i \underline{m}(\gamma)}{\alpha_i}, \quad \text{for all } i = 1, \ldots, k. \qquad \square$$

Now, we have

$$\begin{array}{ccc} F_0 & \subset & F \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ P_k & \subset & P_n \end{array}$$

where p_0 is the u-fold cyclic covering determined by the morphism $\mathrm{H}_1 P_k \xrightarrow{\varphi} \mathbb{Z}/u$. The action of $\langle h_0 \mid h_0^u = 1\rangle$ on F_0 endows $\mathrm{H}_1 F_0$ with a structure of $\mathbb{Z}[t, t^{-1}]$-module; the same holds for $\langle h \mid h^u = 1\rangle$ on $\mathrm{H}_1 F$ as well. The problem is now to compute a presentation matrix of the $\mathbb{Z}[t, t^{-1}]$-module $\mathrm{H}_1 F$; it will be an Alexander matrix of $L(\underline{m})$.

Via homotopy equivalence, it may be assumed that P_k is the wedge of $k-1$ circles $v_1, \ldots, v_{k-1}$ with base point $*$, and that F_0 is a one-dimensional complex with 0-skeleton $p_0^{-1}(*) \simeq \mathbb{Z}/u$. For all $i = 1, \ldots, k-1$, let us note $n_i = \varphi(v_i)$ as in lemma 4.4.2, and $d_i = \gcd(n_i, u)$.[6] Finally, let us denote by x_i the homology class in $\mathrm{H}_1(F_0, p_0^{-1}(*))$ of the lifting of v_i that starts at $0 \in \mathbb{Z}/u \simeq p_0^{-1}(*)$.

The exact sequence of the pair $(F_0, p_0^{-1}(*))$ takes the form

$$0 \longrightarrow \mathrm{H}_1 F_0 \longrightarrow \mathrm{H}_1(F_0, p_0^{-1}(*)) \xrightarrow{\partial} \mathrm{H}_0(p_0^{-1}(*)) \longrightarrow \mathrm{H}_0 F_0 \longrightarrow 0,$$

where $\mathrm{H}_0 F_0 \simeq \mathbb{Z}[t, t^{-1}]/(t^d - 1)$, $\mathrm{H}_0(p_0^{-1}(*)) \simeq \mathbb{Z}[t, t^{-1}]/(t^u - 1)$ and

$$\mathrm{H}_1(F_0, p_0^{-1}(*)) \simeq \bigoplus_{i=1}^{k-1} \mathbb{Z}[t, t^{-1}]/(t^u - 1) \cdot x_i \ .$$

Furthermore, the morphism ∂ is given by $\partial(x_i) = t^{n_i} - 1$. Hence, the given problem boils down to the following:

Technical problem: Give a presentation of the $\mathbb{Z}[t, t^{-1}]$-module $\operatorname{Ker}\psi$, where ψ is the morphism

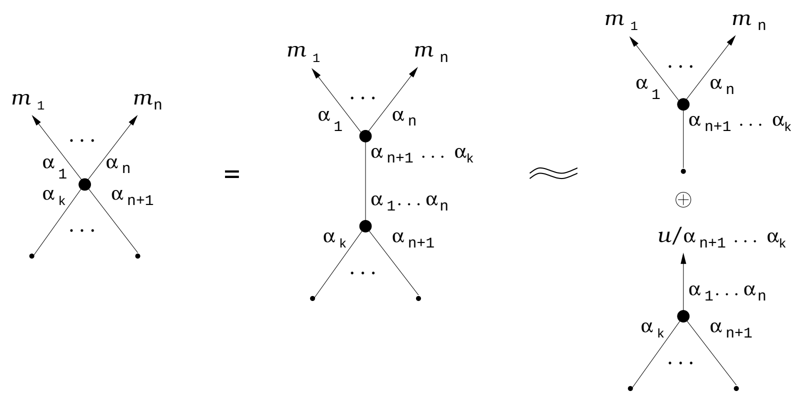
$$\begin{array}{rcl} \bigoplus_{i=1}^{k-1} \mathbb{Z}[t, t^{-1}]/(t^u - 1) \cdot x_i & \xrightarrow{\psi} & \mathbb{Z}[t, t^{-1}]/(t^u - 1) \\ x_i & \longmapsto & t^{n_i} - 1. \end{array}$$

[6] Note that d_i is also equal to $\gcd(m_i, m_i')$; therefore, this notation is consistent (up to sign) with the notations of proposition 1.2.1 and of lemma 4.3.6.

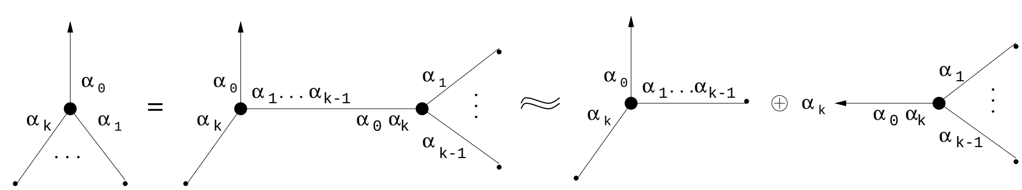
The result applied to ∂ will give a presentation of $\mathrm{H}_1 F_0$.
Furthermore, $p_0^{-1}(v_i)$ consists of d_i connected components, whose homology classes are given by $\left\{ t^j \cdot \frac{t^u-1}{t^{d_i}-1} \cdot x_i \;\middle|\; j = 0, 1, \ldots, d_i - 1 \right\}$. Therefore, a presentation of $\mathrm{H}_1 F$ will be obtained via the presentation of $\mathrm{H}_1 F_0$ by adding the relations $\frac{t^u-1}{t^{d_i}-1} \cdot x_i = 0$ for $i = n+1, \ldots, k$.

It is not very hard to give an answer to this technical problem in full generality; nevertheless, we can save a lot of work by considering non-minimal splicing operations, as in the non-fibered case.

• Reduction 1: It is sufficiant to give a formula for the case of a knot, and the case $k = n+1$. The following splicing gives the general case via theorem 3.3.1.



• Reduction 2: Using inductively the splicing



and theorem 3.3.1, the general case of a knot can be reduced to the special case $(\Sigma(p, q, r), K)$.

Hence, a general formula can be derived from a formula for the two multilinks

$$(\Sigma(\alpha_1, \ldots, \alpha_n, \alpha), m_1 L_1 \cup \ldots \cup m_n L_n) \quad \text{and} \quad (\Sigma(\alpha, \beta, \gamma), K)\,.$$

Let us turn back to our technical problem. We need the following lemmas.

4.4.3 Lemma.
Let $n_1, \ldots, n_r$ be positive integers with $\gcd(n_1, \ldots, n_r) = d$. Then, there exists polynomials $\mu_1, \ldots, \mu_r \in \mathbb{Z}[t]$ such that

$$\mu_1(t) \cdot \frac{t^{n_1} - 1}{t^d - 1} + \ldots + \mu_r(t) \cdot \frac{t^{n_r} - 1}{t^d - 1} = 1,$$

and such that $\mu_1(t), \ldots, \mu_{r-1}(t)$ and $t^d - 1$ don't have any common factor in $\mathbb{Z}[t]$. □

4.4.4 Lemma.
Let $\bigoplus_{i=1}^{k-1} \mathbb{Z}[t,t^{-1}]/(t^u-1)\cdot x_i \xrightarrow{\psi} \mathbb{Z}[t,t^{-1}]/(t^u-1)$ *be the morphism given by* $x_i \longmapsto t^{n_i}-1$. *For* $i=1,\ldots,k-1$, *let us note* $d_i = \gcd(n_i,u)$ *and* $d^i = \gcd(n_1,\ldots,n_i,u) = \gcd(d_1,\ldots,d_i)$. *For* $i=1,\ldots,k-2$, *consider polynomials* $\gamma_i^1,\ldots,\gamma_i^{i+1}\in\mathbb{Z}[t]$ *satisfying*

$$\sum_{j=1}^{i} \gamma_i^j(t)\cdot\frac{t^{n_j}-1}{t^{d^i}-1} + \gamma_i^{i+1}(t)\cdot\frac{t^u-1}{t^{d^i}-1} = 1\,,$$

and such that $\gamma_i^1,\ldots,\gamma_i^i$ *and* $t^{d^i}-1$ *don't have any common factor in* $\mathbb{Z}[t]$. *Then, the* $\mathbb{Z}[t,t^{-1}]$-*module* $Ker\,\psi$ *is presented by*

$$Ker\,\psi = \Big\langle X_1,\ldots,X_{k-1}\ \Big|\ \sum_{j=1}^{i} p_{ij}(t)\cdot X_j = 0\,,\ \ j=1,\ldots,k-1\Big\rangle,$$

where the generators $X_1,\ldots,X_{k-1}$ *are given by*

$$X_i \;=\; \frac{t^{d^{i-1}}-1}{t^{d^i}-1}\cdot x_i \;-\; \frac{t^{n_i}-1}{t^{d^i}-1}\sum_{j=1}^{i-1}\gamma_{i-1}^j(t)\cdot x_j,$$

and the coefficients p_{ij} *are defined recursively by*

$$\begin{aligned} p_{ii}(t) &= \frac{(t^u-1)(t^{d^i}-1)}{(t^{d^{i-1}}-1)}\,, \quad \text{and} \\ p_{ij}(t) &= \frac{t^{d^j}-1}{t^{d^{j-1}}-1}\sum_{\ell=j+1}^{i} p_{i\ell}(t)\cdot\frac{t^{n_\ell}-1}{t^{d^\ell}-1}\cdot\gamma_{\ell-1}^j(t) \quad \text{for } 1\le j<i. \end{aligned}$$

□

With this lemma, it is possible to compute a presentation matrix of $\mathrm{H}_1 F_0$. To obtain an Alexander matrix for $L(\underline{m})$, one has to add the relations $\frac{t^u-1}{t^{d_i}-1}\cdot x_i = 0$ for $i=n+1,\ldots,k$.

4.4.5 Lemma.
The equality $\frac{t^u-1}{t^{d_i}-1}\cdot x_i = \sum_{j=1}^{i}\frac{p_{ij}(t)}{t^{d_i}-1}\cdot X_j$ *holds for all* i. □

Therefore, an Alexander matrix is given via the matrix (p_{ij}) by dividing the i^{th} ligne by $t^{d_i}-1$ for $i=n+1,\ldots,k$.

Using reductions 1 and 2, together with lemmas 4.4.4 and 4.4.5, it is easy to compute an Alexander matrix of $L(\underline{m})$. Here is the final result.

4.4.6 Theorem.
Let $(\Sigma, L(\underline{m})) = (\Sigma(\alpha_1,\ldots,\alpha_k), m_1L_1\cup\ldots\cup m_nL_n)$ *be a fibered Seifert multilink with* $n<k$.[7] *Let* $\beta_1,\ldots,\beta_k$ *be integers such that* $\sum_{j=1}^{k}\beta_j\alpha_1\cdots\widehat{\alpha_j}\cdots\alpha_k = 1$. *Let us note*

- $u = |\underline{m}(\gamma)| = |\sum_{i=1}^{n} m_i\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_k|$;

[7] Since $\Sigma(\alpha_1,\ldots,\alpha_n) = \Sigma(\alpha_1,\ldots,\alpha_n,1)$, it may be assumed that $n<k$.

- $n_0 = -\underline{m}(\gamma)\sum_{i=n+1}^{k}\frac{\beta_i}{\alpha_i}$, and $d_0 = \frac{u}{\alpha_{n+1}\cdots\alpha_k}$;
- $n_i = \frac{m_i - \beta_i \underline{m}(\gamma)}{\alpha_i}$, and $d_i = \gcd(n_i, u)$ for $i = 1, \dots, n-1$;
- $d^i = \gcd(d_0, d_1, \dots, d_i)$ for $i = 0, \dots, n-1$.[8]

Finally, for $i = 1, \dots, n-2$, let $\gamma_i^0, \dots, \gamma_i^{i+1}$ be polynomials in $\mathbb{Z}[t]$ satisfying

$$\sum_{j=0}^{i}\gamma_i^j(t)\cdot\frac{t^{n_j}-1}{t^{d^i}-1} + \gamma_i^{i+1}(t)\cdot\frac{t^u-1}{t^{d^i}-1} = 1\,,$$

and such that $\gamma_i^0, \dots, \gamma_i^i$ and $t^{d^i}-1$ don't have any common factor in $\mathbb{Z}[t]$. Then, the Alexander module of $(\Sigma, L(\underline{m}))$ is given by the direct sum of

$$\bigoplus_{i=n+2}^{k}\mathbb{Z}[t,t^{-1}]\Big/\frac{(t^u-1)(t^{u/\alpha_{n+1}\cdots\alpha_i}-1)}{(t^{u/\alpha_{n+1}\cdots\alpha_{i-1}}-1)(t^{u/\alpha_i}-1)}$$

and of the $\mathbb{Z}[t,t^{-1}]$-module presented by

$$\begin{pmatrix} p_{11} & & & \\ p_{21} & p_{22} & & \\ \vdots & & \ddots & \\ p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,n-1} \end{pmatrix},$$

where the p_{ij} are defined recursively by

- $p_{ii}(t) = \frac{(t^u-1)(t^{d^i}-1)}{(t^{d^{i-1}}-1)}$, and
- $p_{ij}(t) = \frac{t^{d^j}-1}{t^{d^{j-1}}-1}\sum_{\ell=j+1}^{i} p_{i\ell}(t)\cdot\frac{t^{n_\ell}-1}{t^{d^\ell}-1}\cdot\gamma_{\ell-1}^j(t)$ for $1 \le j < i$. □

4.4.7 Corollary.
The Alexander polynomial of a fibered Seifert multilink is given by

$$\Delta^{L(\underline{m})}(t) \doteq (t^d-1)\frac{(t^u-1)^{k-2}}{\prod_{j=n+1}^{k}(t^{u/\alpha_i}-1)}. \qquad \square$$

Eisenbud and Neumann had already this result.

4.4.8 Corollary.
The Alexander module of a Seifert knot $(\Sigma(\alpha_0, \alpha_1, \dots, \alpha_k), L_0)$ is given by

$$\bigoplus_{i=2}^{k}\mathbb{Z}[t,t^{-1}]\Big/\frac{(t^{\alpha_1\cdots\alpha_k}-1)(t^{\alpha_{i+1}\cdots\alpha_k}-1)}{(t^{\alpha_i\cdots\alpha_k}-1)(t^{\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_k}-1)}. \qquad \square$$

[8] Note that $d^{n-1} = d$.

It is known since Burau [6] that the Alexander module associated with an irreducible plane curve singularity determines its topology. No such phenomenon for Seifert knots, as shown by the following example.

4.4.9 Example. The knots $(\Sigma(1,2,3), L_1)$ and $(\Sigma(5,2,3), L_1)$, although topologically distinct, have the same Alexander module $\mathcal{M}_{\mathbb{Z}} \simeq \mathbb{Z}[t,t^{-1}]/(t^2-t+1)$.

Let us conclude this chapter with a couple more examples.

4.4.10 Example. Let L be the (p,q)-torus link with n components in S^3. Theorem 4.4.6 gives the following $(n \times n)$-matrix:

$$\begin{pmatrix} \frac{(t^{npq}-1)(t^n-1)}{(t^{nq}-1)(t^{np}-1)} & & & & \\ & \frac{(t^{npq}-1)(t-1)}{(t^n-1)} & & & \\ & \frac{(t^{npq}-1)(t-1)}{(t^n-1)} & t^{npq}-1 & & \\ & \vdots & \vdots & \ddots & \\ & \frac{(t^{npq}-1)(t-1)}{(t^n-1)} & t^{npq}-1 & \dots & t^{npq}-1 \end{pmatrix}.$$

Hence, the Alexander module of L is simply

$$\mathbb{Z}[t,t^{-1}]\Big/\frac{(t^{npq}-1)(t^n-1)}{(t^{nq}-1)(t^{np}-1)} \oplus \mathbb{Z}[t,t^{-1}]\Big/\frac{(t^{npq}-1)(t-1)}{(t^n-1)} \oplus \Big(\mathbb{Z}[t,t^{-1}]\big/(t^{npq}-1)\Big)^{n-2},$$

if $n \geq 2$, and

$$\mathbb{Z}[t,t^{-1}]\Big/\frac{(t^{pq}-1)(t-1)}{(t^q-1)(t^p-1)},$$

if $n = 1$.

4.4.11 Example. Let us consider the Hopf link $(\Sigma(1,1), m_1L_1 \cup m_2L_2)$ with $\gcd(m_1,m_2) = d$; we get the Alexander matrix (t^d-1) as seen in example 1.3.4.

Chapter 5

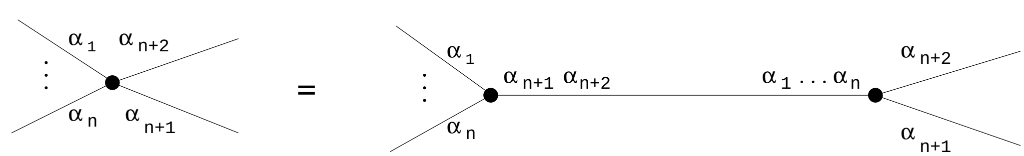
On the Alexander Module of Graph Multilinks

Our first aim was to give a closed formula for the Alexander module over $\mathbb{Z}[t, t^{-1}]$ of any graph multilink. Because of our difficulty in analyzing this invariant via splicing, we are forced to lower our ambitions. This is done in two different ways. The first one consists of restricting the class of multilinks studied: in sections 5.1 and 5.2, we give a closed formula for graph knots and "purely non-fibered" graph multilinks. The second one is scalar extension: working over bigger coefficient rings allows to tackle some difficulties. This method is used in section 5.3 where we give the rank of the Alexander module, and in section 5.4, where the Alexander module over the Novikov ring $\mathbb{Z}[[t]][t^{-1}]$ is computed for any graph multilink. These two strategies are mixed in section 5.5: we compute the Alexander module over the complex numbers of a very wide class of graph multilinks. In particular, the regular link at infinity of any $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ is in this class.

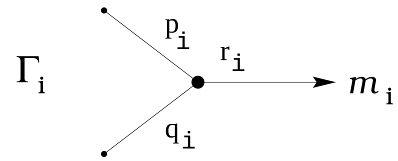
5.1 Graph Knots

Let us present a closed formula for the Alexander module over $\mathbb{Z}[t, t^{-1}]$ of a graph knot. In [11, Chapter 15], Eisenbud and Neumann give an algorithm to compute the Seifert form of a graph knot; in particular, it is possible to compute the Alexander module using this method. Therefore, the result of the present section is not really new. Nevertheless, the techniques involved here are more direct, and the final formula is quite simple; it deserves to be written down.

Consider a graph knot K given by a splice diagram Γ. Using inductively the following transformation, we can replace Γ by an equivalent splice diagram Γ' whose nodes are all of degree 3.



Then, desplice Γ' along every edge joining two nodes and replace every arrowhead vertex with multiplicity zero by a boundary vertex. The connected components of the resulting splice diagram are of the following form: either splice diagrams without any arrowhead, or splice diagrams as below.



Delete every connected component without arrowheads. By theorem 3.3.1, the Alexander module of K is equal to the direct sum of the Alexander modules of the multiknots given by the splice diagrams Γ_i. By corollary 4.4.8, the Alexander module of the multiknot given by Γ_i is

$$\mathbb{Z}[t,t^{-1}]\Big/\frac{(t^{m_ip_iq_i}-1)(t^{m_i}-1)}{(t^{m_ip_i}-1)(t^{m_iq_i}-1)}.$$

Therefore, we have the following theorem.

5.1.1 Theorem.
Using the notations described above, the Alexander module of a graph knot K is given by

$$\mathcal{M}_{\mathbb{Z}}^K = \bigoplus_i \mathbb{Z}[t,t^{-1}]\Big/\frac{(t^{m_ip_iq_i}-1)(t^{m_i}-1)}{(t^{m_ip_i}-1)(t^{m_iq_i}-1)}. \qquad \square$$

It is well known that the Alexander module of an algebraic knot (that is: the knot of an irreducible plane curve singularity) is the sum of cyclic modules (see [1]). By theorem 5.1.1, this remains true for graph knots.

5.2 The Purely Non-Fibered Case

Let us now compute the Alexander module of the **purely non-fibered** graph multilinks, that is, the graph multilinks whose splice components are all non-fibered. We start with an easy and very important lemma.

5.2.1 Lemma.
Two non-fibered Seifert multilinks appear side by side in the minimal splice decomposition of a graph multilink if and only if both weights m_0' and m_0'' that intervene in the splicing are zero.

Proof. Consider $(\Sigma(\alpha_0',\ldots,\alpha_k'), m_0'L_0'\cup\ldots\cup m_n'L_n')$ and $(\Sigma(\alpha_0'',\ldots,\alpha_\ell''), m_0''L_0''\cup\ldots\cup m_r''L_r'')$ two Seifert multilinks spliced along L_0' and L_0''. The weights m_0' and m_0'' must satisfy

$$m_0' = \sum_{j=1}^{r} m_j''\alpha_1''\cdots\widehat{\alpha_j''}\cdots\alpha_\ell'' \quad\text{and}\quad m_0'' = \sum_{i=1}^{n} m_i'\alpha_1'\cdots\widehat{\alpha_i'}\cdots\alpha_k'\,.$$

Hence, we have

$$\begin{cases} \underline{m}'(\gamma') &= \alpha'_1 \cdots \alpha'_k m'_0 + \alpha'_0 m''_0 \\ \underline{m}''(\gamma'') &= \alpha''_0 m'_0 + \alpha''_1 \cdots \alpha''_\ell m''_0 \,. \end{cases}$$

Since the splicing is minimal, the determinant $\alpha'_0\alpha''_0 - \alpha'_1 \cdots \alpha'_k \alpha''_1 \cdots \alpha''_\ell$ of the splicing is not equal to zero. Therefore, the matrix of this linear system is invertible over $\mathbb{Q}$, and $m'_0 = m''_0 = 0$ if and only if $\underline{m}'(\gamma') = \underline{m}''(\gamma'') = 0$. By proposition 4.2.6, this is the case if and only if both Seifert multilinks are non-fibered. □

5.2.2 Theorem.
Let $L(m_1, \ldots, m_n)$ be a graph multilink (with some $m_i \neq 0$) whose splice components are all non-fibered. Let us consider a splice diagram Γ for $L(\underline{m})$. If $\mathcal{N}$ denotes its set of nodes, every element $v \in \mathcal{N}$ corresponds to some non-fibered Seifert multilink; let us note d_v the greatest common divisor of its multiplicities, γ_v its generic Seifert fiber and α_v the product of the weights on the arrowhead edges of Γ adjacent to v.

Then, an Alexander matrix of $L(\underline{m})$ is given by

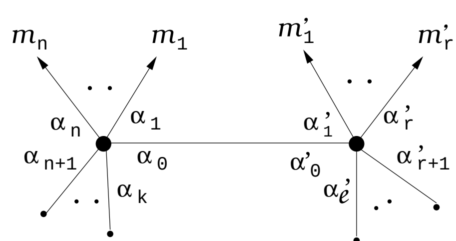
$$v\in\mathcal{N}\left\{ \left(\overbrace{\frac{\ell k(\gamma_v,\gamma_w)}{\alpha_w} \cdot (t^{d_v} - 1)}^{w\in\mathcal{N}} \quad \overbrace{\quad 0 \quad}^{n-1-\#\mathcal{N}} \right) \right. .$$

5.2.3 Remark. If a node $v \in \mathcal{N}$ satisfies $d_v = 0$, we will say that it is a **0-node**. Let us denote by $\mathcal{N}_0$ the set of 0-nodes, and by $\mathcal{N}_1$ the set $\mathcal{N} - \mathcal{N}_0$. Then, the Alexander matrix given above is equivalent to

$$v\in\mathcal{N}_1\left\{ \left(\overbrace{\frac{\ell k(\gamma_v,\gamma_w)}{\alpha_w} \cdot (t^{d_v} - 1)}^{w\in\mathcal{N}_1} \quad \overbrace{\quad 0 \quad}^{n-1-\#\mathcal{N}_1} \right) \right. .$$

This alternative Alexander matrix is perhaps more natural, since the integer $n - 1 - \#\mathcal{N}_1$ is in fact equal to the rank of the Alexander module (see corollary 5.2.7 below).

5.2.4 Example. Consider the general graph link with two splice components,



and assume that both components are non-fibered. Then, an Alexander matrix is given by

$$\begin{pmatrix} \alpha_0\alpha_{n+1}\cdots\alpha_k(t^d-1) & \alpha_1\cdots\alpha_k\alpha'_{r+1}\cdots\alpha'_\ell(t^d-1) & \overbrace{0\ \ldots\ 0}^{n+r-3} \\ \alpha_{n+1}\cdots\alpha_k\alpha'_1\cdots\alpha'_\ell(t^{d'}-1) & \alpha'_0\alpha'_{r+1}\cdots\alpha'_\ell(t^{d'}-1) & 0\ \ldots\ 0 \end{pmatrix},$$

where

$$\begin{aligned} d &= \gcd(m_1, \dots, m_n, \sum_{j=1}^{r} m'_j \alpha'_1 \cdots \widehat{\alpha'_j} \cdots \alpha'_\ell) \\ d' &= \gcd(m'_1, \dots, m'_r, \sum_{i=1}^{n} m_i \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_k). \end{aligned}$$

If the splice is not minimal, that is, if $\alpha_0\alpha'_0 = \alpha_1 \cdots \alpha_k \alpha'_1 \cdots \alpha'_\ell$, then this matrix is equivalent to

$$\begin{pmatrix} \alpha_{n+1} \cdots \alpha_k \alpha'_1 \cdots \alpha'_\ell (t^{\gcd(d,d')} - 1) & \overbrace{0 \, \dots \, 0}^{n+r-2} \end{pmatrix},$$

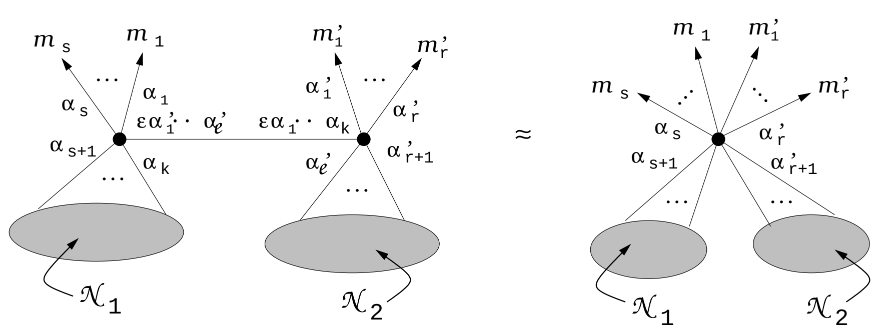
and we get back theorem 4.3.1.

Proof. First of all, let us show that we can restrict ourselves to connected splice diagrams. Indeed, let us suppose that $\Gamma = \Gamma' + \Gamma''$; if $d' > 0$ and $d'' > 0$, we know by proposition 1.2.9 that an Alexander matrix of $L(\underline{m})$ is given by $\mathcal{P} = \begin{pmatrix} \mathcal{P}' & 0 & 0 \\ 0 & \mathcal{P}'' & 0 \end{pmatrix}$, where $\mathcal{P}'$ (resp. $\mathcal{P}''$) is the presentation matrix associated with Γ' (resp. Γ''). The number of columns of zeros is equal to

$$(n' - 1 - \#\mathcal{N}') + (n'' - 1 - \#\mathcal{N}'') + 1 = (n' + n'') - 1 + \#(\mathcal{N}' \cup \mathcal{N}'') = n - 1 - \#\mathcal{N}\,,$$

and we get the Alexander matrix predicted by the theorem. If $d' = 0$ and $d'' > 0$, $\mathcal{P}$ will be given by the direct sum of $\mathcal{P}''$ with a free module of rank n' (recall proposition 1.2.9). The theorem gives this result.

The second step is to show that we can restrict ourselves to minimal (connected) splice diagrams. It is an easy exercice to check that, if two splice diagrams are related by a reduction of type (i) (recall figure 4.2), then the matrices determined by these diagrams are identical. For the reduction of type (ii), we need to check that the matrices determined by the two following diagrams, where $\epsilon = \pm 1$, are equivalent as presentation matrices.



Let us denote by x and x' the nodes appearing in the first diagram, and note $\mathcal{N} = \mathcal{N}_1 \cup \{x, x'\} \cup \mathcal{N}_2$. Also, let us abbreviate $\ell k(\gamma_v, \gamma_w)$ by $\ell(v, w)$. For the first diagram, the theorem gives the matrix

$$
\begin{array}{r}
v\in\mathcal{N}_1\left\{ \vphantom{\frac{\ell}{\alpha}} \right. \\[28pt]
\\
v\in\mathcal{N}_2\left\{ \vphantom{\frac{\ell}{\alpha}} \right.
\end{array}
\begin{pmatrix}
\overbrace{\frac{\ell(v,w)}{\alpha_w}(t^{d_v}-1)}^{w\in\mathcal{N}_1} & \frac{\ell(v,x)}{\alpha_1\cdots\alpha_s}(t^{d_v}-1) & \frac{\ell(v,x)}{\epsilon\alpha'_1\cdots\alpha'_r}(t^{d_v}-1) & \overbrace{\frac{\ell(v,w)}{\alpha_w}(t^{d_v}-1)}^{w\in\mathcal{N}_2} & \overbrace{0}^{n-1-\#\mathcal{N}} \\
\frac{\ell(x,w)}{\alpha_w}(t^{d_x}-1) & \epsilon\alpha'_1\cdots\alpha'_\ell\alpha_{s+1}\cdots\alpha_k(t^{d_x}-1) & \alpha_1\cdots\alpha_k\alpha'_{r+1}\cdots\alpha'_\ell(t^{d_x}-1) & \frac{\ell(x,w)}{\alpha_w}(t^{d_x}-1) & 0 \\
\frac{\ell(x,w)}{\alpha_w}(t^{d_{x'}}-1) & \alpha'_1\cdots\alpha'_\ell\alpha_{s+1}\cdots\alpha_k(t^{d_{x'}}-1) & \epsilon\alpha_1\cdots\alpha_k\alpha'_{r+1}\cdots\alpha'_\ell(t^{d_{x'}}-1) & \frac{\ell(x',w)}{\alpha_w}(t^{d_{x'}}-1) & 0 \\
\frac{\ell(v,w)}{\alpha_w}(t^{d_v}-1) & \frac{\ell(v,x')}{\epsilon\alpha_1\cdots\alpha_s}(t^{d_v}-1) & \frac{\ell(v,x')}{\alpha'_1\cdots\alpha'_r}(t^{d_v}-1) & \frac{\ell(v,w)}{\alpha_w}(t^{d_v}-1) & 0
\end{pmatrix}
$$

which is equivalent to

$$
\begin{array}{r}
v\in\mathcal{N}_1\left\{ \vphantom{\frac{\ell}{\alpha}} \right. \\
\\
v\in\mathcal{N}_2\left\{ \vphantom{\frac{\ell}{\alpha}} \right.
\end{array}
\begin{pmatrix}
\overbrace{\frac{\ell(v,w)}{\alpha_w}(t^{d_v}-1)}^{w\in\mathcal{N}_1} & \frac{\ell(v,x)}{\alpha_1\cdots\alpha_s\alpha'_1\cdots\alpha'_r}(t^{d_v}-1) & \overbrace{\frac{\ell(v,w)}{\alpha_w}(t^{d_v}-1)}^{w\in\mathcal{N}_2} & \overbrace{0}^{n-1-(\#\mathcal{N}-1)} \\
\frac{\ell(x,w)}{\alpha_w}(t^{d}-1) & \alpha'_{r+1}\cdots\alpha'_\ell\alpha_{s+1}\cdots\alpha_k(t^{d}-1) & \frac{\ell(x',w)}{\alpha_w}(t^{d}-1) & 0 \\
\frac{\ell(v,w)}{\alpha_w}(t^{d_v}-1) & \frac{\ell(v,x')}{\alpha_1\cdots\alpha_s\alpha'_1\cdots\alpha'_r}(t^{d_v}-1) & \frac{\ell(v,w)}{\alpha_w}(t^{d_v}-1) & 0
\end{pmatrix},
$$

where $d = \gcd(d_x, d_{x'})$. This is the matrix associated to the second diagram.

We are left with the proof of the theorem for minimal, connected splice diagrams. So, let us suppose that $L(m_1, \ldots, m_n)$ is given by such a diagram.
Recall that given a component L_i of a Seifert multilink $(\Sigma(\alpha_1, \ldots, \alpha_k), m_1L_1 \cup \ldots \cup m_nL_n)$, we note $d_i = \operatorname{sgn}(d_i)\gcd(m_i, m'_i)$, where $m'_i = -\sum_{j\neq i} m_j \ell k(L_i, L_j)$ and $\operatorname{sgn}(d_i) = \operatorname{sgn}(m_i)\cdot \operatorname{sgn}(\alpha_i)$ if $m_i \neq 0$, and $\operatorname{sgn}(d_i) = \operatorname{sgn}(m'_i)$ if $m_i = 0$.

5.2.5 Lemma.
For all $v \in \mathcal{N}$, one has $\sum_{I_v} d_i = 0$, where

$$I_v = \{\ i \mid L_i \textit{ corresponds to an arrowhead edge adjacent to } v\ \}.$$

Proof. Let us fix a node v, that is, a non-fibered Seifert multilink. By lemma 4.3.6, we know that the sum of all the d_i of this multilink is zero. Since the diagram is minimal, lemma 5.2.1 implies that all the d_i involved in splicing (that is, all the d_i of this multilink with i not in I_v) are zero; this proves the lemma. □

Theorem 5.2.2 is a direct consequence of the following proposition.

5.2.6 Proposition.
Let $L(m_1, \ldots, m_n)$ be a purely non-fibered graph multilink (with $d \neq 0$) given by a connected minimal splice diagram Γ; let us note α_i the weight of the edge corresponding to m_iL_i. Let us suppose that

$$\sum_{i=1}^{n_2} d_i = \sum_{i=n_2+1}^{n_3} d_i = \ldots = \sum_{i=n_b+1}^{n} d_i = 0,$$

each sum being minimal with this property and involving only components of the same Seifert multilink [1]*; let us say that* $L_{n_i+1},\ldots,L_{n_{i+1}}$ *are components of* $v_{j(i)}$*, for* $i=1,\ldots,b$*. Choose integers* β_i *such that*

$$\sum_{i=1}^{n_2}\beta_i\,\alpha_1\cdots\widehat{\alpha}_i\cdots\alpha_{n_2}=\ldots=\sum_{i=n_b+1}^{n}\beta_i\,\alpha_{n_b+1}\cdots\widehat{\alpha}_i\cdots\alpha_n=1.$$

Then, the Alexander module of $L(\underline{m})$ *is given by*

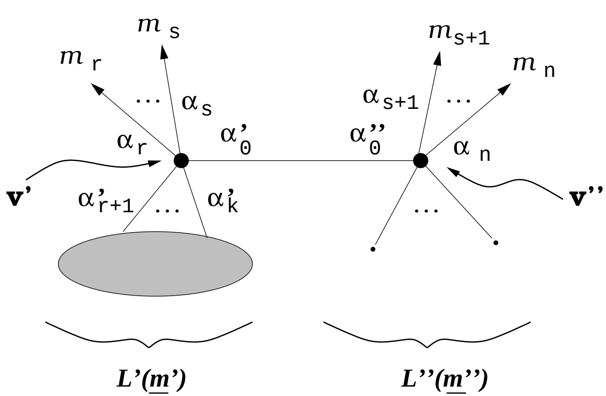
$$\left\langle\ \delta_1^*,\ldots,\delta_b^*,c_{b+1},\ldots,c_{n-1}\ \middle|\ \sum_{i=1}^{b}\frac{\ell k(v,v_{j(i)})}{\alpha_{n_i+1}\cdots\alpha_{n_{i+1}}}(t^{d_v}-1)\cdot\delta_i^*=0\ ,\ v\in\mathcal{N}\right\rangle,$$

where

$$\delta_1=\sum_{i=1}^{n_2}\beta_iL_i,\ ,\ldots,\ \delta_b=\sum_{i=n_b+1}^{n}\beta_iL_i\ ,$$

and * *denotes the image via* $H_1\overline{F}\simeq H^1(\Sigma-\overline{F})\simeq H_1(\Sigma-\overline{F})$.

Proof. We will proceed by induction on $k=\#\mathcal{N}\geq 1$. The case $k=1$ is given by proposition 4.3.7. Let us suppose that proposition 5.2.6 holds for any purely non-fibered graph multilink with less than k splice components, and let us consider $L(\underline{m})$ whose connected, minimal splice diagram Γ has k nodes. Pick a node v'' at the border of Γ and consider $L(\underline{m})$ as the splice of $L'(\underline{m}')$ and $L''(\underline{m}'')$ as depicted below.



Let us note $\mathcal{N}'=\mathcal{N}-\{v''\}$. Since the splice diagram is minimal, it follows from lemma 5.2.1 that both multiplicities appearing via desplicing vanish: $m_0'=m_0''=0$. By induction, the Alexander module of $L'(\underline{m}')$ is given by

$$\mathrm{H}_1\widetilde{X'}\simeq\langle\ \delta_1^*,\ldots,\delta_a^*,M_0',c_1',\ldots,c_r'\ \mid\ \mathcal{R}_v=0\ ,\ v\in\mathcal{N}'\ \rangle\ ,$$

where $r=(s+1)-(a+1)-1=s-a-1$, and

$$\mathcal{R}_v=\sum_{i=1}^{a}\frac{\ell k(v,v_{j(i)})}{\alpha_{n_i+1}\cdots\alpha_{n_{i+1}}}(t^{d_v}-1)\cdot\delta_i^*+\frac{\ell k(v,v')}{\alpha_0'}(t^{d_v}-1)\cdot M_0'\ .$$

[1]This is always possible by lemma 5.2.5.

We have to distinguish two cases, whether $d_{v''} = 0$ or $d_{v''} \neq 0$.

• If $d_{v''} = 0$, we have $m_{s+1} = \ldots = m_n = 0$, so $b = a + n - s$ and $r = n - b - 1$. Proposition 3.2.2 asserts that the Alexander module of $L(\underline{m})$ is given by

$$\mathrm{H}_1\widetilde{X} \simeq \Big(\mathrm{H}_1\widetilde{X}' \oplus \bigoplus_{i=s+1}^{n} \mathbb{Z}[t,t^{-1}]\cdot M_i\Big)\Big/ M_0' \sim \sum_{i=s+1}^{n} \ell k(L_0'', L_i)\cdot M_i \,.$$

It is thus given by the set of generators

$$\{\delta_1^*, \ldots, \delta_a^*, M_{s+1}, \ldots, M_n, c_1', \ldots, c_{n-b-1}'\}$$

with the relations

$$\sum_{i=1}^{a} \frac{\ell k(v, v_{j(i)})}{\alpha_{n_i+1}\cdots\alpha_{n_{i+1}}}(t^{d_v}-1)\cdot\delta_i^* + \sum_{i=s+1}^{n}\frac{\ell k(v,v'')}{\alpha_i}(t^{d_v}-1)\cdot M_i = 0$$

for all $v \in \mathcal{N}'$; this gives the desired presentation.

• On the other hand, if $d_{v''} \neq 0$, we know by the case $k = 1$ (that is, by proposition 4.3.7) that

$$\mathrm{H}_1\widetilde{X''} \simeq \langle\, M_0'', \delta_{a+1}^*, \ldots, \delta_b^*, c_1'', \ldots, c_p'' \;\mid\; \mathcal{R}_{v''} = 0\,\rangle\,,$$

where $p = (n-s+1) - (b-a+1) - 1 = n - s - b + a - 1$ and

$$\mathcal{R}_{v''} = \frac{\ell k(v'', v'')}{\alpha_0''}(t^{d_{v''}}-1)\cdot M_0'' + \sum_{i=a+1}^{b}\frac{\ell k(v'',v'')}{\alpha_{n_i+1}\cdots\alpha_{n_{i+1}}}(t^{d_{v''}}-1)\cdot\delta_i^*.$$

By proposition 3.2.1, we have

$$\mathrm{H}_1\widetilde{X} \;\simeq\; \mathbb{Z}[t,t^{-1}] \oplus (\mathrm{H}_1\widetilde{X}' \oplus \mathrm{H}_1\widetilde{X}'')\Big/(M_0' \sim P_0'',\; P_0' \sim M_0'').$$

Thus, $\mathrm{H}_1\widetilde{X}$ has the set of generators

$$\{\delta_1^*, \ldots, \delta_b^*, c_0, c_1', \ldots, c_r', c_1'', \ldots, c_p''\} = \{\delta_1^*, \ldots, \delta_b^*, c_{b+1}, \ldots, c_{n-1}\},$$

since $1 + r + p = 1 + (s - a - 1) + (n - s - b + a - 1) = n - b - 1$. It is an easy (but tedious) exercise to check that we get the right relations. □

5.2.7 Corollary.
Let $L(\underline{m})$ be a purely non-fibered graph multilink with minimal splice diagram Γ (and $d \neq 0$). For every node $v \in \mathcal{N}$, let us note d_v the greatest common divisor of its multiplicities and γ_v its generic Seifert fiber. Let us also note $\mathcal{N} = \mathcal{N}_0 \sqcup \mathcal{N}_1$, with $\mathcal{N}_0 = \{v \in \mathcal{N} \mid d_v = 0\}$.

Then, the Alexander module of $L(\underline{m})$ over $\mathbb{Q}$ is equal to:

$$\mathcal{M}_{\mathbb{Q}}^{L(\underline{m})} = H_1(\widetilde{X};\mathbb{Q}) \;\simeq\; \bigoplus_{v\in\mathcal{N}_1} \mathbb{Q}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v \;\oplus\; \mathbb{Q}[t,t^{-1}]^{\,n-1-\#\mathcal{N}_1}\,.$$

Proof. Using proposition 1.2.9, we can easily restrict ourselves to connected (minimal) splice diagrams. In this case, proposition 5.2.6 asserts that for some $x_1, \ldots, x_{n-1} \in \mathrm{H}_1\overline{F}$, we have

$$\mathrm{H}_1\widetilde{X} \simeq \left\langle x_1^*, \ldots, x_{n-1}^* \;\middle|\; \sum_{i=1}^{n-1} \ell k(\gamma_v, x_i)(t^{d_v}-1)\cdot x_i^* = 0\,,\; v \in \mathcal{N} \right\rangle .$$

We have the exact sequence of $\mathbb{Z}[t,t^{-1}]$-modules

$$0 \longrightarrow \bigoplus_{v\in\mathcal{N}} \mathbb{Z}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v \xrightarrow{\ i\ } \mathrm{H}_1\widetilde{X} \longrightarrow \mathcal{M} \longrightarrow 0\,,$$

where i is the natural inclusion $\gamma_v \longmapsto \sum_{i=1}^{n-1} \ell k(\gamma_v, x_i)\cdot x_i^*$ and $\mathcal{M}$ is presented by

$$\left\langle x_1^*, \ldots, x_{n-1}^* \;\middle|\; \sum_{i=1}^{n-1} \ell k(\gamma_v, x_i)\cdot x_i^* = 0\,,\; v \in \mathcal{N} \right\rangle .$$

Now, the presentation matrix $\mathcal{P}_{\mathcal{M}} = (\ell k(\gamma_v, x_i))$ of $\mathcal{M}$ is equivalent to

$$v\in\mathcal{N}\left\{\left(\overbrace{\quad \frac{\ell k(\gamma_v,\gamma_w)}{\alpha_w} \quad}^{w\in\mathcal{N}} \quad \overbrace{\quad 0 \quad}^{n-1-\#\mathcal{N}} \right)\right. =: (\,\widetilde{\mathcal{P}}_{\mathcal{M}} \quad 0\,)\,.$$

For $v \in \mathcal{N}$, let us denote by $\alpha(v)$ the product of the weights of all the edges around v that are not components of the Seifert multilink corresponding to v. Finally, for all connecting edge $E \in \mathcal{E}$ (that is, for all edge corresponding to a splicing operation), let us denote by Δ_E the determinant of the splicing.

CLAIM: $\det \widetilde{\mathcal{P}}_{\mathcal{M}} = \prod_{E\in\mathcal{E}} \Delta_E \prod_{v\in\mathcal{N}} \alpha(v)$.
Let us prove this claim by induction on $k = \#\mathcal{N} \geq 1$. It is clearly true for $k=1$; let us suppose that it is true up to $k-1$. With the notations of the figure in the proof of proposition 5.2.6, and setting

$$\widetilde{\mathcal{P}}_{\mathcal{M}'} = {}^{k-2}\left\{\begin{pmatrix} \overbrace{\quad * \quad}^{k-2} & \mathtt{v} \\ \mathtt{w} & \alpha'_{r+1}\cdots\alpha'_k \end{pmatrix}\right.,$$

we know that $\widetilde{\mathcal{P}}_{\mathcal{M}}$ is given by

$$\begin{pmatrix} * & \alpha'_0\cdot\mathtt{v} & \alpha_r\cdots\alpha_s\alpha(v'')\cdot\mathtt{v} \\ \mathtt{w} & \alpha'_0\cdot\alpha'_{r+1}\cdots\alpha'_k & \alpha_r\cdots\alpha_s\alpha'_{r+1}\cdots\alpha'_k\alpha(v'') \\ \frac{1}{\alpha'_0}\alpha_{s+1}\cdots\alpha_n\alpha(v'')\cdot\mathtt{w} & \alpha_{s+1}\cdots\alpha_n\alpha(v'')\cdot\alpha'_{r+1}\cdots\alpha'_k & \alpha''_0\alpha(v'') \end{pmatrix}.$$

Computing the determinant by the last column, we get

$$\begin{aligned} \det\widetilde{\mathcal{P}}_{\mathcal{M}} &= \alpha''_0\alpha(v'')\cdot\alpha'_0\cdot\det\widetilde{\mathcal{P}}_{\mathcal{M}'} - \alpha_r\cdots\alpha_s\alpha'_{r+1}\cdots\alpha'_k\alpha(v'')\cdot\alpha_{s+1}\cdots\alpha_n\alpha(v'')\cdot\det\widetilde{\mathcal{P}}_{\mathcal{M}'} \\ &= \alpha(v'')\cdot(\alpha'_0\alpha''_0 - \alpha_r\cdots\alpha_n\alpha'_{r+1}\cdots\alpha'_k\alpha(v''))\cdot\det\widetilde{\mathcal{P}}_{\mathcal{M}'} \\ &= \alpha(v'')\cdot\Delta_E\cdot\det\widetilde{\mathcal{P}'_{\mathcal{M}}} \end{aligned}$$

and we are done by induction.

Using the claim and the fact that the splice diagram is minimal, we see that $\widetilde{\mathcal{P}}_{\mathcal{M}}$ is invertible over $\mathbb{Q}$. (In fact, $\widetilde{\mathcal{P}}_{\mathcal{M}}$ is invertible over $\mathbb{Q}$ if and only if the splice diagram is minimal.) Hence, $\mathcal{M} \otimes \mathbb{Q} \simeq \mathbb{Q}[t,t^{-1}]^{\,n-1-\#\mathcal{N}}$. Tensoring our exact sequence by $\mathbb{Q}$, we get

$$0 \longrightarrow \bigoplus_{v\in\mathcal{N}} \mathbb{Q}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v \longrightarrow \mathrm{H}_1(\widetilde{X};\mathbb{Q}) \longrightarrow \mathbb{Q}[t,t^{-1}]^{\,n-1-\#\mathcal{N}} \longrightarrow 0\,.$$

Hence,

$$\begin{aligned}\mathrm{H}_1(\widetilde{X};\mathbb{Q}) &\simeq \bigoplus_{v\in\mathcal{N}} \mathbb{Q}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v \,\oplus\, \mathbb{Q}[t,t^{-1}]^{\,n-1-\#\mathcal{N}} \\ &= \bigoplus_{v\in\mathcal{N}_1} \mathbb{Q}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v \,\oplus\, \mathbb{Q}[t,t^{-1}]^{\,n-1-\#\mathcal{N}_1}\,.\end{aligned}$$ □

5.3 The Rank of the Alexander Module

We are now ready to give a formula for the rank over $\mathbb{Z}[t,t^{-1}]$ of the Alexander module $\mathcal{M}_{\mathbb{Z}}^{L(\underline{m})}$ of a graph multilink $L(\underline{m})$.

Let Γ be a minimal splice diagram of $L(\underline{m})$. Each node of Γ corresponds to a Seifert multilink, fibered or non-fibered; we will speak of **fibered nodes** and **non-fibered nodes**. Also, recall that a node is a 0-node if all the multiplicities of the corresponding Seifert multilink are zero. Consider the sub-diagram Γ^{NF} of Γ generated by the non-fibered nodes, that is, desplice Γ along the edges connecting a fibered node with a non-fibered one, and delete all the connected components that correspond to fibered multilinks.

5.3.1 Theorem.
The rank over $\mathbb{Z}[t,t^{-1}]$ of the Alexander module of a graph multilink $L(\underline{m})$ is given by

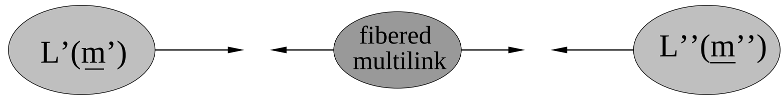
$$rk\,\mathcal{M}_{\mathbb{Z}}^{L(\underline{m})} = \begin{cases} n & \text{if } \underline{m}=\underline{0}; \\ n-r-k+c-1 & \text{else,} \end{cases}$$

where c denotes the number of connected components of Γ (that is, the number of irreducible components of L), r the number of connected components of Γ^{NF}, n the number of arrowheads in Γ^{NF} and k the number of non-fibered nodes that are not 0-nodes.

Proof. If $\underline{m}=\underline{0}$, $\mathcal{M}_{\mathbb{Z}}^{L(\underline{m})}$ is free of rank n, the number of components of L. So, let us assume that $\underline{m}\neq\underline{0}$. We will proceed by double induction on c and r.

• Let us suppose that $L(\underline{m})$ is irreducible (that is, $c=1$). If $r=1$, $L(\underline{m})$ is obtained by splicing a purely non-fibered irreducible graph multilink $L'(\underline{m}')$ (with $\underline{m}'\neq\underline{0}$) with several fibered multilinks. Since the rank of the Alexander module of a fibered multilink is zero, it follows from corollary 5.2.7 and proposition 3.2.4 that $\mathrm{rk}\,\mathcal{M}_{\mathbb{Z}}^{L(\underline{m})} = n-1-k$.

Let us now fix an irreducible graph multilink $L(\underline{m})$ with $r\geq 2$, and let us assume that the formula holds for any irreducible graph multilink with $r'<r$. Then, $L(\underline{m})$ is the result of a splicing of the form illustrated below, with $r'=r-1$, $r''=1$, $\underline{m}'\neq\underline{0}$ and $\underline{m}''\neq\underline{0}$.



Using proposition 3.2.4, the case $r = 1$ and the induction hypothesis, we get the formula

$$\begin{aligned} \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L(\underline{m})} &= \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L'(\underline{m}')} + \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L''(\underline{m}'')} = (n' - (r-1) - k') + (n'' - 1 - k'') \\ &= (n' + n'') - r - (k' + k'') = n - r - k. \end{aligned}$$

This proves the case $c = 1$.

• Let us now fix some $c \geq 2$ and assume that the formula holds for any graph multilink with strictly less then c irreducible components. Then, $L(\underline{m})$ is the disjoint sum of $L'(\underline{m}')$ and $L''(\underline{m}'')$, where $L'(\underline{m}')$ has $c-1$ irreducible components, $L''(\underline{m}'')$ is irreducible, and $\underline{m}' \neq \underline{0}$. If $\underline{m}'' = \underline{0}$, the second part of proposition 1.2.9 and the induction hypothesis give

$$\begin{aligned} \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L(\underline{m})} &= \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L'(\underline{m}')} + \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L''(\underline{m}'')} = (n' - r' - k' + (c-1) - 1) + n'' \\ &= (n' + n'') - (r' + 1) - k' + c - 1 = n - r - k + c - 1. \end{aligned}$$

If $\underline{m}'' \neq \underline{0}$, the first part of proposition 1.2.9, the induction hypothesis and the case $c = 1$ give

$$\begin{aligned} \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L(\underline{m})} &= \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L'(\underline{m}')} + \operatorname{rk} \mathcal{M}_{\mathbb{Z}}^{L''(\underline{m}'')} + 1 \\ &= (n' - r' - k' + (c-1) - 1) + (n'' - r'' - k'') + 1 \\ &= (n' + n'') - (r' + r'') - (k' + k'') + c - 1 = n - r - k + c - 1. \end{aligned}$$

This concludes the proof. □

5.4 Novikov Homology

In 1981, S. P. Novikov [32] introduced Novikov homology in order to construct a generalization of Morse theory valid not only for exact 1-forms on a smooth manifold X, but also for closed ones. In the case of a non-exact integral 1-form, its cohomology class $\mu \in \mathrm{H}^1(X, \mathbb{Z})$ determines an infinite cyclic covering $\widetilde{X}_\mu \longrightarrow X$, and thus, $\mathbb{Z}[t, t^{-1}]$-modules $\mathrm{H}_* \widetilde{X}_\mu$. The Novikov homology of X with respect to μ is given by $\mathrm{H}_* \widetilde{X}_\mu \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[\![t]\!][t^{-1}]$.

If X is the exterior of a link in a homology sphere, we get the following definition: the **Novikov homology** of a multilink $L(\underline{m})$ is the $\mathbb{Z}[\![t]\!][t^{-1}]$-module

$$\widehat{H}_L(\underline{m}) = \mathrm{H}_1 \widetilde{X}(\underline{m}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[\![t]\!][t^{-1}].$$

The ring $\mathbb{Z}[\![t]\!][t^{-1}]$ is called the **Novikov ring**; we will denote it by $\widehat{\Lambda}$. Recall that $\widehat{\Lambda}$ is a principal ring, and that an element of $\widehat{\Lambda}$ is a unit if and only if its lowest coefficient is ± 1.

5.4.1 Example. Let $L(\underline{m})$ be a fibered multilink. Then, the lowest coefficient of $\Delta^{L(\underline{m})}$ is equal to $\det A_F^- = \pm 1$, so the Alexander polynomial is a unit of the ring $\widehat{\Lambda}$. Hence, any Alexander matrix is invertible over $\widehat{\Lambda}$, giving $\widehat{H}_L(\underline{m}) = 0$.

In this section, we compute the Novikov homology of a graph multilink via a splice diagram. As a corollary, we give a majoration for the number of Novikov modules on a given graph link.

Let $L(\underline{m})$ be a graph multilink given by a splice diagram Γ. Let us recall the notations and terminology of the previous section. Each node of Γ corresponds to a Seifert multilink, fibered or non-fibered; we speak of fibered and non-fibered nodes. Also, we say that a node is a 0-node if all the multiplicities of the corresponding Seifert multilink are zero. Consider the sub-diagram Γ^{NF} of Γ generated by the non-fibered nodes, that is, desplice Γ along the edges connecting a fibered node with a non-fibered one, and delete all the connected components that correspond to fibered multilinks. We note c the number of connected components of Γ, r the number of connected components of Γ^{NF}, n the number of arrowheads in Γ^{NF} and k the number of its nodes that are not 0-nodes. For every such node v, let us note α_v the product of the weights on the arrowhead edges in Γ^{NF} adjacent to v.

5.4.2 Theorem.
Let $L(\underline{m})$ be a graph multilink (with $\underline{m} \neq \underline{0}$) given by a splice diagram Γ. Then, its Novikov homology is equal to

$$\widehat{H}_L(\underline{m}) = \mathcal{T} \oplus \widehat{\Lambda}^{n-r-k+c-1},$$

where $\mathcal{T}$ is the $\widehat{\Lambda}$-module presented by the $(k \times k)$-matrix $\mathcal{P} = (p_{vw})$, where $p_{vw} = \frac{\ell k(\gamma_v, \gamma_w)}{\alpha_w}$ if v and w are in the same connected component of Γ^{NF}, and $p_{vw} = 0$ otherwise.
Furthermore, if Γ is minimal, then $\mathcal{T}$ is the torsion submodule of $\widehat{H}_L(\underline{m})$, and $n-r-k+c-1$ is the rank.

Proof. First of all, it may be assumed that Γ is minimal: this is done by checking that if Γ and Γ' are related by a reduction of type (i) or (ii) (recall figure 4.2), then the Novikov homology predicted by the theorem are equal. This argument is very similar to the beginning of the proof of theorem 5.2.2, and thus omitted. Then, we can assume that Γ is connected (that is, $c = 1$), using proposition 1.2.9; here, the computation is exactly the same as at the end of the proof of theorem 5.3.1.

So, let us suppose that Γ is a connected, minimal splice diagram. We will proceed by induction on $r \geq 0$, the number of connected components of Γ^{NF}. Of course, r is zero if and only if $L(\underline{m})$ is fibered; in this case, the Novikov homology is trivial and the theorem is true. Now, let us suppose that $r = 1$; this means that $L(\underline{m})$ is made of one purely non-fibered multilink L^{NF} spliced with fibered multilinks. Proposition 5.2.6 gives the Alexander module of L^{NF} over $\mathbb{Z}[t, t^{-1}]$; let us denote by $\mathcal{P}^{NF}$ the matrix described in this theorem. We need to understand the effect on the Novikov homology of splicing with fibered multilinks.

By lemma 5.2.1, and since Γ is minimal, these splicings will be of the type $m > 0$. Recall that, by proposition 3.2.8, if $L(\underline{m}) = L'(\underline{m}')\underset{m'L' \quad m''L''}{\overline{\qquad\qquad}} L''(\underline{m}'')$ with $\gcd(m', m'') = m > 0$, and if $\mathcal{P}'$ (resp. $\mathcal{P}''$) is a presentation matrix of $\mathrm{H}_1\widetilde{X}'$ (resp. $\mathrm{H}_1\widetilde{X}''$), a presentation matrix of $\mathrm{H}_1\widetilde{X}$ is given by

$$\begin{pmatrix} \mathcal{P}' & 0 & 0 \\ 0 & \mathcal{P}'' & 0 \\ \ell k(i_+T', -) & \ell k(i_+T'', -) & 0 \\ * & * & \frac{(t^m-1)(t^d-1)}{(t^{d'}-1)(t^{d''}-1)} \end{pmatrix}.$$

Working over $\mathbb{Z}[t,t^{-1}]$, this is not a very satisfactory formula; but now, we are working over the Novikov ring $\widehat{\Lambda}$, so the polynomial $p(t) = \frac{(t^m-1)(t^d-1)}{(t^{d'}-1)(t^{d''}-1)}$ is a unit. Hence, the above matrix is equivalent to

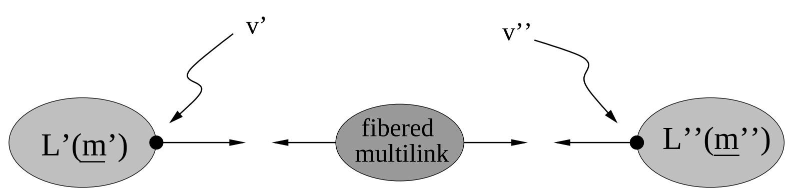
$$\mathcal{P} = \begin{pmatrix} \mathcal{P}' & 0 \\ 0 & \mathcal{P}'' \\ \ell k(i_+T',-) & \ell k(i_+T'',-) \end{pmatrix}.$$

Let us apply this formula to the case we are interested in, that is: $L'(\underline{m}') = L^{NF}$ and $L''(\underline{m}'')$ fibered. If the splicing corresponds to an edge linking v to some fibered node, a presentation matrix is given by

$$\mathcal{P} \sim \begin{pmatrix} \mathcal{P}^{NF} & 0 \\ 0 & I \\ \ell k(\gamma_v,-) & * \end{pmatrix} \sim \begin{pmatrix} \mathcal{P}^{NF} \\ \ell k(\gamma_v,-) \end{pmatrix}.$$

Now, the ligne of $\mathcal{P}^{NF}$ corresponding to v is equal to $\ell k(\gamma_v,-)\cdot(t^{d_v}-1)$. Since v is not a 0-node, this is equivalent (over $\widehat{\Lambda}$) to $\ell k(\gamma_v,-)$. Therefore, the operation of splicing a given purely non-fibered multilink with a fibered multilink does not change the Novikov homology. In other words, the Novikov homology of $L(\underline{m})$ (with $r=1$) is equal to the Novikov homology of L^{NF}. By remark 5.2.3, the Alexander module over $\mathbb{Z}[t,t^{-1}]$ of L^{NF} is equal to $\widetilde{\mathcal{T}} \oplus \mathbb{Z}[t,t^{-1}]^{n-k-1}$, where $\widetilde{\mathcal{T}}$ is presented by the $(k\times k)$-matrix $\left(\frac{\ell k(\gamma_v,\gamma_w)}{\alpha_w}\cdot(t^{d_v}-1)\right)$. Since $d_v \neq 0$, the Novikov homology is given by $\mathcal{T} \oplus \widehat{\Lambda}^{n-k-1}$, where $\mathcal{T}$ is presented by $\left(\frac{\ell k(\gamma_v,\gamma_w)}{\alpha_w}\right)$. This settles the case $r=1$.

Let us now suppose that the theorem holds for any graph multilink with Γ^{NF} having at most $r-1$ connected components, and let us consider $L(\underline{m})$ with Γ^{NF} having $r \geq 2$ connected components and k non-0-nodes. Since $\Gamma = \Gamma(L(\underline{m}))$ is a connected tree, it is always possible to desplice it as illustrated below, where v' and v'' are non-fibered non-0-nodes, Γ'^{NF} has $r-1$ connected components and k' non-0-nodes, and Γ''^{NF} is connected with $k'' = k-k'$ non-0-nodes.



By induction and by the case $r=1$, we know the Novikov homology of $L'(\underline{m}')$ and $L''(\underline{m}'')$; let us denote presentation matrices by $\mathcal{P}'$ and $\mathcal{P}''$. Applying the same method as in the case $r=1$, we get the presentation matrix

$$\begin{pmatrix} \mathcal{P}' & 0 & 0 \\ 0 & I & 0 \\ \ell k(\gamma_{v'},-) & * & 0 \\ 0 & 0 & \mathcal{P}'' \\ \ell\cdot\ell k(\gamma_{v'},-) & * & \ell k(\gamma_{v''},-) \end{pmatrix} \sim \begin{pmatrix} \mathcal{P}' & 0 & 0 \\ 0 & I & 0 \\ \ell k(\gamma_{v'},-) & * & 0 \\ 0 & 0 & \mathcal{P}'' \\ 0 & * & \ell k(\gamma_{v''},-) \end{pmatrix} \sim \begin{pmatrix} \mathcal{P}' & 0 \\ \ell k(\gamma_{v'},-) & 0 \\ 0 & \mathcal{P}'' \\ 0 & \ell k(\gamma_{v''},-) \end{pmatrix},$$

which is equivalent to $\mathcal{P}' \oplus \mathcal{P}''$. Since the set of non-0-nodes of Γ is equal to the disjoint union of the sets of non-0-nodes of Γ' and of Γ'', and since

$$(n'-(r-1)-k') + (n''-1-k'') = (n'+n'') - r - (k'+k'') = n-r-k,$$

we are done by induction.

Finally, we know by theorem 5.3.1 that if Γ is minimal, the rank of the Alexander module over $\mathbb{Z}[t,t^{-1}]$ is equal to $n-r-k+c-1$. This implies that the rank of $\widehat{H}_L(\underline{m})$ is also equal to $n-r-k+c-1$. Since the ring $\widehat{\Lambda}$ is principal, it follows that $\mathcal{T}$ is the torsion submodule of $\widehat{H}_L(\underline{m})$. □

5.4.3 Corollary.
Let L be a graph link with n components. Then, there exists a finite number of hyperplanes in $\mathbb{Z}^n$, defined by homogeneous linear equations over the integers, such that the Novikov homology $\widehat{H}_L(\underline{m})$ only depends on which hyperplanes $\underline{m}$ belongs.

Proof. Given a graph link L, its Novikov homology depends only on whether the nodes are fibered, non-fibered or 0-nodes. These conditions correspond to hyperplanes (or intersections of hyperplanes for 0-nodes) in $\mathbb{Z}^n$. □

See [34, Theorem 3] for an analogous result in another context.

5.4.4 Corollary.
Let L be a graph link with c irreducible components and b splice components. Then, the number of non-isomorphic Novikov modules (corresponding to the different multiplicities) is bounded above by $3^b - 2(b-c)$.

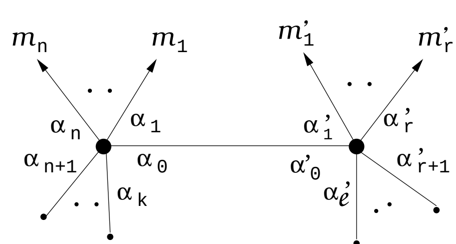
Proof. Consider a minimal splice diagram Γ for L; this splice diagram has c connected components and b nodes. The module $\widehat{H}_L(\underline{m})$ only depends on whether the nodes are fibered, non-fibered or 0-nodes: this gives 3^b possibilities. But a node adjacent to a 0-node cannot be fibered: this takes away $2 \cdot \#\{\text{ splicing }\} = 2(b-c)$ possible cases. □

5.4.5 Example. The general Seifert multilink, as given in figure 4.1, has the following Novikov homology

$$\widehat{H}_L((\underline{m})) = \begin{cases} 0 & \text{if } \underline{m} \in \mathbb{Z}^n - V; \\ \hat{\Lambda}/(\alpha_{n+1}\cdots\alpha_k) \oplus \hat{\Lambda}^{n-2} & \text{if } \underline{m} \in V - \{0\}; \\ \hat{\Lambda}^n & \text{if } \underline{m} = 0, \end{cases}$$

where $V = \{(m_1,\ldots,m_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n m_i\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_k = 0\}$.

5.4.6 Example. Let us study the general graph link with two Seifert splice components, as already described.



Consider the following hyperplanes in $\mathbb{Z}^{n+r}$:

$$\begin{aligned} H &= V\Big(\sum_{i=1}^{n} m_i\alpha_0\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_k \;+\; \sum_{j=1}^{r} m'_j\alpha_1\cdots\alpha_k\alpha'_1\cdots\widehat{\alpha'_j}\cdots\alpha'_\ell\Big) \\ H' &= V\Big(\sum_{i=1}^{n} m_i\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_k\alpha'_1\cdots\alpha'_\ell \;+\; \sum_{j=1}^{r} m'_j\alpha'_0\alpha'_1\cdots\widehat{\alpha'_j}\cdots\alpha'_\ell\Big). \end{aligned}$$

Also, let us note

$$\begin{aligned} E &= V(m_1)\cap\ldots\cap V(m_n)\cap V\Big(\sum_{j=1}^{r} m'_j\alpha'_1\cdots\widehat{\alpha'_j}\cdots\alpha'_\ell\Big)\subset H \\ E' &= V(m'_1)\cap\ldots\cap V(m'_r)\cap V\Big(\sum_{i=1}^{n} m_i\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_k\Big)\subset H'. \end{aligned}$$

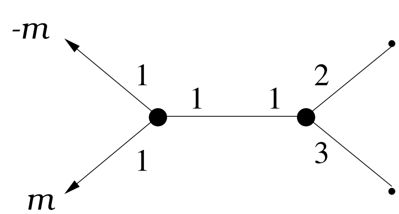
Then, the Novikov homology is given by

$$\widehat{H}_L(\underline{m}) = \begin{cases} 0 & \text{if } \underline{m}\in\mathbb{Z}^{n+r}-(H\cup H'); \\ \hat{\Lambda}/(\alpha_{n+1}\cdots\alpha_k)\;\oplus\;\hat{\Lambda}^{n-1} & \text{if } \underline{m}\in H-(H\cap H'); \\ \hat{\Lambda}/(\alpha'_{r+1}\cdots\alpha'_\ell)\;\oplus\;\hat{\Lambda}^{r-1} & \text{if } \underline{m}\in H'-(H\cap H'); \\ \mathcal{P}\;\oplus\;\hat{\Lambda}^{n+r-3} & \text{if } \underline{m}\in H\cap H'-(E\cup E'); \\ \hat{\Lambda}/(\alpha'_{r+1}\cdots\alpha'_\ell\cdot a)\;\oplus\;\hat{\Lambda}^{n+r-2} & \text{if } \underline{m}\in E-\{\underline{0}\}; \\ \hat{\Lambda}/(\alpha_{n+1}\cdots\alpha_k\cdot a')\;\oplus\;\hat{\Lambda}^{n+r-2} & \text{if } \underline{m}\in E'-\{\underline{0}\}; \\ \hat{\Lambda}^{n+r} & \text{if } \underline{m}=\underline{0}, \end{cases}$$

where $a=\gcd(\alpha'_0,\alpha_{n+1}\cdots\alpha_k)$, $a'=\gcd(\alpha_0\alpha'_{r+1}\cdots\alpha'_\ell)$ and $\mathcal{P}$ is the $\hat{\Lambda}$-module presented by

$$\begin{pmatrix} \alpha_0\alpha_{n+1}\cdots\alpha_k & \alpha_1\cdots\alpha_k\alpha'_{r+1}\cdots\alpha'_\ell \\ \alpha_{n+1}\cdots\alpha_k\alpha'_1\cdots\alpha'_\ell & \alpha'_0\alpha'_{r+1}\cdots\alpha'_\ell \end{pmatrix}.$$

5.4.7 Example. It is false to believe that a graph multilink $L(\underline{m})$ is fibered exactly when $\widehat{H}_L(\underline{m})$ is trivial. Indeed, the following splice diagram represents a non-fibered link in S^3 whose Alexander module is equal to the Alexander module of the Hopf link; in particular, its Novikov homology vanishes. This example also shows that a non-fibered graph link in S^3 can have unimodular Seifert forms (compare with corollary 4.3.5).



5.5 On the Alexander Module over $\mathbb{C}[t,t^{-1}]$

One of the main goals of Eisenbud and Neumann was to "find the Jordan normal form of the monodromy from the characteristic pairs" [11, p. 9]. Equivalently, the problem is to compute the Alexander module over $\mathbb{C}[t,t^{-1}]$ of an algebraic multilink from a splice diagram. This question is "completely settled for algebraic links, though the obvious generalizations to iterated torus links and graph links are much less satisfactorily resolved" [11, p. 10]. In this section, we give a closed formula for the Alexander module over $\mathbb{C}[t,t^{-1}]$ of a very large class of graph multilinks (see the restriction below). In particular, the regular link at infinity of any $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ is in this class.

Let us first recall Eisenbud and Neumann's result in the fibered case. Let $L(\underline{m})$ be a fibered graph multilink with monodromy h given by a minimal splice diagram Γ. Let us denote by $\mathcal{N}$ the set of nodes of Γ, by $\mathcal{E}$ the set of edges connecting two nodes, and by $\mathcal{V}$ the set of non-arrowhead vertices of Γ. For every node $v \in \mathcal{N}$, let us note d_v the greatest common divisor of the multiplicities of the corresponding Seifert multilink. Finally, for every edge $E \in \mathcal{E}$, let d_E be the greatest common divisor of the multiplicities involved in the corresponding splicing.

5.5.1 Theorem. (Eisenbud-Neumann)
The Alexander module $\mathcal{M}_{\mathbb{C}}^{L(\underline{m})}$ is determined by the following properties.

- *The Jordan normal form of h_* consists of 1×1 and 2×2 Jordan blocks.*
- *The characteristic polynomial of h_* is equal to*

$$\Delta(t) = (t^d - 1) \prod_{v \in \mathcal{V}} (t^{|\underline{m}(v)|} - 1)^{\delta_v - 2}.$$

- *The eigenvalues corresponding to the 2×2 Jordan blocks are the roots of $\Delta'(t)$, where $\Delta'(t)$ is equal to*

$$q(t) = (t^d - 1) \frac{\prod_{E \in \mathcal{E}} (t^{d_E} - 1)}{\prod_{v \in \mathcal{N}} (t^{d_v} - 1)}$$

if $L(\underline{m})$ has "uniform twists", and $\Delta'(t)$ is a quotient of $q(t)$ in general.[2] □

Now, let $L(\underline{m})$ be a graph multilink (with $\underline{m} \neq \underline{0}$) given by a minimal splice diagram Γ. Via proposition 1.2.9, it may be assumed that Γ is connected (that is, $L(\underline{m})$ is irreducible). Let us desplice Γ along $\mathcal{E}'$, the set of edges connecting a fibered node and a non-fibered one. This defines $\Gamma^{NF} = \bigsqcup_{i=1}^{r} \Gamma_i^{NF}$ purely non-fibered with n arrowhead edges, and Γ^F corresponding to fibered multilinks.

RESTRICTION: We will assume that Γ^F is connected.

Let us note d the greatest common divisor of the multiplicities in Γ^F, ℓ the number of arrowhead edges in Γ^F, $\mathcal{V}$ the set of non-arrowhead vertices of Γ^F, and $\mathcal{N}_1$ the set of non-fibered nodes that are not 0-nodes. Finally, let us note

$$\epsilon(t) = \begin{cases} \gcd\left(t^d - 1, \{\frac{t^{d_E}-1}{t^{d_v}-1}\}_{(v,E)} \ ; \ E \in \mathcal{E}' \text{ is adjacent to } v \in \mathcal{N}_1\right) & \text{if } \#\mathcal{E}' = \ell; \\ 1 & \text{if } \#\mathcal{E}' < \ell. \end{cases}$$

5.5.2 Theorem.
The Alexander module $\mathcal{M}_{\mathbb{C}}^{L(\underline{m})}$ is determined by the following properties.

[2] Every algebraic multilink has uniform twists, as well as every multilink at infinity.

- *The rank of* $\mathcal{M}_{\mathbb{C}}^{L(\underline{m})}$ *is equal to* $n - \#\mathcal{N}_1 - r$.
- *The order ideal of the torsion submodule* $\mathcal{T} = Tor\, \mathcal{M}_{\mathbb{C}}^{L(\underline{m})}$ *is generated by*

$$\widetilde{\Delta}(t) = (t^d - 1) \cdot \epsilon(t) \cdot \frac{\prod_{v \in \mathcal{V}} (t^{|\underline{m}(v)|} - 1)^{\delta_v - 2} \prod_{v \in \mathcal{N}_1} (t^{d_v} - 1)}{\prod_{E \in \mathcal{E}'} (t^{d_E} - 1)}.$$

- *The Jordan normal form of* t *restricted to* $\mathcal{T}$ *has blocks of dimension at most* 2*; the eigenvalues corresponding to the* 2×2 *blocks are the roots of* $\Delta'(t)$.

Before starting the proof of the theorem, let us recall an important fact of chapter 1. The Alexander module $\mathrm{H}_1\widetilde{X}$ of a multilink $L(\underline{m})$ can be understood as the quotient of $\mathrm{H}_1(\Sigma - \overline{F}) \otimes \mathbb{Z}[t, t^{-1}]$ by the relation $i_+\alpha = t \cdot i_-\alpha$ for $\alpha \in \mathrm{H}_1 F$, where F is any good Seifert surface for $L(\underline{m})$. In particular, an element $T \in \mathrm{H}_1 F$ gives an element $i_+ T \in \mathrm{H}_1 \widetilde{X}$ via the morphism

$$\mathrm{H}_1 F \xrightarrow{i_+} \mathrm{H}_1(\Sigma - \overline{F}) \hookrightarrow \mathrm{H}_1(\Sigma - \overline{F}) \otimes \mathbb{Z}[t, t^{-1}] \twoheadrightarrow \mathrm{H}_1 \widetilde{X}.$$

Furthermore, if $L(\underline{m})$ is fibered with monodromy $F \xrightarrow{h} F$, i_+ is an isomorphism and the sequence of morphisms given above fits into the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}_1 F & \longrightarrow & \mathrm{H}_1 F \otimes \mathbb{Z}[t,t^{-1}] & \longrightarrow & (\mathrm{H}_1 F \otimes \mathbb{Z}[t,t^{-1}])/(t \cdot x = h_*(x)) \\ {\scriptstyle i_+}\downarrow \simeq & & \downarrow \simeq & & {\scriptstyle \varphi}\downarrow \simeq \\ \mathrm{H}_1(\Sigma - \overline{F}) & \longrightarrow & \mathrm{H}_1(\Sigma - \overline{F}) \otimes \mathbb{Z}[t,t^{-1}] & \longrightarrow & (\mathrm{H}_1(\Sigma - \overline{F}) \otimes \mathbb{Z}[t,t^{-1}])/(i_+\alpha = t \cdot i_-\alpha), \end{array}$$

where φ is an isomorphism of $\mathbb{Z}[t, t^{-1}]$-modules.

The main tool for the proof of the theorem is the following lemma.

5.5.3 Lemma.
Let $L(\underline{m})$ *be the splice* $L'(\underline{m}')\underset{m'L'_0 \quad m''L''_0}{\overline{\qquad\qquad}}L''(\underline{m}'')$ *where* $L''(\underline{m}'')$ *is a purely non-fibered graph multilink and* $(m', m'') \neq (0, 0)$*. If* $\widetilde{X}$ *(resp.* $\widetilde{X}'$, $\widetilde{X}''$*) denotes the total space of the infinite cyclic covering associated with* $L(\underline{m})$ *(resp.* $L'(\underline{m}')$, $L''(\underline{m}'')$*), then*

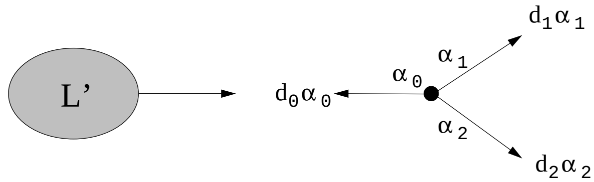
$$H_1(\widetilde{X}; \mathbb{Q}) \;\simeq\; \left(H_1(\widetilde{X}'; \mathbb{Q}) \oplus H_1(\widetilde{X}''; \mathbb{Q}) \right) \Big/ \; i_+ T' + i_+ T'' \;,$$

where T' *(resp.* T''*) is a boundary component of a Seifert surface for* $L'(\underline{m}')$ *near* L'_0 *(resp. for* $L''(\underline{m}'')$ *near* L''_0*).*

Proof. Using lemma 5.2.1 and proposition 3.2.1, the general case can be reduced to the case where $L''(\underline{m}'')$ is a non-fibered Seifert multilink $L''(\underline{m}'') = (\Sigma''(\alpha''_1, \ldots, \alpha''_k), m''_1 L''_1 \cup \ldots \cup m''_n L''_n)$. By proposition 3.2.2, we can further assume that $k = n$. Finally, using proposition 3.2.1 and the fact that the present lemma is true when splicing two non-fibered Seifert multilinks, we just need to prove the lemma for $k = n = 3$, that is, for the splice illustrated below, where $d_0 \neq 0$ and $d_0 + d_1 + d_2 = 0$.

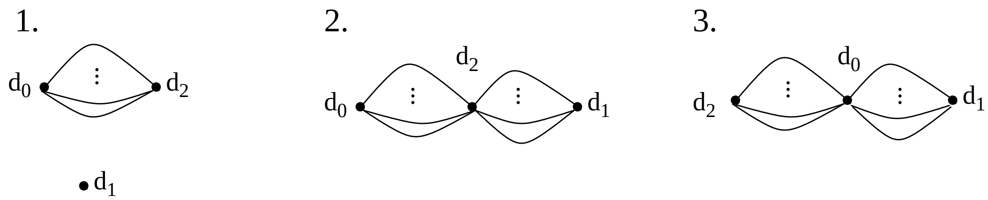
Let us note $d' = \gcd(m'_1, \ldots, m'_r)$, $d = \gcd(d_0, d_1, d_2)$; without loss of generality, it may be assumed that $\gcd(d', d) = 1$. We have to consider three cases:

1. $d_1 = 0$ (or $d_2 = 0$)



2. $\operatorname{sgn}(d_0) = \operatorname{sgn}(d_1)$ (or $\operatorname{sgn}(d_2)$)
3. $\operatorname{sgn}(d_0) = -\operatorname{sgn}(d_1) = -\operatorname{sgn}(d_2)$

The corresponding vertical Seifert surfaces in the orbit space are as follows.



The first case is trivial: we just need to use proposition 3.2.5, and to check that in this case, the polynomial $p(t)$ is equal to 1. Here, the result is even true over $\mathbb{Z}[t,t^{-1}]$. For the second case, the proof is quite tedious; let us give a brief sketch. Given a Seifert surface F' (with d' connected components) for $L'(\underline{m}')$, one can construct a good Seifert surface F for $L(\underline{m})$ as follows:

(i) paste a vertical Seifert surface for $L''(\underline{m}'')$ (made of $|d_2|$ rings) with F' along the $|d_0|$ obvious boundary components;

(ii) make $d' + d_1 - 1$ surgeries, in order to get a connected Seifert surface.

It is not difficult to find a natural basis of $\mathrm{H}_1(F;\mathbb{Q})$ (resp. $\mathrm{H}_1(\overline{F};\mathbb{Q})$) from a basis of $\mathrm{H}_1(F';\mathbb{Q})$ (resp. $\mathrm{H}_1(\overline{F}';\mathbb{Q})$). The crucial point here is that we don't create any big cycle by pasting rings with F'.[3] Then, one can compute the Seifert matrices associated with these surfaces; using theorem 1.2.7, it is an easy (but long) exercice to check that the lemma is true. The third case is very similar. □

Proof of the theorem. If $\Gamma = \Gamma^F$, the multilink is fibered and the result is given by theorem 5.5.1. Let us now suppose that the ℓ-component multilink given by Γ^F is spliced along r components (for some $1 \leq r \leq \ell$) to purely non-fibered multilinks. Let us note $\mathcal{E}' = \{E_1, \ldots, E_r\}$ the edges of Γ corresponding to these splicings, and v_i the non-fibered node adjacent to E_i. By lemma 5.2.1, d_{E_i} is not equal to zero, so we can apply lemma 5.5.3 r times. This leads to the isomorphism

$$\mathcal{M}_{\mathbb{C}} \simeq \left(\mathcal{M}_{\mathbb{C}}^F \oplus \mathcal{M}_{\mathbb{C}}^1 \oplus \ldots \oplus \mathcal{M}_{\mathbb{C}}^r\right) / \; i_+T_j + \gamma_{v_j} \;\; (j = 1, \ldots, r)\,,$$

[3]Unfortunately, such big cycle appear when splicing a non-fibered Seifert multilink with several fibered ones; this is the reason of the restriction in the statement of theorem 5.5.2.

where $\mathcal{M}_{\mathbb{C}}$ (resp. $\mathcal{M}^F_{\mathbb{C}}$, $\mathcal{M}^i_{\mathbb{C}}$) denotes the Alexander module of the multilink given by Γ (resp. Γ^F, Γ^{NF}_i), and where γ_{v_i} denotes the generic Seifert fiber of the Seifert fibration corresponding to the node v_i.

Since Γ^F is fibered, its Alexander module is given by $\mathrm{H}_1 F$, with t acting as the monodromy h_*. Furthermore, the monodromy permutes cyclically the d_{E_i} border components of the fiber near L_i, for $i = 1, \ldots, \ell$. If $d = 1$, we know by lemma 2.1.2 that $\mathrm{H}_1 F$ factors into $\widetilde{H} \oplus B$, where $B = \bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{d_{E_i}} \mathbb{Z} T^i_j \Big/ \sum_{i,j} T^i_j$. This leads to a factorization of the Alexander module $\mathcal{M}^F_{\mathbb{Z}} = \widetilde{\mathcal{M}}^F_{\mathbb{Z}} \oplus \mathcal{B}^F_{\mathbb{Z}}$, where

$$\mathcal{B}^F_{\mathbb{Z}} = \Big(\bigoplus_{i=1}^{\ell} \mathbb{Z}[t, t^{-1}]/(t^{d_{E_i}} - 1) \cdot T_i \Big) \Big/ \sum_{i=1}^{\ell} \sum_{j=1}^{d_{E_i}} t^j \cdot T_i.$$

Since $\sum_{j=1}^{d_{E_i}} t^j = \frac{t^{d_{E_i}-1}}{t-1}$, and using corollary 1.1.3, we get the factorization $\mathcal{M}^F_{\mathbb{Z}} = \widetilde{\mathcal{M}}^F_{\mathbb{Z}} \oplus \mathcal{B}^F_{\mathbb{Z}}$, where

$$\mathcal{B}^F_{\mathbb{Z}} = \Big(\bigoplus_{i=1}^{\ell} \mathbb{Z}[t, t^{-1}]/(t^{d_{E_i}} - 1) \cdot T_i \Big) \Big/ \sum_{i=1}^{\ell} \frac{t^{d_{E_i}} - 1}{t^d - 1} \cdot T_i.$$

Finally, working over $\mathbb{C}[t, t^{-1}]$, and interpretating the Alexander module as a quotient of $\mathrm{H}_1(\Sigma - \overline{F}) \otimes \mathbb{C}[t, t^{-1}]$, this leads to $\mathcal{M}^F_{\mathbb{C}} = \widetilde{\mathcal{M}}^F_{\mathbb{C}} \oplus \mathcal{B}^F_{\mathbb{C}}$, where

$$\mathcal{B}^F_{\mathbb{C}} = \Big(\bigoplus_{j=1}^{\ell} \mathbb{C}[t, t^{-1}]/(t^{d_{E_j}} - 1) \cdot i_+ T_j \Big) \Big/ \sum_{j=1}^{\ell} \frac{t^{d_{E_j}} - 1}{t^d - 1} \cdot i_+ T_j.$$

Therefore, a presentation matrix of $\mathcal{B}^F_{\mathbb{C}}$ is given by

$$\begin{pmatrix} t^{d_{E_1}} - 1 & & & & \\ & t^{d_{E_2}} - 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & t^{d_{E_\ell}} - 1 \\ \frac{t^{d_{E_1}}-1}{t^d-1} & \frac{t^{d_{E_2}}-1}{t^d-1} & \ldots & \ldots & \frac{t^{d_{E_\ell}}-1}{t^d-1} \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} t^{d_{E_1}} - 1 & & & & & \\ & t^{d_{E_2}} - 1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & t^{d_{E_{\ell-1}}} - 1 & \\ \frac{t^{d_{E_1}}-1}{t^d-1} & \frac{t^{d_{E_2}}-1}{t^d-1} & \ldots & \ldots & \frac{t^{d_{E_{\ell-1}}}-1}{t^d-1} & \frac{t^{d_{E_\ell}}-1}{t^d-1} \end{pmatrix}.$$

This leads to the factorization $\Delta(t) = \widetilde{\Delta}^F(t) \cdot \frac{1}{t^d-1} \prod_{j=1}^{\ell} (t^{d_{E_j}} - 1)$. Also, it is an exercise to check that in fact, $\mathcal{B}^F_{\mathbb{C}}$ is given by

$$\mathcal{B}^F_{\mathbb{C}} \simeq \bigoplus_{j=1}^{n-1} \mathbb{C}[t, t^{-1}] \Big/ \Big(\frac{(t^{d^j} - 1)(t^{d_{E_{j+1}}} - 1)}{(t^{d^{j+1}} - 1)} \Big),$$

where d^j denotes the greatest common divisor of $d_{E_1},\dots,d_{E_j}$. Since $\gcd(d^j, d_{E_{j+1}}) = d^{j+1}$, the polynomial $\frac{(t^{d_j}-1)(t^{d_{E_{j+1}}}-1)}{(t^{d_{j+1}}-1)}$ has only simple roots. Therefore, the submodule $\mathcal{B}_{\mathbb{C}}^F$ contributes only to the one-dimensional blocks of the Jordan normal form of the monodromy.

On the other hand, we know by corollary 5.2.7 that for $j=1,\dots,r$,

$$\mathcal{M}_{\mathbb{C}}^j \simeq \bigoplus_{v\in\mathcal{N}_1(j)} \mathbb{C}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v \oplus \mathbb{C}[t,t^{-1}]^{\,n(j)-1-\#\mathcal{N}_1(j)},$$

where $\mathcal{N}_1(j)$ is the set of non-0-nodes of Γ_j^{NF}, and $n(j)$ the number of its arrowhead edges.

Putting all this together, we have the isomorphism

$$\begin{aligned}
\mathcal{M}_{\mathbb{C}} &\simeq \left(\mathcal{M}_{\mathbb{C}}^F \oplus \mathcal{M}_{\mathbb{C}}^1 \oplus \dots \oplus \mathcal{M}_{\mathbb{C}}^r\right) / \; i_+T_j + \gamma_{v_j} \\
&\simeq \mathbb{C}[t,t^{-1}]^{n-\#\mathcal{N}_1-r} \oplus \widetilde{\mathcal{M}}_{\mathbb{C}}^F \oplus \Big(\mathcal{B}_{\mathbb{C}}^F \oplus \bigoplus_{v\in\mathcal{N}_1} \mathbb{C}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v\Big)\Big/ \; i_+T_j + \gamma_{v_j} \\
&\simeq \mathbb{C}[t,t^{-1}]^{n-\#\mathcal{N}_1-r} \oplus \widetilde{\mathcal{M}}_{\mathbb{C}}^F \oplus \bigoplus_{v\in\mathcal{N}_1-\{v_1,\dots,v_r\}} \mathbb{C}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v \oplus \mathcal{B}_{\mathbb{C}},
\end{aligned}$$

where $\mathcal{B}_{\mathbb{C}}$ is presented by

$$\begin{pmatrix}
t^{d_{E_1}}-1 & & & & \\
& t^{d_{E_2}}-1 & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & t^{d_{E_\ell}}-1 \\
\frac{t^{d_{E_1}}-1}{t^d-1} & \frac{t^{d_{E_2}}-1}{t^d-1} & \dots & \dots & \frac{t^{d_{E_\ell}}-1}{t^d-1} \\
t^{d_{v_1}}-1 & & & & \\
& t^{d_{v_2}}-1 & & & \\
& & \ddots & & \\
& & & t^{d_{v_r}}-1 & \\
& & & & 0
\end{pmatrix}.$$

Since d_{v_j} divides d_{E_j} for all $j=1\dots,r$, it is equivalent to the matrix

$$\begin{pmatrix}
t^{d_{v_1}}-1 & & & & & & \\
& t^{d_{v_2}}-1 & & & & & \\
& & \ddots & & & & \\
& & & t^{d_{v_r}}-1 & & & \\
& & & & t^{d_{E_{r+1}}}-1 & & \\
& & & & & \ddots & \\
& & & & & & t^{d_{E_\ell}}-1 \\
\frac{t^{d_{E_1}}-1}{t^d-1} & \frac{t^{d_{E_2}}-1}{t^d-1} & \dots & \frac{t^{d_{E_r}}-1}{t^d-1} & \frac{t^{d_{E_{r+1}}}-1}{t^d-1} & \dots & \frac{t^{d_{E_\ell}}-1}{t^d-1}
\end{pmatrix}.$$

If $r < \ell$, this matrix is equivalent to the square matrix

$$\begin{pmatrix} t^{d_{v_1}}-1 & & & & & & & \\ & t^{d_{v_2}}-1 & & & & & & \\ & & \ddots & & & & & \\ & & & t^{d_{v_r}}-1 & & & & \\ & & & & t^{d_{E_{r+1}}}-1 & & & \\ & & & & & \ddots & & \\ & & & & & & t^{d_{E_{\ell-1}}}-1 & \\ \frac{t^{d_{E_1}}-1}{t^d-1} & \frac{t^{d_{E_2}}-1}{t^d-1} & \dots & & & \dots & \frac{t^{d_{E_{\ell-1}}}-1}{t^d-1} & \frac{t^{d_{E_\ell}}-1}{t^d-1} \end{pmatrix}.$$

Therefore, the determinant of a presentation matrix of the torsion submodule $\mathcal{T} = \widetilde{\mathcal{M}}^F_{\mathbb{C}} \oplus \bigoplus_{v\in\mathcal{N}_1-\{v_1,\dots,v_r\}} \mathbb{C}[t,t^{-1}]/(t^{d_v}-1)\cdot\gamma_v \oplus \mathcal{B}_{\mathbb{C}}$ is given by

$$\widetilde{\Delta}(t) = \widetilde{\Delta}^F(t)\cdot \prod_{v\in\mathcal{N}_1-\{v_1,\dots,v_r\}} (t^{d_v}-1)\cdot\prod_{j=1}^{r}(t^{d_{v_j}}-1)\cdot\frac{1}{t^d-1}\prod_{j=r+1}^{\ell}(t^{d_{E_j}}-1).$$

Since $\Delta(t) = \widetilde{\Delta}^F(t)\cdot\frac{1}{t^d-1}\prod_{j=1}^{\ell}(t^{d_{E_j}}-1)$, we get

$$\widetilde{\Delta}(t) = \frac{\Delta(t)\cdot\prod_{v\in\mathcal{N}_1}(t^{d_v}-1)}{\prod_{E\in\mathcal{E}'}(t^{d_E}-1)}.$$

The result is then obtained via Eisenbud and Neumann's formula (theorem 5.5.1).

On the other hand, if $r=\ell$, $\mathcal{B}_{\mathbb{C}}$ is presented by

$$\mathcal{P} = \begin{pmatrix} t^{d_{v_1}}-1 & & & \\ & t^{d_{v_2}}-1 & & \\ & & \ddots & \\ & & & t^{d_{v_\ell}}-1 \\ \frac{t^{d_{E_1}}-1}{t^d-1} & \frac{t^{d_{E_2}}-1}{t^d-1} & \dots & \frac{t^{d_{E_\ell}}-1}{t^d-1} \end{pmatrix},$$

and $\widetilde{\Delta}(t) = \widetilde{\Delta}^F(t)\cdot\prod_{v\in\mathcal{N}_1-\{v_1,\dots,v_r\}}(t^{d_v}-1)\cdot\Delta_0\mathcal{P}$. It is an easy computation to check that

$$\Delta_0\mathcal{P} = \frac{1}{t^d-1}\cdot\prod_{j=1}^{\ell}(t^{d_{v_j}}-1)\cdot\gcd\Big(t^d-1,\frac{t^{d_{E_1}}-1}{t^{d_{v_1}}-1},\dots,\frac{t^{d_{E_\ell}}-1}{t^{d_{v_\ell}}-1}\Big),$$

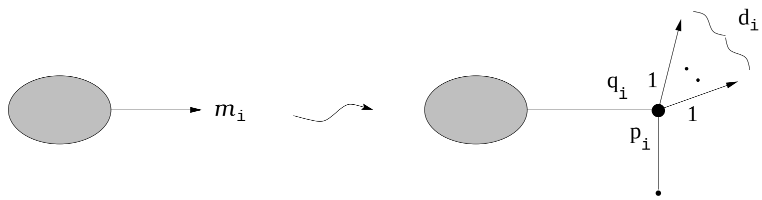
and we get

$$\begin{aligned}\widetilde{\Delta}(t) &= \widetilde{\Delta}^F(t)\cdot\prod_{v\in\mathcal{N}_1-\{v_1,\dots,v_r\}}(t^{d_v}-1)\cdot\prod_{j=1}^{\ell}(t^{d_{v_j}}-1)\cdot\frac{1}{t^d-1}\cdot\epsilon(t)\\ &= \frac{\Delta(t)\cdot\epsilon(t)\cdot\prod_{v\in\mathcal{N}_1}(t^{d_v}-1)}{\prod_{E\in\mathcal{E}'}(t^{d_E}-1)}.\end{aligned}$$

This concludes the proof of the theorem. □

5.6 The Monodromy of the Link at Infinity

Given a reduced polynomial application $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$, let us denote by $\mathcal{L}(f,\infty)$ its regular link at infinity. In [29], Neumann proved that $\mathcal{L}(f,\infty)$ can be seen as the border of the fiber of some multilink $L(m_1,\ldots,m_n)$, with $m_i \geq 1$ for all i. This means that a splice diagram of $\mathcal{L}(f,\infty)$ can be obtained from a splice diagram of $L(\underline{m})$ by making the following operation for each component of the multilink, where $m_i' = -\sum_{j\neq i} m_j \ell k(L_i,L_j)$, $d_i = \gcd(m_i,m_i') > 0$, $p_i = \frac{m_i}{d_i}$ and $q_i = \frac{m_i'}{d_i}$.



Neumann also showed how to construct a splice diagram for $L(\underline{m})$; this multilink being fibered, it is possible to compute $\mathcal{M}_{\mathbb{C}}^{L(\underline{m})}$, its Alexander module over $\mathbb{C}[t,t^{-1}]$ using theorem 5.5.1. By theorem 5.5.2, it is now easy to compute $\mathcal{M}_{\mathbb{C}}^{\mathcal{L}}$, the Alexander module of $\mathcal{L}(f,\infty)$ over $\mathbb{C}[t,t^{-1}]$.

5.6.1 Theorem.
Let $L(m_1,\ldots,m_n)$ be the multilink associated with $\mathcal{L}(f,\infty)$; let us note $d_i = \gcd(m_i,m_i')$, $d = \gcd(m_1,\ldots,m_n)$, $\Delta(t)$ the characteristic polynomial of the monodromy of $L(\underline{m})$, and $\Delta'(t)$ the polynomial corresponding to the 2×2 Jordan blocks. Then, the Alexander module $\mathcal{M}_{\mathbb{C}}^{\mathcal{L}}$ is given by the following properties:

- *$rk\,\mathcal{M}_{\mathbb{C}}^{\mathcal{L}} = \sum_{i=1}^n (d_i - 1)$;*
- *the order ideal of the torsion of $\mathcal{M}_{\mathbb{C}}^{\mathcal{L}}$ is generated by*

$$\widetilde{\Delta}(t) = \frac{\Delta(t)}{\frac{t-1}{t^d-1}\prod_{i=1}^n \frac{t^{d_i}-1}{t-1}}\ ;$$

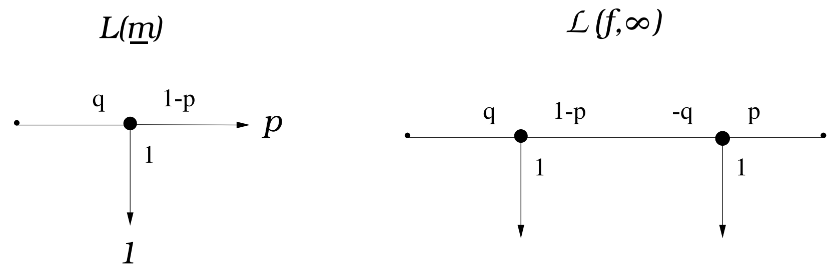
- *the Jordan normal form of the monodromy restricted to the torsion of $\mathcal{M}_{\mathbb{C}}^{\mathcal{L}}$ has blocks of dimension at most two; the 2×2-blocks correspond to the roots of $\Delta'(t)$.* □

If all the fibers $f^{-1}(c)$ are regular at infinity, this theorem says nothing. In such a case indeed, all the multiplicities m_i are equal to one, and $\mathcal{L}(f,\infty)$ is nothing but $L(\underline{m})$. On the other hand, if some fiber $f^{-1}(c)$ is not regular at infinity, then some multiplicity m_i is greater then one, and the result is interesting.

5.6.2 Example. Consider the polynomial

$$f(x,y) = x + x^p y^q, \quad \text{with} \quad p \geq 2,\ q \geq 1,\ \gcd(p,q) = \gcd(p-1,q) = 1.$$

This polynomial has no singular value, and the only irregular fiber is $f^{-1}(0)$. Here are the splice diagrams of the multilink at infinity and of the regular link at infinity of f (see [29, p.451]).



The multilink $L(\underline{m})$ is a fibered Seifert multilink: its Alexander module over $\mathbb{Z}[t,t^{-1}]$ equal to $\mathbb{Z}[t,t^{-1}]/(t^q-1)$ (for example, use theorem 4.4.6). In particular, its monodromy is of finite order q. Now, we can compute the Alexander module over $\mathbb{C}[t,t^{-1}]$ of $\mathcal{L}(f,\infty)$; the result is

$$\mathcal{M}_{\mathbb{C}}^{\mathcal{L}} = \mathbb{C}[t,t^{-1}]/(t^q-1).$$

So in this class of examples, the Alexander module over $\mathbb{C}[t,t^{-1}]$ of $L(\underline{m})$ and $\mathcal{L}(f,\infty)$ are isomorphic.

If $q=1$, the regular link at infinity is the non-fibered Seifert link $(\Sigma(1,-1,p), L_1 \cup L_2)$. By theorem 1.2.7, the Alexander module over $\mathbb{Z}[t,t^{-1}]$ of this link is

$$\mathcal{M}_{\mathbb{Z}}^{\mathcal{L}} = \mathbb{Z}[t,t^{-1}]/p(t-1).$$

In particular, the Alexander module of the regular link at infinity of the Broughton polynomial $f(x,y)=x+x^2y$ is given by $\mathbb{Z}[t,t^{-1}]/2(t-1)$.

Chapter 6

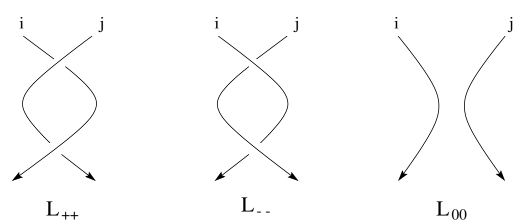
The Conway Potential Function of a Graph Link

6.1 The Conway Potential Function

In 1970, Conway [9] introduced a new invariant of links called the **potential function**. Given an oriented ordered link $L = L_1 \cup \ldots \cup L_n$ in S^3, its potential function is a well defined rational function $\nabla_L(t_1, \ldots, t_n)$ which satisfies

$$\nabla_L(t_1,\ldots,t_n) \doteq \begin{cases} \frac{1}{t_1 - t_1^{-1}}\,\Delta_L(t_1^2) & \text{if } n = 1; \\ \Delta_L(t_1^2,\ldots,t_n^2) & \text{if } n > 1. \end{cases}$$

Thus, this invariant is basically the multivariable Alexander polynomial without the ambiguity concerning multiplication by units of Λ_n. This might seem a minor improvement. However, the potential function has a very remarkable new property: it can be computed directly from a link diagram using "skein-type" formulas. For example, if L_{++}, L_{--} and L_{00} differ by the following local operation,



then we have the equality

$$\nabla_{L_{++}} + \nabla_{L_{--}} = (t_i t_j + t_i^{-1} t_j^{-1}) \cdot \nabla_{L_{00}}.$$

(For a characterization of ∇_L using skein-type relations, see [27].) Thus, Conway pointed out a preferred representative of the Alexander polynomial, and gave a very easy method to compute it. Unfortunately, his paper contains neither a precise definition of the potential function, nor a proof of its unicity.

As a particular case of the potential function, Conway defined what he called the **reduced polynomial** $\mathbf{D_L(t)}$ of a non-ordered oriented link L. (This was later called the **Conway**

polynomial $\boldsymbol{\Omega_L(t)}$ by Kauffmann, and the **reduced potential function** $\overline{\nabla}(\mathbf{t})$ by Hartley; we will follow Kauffman's terminology and notation.) It is given by

$$\Omega_L(t) = (t - t^{-1}) \cdot \nabla_L(t, \dots, t).$$

In 1981, Kauffman [20] found a very simple geometric construction of the Conway polynomial, namely

$$\Omega_L(t) = \det\left(t^{-1}A - tA^T\right),$$

where A is any Seifert matrix of the link L and A^T its transposed. Finally, in 1983, Hartley [14] gave a definition of the multivariable potential function ∇_L for any ordered oriented link in S^3. This definition was later extended by Turaev [46] to links in a $\mathbb{Z}$-homology 3-sphere, and by Boyer and Lines [4] to links in a $\mathbb{Q}$-homology 3-sphere.

Let us summarize the results that we need in the following theorem. We refer to [4] for the proofs.

6.1.1 Theorem.
Given an oriented ordered link $L = L_1 \cup \ldots \cup L_n$ in a $\mathbb{Z}$-homology sphere Σ, there exists a well defined invariant ∇_L related to the multivariable Alexander polynomial of L by the equality

$$\nabla_L(t_1, \dots, t_n) \doteq \begin{cases} \frac{1}{t_1 - t_1^{-1}}\, \Delta_L(t_1^2) & \text{if } n = 1; \\ \Delta_L(t_1^2, \dots, t_n^2) & \text{if } n > 1. \end{cases} \tag{6.1}$$

Furthermore, ∇_L satisfies the symmetry formula

$$\nabla_L(t_1^{-1}, \dots, t_n^{-1}) = (-1)^n\, \nabla_L(t_1, \dots, t_n). \tag{6.2}$$

Also, if $L' = (-L_1) \cup L_2 \cup \ldots \cup L_n$, then

$$\nabla_{L'}(t_1, t_2, \dots, t_n) = -\nabla_L(t_1^{-1}, t_2, \dots, t_n). \tag{6.3}$$

Finally, if $L' = L_2 \cup \ldots \cup L_n$, we have the following analogue of the Torres formula (compare proposition 2.1.1 and corollary 2.2.5):

$$\nabla_L(1, t_2, \dots, t_n) = (t_2^{\ell_{12}} \cdots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \cdots t_n^{-\ell_{1n}}) \cdot \nabla_{L'}(t_2, \dots, t_n), \tag{6.4}$$

where ℓ_{ij} stands for the linking number $\ell k(L_i, L_j)$. □

6.2 The Formula for Graph Links

In 1999, Neumann [31] succeeded in computing the Conway polynomial Ω_L of any fibered graph link in S^3 (that is: of any fibered solvable link). Let us recall his argument very briefly.

Let L be an oriented graph link given by a splice diagram Γ. The Alexander polynomial of L (viewed as a multilink with multiplicities ± 1) is computed in [11]:

$$\Delta^L(t) = (t-1) \prod_v (t^{\ell_v} - 1)^{\delta_v - 2},$$

where the product is over all non-arrowhead vertices of Γ, δ_v is the valency of the vertex v, and ℓ_v denotes the linking number of L with the virtual component corresponding to v.[1] By the equalities

$$\Omega_L(t) = \pm t^\nu \Delta^L(t^2)\ , \quad \text{and} \quad \Omega_L(t^{-1}) = (-1)^{n-1}\Omega_L(t),$$

it follows

$$\Omega_L(t) = \epsilon(L)(t - t^{-1}) \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v}\ ,$$

where $\epsilon(L) = \pm 1$, and the only issue is to determine the sign $\epsilon(L)$.

If L is a fibered link, its Seifert matrix A is unimodular. Therefore, the leading coefficient of $\Omega_L(t) = \det\left(t^{-1}A - tA^T\right)$ is given by $\det(-A)$. For fibered links in S^3, Lee Rudolph defined an integer invariant λ called the **enhanced Milnor number**, which is known to satisfie the following formula (see [30]):

$$(-1)^\lambda = \det(-A). \qquad (*)$$

Neumann proves that $\lambda \equiv k_- + j_- \pmod 2$, where k_- is the number of (-1)-weighted arrowheads and j_- the number of non-arrowhead vertices v for which ℓ_v is negativ and δ_v is odd. This leads to

$$\det(-A) = (-1)^{k_- + j_-}, \qquad (**)$$

giving the sign $\epsilon(L) = (-1)^{k_-}$. So the formula is

$$\Omega_L(t) = (-1)^{k_-}(t - t^{-1}) \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v}. \qquad (***)$$

These results lead Neumann to the following questions:

- Is the formula $(***)$ true for any graph link in a homology sphere ?
- Does the equality $(**)$ hold for fibered graph links in a homology sphere ?
- Is the equation $(*)$ still valid for fibered graph links in a homology sphere ?

The aim of this chapter is to present a formula for the multivariable potential function of any (fibered or non-fibered) graph link in a homology sphere. As a consequence, we give a positive answer to all three of Neumann's questions (see corollaries 6.2.8, 6.2.9 and 6.2.11).

First of all, we need a formula for the multivariable Alexander polynomial of a graph link L. This is given by [11, Theorem 12.1].

6.2.1 Theorem. (Eisenbud-Neumann)
If L is a graph link with n components given by a splice diagram Γ, its multivariable Alexander polynomial is equal to

$$\Delta_L(t_1,\dots,t_n) \doteq \begin{cases} (t_1 - 1)\prod_v (t_1^{\ell_{1v}} - 1)^{\delta_v - 2} & \text{if } n = 1; \\ \prod_v (t_1^{\ell_{1v}} \cdots t_n^{\ell_{nv}} - 1)^{\delta_v - 2} & \text{if } n \geq 2, \end{cases}$$

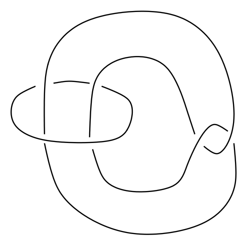
where the product is over all non-arrowhead vertices v of Γ, δ_v is the valency of the vertex v, and ℓ_{iv} denotes the linking number of L_i with the virtual component corresponding to v. □

[1] This formula follows easily from theorem 3.2.9 and corollaries 4.3.2 and 4.4.7.

In proving this theorem, Eisenbud and Neumann also show the following remarkable result.

6.2.2 Theorem. (Eisenbud-Neumann)
If L is a graph link, then $\Delta_L(t_1,\ldots,t_n) = 0$ if and only if L is **algebraically split**, *that is: after reindexing if necessary, there is an index $1 \leq q < n$ such that $\ell_{ij} = 0$ whenever $1 \leq i \leq q < j \leq n$.*

6.2.3 Remark. This is a very striking property of graph links. For example, it implies that the Alexander polynomial Δ_L of a 2-component graph link is zero if and only if the linking number ℓ of the components is zero. For general 2-component links L, if Δ_L vanishes, then $\ell = 0$ (by the Torres formula). But the converse is false: the Whitehead link (illustrated below) has Alexander polynomial $(t_1 - 1)(t_2 - 1)$, although $\ell = 0$. As a matter of fact, it is still an open question how to characterize geometrically 2-component links with vanishing Alexander polynomial (see [12, Problem 16]).



Using theorem 6.2.1 along with the equations (6.1) and (6.2), it is easy to compute the Conway potential function up to sign.

6.2.4 Proposition.
The potential function of a graph link L in a homology sphere is given by

$$\nabla_L(t_1,\ldots,t_n) = \epsilon(L) \cdot \prod_v (t_1^{\ell_{1v}} \cdots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2},$$

where $\epsilon(L)$ is equal to $+1$ or -1.

Proof. By theorem 6.2.1 and equation (6.1), we have

$$\begin{aligned}\nabla_L(t_1,\ldots,t_n) &= \begin{cases} \frac{\epsilon(L)}{t_1 - t_1^{-1}} \cdot t_1^{\nu_1} \cdot (t_1 - t_1^{-1}) \cdot \prod_v (t_1^{\ell_{1v}} - t_1^{-\ell_{1v}})^{\delta_v - 2} & \text{if } n = 1; \\ \epsilon(L) \cdot t_1^{\nu_1} \cdots t_n^{\nu_n} \cdot \prod_v (t_1^{\ell_{1v}} \cdots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2} & \text{if } n \geq 2, \end{cases} \\ &= \epsilon(L) \cdot t_1^{\nu_1} \cdots t_n^{\nu_n} \cdot \prod_v (t_1^{\ell_{1v}} \cdots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2},\end{aligned}$$

for some integers $\nu_1, \ldots, v_n$ and some sign $\epsilon(L) = \pm 1$. The symmetry formula (6.2) implies that $\nu_1 = \ldots = \nu_n = 0$, giving the proposition. □

Therefore, the only problem is to determine the sign $\epsilon(L)$.

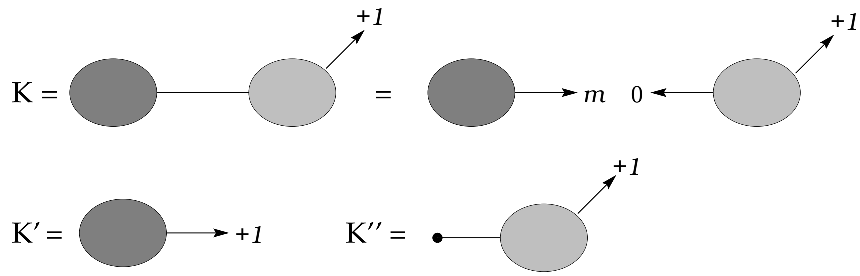
6.2.5 Lemma.
Let K be a graph knot with $(+1)$-weighted arrowhead. Then $\epsilon(K) = +1$, that is:

$$\nabla_K(t) = \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2}.$$

Proof. Using Kauffman's construction of the Conway polynomial

$$\Omega_K(t) = (t - t^{-1}) \cdot \nabla_K(t) = \det\left(t^{-1}A - tA^T\right),$$

it is easy to show the following statement: if K, K' and K'' are knots related by the splicing operation illustrated below,



then their Conway polynomials satisfy

$$\Omega_K(t) = \Omega_{K'}(t^m) \cdot \Omega_{K''}(t).$$

(The argument is very similar to the methods of section 3.3; see also [24, p.62]). Therefore, if the lemma holds for K' and K'', it also holds for K. Indeed, if $m \neq 0$, we have

$$\Omega_K(t) = (t^m - t^{-m}) \prod_{v \in \mathcal{V}'} (t^{m\ell'_v} - t^{-m\ell'_v})^{\delta_v - 2} \cdot (t - t^{-1}) \prod_{v \in \mathcal{V}''} (t^{\ell''_v} - t^{-\ell''_v})^{\delta_v - 2},$$

where $\mathcal{V}'$ (resp. $\mathcal{V}''$) denotes the set of non-arrowhead vertices of K' (resp. K''), and ℓ'_v (resp. ℓ''_v) the linking number of K' (resp. K'') with the vertex v. Clearly, $\ell_v = \ell''_v$ for all $v \in \mathcal{V}''$, $\ell_v = m\,\ell'_v$ for all $v \in \mathcal{V}'$, and $\mathcal{V}' \cup \mathcal{V}'' = \mathcal{V} \cup \{w\}$ with $\ell''_w = m$ and $\delta_w = 1$. Therefore, we get $\Omega_K(t) = (t - t^{-1}) \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2}$. On the other hand, if $m = 0$, we have

$$\Omega_K(t) = \Omega_{K'}(1) \cdot \Omega_{K''}(t) = 1 \cdot (t - t^{-1}) \prod_{v \in \mathcal{V}''} (t^{\ell''_v} - t^{-\ell''_v})^{\delta_v - 2}.$$

Now,

$$\prod_{v \in \mathcal{V}} (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2} = (t^0 - t^0)^{\sum_{v \in \mathcal{V}'} (\delta_v - 2) + 1} \cdot \prod_{v \in \mathcal{V}''} (t^{\ell''_v} - t^{-\ell''_v})^{\delta_v - 2}.$$

Since the splice diagram of K' is a tree and the sum is on every vertex of this tree except one vertex of degree 1, we have

$$\sum_{v \in \mathcal{V}'} (\delta_v - 2) + 1 = 2 + \sum_{v \text{ vertex}} (\delta_v - 2) = 2 + 2 \cdot \#\{\,\text{edges}\,\} - 2 \cdot \#\{\,\text{vertices}\,\} = 0,$$

and the lemma holds for K.

Via splicing, we can therefore restrict ourselves to the case of a Seifert knot $K' = (\Sigma(r,p,q), L)$. Let us also note $K = (\Sigma(1,p,q), L)$; this knot K is simply a (p,q)-torus knot in S^3. It is well known that

$$\Omega_K(t) = (t - t^{-1}) \cdot \frac{(t^{pq} - t^{-pq})}{(t^p - t^{-p})(t^q - t^{-q})}.$$

(As a matter of fact, even the Seifert matrix of K is well known, see e.g. [11, p. 119]). Hence, the lemma holds for K, and we are left with the proof that $\Omega_{K'}(t) = \Omega_K(t)$. By proposition 6.2.4, we know that they are equal up to sign. By Kauffman's construction (and using the fact that K and K' are fibered), we just need to check that $\det(A) = \det(A')$, where A (resp. A') is a Seifert matrix for K (resp. K'). If the monodromy of K is given by H, the monodromy of K' is H^r (see [11, p. 118]). It follows

$$A' = (A - A^T)(I - H^r)^{-1} = A(I - H)(I - H^r)^{-1}. \qquad (\diamondsuit)$$

Now, the characteristic polynomials of H and H^r are equal up to sign. Furthermore, H and H^r are diagonalizable over $\mathbb{C}$. Therefore, H and H^r are conjugate; in particular $\det(I-H) = \det(I - H^r)$. By the equation ($\diamondsuit$), we have $\det(A) = \det(A')$ and the lemma is proved. □

6.2.6 Lemma.
Consider $L = L_1 \cup \ldots \cup L_n$. If $L - L_i$ is algebraically split for all i, then L is algebraically split.

Proof. Let us associate to L a graph G_L as follows: the vertices of G_L correspond to the components of L, and two vertices are linked with an edge if the linking number of the corresponding components is not equal to zero. Clearly, L is algebraically split if and only if G_L is not connected. Given a vertex v of a graph G, let us denote by $G - v$ the subgraph obtained by deleting the vertex v and every edge adjacent to v. We are left with the proof of the following assertion: given a graph G, if $G - v$ is not connected for any vertex v of G, then G is not connected. In other words: if G is a connected graph, there exists a vertex v such that $G - v$ is connected. This last statement is very easy to prove: given G a connected graph, let T be a maximal subtree of G. Since T is a tree, it has at least one vertex v of degree one (in fact: it has at least two such vertices). Then, $T - v$ is connected, as well as $G - v$. □

We are now ready to state and prove our main result.

6.2.7 Theorem.
Let L be a graph link with n components given by a splice diagram Γ. Then, its Conway potential function is equal to

$$\nabla_L(t_1, \ldots, t_n) = (-1)^{k_-} \prod_v (t_1^{\ell_{1v}} \cdots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2},$$

where the product is over all non-arrowhead vertices v of Γ, δ_v is the valency of the vertex v, ℓ_{iv} denotes the linking number of L_i with the virtual component corresponding to v, and k_- is equal to the number of (-1)-weighted arrowheads.

Proof. By formula (6.3), it may be assumed that all the arrowheads have weight $(+1)$. Using the notation of proposition 6.2.4, we have to check that if $k_- = 0$, then $\epsilon(L) = +1$. Let us give a proof by induction on $n \geq 1$.

The case $n = 1$ is settled by lemma 6.2.5. Let us fix $n \geq 2$, and assume that $\epsilon(L') = +1$ for all graph link L' with $\leq n - 1$ arrowheads, all $(+1)$-weighted. Let $L = L_1 \cup \ldots \cup L_n$ be a graph link with $k_- = 0$. If $\nabla_{L-L_i} = 0$ for all i, then (by formula (6.1) and theorem 6.2.2)

$L - L_i$ is algebraically split for all i. By lemma 6.2.6, it follows that L is algebraically split, so that $\nabla_L = 0$. In this case, there is nothing to prove. Therefore, it may be assumed (after possible renumbering) that $\nabla_{L-L_1} \neq 0$. Let us note $L' = L - L_1$. If $\ell_{12} = \ldots = \ell_{1n} = 0$, then $L = L_1 \sqcup L'$ is algebraically split, so $\nabla_L = 0$. Without loss of generality, it may be assumed that $\ell_{12} \cdots \ell_{1n} \neq 0$. To summarize, it may be assumed that

$$(t_2^{\ell_{12}} \cdots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \cdots t_n^{-\ell_{1n}}) \cdot \nabla_{L'}(t_2, \ldots, t_n) \neq 0.$$

The key ingredient of the induction step is the Torres formula (6.4). By proposition 6.2.4, we have

$$\nabla_L(1, t_2, \ldots, t_n) = \epsilon(L) \cdot \prod_{v \in \mathcal{V}} (t_2^{\ell_{2v}} \cdots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2}.$$

On the other hand, by the Torres formula,

$$\nabla_L(1, t_2, \ldots, t_n) = (t_2^{\ell_{12}} \cdots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \cdots t_n^{-\ell_{1n}}) \cdot \nabla_{L'}(t_2, \ldots, t_n)\ ,$$

which, by induction, is equal to

$$(t_2^{\ell_{12}} \cdots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \cdots t_n^{-\ell_{1n}}) \cdot \prod_{v \in \mathcal{V}'} (t_2^{\ell_{2v}} \cdots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2}.$$

Now, the set of vertices $\mathcal{V}'$ of L' is equal to $\mathcal{V} \cup \{v_1\}$, where v_1 is the vertex corresponding to the component L_1. Therefore, $\delta_{v_1} = 1$, $\ell_{v_1 v} = \ell_{1v}$, and we have the equality

$$\epsilon(L) \cdot \prod_{v \in \mathcal{V}} (t_2^{\ell_{2v}} \cdots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2} = \prod_{v \in \mathcal{V}} (t_2^{\ell_{2v}} \cdots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2}.$$

Since

$$\prod_{v \in \mathcal{V}} (t_2^{\ell_{2v}} \cdots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2} = (t_2^{\ell_{12}} \cdots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \cdots t_n^{-\ell_{1n}}) \cdot \nabla_{L'}(t_2, \ldots, t_n),$$

which is assumed to be non-zero, $\epsilon(L)$ is equal to $+1$. □

6.2.8 Corollary.
The Conway polynomial of a graph link L is given by

$$\Omega_L(t) = (-1)^{k_-}(t - t^{-1}) \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2}.$$

Proof. Use the fact that $\Omega_L(t) = (t - t^{-1}) \cdot \nabla_L(t, \ldots, t)$. □

6.2.9 Corollary.
Let A be a Seifert matrix for a fibered graph link L; then

$$\det(-A) = (-1)^{k_- + j_-},$$

with j_- the number of non-arrowhead vertices v for which ℓ_v is negativ and δ_v is odd.

Proof. If L is fibered, A is unimodular, so $\det(-A)$ is the leading coefficient of $\Omega_L(t) = \det(t^{-1}A - tA^T)$. By corollary 6.2.8, this is the leading coefficient of $(-1)^{k_-}\prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2}$, which is clearly $(-1)^{k_- + j_-}$. □

6.2.10 Corollary.
Let A be a Seifert matrix for a graph knot; then $\det(A) = +1$.

Proof. It is easy to check that a Seifert matrix for a graph knot K is equal to a Seifert matrix for the knot K' obtained from K by deleting every non-fibered splice component. Therefore, it may be assumed that K is fibered. Using corollary 6.2.9 and the fact that K is a knot, we have

$$\det(A) = \det(-A) = \operatorname{sgn}(K) \cdot (-1)^{j_-},$$

where $\operatorname{sgn}(K)$ is the sign of the arrowhead-vertex corresponding to K. We can prove that $\operatorname{sgn}(K) \cdot (-1)^{j_-}$ is multiplicative under splicing, that is: $\operatorname{sgn}(K) \cdot (-1)^{j_-} = \operatorname{sgn}(K') \cdot \operatorname{sgn}(K'') \cdot (-1)^{j'_- + j''_-}$, where K, K' and K'' are as in the proof of lemma 6.2.5. Therefore, we just need to check that $\operatorname{sgn}(K) \cdot (-1)^{j_-} = +1$ if K is the Seifert knot $(\Sigma(p, q, r), \operatorname{sgn}(K) \cdot K)$. If A is a Seifert matrix for K, then A^T is a Seifert matrix for $-K$; therefore, it may also be assumed that $\operatorname{sgn}(K) = +1$. We then get

$$j_- = \begin{cases} 0 & \text{if } p, q > 0; \\ 2 & \text{else.} \end{cases}$$

This concludes the proof. □

6.2.11 Corollary.
Let A be a Seifert matrix for a fibered graph link L; then

$$\det(-A) = (-1)^{\lambda},$$

where λ is the enhanced Milnor number of L.

Proof. By [30, Theorem 6.1. and §10], we have the equality

$$\lambda = \ell k(P \cup L_+, N \cup L_-),$$

where the notations are as in [31]. By corollary 6.2.9, it remains to show that

$$\ell k(P \cup L_+, N \cup L_-) \equiv k_- + j_- \pmod 2.$$

The argument is exactly as in [31]. □

6.2.12 Remark. Of course, corollaries 6.2.9 and 6.2.11 are false for non-fibered graph links. For example, a Seifert matrix A for the trivial 2-component link satisfies $\det(-A) = 0$.

Appendix A

Résumé de la thèse en français

A.1 Introduction

Un **multi-entrelacs** est un entrelacs orienté $L = L_1 \cup \ldots \cup L_n$ dans une 3-sphère d'homologie orientée Σ, dont chaque composante L_i est munie d'un entier m_i appelé **poids**. On utilisera la convention naturelle suivante: $m_i L_i = (-m_i)(-L_i)$, où $-L_i$ représente L_i avec l'orientation opposée. Bien entendu, un multi-entrelacs dont tous les poids sont ± 1 est simplement un entrelacs orienté.

D'une façon plus formelle, on peut définir un multi-entrelacs comme la donnée d'un entrelacs orienté L et d'une classe d'homologie $\underline{m} = (m_1, \ldots, m_n)$ dans $\mathrm{H}_1 L$. Or, des théorèmes classiques de James Alexander et de Henry Whitehead impliquent que ce $\mathbb{Z}$-module est isomorphe au groupe des classes d'homotopie d'applications $X \xrightarrow{\varphi} S^1$, où X désigne le complémentaire de L. On dira que le multi-entrelacs est **fibré** s'il existe une fibration localement triviale dans la classe d'homotopie correspondant aux poids. Dans le cas d'un entrelacs orienté, on retombe évidemment sur la définition usuelle d'un entrelacs fibré. En outre, certaines techniques standards de topologie algébrique montrent que les classes d'homotopie d'applications de X dans le cercle sont en bijection avec les revêtements infinis cycliques de X; ainsi, l'ensemble des structures de multi-entrelacs sur un entrelacs L est en bijection avec l'ensemble des revêtements infinis cycliques du complémentaire X de L. Le choix d'un système de poids $\underline{m}$ détermine donc un revêtement $\widetilde{X}(\underline{m}) \xrightarrow{p} X$. Si t désigne un générateur du groupe (infini cyclique) des transformations du revêtement, l'homologie $\mathrm{H}_1 \widetilde{X}(\underline{m})$ est munie d'une structure naturelle de module sur $\mathbb{Z}[t, t^{-1}]$, l'anneau des polynômes de Laurent à coefficients entiers. Il s'agit du **module d'Alexander** du multi-entrelacs. Le déterminant d'une matrice de présentation carrée de ce module est appelé le **polynôme d'Alexander** du multi-entrelacs. Lorsque tous les poids sont ± 1, ces définitions coïncident avec les définitions du module et du polynôme d'Alexander d'un entrelacs orienté.

Dans cette thèse, nous proposons tout d'abord une étude systématique du module d'Alexander des multi-entrelacs et une méthode de calcul originale de cet invariant. Dans un second temps, nous analysons les implications de ces premiers résultats pour le polynôme d'Alexander. Enfin, cette technique de calcul est implémentée sur certaines classes de multi-entrelacs particulièrement intéressants: les multi-entrelacs seifertiques, et certains multi-entrelacs de Waldhausen.

Le concept de multi-entrelacs a été dégagé en 1985 par DAVID EISENBUD et WALTER NEUMANN dans leur livre *Three-dimensional link theory and invariants of plane curve singularities* [11]. Il s'agit certes d'une généralisation naturelle de la notion d'entrelacs orienté; cependant, les raisons qui ont motivé l'introduction des multi-entrelacs ne sont pas évidentes au premier abord. Nous allons donc présenter, de façon relativement informelle, trois domaines où l'élaboration d'une théorie cohérente nécessite l'utilisation des multi-entrelacs.

Singularités de courbes planes complexes

Considérons $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ une application polynomiale, et V la courbe algébrique plane donnée par

$$V = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}.$$

Le but est d'étudier la topologie de V au voisinage d'un point donné $a = (x_0, y_0) \in V$. Si le gradient de f ne s'annule pas en a, on dit que a est un **point régulier** de f. Dans ce cas, le théorème des fonctions implicites implique que V est une surface lisse au voisinage de a; il n'y a donc rien à étudier du point de vue topologique. En revanche, si $\frac{\partial f}{\partial x}(a) = \frac{\partial f}{\partial y}(a) = 0$, on parle de **point singulier** de f, et la topologie de V au voisinage de a est très intéressante.

L'idée fondamentale de cette théorie est due à WILHELM WIRTINGER et à son étudiant de thèse KARL BRAUNER [5]: il s'agit de considérer, pour un $\epsilon > 0$ suffisamment petit, l'intersection K_ϵ de V et d'une 3-sphère $S_\epsilon = \partial D_\epsilon \subset \mathbb{C}^2$ de rayon ϵ centrée en a. Il s'avère que K_ϵ est une variété lisse de dimension 1, en d'autres termes un entrelacs: on parle de **l'entrelacs de la singularité**.[1] De plus, la topologie de $V \cap D_\epsilon$ est déterminée par cet entrelacs; plus précisément, JOHN MILNOR [25, Theorem 2.10] démontre que pour tout ϵ suffisamment petit, la paire $(D_\epsilon, V \cap D_\epsilon)$ est homéomorphe au cône sur la paire (S_ϵ, K_ϵ). Par exemple, l'origine est un point singulier de $f(x,y) = xy$, et l'on voit facilement que l'entrelacs de cette singularité est l'entrelacs de Hopf; ainsi, au voisinage de l'origine, le sous-ensemble algébrique $V = \{(x,y) \in \mathbb{C}^2 \mid xy = 0\}$ est homéomorphe au cône sur l'entrelacs de Hopf. Un autre exemple simple est donné par l'application $f(x,y) = x^p + y^q$, où p et q sont deux entiers premiers entre eux strictement supérieurs à 1: l'origine est un point singulier, et l'entrelacs associé est le noeud torique de type (p,q).

Un autre résultat fondamental est donné par le fameux théorème de fibration de Milnor [25, Theorem 4.8]. Dans notre contexte, il s'énonce comme suit: pour tout ϵ suffisamment petit, l'application $\phi = \frac{f}{|f|}\colon S_\epsilon - K_\epsilon \longrightarrow S^1$ est une fibration différentiable localement triviale. On appelle ϕ la **fibration de Milnor** et $F = \phi^{-1}(1)$ la **fibre de Milnor**. A partir des exemples ci-dessus, on voit que l'entrelacs de Hopf et les noeuds toriques fibrent sur le cercle. La fibre de Milnor fournit une surface de Seifert pour l'entrelacs correspondant.

Soit à présent a un point singulier d'une application $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$; via translation, on peut ramener a à l'origine. Considérons f comme élément de l'anneau $\mathbb{C}\{x,y\}$ des séries convergentes à coefficients complexes; comme cet anneau est factoriel, f se décompose de façon unique en produit de facteurs irréductibles dans $\mathbb{C}\{x,y\}$

$$f = f_1^{m_1} \cdot f_2^{m_2} \cdots f_n^{m_n},$$

[1] Un entrelacs qui se réalise comme l'entrelacs d'une singularité et un **entrelacs algébrique**.

avec $m_i \geq 1$. On dit que f est **réduite** si toutes les multiplicités m_i sont égales à 1; cela correspond au fait que la singularité est isolée. Pour tout i, l'espace $L_i = \{(x,y) \in S_\epsilon \mid f_i(x,y) = 0\}$ est connexe. Comme K_ϵ est égal à l'union $L_1 \cup L_2 \cup \ldots \cup L_n$, le nombre de composantes de l'entrelacs de la singularité est donc égal au nombre de facteurs irréductibles de f dans $\mathbb{C}\{x,y\}$, ou **branches** de f.

En tant qu'ensemble, l'entrelacs K_ϵ ne dépend certes pas des entiers m_i. En revanche, la fibration de Milnor en dépend: la classe d'homotopie de $\phi\colon S_\epsilon - K_\epsilon \longrightarrow S^1$ est la classe donnée par $\underline{m} = (m_1, \ldots, m_n)$. En d'autres termes, le théorème de fibration de Milnor nous dit que le multi-entrelacs $K_\epsilon = m_1 L_1 \cup \ldots \cup m_n L_n$ est fibré. De façon équivalente, la fibre de Milnor est une surface de Seifert pour le multi-entrelacs $K_\epsilon = m_1 L_1 \cup \ldots \cup m_n L_n$.[2] Enfin, le module d'Alexander associé à la singularité est le module d'Alexander de ce même multi-entrelacs.

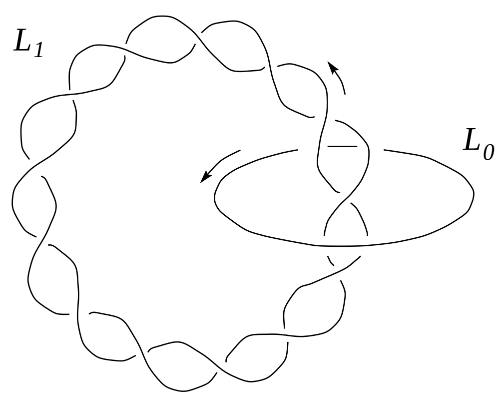
D'une façon générale, on peut donc associer naturellement un multi-entrelacs à une singularité de courbe plane. Si cette singularité est isolée, le multi-entrelacs est simplement un entrelacs orienté.

Additivité d'invariants d'entrelacs

Le paragraphe précédent laisse penser que les multi-entrelacs n'apparaissent que lorsque la singularité est non isolée. En vérité, ils entrent en ligne de compte pour le calcul du polynôme d'Alexander d'une singularité même si cette dernière est isolée. Pour commencer, nous allons tenter d'expliquer ce fait sur un exemple. Considérons la singularité donnée par l'application polynomiale

$$f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7.$$

On peut montrer que f est irréductible dans $\mathbb{C}\{x,y\}$; en particulier, l'origine est un point singulier isolé de f. De plus, le noeud K de cette singularité est obtenu de la façon suivante. Soient K' un noeud de trèfle orienté, et L l'entrelacs orienté illustré ci-dessous formé d'un noeud torique L_1 de type $(13,2)$ et d'une composante triviale L_0 enlaçant deux fois la première.



Considérons des voisinages tubulaires $\mathcal{N}(L_0)$ et $\mathcal{N}(K')$ munis de parallèles et de méridiens standards $P, M \subset \partial\mathcal{N}(L_0)$, et $P', M' \subset \partial\mathcal{N}(K')$. Alors, K est l'image de L_1 par l'application

$$S^3 - \mathcal{N}(L_0) \hookrightarrow (S^3 - \mathcal{N}(L_0)) \cup_h (S^3 - \mathcal{N}(K')),$$

[2] La définition d'une surface de Seifert pour un multi-entrelacs est donnée en section 1.2.

où $h: \partial\mathcal{N}(L_0) \longrightarrow \partial\mathcal{N}(K')$ est un homéomorphisme envoyant P sur M' et M sur P'. On dit que K est obtenu par **épissure** (ou **satellisation**) de L et K' le long des **composantes de recollement** L_0 et K'. Il faut imaginer K comme un noeud torique de type $(13, 2)$ construit sur un tore noué en un noeud de trèfle.

D'une façon plus générale, tout entrelacs algébrique est un **entrelacs torique itéré**: il s'obtient par épissure à partir d'un certain nombre d'entrelacs toriques, le nombre de ces entrelacs toriques étant donné par le nombre de paires caractéristiques de Puiseux de la singularité.[3] L'idée est alors d'utiliser cette décomposition canonique pour calculer certains invariants de la singularité, comme son polynôme d'Alexander (c'est-à-dire, le polynôme caractéristique de la monodromie).

Or, pour avoir une formule d'additivité du polynôme d'Alexander via épissure, il est nécessaire d'introduire des poids sur les composantes de recollement: le poids d'une composante de recollement est égal au coefficient d'enlacement de l'autre composante de recollement avec le reste de l'entrelacs dont elle fait partie. Le polynôme d'Alexander (convenablement normalisé) des multi-entrelacs ainsi obtenus est multiplicatif via épissure. Dans l'exemple précédent, le polynôme d'Alexander de K est égal au produit des polynômes d'Alexander du multi-entrelacs $L_1 \cup 0L_0$ et du multi-entrelacs $2K'$. De cette manière, on ramène le problème du calcul du polynôme d'Alexander d'une singularité quelconque au cas très simple d'un multi-entrelacs torique.

La nécessité de l'introduction de poids lors du processus d'épissure est aussi apparue dans le contexte plus topologique de la théorie des noeuds. En effet, Herbert Seifert [42] a démontré en 1950 l'assertion suivante: soient K' un noeud dans S^3, $\mathcal{N}(K')$ un voisinage tubulaire fermé de K', et $f: \mathcal{N}(K') \longrightarrow S^1 \times D^2$ un homéomorphisme préservant l'orientation, qui envoie K' sur $S^1 \times \{0\}$ et un parallèle standard de $\mathcal{N}(K')$ sur $S^1 \times \{1\}$; si K est un noeud dans l'intérieur de $\mathcal{N}(K')$ avec $K \sim m \cdot K'$ dans $\mathrm{H}_1(\mathcal{N}(K'))$, leurs polynômes d'Alexander satisfont l'égalité

$$\Delta_K(t) = \Delta_{K'}(t^m) \cdot \Delta_{f(K)}(t).$$

Cette formule fut généralisée au cas multivarié par Guillermo Torres [44] en 1953. On désigne habituellement ce résultat sous le nom de *formule de Seifert-Torres* (voir le corollaire 3.2.11).

En résumé, si l'on désire calculer le polynôme d'Alexander d'un entrelacs obtenu par épissure d'autres entrelacs, il est utile de les considérer comme des multi-entrelacs: on a alors une formule très simple reliant le polynôme d'Alexander des différents multi-entrelacs considérés. Cette technique de calcul est particulièrement indiquée pour les multi-entrelacs algébriques: ces derniers s'expriment canoniquement comme l'épissure de multi-entrelacs toriques.

Cependant, l'épissure ne se limite pas à un artifice utile dans le cadre algébrique. Les théorèmes de décomposition de William Jaco, Peter Shalen et Klaus Johansson [17, 18] et d'hyperbolisation de William Thurston impliquent que tout entrelacs L dans S^3 s'exprime de façon canonique comme épissure d'entrelacs toriques et d'entrelacs hyperboliques: ce sont les **composantes d'épissure** de L. De plus, la donnée d'une structure de multi-entrelacs sur L (par exemple: une orientation) détermine une structure de multi-entrelacs sur chaque composante d'épissure. Il est ensuite possible de démontrer toute une

[3]Si l'exemple présenté ci-dessus se retrouve partout, c'est qu'il s'agit de l'exemple le plus simple avec deux paires caractéristiques de Puiseux.

série de résultats reliant certaines propriétés du multi-entrelacs aux propriétés correspondantes des composantes d'épissure, vues comme multi-entrelacs (voir [11]). Par exemple, un multi-entrelacs est fibré si et seulement si il est irréductible et toutes ses composantes d'épissure sont des multi-entrelacs fibrés. Bien entendu, ce résultat est faux si l'on ne considère pas les composantes d'épissure comme des multi-entrelacs.

Courbes planes et entrelacs à l'infini

Les travaux de Walter Neumann et Lee Rudolph [28, 29] ont montré que les multi-entrelacs apparaissent aussi dans l'étude topologique globale des courbes algébriques dans $\mathbb{C}^2$. Soit $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ une application polynômiale; une fibre $f^{-1}(c)$ est dite **régulière** s'il existe un voisinage D de c dans $\mathbb{C}$ tel que $f|: f^{-1}(D) \longrightarrow D$ est une fibration localement triviale.[4] On pourrait croire qu'il suffit pour cela que $f^{-1}(c)$ soit non singulière; il n'en est rien. Le fameux polynôme de Broughton $g(x,y) = x + x^2y$ est le premier contre-exemple: g n'a aucune valeur singulière et pourtant, $g^{-1}(0)$ n'est pas régulière. En fait, la condition additionelle suivante est requise: une fibre $f^{-1}(c)$ est dite **régulière à l'infini** s'il existe un voisinage D de c dans $\mathbb{C}$ et un compact K dans $\mathbb{C}^2$ tels que f restreinte à $f^{-1}(D) - K$ est une fibration localement triviale. Hà Huy Vui et Lê Dung Trang [13] ont démontré que $f^{-1}(c)$ est régulière si et seulement si elle est non singulière et régulière à l'infini.

Etant donnée $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ et c dans $\mathbb{C}$, l'intersection de la fibre $f^{-1}(c)$ avec une sphère S^3_R suffisamment grande centrée en l'origine définit un entrelacs appelé **l'entrelacs à l'infini** de $f^{-1}(c)$. Il s'avère que pour deux fibres régulières à l'infini $f^{-1}(c)$ et $f^{-1}(c')$, les deux entrelacs à l'infini correspondants sont isotopes: ils définissent **l'entrelacs régulier à l'infini** de f, noté $\mathcal{L}(f,\infty)$. Cet entrelacs est d'un type bien particulier: il existe un multi-entrelacs fibré $L(\underline{m})$ avec poids $m_i \geq 1$, appelé **multi-entrelacs à l'infini** de f, tel que $\mathcal{L}(f,\infty)$ est égal à l'intersection de la fibre du multi-entrelacs avec un voisinage tubulaire de L.

Voici un premier résultat remarquable. Toutes les fibres $f^{-1}(c)$ sont régulières à l'infini si et seulement si tous les poids du multi-entrelacs sont 1 (ce qui est équivalent à demander que le multi-entrelacs $L(\underline{m})$ soit simplement l'entrelacs orienté $\mathcal{L}(f,\infty)$). Notons que cette condition est satisfaite si et seulement si $\mathcal{L}(f,\infty)$ est un entrelacs fibré. En résumé, le défaut de régularité à l'infini d'une application polynômiale f est mesuré par les poids d'un multi-entrelacs.

Citons encore le théorème principal de [29]. La topologie d'une courbe algébrique régulière $V \subset \mathbb{C}^2$, vue comme variété plongée, est determinée par le multi-entrelacs à l'infini associé $L(\underline{m})$. Plus précisément, V est proprement isotope à la surface plongée obtenue en attachant un col au bord de F, la fibre de $L(\underline{m})$.

A.2 Résumé des résultats

Chapitre 1: Le module d'Alexander des multi-entrelacs

Rappelons qu'un **multi-entrelacs** est un entrelacs orienté $L = L_1 \cup \ldots \cup L_n$ dans une 3-sphère d'homologie orientée Σ, dont chaque composante L_i est munie d'un entier m_i appelé

[4] Notons que pour une application polynômiale f fixée, il n'existe qu'un nombre fini de valeurs complexes c telles que $f^{-1}(c)$ n'est pas régulière.

poids. On utilisera la notation

$$L(\underline{m}) = L(m_1, \ldots, m_n) = m_1 L_1 \cup \ldots \cup m_n L_n\,,$$

et la convention que $m_i L_i = (-m_i)(-L_i)$, où $-L_i$ représente L_i avec l'orientation opposée. Lorsque tous les poids sont ± 1, on obtient simplement un entrelacs orienté: on parle alors du **cas usuel**. Si l'on note X l'extérieur d'un entrelacs L, une suite de poids $\underline{m}$ peut être considérée comme un élément des $\mathbb{Z}$-modules isomorphes suivants:

$$\mathrm{H}_1 L \simeq \mathrm{H}^1 X \simeq \mathrm{Hom}(\mathrm{H}_1 X, \mathbb{Z}) \simeq \mathrm{Hom}(\pi_1 X, \mathbb{Z}) \simeq [X, S^1].$$

Ainsi, munir un entrelacs d'une structure de multi-entrelacs $\underline{m}$ revient à choisir un revêtement infini cyclique $\widetilde{X}(\underline{m}) \xrightarrow{p} X$: c'est le revêtement induit $\phi^* \exp$, où $\phi \colon X \longrightarrow S^1$ est une application quelconque dans la classe d'homotopie $\underline{m} \in [X, S^1]$, et $\exp \colon \mathbb{R} \longrightarrow S^1$ est l'exponentielle.

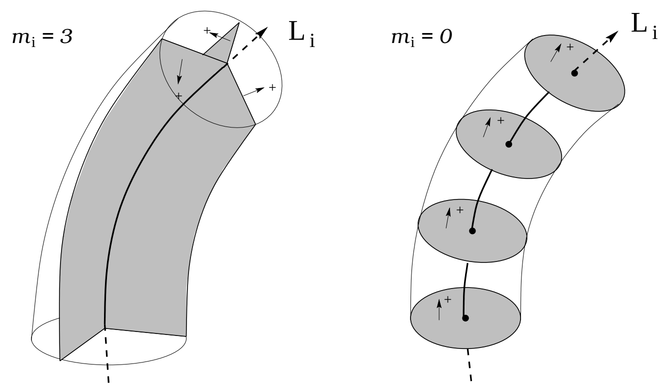
Comme nous l'avons déjà mentionné en introduction, le choix d'un générateur t du groupe infini cyclique des transformations du revêtement munit $\mathrm{H}_* \widetilde{X}(\underline{m})$ d'une structure de module sur $\mathbb{Z} < t >= \mathbb{Z}[t, t^{-1}]$. Clairement, les modules $\mathrm{H}_i \widetilde{X}(\underline{m})$ sont triviaux pour $i \geq 3$; de plus, on vérifie que $\mathrm{H}_2 \widetilde{X}(\underline{m})$ est libre de rang égal au rank de $\mathrm{H}_1 \widetilde{X}(\underline{m})$ (proposition 1.1.4), et que $\mathrm{H}_0 \widetilde{X}(\underline{m})$ est donné par $\mathbb{Z}[t, t^{-1}]/(t^d - 1)$, où d est le plus grand commun diviseur des poids (corollaire 1.1.2). Ainsi, le seul module intéressant est $\mathrm{H}_1 \widetilde{X}(\underline{m})$: c'est le **module d'Alexander** de $L(\underline{m})$. Une matrice de présentation de ce module est une **matrice d'Alexander** de $L(\underline{m})$. Comme il existe toujours une matrice d'Alexander carrée (proposition 1.1.5), on peut définir le **polynôme d'Alexander** de $L(\underline{m})$ comme le déterminant d'une telle matrice. Cet invariant, noté $\Delta^{L(\underline{m})}$, est défini modulo les unités de $\mathbb{Z}[t, t^{-1}]$, c'est-à-dire, à multiplication par $\pm t^\nu$ près. Dorénavant, on notera $\doteq$ l'égalité modulo les unités de l'anneau.

Une classe intéressante de multi-entrelacs est donnée par les **multi-entrelacs fibrés**: il s'agit des multi-entrelacs $L(\underline{m})$ tels qu'il existe une fibration localement triviale $\phi \colon X \longrightarrow S^1$ dans la classe d'homotopie $\underline{m}$. Dans ce cas, l'espace $\widetilde{X}(\underline{m})$ est homéomorphe au produit $F \times \mathbb{R}$, où $F := \phi^{-1}(1)$ est la **fibre** de $L(\underline{m})$. De plus, un générateur t du groupe du revêtement agit sur ce produit comme $(x, z) \longmapsto (h(x), z + 1)$, où $h \colon F \longrightarrow F$ est un homéomorphisme défini à isotopie près. Il s'agit de la **monodromie géométrique** de $L(\underline{m})$; le terme de **monodromie algébrique** désigne l'homomorphisme induit $h_* \colon \mathrm{H}_1 F \longrightarrow \mathrm{H}_1 F$.[5]

Soit $L(\underline{m})$ un multi-entrelacs fibré de fibre F, et soit $\overline{F}$ l'union $F \cup L$ (où l'on considère F comme surface ouverte dans $\Sigma - L$). Il est facile de prouver les assertions suivantes:

- le nombre de composantes connexes de F est égal à d, le plus grand commun diviseur des poids (proposition 1.1.7);
- les $\mathbb{Z}$-modules $\mathrm{H}_1 F$ et $\mathrm{H}_1 \overline{F}$ sont libres de même rang (proposition 1.1.8);
- une matrice d'Alexander est donnée par $H^T - tI$, où H désigne une matrice de la monodromie algébrique (proposition 1.1.9).

Dans la section 1.2, on montre que ces résultats demeurent vrais dans le cas général si l'on remplace la fibre F par une **surface de Seifert** pour $L(\underline{m})$, c'est à dire, une surface orientée



$F \hookrightarrow \Sigma - L$ qui, au voisinage d'une composante L_i de poids m_i, se comporte selon l'illustration ci-dessus.

Les résultats précis sont les suivants. Pour tout multi-entrelacs $L(\underline{m})$, il existe une surface de Seifert F avec $d = \text{pgcd}(\underline{m})$ composantes connexes (proposition 1.2.2): on parle de **bonne surface de Seifert**. De plus, si l'on note $\overline{F}$ l'union $F \cup L$, les $\mathbb{Z}$-modules $\mathrm{H}_1 F$ et $\mathrm{H}_1 \overline{F}$ sont libres de même rang (corollaire 1.2.4)[6]. Pour la troisième assertion, l'idée est de définir une généralisation de la forme de Seifert associée à un entrelacs orienté. Etant donnée F une bonne surface de Seifert pour $L(\underline{m})$, on appelle **matrices de Seifert** de $L(\underline{m})$ associées à F des matrices A_F^+, A_F^- des formes bilinéaires

$$\mathrm{H}_1 F \times \mathrm{H}_1 \overline{F} \longrightarrow \mathbb{Z}$$

données par $(x, y) \longmapsto \ell k(i_+ x, y)$ et $(x, y) \longmapsto \ell k(i_- x, y)$, où ℓk désigne le coefficient d'enlacement et $i_+ x$ (resp. $i_- x$) le cycle x poussé dans la direction positive (resp. négative) normale à F. Dans le cas usuel, l'inclusion $F \subset \overline{F}$ est une équivalence d'homotopie; on a donc un isomorphisme canonique $\mathrm{H}_1 F = \mathrm{H}_1 \overline{F}$ et l'on vérifie facilement que $A_F^+ = (A_F^-)^T$. Ce n'est pas le cas en général: malgré le fait que $\mathrm{H}_1 F$ et $\mathrm{H}_1 \overline{F}$ soient libres de même rang, il n'y a pas d'isomorphisme canonique entre ces deux modules. Ainsi, on a la liberté de choisir deux bases (l'une pour $\mathrm{H}_1 F$, l'autre pour $\mathrm{H}_1 \overline{F}$) et il est nécessaire de considérer les deux matrices de Seifert A_F^+ et A_F^-.

Dans le cas fibré, on démontre que ces matrices sont unimodulaires, et qu'une matrice de la monodromie est donnée par $H = (A_F^+ \cdot (A_F^-)^{-1})^T$ (proposition 1.2.5). En particulier, une matrice d'Alexander est donnée par

$$H^T - tI = A_F^+ \cdot (A_F^-)^{-1} - tI, \quad \text{matrice équivalente à} \quad A_F^+ - tA_F^-.$$

C'est ce dernier résultat qui s'avère toujours correct: étant donnée une bonne surface de Seifert F pour un multi-entrelacs $L(\underline{m})$, $A_F^+ - tA_F^-$ est une matrice d'Alexander de $L(\underline{m})$ (théorème 1.2.7). En particulier, le polynôme d'Alexander est donné par $\Delta^{L(\underline{m})}(t) \doteq \det(A_F^+ - tA_F^-)$.

Dans la section 1.3, le "free differential calculus" de Fox est appliqué aux multi-entrelacs. On en tire un algorithme pour calculer une matrice d'Alexander d'un multi-entrelacs $L(\underline{m})$ dans

[5] Notons que cette définition n'est pas communément admise: nous utilisons la convention des "topologues" $(x, z) \longmapsto (h(x), z + 1)$. La monodromie des "géomètres" est définie par $(x, z) \longmapsto (h(x), z - 1)$.

[6] Sauf dans certains cas dégénérés.

S^3 à partir d'une projection de $L(\underline{m})$ (proposition 1.3.3). Dans le cas usuel, cet algorithme n'est autre que le marquage d'Alexander.

Finalement, la section 1.4 est consacrée à l'étude des invariants de torsion des multi-entrelacs. Etant donné un multi-entrelacs $L(\underline{m})$ d'extérieur X, et k un entier supérieur ou égal à 1, le $\mathbf{k^e}$ **revêtement cyclique** de $L(\underline{m})$ est le revêtement $\widetilde{X}_k \longrightarrow X$ déterminé par l'homomorphisme $\pi_1 X \xrightarrow{\underline{m}} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/k$. Comme pour le cas $k=0$, le seul module intéressant est $\mathrm{H}_1\widetilde{X}_k$; son écriture comme produit de $\mathbb{Z}$-modules cycliques fournit une suite d'entiers appelés les $\mathbf{k^e}$ **nombres de torsion** de $L(\underline{m})$. Sans surprise, les matrices de Seifert $A_F^{\pm}$ permettent de calculer ces invariants; plus précisément, la matrice

$$k{-}1 \text{ blocs} \left\{ \begin{pmatrix} \overbrace{\quad}^{n} & \overbrace{\begin{matrix} A^+ & & & A^- \\ -A^- & A^+ & & A^- \\ & \ddots & \ddots & \vdots \\ & & -A^- & A^+ + A^- \end{matrix}}^{k-1 \text{ blocs}} \\ 0 & \end{pmatrix} \right.$$

est une matrice de présentation du $\mathbb{Z}$-module $\mathrm{H}_1\widetilde{X}_k$ (théorème 1.4.3).

Chapitre 2: Le polynôme d'Alexander

A l'aide des surfaces qui portent son nom, H. Seifert [41] a caractérisé, parmi tous les polynômes de Laurent à coefficients entiers, ceux qui sont polynômes d'Alexander d'un noeud. Dans le chapitre 2, nous tentons de marcher sur ses pas: il s'agit d'étudier en détail les modules d'homologie $\mathrm{H}_1 F$ et $\mathrm{H}_1 \overline{F}$ ainsi que les formes de Seifert associées pour en tirer des propriétés générales des polynômes $\Delta^{L(\underline{m})}$. Le résultat principal est le suivant.

Soient $L(m_1, \dots, m_n)$ un multi-entrelacs, $\Delta^{L(\underline{m})}(t)$ son polynôme d'Alexander et d le plus grand commun diviseur de ses poids (que l'on supposera non nul); notons encore $\ell_{ij} = \ell k(L_i, L_j)$ si $i \neq j$, $\ell_{ii} = 0$, $m_i' = \sum_j m_j \ell_{ij}$ et $d_i = \mathrm{pgcd}(m_i, m_i')$. Alors, il existe un unique polynôme $\nabla^{L(\underline{m})}(t)$ dans $\mathbb{Z}[t, t^{-1}]$ qui satisfait les propriétés suivantes (théorème 2.1.6):

- $\Delta^{L(\underline{m})}(t) \doteq \frac{1}{t^d - 1} \prod_{i=1}^n (t^{d_i} - 1) \cdot \nabla^{L(\underline{m})}(t)$;
- $\nabla^{L(\underline{m})}(t^{-1}) = \nabla^{L(\underline{m})}(t)$, et le coefficient dominant de $\nabla^{L(\underline{m})}(t)$ est positif.

De plus, ce polynôme satisfait:

- $|\nabla^{L(\underline{m})}(1)| = \frac{d^2\ D}{d_1 \cdots d_n m_1 \cdots m_n}$, où D est un $(n-1)$-mineur quelconque de la matrice

$$\begin{pmatrix} -\sum_j m_1 m_j \ell_{1j} & m_1 m_2 \ell_{12} & \dots & m_1 m_n \ell_{1n} \\ m_1 m_2 \ell_{12} & -\sum_j m_2 m_j \ell_{2j} & \dots & m_2 m_n \ell_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_1 m_n \ell_{1n} & m_2 m_n \ell_{2n} & \dots & -\sum_j m_n m_j \ell_{nj} \end{pmatrix};$$

- $\nabla^{L(m_1, \dots, m_{n-1}, 0)}(t) = \nabla^{L(m_1, \dots, m_{n-1})}(t)$.

En particulier, il existe un représentant canonique (au signe près) du polynôme d'Alexander, lequel n'était a priori défini que modulo $\pm t^\nu$. On démontre également que pour tout multi-entrelacs $L(\underline{m})$, ce représentant satisfait $\Delta^{L(\underline{m})} = \Delta^{L(-\underline{m})}$ (corollaire 2.1.7). Néanmoins, la caractérisation complète des polynômes d'Alexander de multi-entrelacs semble hors d'atteinte.

A présent, considérons un entrelacs orienté $L = L_1 \cup \ldots \cup L_n$ dans une 3-sphère d'homologie, et notons X son extérieur. L'homomorphisme de Hurewicz $\pi_1 X \longrightarrow \mathrm{H}_1 X$ induit un revêtement $\widehat{X} \longrightarrow X$, appelé le revêtement universel abélien de X. Les méridiens $t_1, \ldots, t_n$ de L forment une $\mathbb{Z}$-base de $\mathrm{H}_1 X$, ce qui munit $\mathrm{H}_1 \widehat{X}$ d'une structure naturelle de module sur $\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. On définit le **polynôme d'Alexander multivarié** de L comme le plus grand commun diviseur Δ_L des éléments de l'idéal d'ordre de $\mathrm{H}_1 \widehat{X}$. Le problème de la caractérisation de ces polynômes a été posé par Ralph Fox en 1962 [12, Problem 2]. A l'époque, les seules conditions connues étaient les fameuses *conditions de Torres* (voir [44, 45]). Ces dernières sont définies récursivement à partir des conditions de Seifert via une propriété de symétrie et la *formule de Torres* que voici:

$$\Delta_L(t_1, \ldots, t_{n-1}, 1) \doteq \begin{cases} (t_1^{\ell_1} \cdots t_{n-1}^{\ell_{n-1}} - 1) \cdot \Delta_{L'}(t_1, \ldots, t_{n-1}) & \text{si } n > 2; \\ \frac{t_1^{\ell_1} - 1}{t_1 - 1} \cdot \Delta_{L'}(t_1) & \text{si } n = 2, \end{cases}$$

où $L = L_1 \cup \ldots \cup L_n$, $\ell_i = \ell k(L_i, L_n)$, et $L' = L_1 \cup \ldots \cup L_{n-1}$. Force est de constater que très peu de progrès ont été réalisés depuis: les conditions de Torres demeurent (presque) les seules conditions nécessaires connues.

Or, le polynôme d'Alexander d'un multi-entrelacs $L(\underline{m})$ est relié au polynôme d'Alexander multivarié de l'entrelacs orienté sous-jacent L par la formule

$$\Delta^{L(\underline{m})}(t) \doteq \begin{cases} \Delta_L(t^{m_1}) & \text{si } n = 1; \\ (t^d - 1) \cdot \Delta_L(t^{m_1}, \ldots, t^{m_n}) & \text{si } n > 1. \end{cases}$$

En utilisant ce "dictionnaire", on peut traduire le théorème principal du chapitre 2 en un certain nombre de résultats portant sur le polynôme multivarié. En particulier, on obtient immédiatement des théorèmes d'Hosokawa [16] (corollaire 2.2.2), de Kidwell [21] (corollaire 2.2.3), ainsi que les conditions de Torres (corollaires 2.2.5 et 2.2.6). Malheureusement, il s'avère que les conditions données sur $\Delta^{L(\underline{m})}$ sont exactement équivalentes aux conditions de Torres sur Δ_L (proposition 2.2.8). Néanmoins, il s'agit d'une preuve entièrement nouvelle (et géométrique) de ces formules, qui fournit un angle d'attaque original pour le problème de Fox.

Rétrospectivement, il n'est pas surprenant que la caractérisation des polynômes d'Alexander de multi-entrelacs soit si ardue: une réponse à cette question fournirait une solution au deuxième problème de Fox, qui constitue certainement une des interrogations majeures dans ce domaine.

Chapitre 3: Module d'Alexander et épissure

Rappelons la définition de l'épissure, déjà mentionnée en introduction. Soient $L' = L'_0 \cup L'_1 \cup \ldots \cup L'_n$ et $L'' = L''_0 \cup L''_1 \cup \ldots \cup L''_r$ deux entrelacs dans des sphères d'homologie Σ' et Σ''. Choisissons des voisinages tubulaires $\mathcal{N}(L'_0)$ et $\mathcal{N}(L''_0)$ munis de parallèles et de méridiens standards $P', M' \subset \partial\mathcal{N}(L_0')$, et $P'', M'' \subset \partial\mathcal{N}(L_0'')$; posons $\Sigma = (\Sigma' - \mathcal{N}(L_0')) \cup_h (\Sigma'' - \mathcal{N}(L_0''))$, où $h\colon \partial\mathcal{N}(L_0') \longrightarrow \partial\mathcal{N}(L_0'')$ est un homéomorphisme envoyant P' sur M'' et M' sur P''. Il est facile de vérifier que Σ est une sphère d'homologie. L'entrelacs $L = L_1' \cup \ldots \cup L_n' \cup L_1'' \cup \ldots \cup L_r''$ dans Σ est appelé **l'épissure** de L' et L'' le long de L_0' et L_0'' [43, 11]; il est noté

$$L = L' \underset{L_0' \quad L_0''}{\overline{\qquad\quad}} L''.$$

Réciproquement, considérons un entrelacs L dans une sphère d'homologie Σ, et T un tore plongé dans $\Sigma - L$; alors, L est le résultat d'une unique épissure $L' \underset{L_0' \quad L_0''}{\overline{\qquad}} L''$ le long de T. On dit que L a été **décomposé** le long de T.

Supposons à présent qu'un entrelacs non orienté L d'extérieur X soit le résultat de l'épissure de deux entrelacs non orientés L' et L'', d'extérieurs X' et X''. Par définition, X est égal à l'union $X' \cup X''$ le long d'un certain tore. Ainsi, toute structure de multi-entrelacs $\underline{m}$ sur L (par exemple, une orientation de L) induit des structures de multi-entrelacs $\underline{m}'$ sur L' et $\underline{m}''$ sur L'': ce sont tout simplement les restrictions de la classe de cohomologie $\underline{m} \in \mathrm{H}^1 X = \mathrm{H}^1(X' \cup X'')$ à $\mathrm{H}_1 X'$ et $\mathrm{H}_1 X''$. Un des points cruciaux (et l'une des raisons d'être des multi-entrelacs) est que même si l'on part avec des poids ± 1, on peut parfaitement aboutir à des multi-entrelacs avec poids supérieurs à un.

Notons encore que l'épissure $L'(\underline{m}') \underset{m_0' L_0' \quad m_0'' L_0''}{\overline{\qquad\qquad}} L''(\underline{m}'')$ de deux multi-entrelacs est définie si et seulement si

$$m_0' = \sum_j m_j'' \ell k(L_0'', L_j'') \quad \text{ et } \quad m_0'' = \sum_i m_i' \ell k(L_0', L_i').$$

Géométriquement, cela correspond au fait que deux surfaces de Seifert F' pour $L'(\underline{m}')$ et F'' pour $L''(\underline{m}'')$ peuvent se recoller le long du tore d'épissure pour former une surface de Seifert $F = F' \cup F''$ pour $L(\underline{m})$.

Le but du chapitre 3 est de trouver une formule qui relie le module d'Alexander de deux multi-entrelacs $L'(\underline{m}')$ et $L''(\underline{m}'')$ au module d'Alexander du multi-entrelacs donné par l'épissure $L'(\underline{m}') \underset{m_0' L_0' \quad m_0'' L_0''}{\overline{\qquad\qquad}} L''(\underline{m}'')$. Notons $d' = \mathrm{pgcd}(\underline{m}')$, $d'' = \mathrm{pgcd}(\underline{m}'')$, $d = \mathrm{pgcd}(\underline{m})$ (qui est égal au $\mathrm{pgcd}(d', d'')$), et $m = \mathrm{pgcd}(m_0', m_0'')$. Le problème est parfaitement résolu si $m = 0$ (propositions 3.2.1 et 3.2.2), $m = 1$ (corollaire 3.2.6), et si $m = d'$ ou $m = d''$. En revanche, le cas général n'est pas réglé: si l'on note $\mathcal{M}$, $\mathcal{M}'$ et $\mathcal{M}''$ les modules d'Alexander respectifs des multi-entrelacs $L(\underline{m})$, $L'(\underline{m}')$ et $L''(\underline{m}'')$, on obtient par Mayer-Vietoris une suite exacte de la forme

$$0 \longrightarrow (\mathcal{M}' \oplus \mathcal{M}'')/\sim \longrightarrow \mathcal{M} \longrightarrow \mathbb{Z}[t, t^{-1}]/(p(t)) \longrightarrow 0,$$

où $p(t) = \frac{(t^m - 1)(t^d - 1)}{(t^{d'} - 1)(t^{d''} - 1)}$ (proposition 3.2.5). En général, la suite n'est pas scindée, et il est extrêmement ardu d'en tirer une formule donnant $\mathcal{M}$ en fonction de $\mathcal{M}'$ et $\mathcal{M}''$. La grande difficulté de ce problème (ou sa grande richesse) peut être interprétée géométriquement de la façon suivante: si F' et F'' sont des surfaces de Seifert respectives pour $L'(\underline{m}')$ et $L''(\underline{m}'')$, l'entier m représente le nombre de composantes connexes de l'intersection $F' \cap F''$. Lorsque m est plus petit que 2, l'homologie de $F = F' \cup F''$ s'obtient facilement à partir de l'homologie de F' et F''. En revanche, si m est supérieur ou égal à 2, certains “grands cycles” peuvent apparaître sur F; il est très difficile de les décrire. En fait, le polynôme $p(t)$ est nul si et seulement si il n'y a pas de grands cycles sur F.

Néanmoins, on obtient la multiplicativité de $\Delta^{L(\underline{m})}$ via épissure (théorème 3.2.9, déjà bien connu d'Eisenbud et Neumann), ainsi qu'un résultat analogue pour le polynôme d'Alexander multivarié Δ_L (théorème 3.2.10). En corollaire, on retombe sur la formule de Seifert-Torres pour les entrelacs satellites (corollaire 3.2.11).

Dans la section 3.3, l'épissure d'un (multi-)noeud $m'L'$ et d'un multi-entrelacs $L''(\underline{m}'')$ est considérée. Ce cas ne présente pas de grande difficulté: le module d'Alexander $\mathcal{M}$ du résultat de l'épissure est donné par

$$\mathcal{M} = \begin{cases} \mathcal{M}' \oplus \mathcal{M}''_0 & \text{si } m' \neq 0; \\ \mathcal{M}''_0 & \text{si } m' = 0, \end{cases}$$

où $\mathcal{M}'$ désigne le module d'Alexander du multi-noeud et $\mathcal{M}''_0$ le module d'Alexander du multi-entrelacs auquel on a ôté la composante le long de laquelle est effectuée l'épissure (théorème 3.3.1).

Chapitre 4: Le module d'Alexander des entrelacs seifertiques

Le théorème de décomposition de Jaco-Shalen-Johansson [17, 18] et le théorème d'hyperbolisation de Thurston (voir [3]) sont deux des résultats les plus remarquables en topologie de dimension trois. Depuis l'avènement de ces théories, on sait que les 3-variétés "suffisamment grandes" se décomposent de façon canonique en morceaux munis de structures géométriques. Plus précisément: si M est une 3-variété à bord, irréductible, connexe, orientable et compacte, il existe une famille finie de tores essentiels dans M telle que chaque composante connexe de la variété M découpée le long de ces tores est soit seifertique, soit hyperbolique; de plus, il existe une famille minimale avec cette propriété, unique à isotopie près.

Lorsque M est l'extérieur d'un entrelacs irréductible L dans une sphère d'homologie, la notion d'épissure permet de traduire ce théorème comme suit: il existe une unique famille minimale de tores essentiels dans M telle que si l'on décompose L le long de ces tores, chaque entrelacs obtenu est soit seifertique, soit hyperbolique. Ces entrelacs sont appelés les **composantes d'épissure** de L. Comme on l'a vu, chaque composante d'épissure hérite d'une structure naturelle de multi-entrelacs. On obtient donc: tout multi-entrelacs irréductible se décompose de façon canonique en multi-entrelacs seifertiques ou hyperboliques. Dans ce contexte, l'introduction de la structure de multi-entrelacs est motivée par l'additivité d'une foule d'invariants via cette décomposition canonique: le polynôme d'Alexander, on l'a vu, mais aussi la "fibrabilité". En effet, un multi-entrelacs est fibré si et seulement si il est irréductible et chacune de ses composantes d'épissure est fibrée.

Il s'avère que les multi-entrelacs algébriques sont ce qu'on appelle des **multi-entrelacs de Waldhausen** (ou **multi-entrelacs graphés**): toutes leurs composantes d'épissure sont seifertiques. Ces multi-entrelacs forment une classe très intéressante: en se basant sur des résultats de Seifert, Eisenbud et Neumann en ont donné une classification au moyen d'arbres décorés appelés **diagrammes d'épissure**. Ces diagrammes sont extrêmement pratiques à plusieurs égards: ils permettent entre autres de calculer des coefficients d'enlacement, et de déterminer si le multi-entrelacs est fibré. Ces résultats sont rappelés en section 4.2.

La question à laquelle est consacré l'essentiel du reste de cette thèse est la suivante: comment calculer le module d'Alexander d'un multi-entrelacs de Waldhausen à partir de son diagramme d'épissure ? Le premier pas consiste à tenter d'y répondre pour les multi-entrelacs "élémentaires", c'est-à-dire, les multi-entrelacs seifertiques. Ce problème est complètement résolu dans le chapitre 4. Les techniques de calcul diffèrent cependant du tout au tout, selon que le multi-entrelacs fibre ou non.

La section 4.3 est consacrée au cas non fibré. Ici, le résultat s'avère étonnement simple: si $L(\underline{m})$ est un multi-entrelacs seifertique non fibré à n composantes, avec $d = \text{pgcd}(\underline{m})$, le module d'Alexander est donné par

$$\mathbb{Z}[t, t^{-1}]/\alpha\,(t^d - 1) \;\oplus\; \mathbb{Z}[t, t^{-1}]^{\,n-2},$$

avec α un certain entier (théorème 4.3.1). Ce résultat est d'autant plus intéressant que le polynôme d'Alexander d'un tel multi-entrelacs est presque toujours nul, et ne donne donc que très peu d'information. L'idée de la démonstration est la suivante. Le fait que $L(\underline{m})$ ne fibre pas implique qu'il existe une surface de Seifert F en anneaux pour $L(\underline{m})$. Il est facile de calculer $\mathrm{H}_1 F$ et $\mathrm{H}_1 \overline{F}$, ainsi que des matrices de Seifert associées. Néanmoins, cette surface de Seifert n'est pas bonne en général: il faut encore procéder à des chirurgies pour en réduire le nombre de composantes connexes. En utilisant les résultats du chapitre 1, on en déduit une matrice d'Alexander pour $L(\underline{m})$.

Dans le cas fibré en revanche (section 4.4), le module d'Alexander est un invariant très riche. Il est possible de donner une matrice d'Alexander, mais celle-ci s'avère très compliquée (théorème 4.4.6). La technique de la démonstration n'est pas aussi originale que dans le cas non fibré: il s'agit d'une généralisation naturelle d'un calcul de F. Michel et C. Weber [26] basé sur un théorème de F. Waldhausen [47]. En corollaire, on obtient quelques résultats (connus), comme le polynôme d'Alexander d'un multi-entrelacs seifertique fibré (corollaire 4.4.7), et la décomposition en modules cycliques du module d'Alexander d'un noeud seifertique (corollaire 4.4.8).

Chapitre 5: Sur le module d'Alexander des entrelacs de Waldhausen

Le problème posé au chapitre 4 était de calculer le module d'Alexander d'un multi-entrelacs de Waldhausen à partir de son diagramme d'épissure. On l'a vu, la question est réglée dans le cas seifertique. En revanche, le cas général se révèle hors d'atteinte, et ce pour une raison très simple: au chapitre 3, nous n'avons pas réussi à fournir une formule close décrivant le comportement du module d'Alexander via épissure. Ce problème n'ayant été résolu que dans certains cas favorables, la question initiale ne trouve ici de réponse que pour certains multi-entrelacs bien particuliers: les noeuds de Waldhausen (théorème 5.1.1) et les multi-entrelacs de Waldhausen dont aucune composante d'épissure ne fibre (théorème 5.2.2). En outre, une formule générale est donnée pour le rang du module d'Alexander (théorème 5.3.1).

Cependant, il existe un moyen de simplifier ce problème: il s'agit de travailler sur un anneau plus grand, en d'autres termes: d'étendre les scalaires. Une façon naturelle de le faire a été proposée par S.P. Novikov [32] (dans un contexte complètement différent): **l'homologie de Novikov** d'un multi-entrelacs $L(\underline{m})$ est le $\mathbb{Z}[\![t]\!][t^{-1}]$-module

$$\widehat{H}_L(\underline{m}) = \mathrm{H}_1 \widetilde{X}(\underline{m}) \;\otimes_{\mathbb{Z}[t,t^{-1}]}\; \mathbb{Z}[\![t]\!][t^{-1}].$$

L'anneau principal $\mathbb{Z}[\![t]\!][t^{-1}]$ est appelé **l'anneau de Novikov**. Ses unités sont exactement les éléments dont le premier coefficient non nul est égal à ± 1; en particulier, le polynôme d'Alexander d'un multi-entrelacs fibré est une unité de l'anneau de Novikov, si bien que l'homologie de Novikov d'un multi-entrelacs fibré est triviale. De même, le polynôme $p(t) = \frac{(t^m-1)(t^d-1)}{(t^{d'}-1)(t^{d''}-1)}$, qui est à l'origine de tous nos soucis, est une unité de $\mathbb{Z}[\![t]\!][t^{-1}]$. Ainsi, il est possible (à l'aide des chapitres précédents) de trouver une formule pour l'homologie de Novikov de n'importe quel multi-entrelacs de Waldhausen (théorème 5.4.2). En corollaire, on obtient le résultat suivant: soit L un entrelacs de Waldhausen avec c composantes irréductibles et k composantes d'épissure; alors, le nombre de modules de Novikov (correspondants aux différents poids $\underline{m}$) est borné par $3^k - 2(k-c)$.

Dans l'introduction de leur livre, Eisenbud et Neumann avaient formulé quatre problèmes liés aux multi-entrelacs de Waldhausen. Le second peut s'énoncer de la façon suivante: calculer le module d'Alexander sur $\mathbb{C}[t, t^{-1}]$ d'un multi-entrelacs de Waldhausen à partir de son diagramme d'épissure. L'anneau $\mathbb{C}[t, t^{-1}]$ fournit une autre extension des scalaires naturelle, qui simplifie sensiblement le problème. En effet, un module de type fini sur $\mathbb{C}[t, t^{-1}]$ est entièrement déterminé par son rang et par la forme normale de Jordan de l'endomorphisme donné par la multiplication par t restreinte au sous-module de torsion. Ce problème est résolu par Eisenbud et Neumann dans le cas fibré, où le rang est nul et la multiplication par t n'est autre que la monodromie algébrique. En revanche, le cas non fibré n'est pas considéré par ces deux auteurs. En section 5.5, nous calculons le module d'Alexander sur $\mathbb{C}[t, t^{-1}]$ d'une vaste classe de multi-entrelacs de Waldhausen (théorème 5.5.2). En deux mots, il s'agit de tous les multi-entrelacs de Waldhausen tels que le sous-diagramme d'épissure constitué des composantes fibrées soit connexe. Cette classe est suffisamment générale pour contenir tous les entrelacs apparaissant comme entrelacs réguliers à l'infini d'une application polynomiale $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$. En particulier, on obtient donc le module d'Alexander sur $\mathbb{C}[t, t^{-1}]$ de tout entrelacs régulier à l'infini (théorème 5.6.1).

Chapitre 6: La fonction potentiel d'un entrelacs de Waldhausen

En 1970, Conway [9] a introduit un nouvel invariant d'entrelacs orientés appelé la **fonction potentiel**. Etant donné un entrelacs orienté $L = L_1 \cup \ldots \cup L_n$, sa fonction potentiel est une fonction rationnelle $\nabla_L(t_1, \ldots, t_n)$ reliée au polynôme d'Alexander multivarié par la formule suivante:

$$\nabla_L(t_1, \ldots, t_n) \doteq \begin{cases} \frac{1}{t_1 - t_1^{-1}} \Delta_L(t_1^2) & \text{si } n = 1; \\ \Delta_L(t_1^2, \ldots, t_n^2) & \text{si } n > 1. \end{cases} \tag{$*$}$$

Ainsi, cet invariant permet de choisir un représentant canonique du polynôme d'Alexander multivarié (défini quant à lui modulo $\pm t_1^{\nu_1} \cdots t_n^{\nu_n}$). La fonction potentiel possède certaines propriétés remarquables. Entre autres, on peut la calculer à partir d'une projection de l'entrelacs via des "relations d'échange". De plus, la formule de Torres pour Δ_L se traduit en la formule suivante, beaucoup plus naturelle: soit $L = L_1 \cup \ldots \cup L_n$ un entrelacs orienté, et L' le sous-entrelacs $L_1 \cup \ldots \cup L_{n-1}$; alors leurs fonctions potentiel respectives satisfont l'égalité

$$\nabla_L(t_1, \ldots, t_{n-1}, 1) = (t_1^{\ell_1} \cdots t_{n-1}^{\ell_{n-1}} - t_1^{-\ell_1} \cdots t_{n-1}^{-\ell_{n-1}}) \cdot \nabla_{L'}(t_1, \ldots, t_{n-1}), \tag{$**$}$$

où ℓ_i désigne le coefficient d'enlacement $\ell k(L_i, L_n)$.

Notons encore que ce que l'on appelle communément le **polynôme de Conway** est défini par la formule

$$\Omega_L(t) = (t - t^{-1}) \cdot \nabla_L(t, \ldots, t).$$

Ce fameux invariant n'est donc qu'un cas très particulier de la fonction potentiel de Conway.

Il serait très intéressant de savoir calculer la fonction potentiel d'un entrelacs de Waldhausen à partir de son diagramme d'épissure. En 1999, W. Neumann [31] a réussi à calculer le polynôme de Conway Ω_L de tout entrelacs de Waldhausen fibré dans S^3; et il pose la question de la validité des formules obtenues si l'on considère des entrelacs de Waldhausen quelconques (fibrés ou non) dans des sphères d'homologie quelconques.

Dans le chapitre 6, nous apportons une réponse affirmative aux interrogations de Neumann concernant Ω_L. Mieux: nous calculons la fonction potentiel de n'importe quel entrelacs de Waldhausen dans une sphère d'homologie (théorème 6.2.7). En fait, le polynôme d'Alexander multivarié d'un entrelacs de Waldhausen est connu depuis Eisenbud-Neumann. Via la formule $(*)$ et une propriété de symétrie de ∇_L, il est facile de calculer la fonction potentiel d'un entrelacs de Waldhausen au signe près. Toute la difficulté est de déterminer ce signe. L'idée est alors de procéder par induction sur le nombre de composantes de l'entrelacs, en utilisant la formule de Torres $(**)$ pour le pas de récurrence. La formule finale est extrêmement simple. En corollaires immédiats, on obtient les réponses aux questions de Neumann (corollaires 6.2.8, 6.2.9 et 6.2.11), ainsi que le résultat amusant suivant: toute matrice de Seifert d'un noeud de Waldhausen est de déterminant $+1$ (corollaire 6.2.10).

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Index