ANDREWS–GORDON TYPE IDENTITIES WITH PARITY RESTRICTIONS THROUGH PARTICLE MOTION

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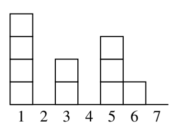
Abstract. In this paper, we use the particle motion bijection introduced by Warnaar and developed by the two authors, Jouhet and Konan, to study q-series and partition identities of the Andrews–Gordon type with parity restrictions. These restrictions are of the type "even (resp. odd) parts appear an even number of times". We prove q-series identities where a multisum equals a sum of products, which generalise identities of Andrews and Kim–Yee in a similar way that Stanton's identities generalised the Andrews–Gordon identities. As a consequence of our results, we obtain a simple proof of a recent identity of Chern–Li–Stanton–Xue–Yee related to Ariki–Koike algebras.

1. Introduction

A *partition* of a positive integer n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers whose sum is n. The integer n is called the *weight* of the partition λ and is also denoted $|\lambda|$, and ℓ is called its *length*. Every partition λ can also be written in terms of its *frequency sequence* $f = (f_1, f_2, \cdots)$, where f_j denotes the number of occurrences of the part j in λ. Its weight is $|f| = \sum_{k\geq 1} kf_k$. In this paper, we will often write partitions in terms of their frequency sequences and represent them graphically with columns of boxes on the x-axis.

Example 1.1. The partition $\lambda = (6, 5, 5, 5, 3, 3, 1, 1, 1, 1)$ has frequency sequence $f = (4, 0, 2, 0, 3, 1, 0, 0, \ldots)$ which can be represented by the picture below.



Using the *q-Pochhammer symbols* (see [GR04])

$$(a; q)_n := (a)_n = \prod_{k=0}^{n-1}(1 - aq^k),$$
$$(a_1, \ldots, a_m; q)_n := (a_1; q)_n \cdots (a_m; q)_n,$$

for $n \in \mathbb{N} \cup \{+\infty\}$, the *Rogers–Ramanujan identities* [RR19] state that for $a \in \{0, 1\}$,

$$\sum_{n\geq 0} \frac{q^{n^2+(1-a)n}}{(q)_n} = \frac{1}{(q^{2-a}, q^{3+a}; q^5)_\infty}. \tag{1.1}$$

As combinatorial identities, they assert that for $a \in \{0, 1\}$ and for all $n \in \mathbb{N}$, the number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n such that $\lambda_i - \lambda_{i+1} \geq 2$ for all i, where the part 1 appears at most a times, is equal to the number of partitions of n into parts congruent to $\pm(2 - a)$ modulo 5. Equivalently, in terms of frequency sequences, they can be written as follows: for $a \in \{0, 1\}$ and for all $n \in \mathbb{N}$, the number of partitions f of n such that $f_i + f_{i+1} \leq 1$ for all i and $f_1 \leq a$, equals the number of partitions of n into parts congruent to $\pm(2 - a)$ modulo 5.

The Rogers–Ramanujan identities are undoubtedly among the most famous partition/q-series identities and appear in several fields of mathematics: they revealed the role of vertex operator algebras in representation theory [LW84, LW85], arise naturally in the theory of arc spaces in algebraic geometry [BMS13], and played a key role in the exact solution of the hard hexagonal model in mathematical physics [Bax81], to name only a few examples. For more detail about these identities, see the book [Sil18].

The Rogers–Ramanujan identities were extended combinatorially by Gordon [Gor61].

Theorem 1.2 (Gordon's identities [Gor61])**.** *For integers $k \geq 1$ and $0 \leq r \leq k$, the number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n such that $\lambda_i - \lambda_{i+k} \geq 2$ for all i, and where the part 1 appears at most $(k - r)$ times, is equal to the number of partitions of n into parts not congruent to $0, \pm(k - r + 1)$ modulo $2k + 3$.*

Equivalently, the difference condition above can be rephrased in terms of frequencies as follows:

(1.2) partitions of n whose frequency sequence $(f_i)_{i\geq 1}$ satisfies $f_i + f_{i+1} \leq k$ for all i, and $f_1 \leq k - r$.

Andrews [And74] then found a q-series counterpart to the Gordon identities, obtaining what is now known as the Andrews–Gordon identities.

Theorem 1.3 (Andrews–Gordon identities [And74])**.** *Let $k \geq 1$ and $0 \leq r \leq k$ be two integers. Then*

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \frac{(q^{k+1-r}, q^{k+2+r}, q^{2k+3}; q^{2k+3})_\infty}{(q)_\infty}. \tag{1.3}$$

While it is clear that the right-hand side of (1.3) is the generating function for partitions into parts not congruent to $0, \pm(k-r+1)$ modulo $2k+3$, it is much harder to prove that the left-hand side is the generating function for partitions with difference/frequency conditions. This fact was originally proved by Andrews [And74] using recurrences, and it took more than twenty years until a bijective proof was found by Warnaar [War97]. Note that this bijection only shows the correspondence between the sum side of (1.3) and the frequency conditions; finding a bijective proof of the Andrews–Gordon identities themselves is still an open problem.

Warnaar's bijection relies on *particle motion* on frequency sequence diagrams starting from a minimal partition satisfying the difference conditions. It was applied on usual frequency sequences $f = (f_1, f_2, \cdots)$. In [DJK24], the first author, Jouhet and Konan generalised Warnaar's approach by allowing parts of size 0, hence considering frequency sequences of the form $f = (f_0, f_1, \cdots)$. For the purposes of this paper, we need to consider parts of size 0 and -1. For simplicity and generality, we describe the theory in an even more general context by applying particle motion to generalised frequency sequences allowing parts of any size. So from now on, a *partition* will be a finite non-increasing sequence of *non-negative* integers, and a *generalised frequency sequence* will be a sequence $(f_i)_{i\in\mathbb{Z}}$ such that only finitely many of the f_u's are non-zero. The principle of Warnaar's particle motion and its generalisations is described in Section 2, together with applications. For instance, in [DJJ25], the two authors and Jouhet used their generalisation of particle motion to prove this theorem of Stanton, which generalises Theorem 1.3.

Theorem 1.4. [Sta18, Theorem 3.2] *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{k+1-r+j-2s}, q^{k+2+r-j+2s}, q^{2k+3}; q^{2k+3})_\infty}{(q)_\infty}. \tag{1.4}$$

In this paper, our goal is to use particle motion to prove q-series identities related to partitions with parity conditions.

The study of partition identities with the frequency conditions (1.2) but with added parity conditions originated from the work of Andrews [And10], and was followed by several others since [Kur10, KY13, KL24, CLS⁺24]. Let us first describe Andrews' result, slightly reformulated to fit the context of generalised frequency sequences.

Definition 1.5. Let $\mathcal{W}_{k,a}$ (resp. $\overline{\mathcal{W}}_{k,a}$) denote the set of generalised frequency sequences $(f_i)_{i\in\mathbb{Z}}$ such that

- $f_i + f_{i+1} \leq k$ for all $i > 0$,
- $f_i = 0$ for all $i \leq 0$,
- $f_1 \leq a$,
- f_i is even if i is even (resp. odd). That is, even (resp. odd) parts appear an even number of times.

Andrews proved the following two theorems using recurrence equations.

Theorem 1.6 (Andrews [And10])**.** *For $k \geq a \geq 0$ and $k \equiv a \pmod 2$,*

$$\sum_{f\in\mathcal{W}_{k,a}} q^{|f|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+2s_{a+1}+2s_{a+3}+\cdots+2s_{k-1}}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{(-q;q^2)_\infty(q^{a+1}, q^{2k+3-a}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Theorem 1.7 (Andrews [And10])**.** *For $k \geq a \geq 1$, k even and a odd,*

$$\sum_{f\in\overline{\mathcal{W}}_{k,a}} q^{|f|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+(s_1-s_2+s_3-\cdots+s_{a-2}-s_{a-1})+(s_a+s_{a+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{(-q^2;q^2)_\infty(q^{a+1}, q^{2k+3-a}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Then, using Gordon marking, Kurşungöz [Kur10] was able to give a formula for the generating function of $\mathcal{W}_{k,a}(n)$ for $k \not\equiv a \pmod 2$ and $\overline{\mathcal{W}}_{k,a}(n)$ for k odd and a even as a sum. We point out that in these particular cases, as well as for the sum side of the Andrews–Gordon identities, Gordon marking is equivalent to particle motion. Then Kim and Yee [KY13] reproved these sum sides and were able to give product (or sum of products) expressions as well, using recurrences and q-series identities. Their results are the following.

Theorem 1.8 (Kim–Yee [KY13], sum side by Kurşungöz [Kur10])**.** *For $k \geq a \geq 0$ and $k \not\equiv a \pmod 2$,*

$$\sum_{f\in\mathcal{W}_{k,a}} q^{|f|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+2s_{a+1}+2s_{a+3}+\cdots+2s_k}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{(-q^3;q^2)_\infty(q^{a+2},q^{2k+2-a},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty} + q\cdot\frac{(-q^3;q^2)_\infty(q^{a},q^{2k+4-a},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Theorem 1.9 (Kim–Yee [KY13], sum side by Kurşungöz [Kur10])**.** *For $k \geq a \geq 0$, k odd and a even,*

$$\sum_{f\in\overline{\mathcal{W}}_{k,a}} q^{|f|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+(s_1-s_2+s_3-\cdots+s_{a-1}-s_a)+(s_{a+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{(-q^2;q^2)_\infty(q^{a+2},q^{2k+2-a},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Note that for $k \geq a \geq 1$ and a odd, $\overline{\mathcal{W}}_{k,a} = \overline{\mathcal{W}}_{k,a-1}$. The cases where $k \equiv a \pmod 2$ thus follow immediately from Theorems 1.7 and 1.9. That is, if a and k are even, then

$$\sum_{f\in\overline{\mathcal{W}}_{k,a}} q^{|f|} = \sum_{f\in\overline{\mathcal{W}}_{k,a+1}} q^{|f|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+(s_1-s_2+s_3-\cdots+s_{a-1}-s_a)+(s_{a+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{(-q^2;q^2)_\infty(q^{a+2},q^{2k+2-a},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

If a and k are odd, then

$$\sum_{f\in\overline{\mathcal{W}}_{k,a}} q^{|f|} = \sum_{f\in\overline{\mathcal{W}}_{k,a-1}} q^{|f|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+(s_1-s_2+s_3-\cdots+s_{a-2}-s_{a-1})+(s_{a}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{(-q^2;q^2)_\infty(q^{a+1},q^{2k+3-a},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Thus all the above results can be unified as follows.

Theorem 1.10 (Unification of Theorems 1.6–1.9)**.** *For all integers $k \geq a \geq 0$,*

$$\sum_{f\in\mathcal{W}_{k,a}} q^{|f|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+2s_{a+1}+2s_{a+3}+\cdots+(1-(-1)^{k+a})s_k}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} \tag{1.5}$$
$$= \frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\left((q^{k-2\lfloor\frac{k-a}{2}\rfloor+1},q^{k+2\lfloor\frac{k-a}{2}\rfloor+3},q^{2k+4};q^{2k+4})_\infty + q(q^{k-2\lfloor\frac{k-a+1}{2}\rfloor+1},q^{k+2\lfloor\frac{k-a+1}{2}\rfloor+3},q^{2k+4};q^{2k+4})_\infty\right),$$

and

$$\sum_{f\in\overline{\mathcal{W}}_{k,a}} q^{|f|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+(s_1-s_2+s_3-\cdots+(-1)^{a+1}s_a)+(s_{a+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} \tag{1.6}$$
$$= \frac{(-q^2;q^2)_\infty(q^{2\lfloor\frac{a+2}{2}\rfloor},q^{2k+4-2\lfloor\frac{a+2}{2}\rfloor},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Our main purpose in this paper is to prove the following identities, which generalise Theorem 1.10 in the same way that Stanton's Theorem 1.4 generalises the Andrews–Gordon identities. We achieve this using particle motion.

Theorem 1.11. *For non-negative integers j, r, and k with $j + r \leq k$, let $\mathcal{Z}^o_{j,r,k}$ be the set of all frequency sequences $(f_i)_{i\in\mathbb{Z}}$ such that*

- $f_i + f_{i+1} \leq k$ *for all* $i \geq 0$,
- $f_i = 0$ *for all* $i < 0$,
- $f_0 \leq j$ *and* $2f_0 + f_1 \leq k - r + j$,
- f_i *is even if* i *is odd. That is, odd parts appear an even number of times.*

Then

$$\sum_{f \in \mathcal{Z}^o_{j,r,k}} q^{|f|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)+(s_{j+1}-s_{j+2}+s_{j+3}-\cdots\pm s_{k-r})+(s_{k-r+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \sum_{s=0}^{j} \frac{(-q^2;q^2)_\infty (q^{2\lfloor \frac{k+2-r+j-2s}{2} \rfloor}, q^{2k+4-2\lfloor \frac{k+2-r+j-2s}{2} \rfloor}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

When $j = 0$, Theorem 1.11 reduces to (1.6) after setting $r = k - a$.

Theorem 1.12. *For non-negative integers* a, b, *and* k *with* $2a + 2b \leq k$, *let* $\mathcal{Z}^e_{a,b,k}$ *be the set of all frequency sequences* $(f_i)_{i \in \mathbb{Z}}$ *such that*

- $f_i + f_{i+1} \leq k$ *for all* $i \geq 0$,
- $f_i = 0$ *for all* $i < 0$,
- $f_0 \leq 2a$ *and* $2f_0 + f_1 \leq k - 2b + 2a$,
- f_i *is even if* i *is even. That is, even parts appear an even number of times.*

Then

$$\sum_{f \in \mathcal{Z}^e_{a,b,k}} q^{|f|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-2(s_2+s_4+\cdots+s_{2a})+2(s_{k-2b+1}+s_{k-2b+3}+\cdots+s_{k-1})}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \sum_{s=0}^{a} \frac{(-q;q^2)_\infty (q^{k+1+2a-2b-4s}, q^{k+3-2a+2b+4s}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Theorem 1.13. *For non-negative integers* a, b, *and* k *with* $2a + 2b - 1 \leq k$, *let* $\widetilde{\mathcal{Z}}^e_{a,b,k}$ *be the set of all frequency sequences* $(f_i)_{i \in \mathbb{Z}}$ *such that*

- $f_i + f_{i+1} \leq k$ *for all* $i \geq 0$,
- $f_i = 0$ *for all* $i < 0$,
- $f_0 \leq 2a$ *and* $2f_0 + f_1 \leq k - 2b + 2a + 1$,
- f_i *is even if* i *is even. That is, even parts appear an even number of times.*

Then

$$\sum_{f \in \widetilde{\mathcal{Z}}^e_{a,b,k}} q^{|f|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-2(s_2+s_4+\cdots+s_{2a})+2(s_{k-2b+2}+s_{k-2b+4}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{s=0}^{a} \left[(q^{k+3+2a-2b-4s}, q^{k+1-2a+2b+4s}, q^{2k+4}; q^{2k+4})_\infty + q(q^{k+1+2a-2b-4s}, q^{k+3-2a+2b+4s}, q^{2k+4}; q^{2k+4})_\infty \right].$$

When $a = 0$, Theorems 1.12 and 1.13 reduce to Theorems 1.6 and 1.8, respectively.

Other nice corollaries follow from them as well. Indeed, Theorem 1.11 implies the following.

Corollary 1.14. *For integers* k *and* a *with* $k \geq 1$ *and* $0 \leq a \leq k$,

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_a)+(s_{a+1}-s_{a+2}+\cdots+(-1)^{k-a-1}s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \sum_{i=0}^{a} \frac{(-q^2;q^2)_\infty (q^{2\lfloor \frac{k-i+2}{2} \rfloor}, q^{2k+4-2\lfloor \frac{k-i+2}{2} \rfloor}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

And Theorems 1.12 and 1.13 imply the following.

Corollary 1.15. *For integers* k *and* a *with* $k \geq 1$ *and* $0 \leq 2a \leq k$,

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-2(s_2+s_4+\cdots+s_{2a})}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{i=0}^{a} \frac{(-q;q^2)_\infty (q^{k+1-2i}, q^{k+3+2i}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

The interest for these theorems with parity restrictions became even greater recently, as it was discovered that they have some connection to the representation theory of Ariki–Koike algebras, introduced independently by Ariki–Koike [AK94] and Broué and Malle [BM93]. Ariki and Mathas [AM00] showed that simple modules of Ariki–Koike algebras are labelled by Kleshchev multipartitions, and that their generating function is

$$\frac{(q^{a+1},q^{k+1-a},q^{k+2};q^{k+2})_\infty}{(q)_\infty(q;q^2)_\infty}, \tag{1.7}$$

which is exactly the product side of Theorem 1.9 where q is replaced by $q^{1/2}$. Interestingly, this product is also the principal specialisation of the character of standard modules of level k for the affine Kac–Moody Lie algebra $A_1^{(1)}$, as studied by Meurman and Primc [MP99] and the first author, Hardiman and Konan [DHK25].

By studying the connection between (1.7) and Ariki–Koike algebras, Chern, Li, Stanton, Xue, and Yee [CLS⁺24] discovered and proved the following q-series identity, which was later reproved by Kanade and Lovejoy [KL24] in a more simple manner using the machinery of Bailey pairs.

Theorem 1.16 ([KL24, CLS⁺24])**.** *Let $k \geq 1$ and $0 \leq a \leq k-1$. Then*

$$\begin{aligned}\frac{(q^{a+1},q^{k+1-a},q^{k+2};q^{k+2})_\infty}{(q)_\infty(q;q^2)_\infty} &= \sum_{n_k,\dots,n_1\geq 0}\frac{q^{\binom{n_k+1}{2}+\cdots+\binom{n_1+1}{2}}}{(q)_{n_k}}\prod_{i=1}^{k-1}\begin{bmatrix}n_{i+1}+\delta_{a,i}\\ n_i\end{bmatrix}_q\\ &= \sum_{\substack{N_k\geq N_{k-1}\geq\cdots\geq N_{a+1}\\ N_{a+1}+1\geq N_a\geq\cdots\geq N_2\geq 0}}\frac{q^{\sum_{i=2}^k\binom{N_i+1}{2}}(-q;q)_{N_{2+\delta_{a+1,2}}}\left(1-q^{N_{a+1}+1}\right)}{\left(1-q^{N_{a+1}-N_a+1}\right)\prod_{i=2}^k(q;q)_{N_i-N_{i-1}}},\end{aligned} \tag{1.8}$$

where $N_1=0$ and $N_{k+1}=\infty$.

Here we used the q-binomial coefficients, defined for all $0\leq j\leq n$ by

$$\begin{bmatrix}n\\ j\end{bmatrix}_q := \frac{(q)_n}{(q)_j(q)_{n-j}}.$$

Particle motion provides another simple proof of Theorem 1.16, as it follows directly from Theorem 1.11.

Indeed, setting $s_i = n_{k+1-i}$ for $i\in\{1,\dots,a\}$ and $s_i = n_{k+1-i}+1$ for $i\in\{a+1,\dots,k\}$, Equation (1.8) can be rewritten as

$$\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{q^{(s_1^2+\cdots+s_k^2-s_1-\cdots-s_{k-a}+s_{k-a+1}+\cdots+s_k)/2}(1-q^{s_{k-a}})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \frac{\left(q^{a+1},q^{k+1-a},q^{k+2};q^{k+2}\right)_\infty}{(q;q)_\infty\,(q;q^2)_\infty}. \tag{1.9}$$

Define

$$\mathrm{AK}_{a,k}(q) := \sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_{k-a}+s_{k-a+1}+\cdots+s_k}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}.$$

Setting $j=k-a$ and $r=a$ in Theorem 1.11 yields the following.

Corollary 1.17.

$$\mathrm{AK}_{a,k}(q) = \sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_{k-a}+s_{k-a+1}+\cdots+s_k}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{i=a}^{k}\frac{(q^{2i+2},q^{2k+2-2i},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty(q^2;q^4)_\infty}.$$

Theorem 1.16 (in the form (1.9)) then follows directly by taking difference $\mathrm{AK}_{a,k}(q)-\mathrm{AK}_{a+1,k}(q)$ in Corollary 1.17 and replacing q by $q^{1/2}$.

Remark 1.18. This is reminiscent of a proof strategy in [ADJM23] where the identity

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0}\frac{q^{s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i}(1-q^{s_i})}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}(q)_{s_{r-1}}} = \frac{(q^{r-i},q^{r+i+1},q^{2r+1};q^{2r+1})_\infty}{(q)_\infty},$$

valid for all integers $r>0$ and $0\leq i\leq r-1$, was proved by taking the difference of the cases i and $i-1$ in this identity of Bressoud [Bre80, (3.3)]:

$$\sum_{s_1\geq\cdots\geq s_{r-1}\geq 0}\frac{q^{s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i}}{(q)_{s_1-s_2}\dots(q)_{s_{r-2}-s_{r-1}}(q)_{s_{r-1}}} = \sum_{k=0}^{i}\frac{(q^{r-i+k},q^{r+i-k+1},q^{2r+1};q^{2r+1})_\infty}{(q)_\infty}.$$

The paper is organised as follows. In Section 2, we introduce particle motion and prove some of its properties, including some related to parity. In Sections 3, 4 and 5, we use particle motion and some enumeration techniques to prove Theorems 1.11, 1.12 and 1.13, respectively, together with their corollaries. We conclude with some open problems.

2. Particle motion and its properties

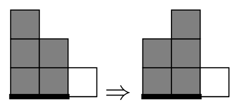
We describe the principle of Warnaar's particle motion adapted to the context of generalised frame sequences, following the formulation of [DJJ25].

Let $f = (f_i)_{i\in\mathbb{Z}}$ be a generalised frequency sequence and u be an index such that, letting $h := f_u + f_{u+1}$,

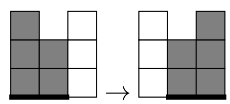
$$f_v + f_{v+1} \leq h \text{ for all } v \geq u.$$

Then the process applied at index u depends on whether $f_{u+1} + f_{u+2} < h$ or $f_{u+1} + f_{u+2} = h$:

(1) If $f_{u+1} + f_{u+2} < h$, then transform (f_u, f_{u+1}) to $(f_u - 1, f_{u+1} + 1)$, leave u unchanged (*particle motion*).



(2) If $f_{u+1} + f_{u+2} = h$, then leave (f_u, f_{u+1}) unchanged, but change u to $u + 1$ (*focus shift*).



Let $\mathsf{pm}_u^{(m)}(f)$ denote the frequency sequence obtained from (f_u, f_{u+1}) after m particle motions (not counting focus shifts) have been applied starting from index u. Moreover, if $\mathsf{pm}_u^{(m)}(f) = (\overline{f}_i)_{i\in\mathbb{Z}}$ and the final focus is on the pair $(\overline{f}_v, \overline{f}_{v+1})$, then we say that the pair (f_u, f_{u+1}) *moves to* $(\overline{f}_v, \overline{f}_{v+1})$.

Note that by construction, $(\overline{f}_i)_{i\in\mathbb{Z}} = \mathsf{pm}_u^{(m)}(f)$ satisfies $\overline{f}_i + \overline{f}_{i+1} \leq h$ for all $i \geq u$. Each particle motion increases the weight of the partition by 1 (as it removes a part u and adds a part $u + 1$), while focus shifts leave the partition unchanged. Both leave the length of the partition unchanged.

For example, if $f = (\dots, 0, f_1 = 4, f_2 = 0, 2, 1, 3, 1, 0, \dots)$, then $\mathsf{pm}_1^{(5)}(f) = (\dots, 0, \overline{f}_1 = 2, 1, 3, 1, 1, 3, 0, \dots)$, and the pair (f_1, f_2) moves to $(\overline{f}_5, \overline{f}_6)$. The process is shown in Figure 1.

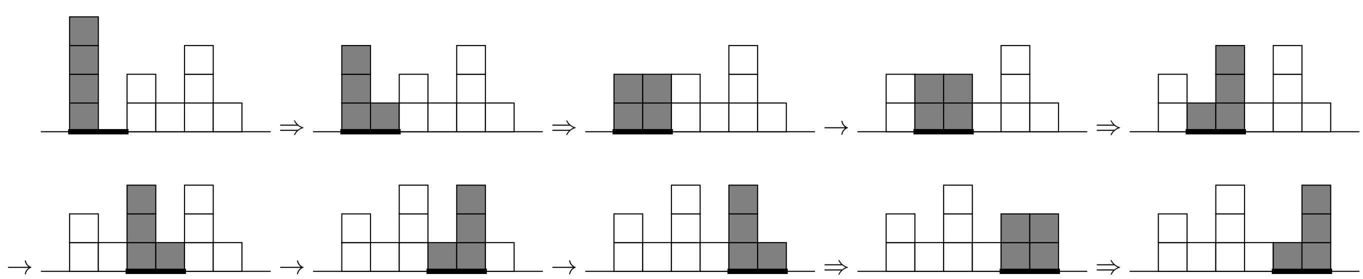


Figure 1. Illustration of applying 5 particle motions starting from index 1 in the frequency sequence $f = (\dots, 0, f_1 = 4, f_2 = 0, 2, 1, 3, 1, 0, \dots)$. The symbol $\Rightarrow$ indicates a particle motion, and $\rightarrow$ indicates a focus shift.

To obtain the left-hand side of (1.3), Warnaar's idea is to start with a minimal partition satisfying (1.2) whose weight is $s_1^2 + \cdots + s_k^2 + s_{k-r+1} + \cdots + s_k$, generating the numerator of (1.3). Then, noting that $\frac{1}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}$ generates k-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ where for all i, $\lambda^{(i)}$ has $s_i - s_{i+1}$ *non-negative* parts (after setting $s_{k+1} := 0$), the parts of $\boldsymbol{\lambda}$ are inserted into the minimal partition via particle motions. More precisely, Warnaar's result is equivalent to the following.

Theorem 2.1 (Warnaar [War97], reformulated)**.** *For integers $k \geq 1$ and $0 \leq r \leq k$, let*

$$\mathcal{O}_{r,k} := \{(\boldsymbol{\lambda}, \tilde{\mathrm{fs}}_{r,k}(\boldsymbol{\lambda})) : \boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)}) \text{ is a } k\text{-tuple of partitions into non-negative parts}\},$$

where

$$\tilde{\mathrm{fs}}_{r,k}(\boldsymbol{\lambda}) := (\dots,0,0,\underbrace{f_1=k-r, f_2=r,\dots,k-r,r}_{s_k \text{ pairs}},\dots,$$
$$\underbrace{\min(k-r,i),\max(0,i-k+r),\dots,\min(k-r,i),\max(0,i-k+r)}_{s_i-s_{i+1}\text{ pairs}},\dots,\underbrace{1,0,\dots,1,0}_{s_1-s_2\text{ pairs}},0,0,\dots),$$

and for all i, $\lambda^{(i)}$ has length $s_i - s_{i+1}$ (after setting $s_{k+1}=0$).

Let

$$\mathcal{B}_{r,k} := \{(f_i)_{i\in\mathbb{Z}} : f_i + f_{i+1} \le k \text{ for all } i,\ f_1 \le k-r, \text{ and } f_i = 0 \text{ for all } i \le 0\}.$$

Define

$$\tilde{\Lambda}(\boldsymbol{\lambda}, \tilde{\mathrm{fs}}_{r,k}(\boldsymbol{\lambda})) = \left(\mathsf{pm}_1^{(\lambda_0)} \circ \mathsf{pm}_3^{(\lambda_1)} \circ \cdots \circ \mathsf{pm}_{2s_1-3}^{(\lambda_{s_1-2})} \circ \mathsf{pm}_{2s_1-1}^{(\lambda_{s_1-1})}\right)\left(\tilde{\mathrm{fs}}_{r,k}(\boldsymbol{\lambda})\right),$$

where we renamed

$$(\lambda_{s_1-1},\dots,\lambda_1,\lambda_0) := (\lambda_1^{(1)},\dots,\lambda_{s_1-s_2}^{(1)},\dots,\lambda_1^{(k-1)},\dots,\lambda_{s_{k-1}-s_k}^{(k-1)},\lambda_1^{(k)},\dots,\lambda_{s_k}^{(k)}).$$

Then $\tilde{\Lambda}$ is a weight-preserving bijection between $\mathcal{O}_{r,k}$ and $\mathcal{B}_{r,k}$.

The partitions of $\mathcal{B}_{r,k}$ being exactly those satisfying the frequency conditions (1.2), and those of $\mathcal{O}_{r,k}$ being generated by the left-hand side of (1.3), this proves bijectively that the generating function for partitions satisfying (1.2) is indeed the left-hand side of (1.3).

Here we define a map Λ which generalises Warnaar's map $\tilde{\Lambda}$, and which we will use in different contexts.

Definition 2.2. Let $\mathcal{P}$ be the set of pairs $(\boldsymbol{\lambda}, (f_i)_{i\in\mathbb{Z}})$, where $\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(k)})$ is a k-tuple of partitions (for some $k \ge 0$) into non-negative parts of total length ℓ, and $(f_i)_{i\in\mathbb{Z}}$ is a generalised frequency sequence where there exists some index u such that

- for all $i \notin \{u, u+1, \dots, u+2\ell-1\}$, $f_i = 0$,
- for all $i \in \{0,\dots,\ell-1\}$, $f_{u+2i} + f_{u+2i+1} \ge f_{u+2i+2} + f_{u+2i+3}$.

Let ℓ_i be the length of $\lambda^{(i)}$ and, as before, rename

$$(\lambda_{\ell-1},\dots,\lambda_1,\lambda_0) := (\lambda_1^{(1)},\dots,\lambda_{\ell_1}^{(1)},\dots,\lambda_1^{(k)},\dots,\lambda_{\ell_k}^{(k)}).$$

Then the map Λ is defined on $\mathcal{P}$ as follows: $\Lambda(\boldsymbol{\lambda}, (f_i)_{i\in\mathbb{Z}})$ is the generalised frequency sequence obtained from $(f_i)_{i\in\mathbb{Z}}$ by applying, for all $i \in \{0,\dots,\ell-1\}$, λ_i particle motions at index $u+2i$, starting from the right (i.e. starting with $i=\ell-1$ and ending with $i=0$). In other words,

$$\Lambda(\boldsymbol{\lambda}, (f_i)_{i\in\mathbb{Z}}) = \left(\mathsf{pm}_u^{(\lambda_0)} \circ \mathsf{pm}_{u+2}^{(\lambda_1)} \circ \cdots \circ \mathsf{pm}_{u+2\ell-2}^{(\lambda_{\ell-1})}\right)((f_i)_{i\in\mathbb{Z}}).$$

In [DJK24], the first author, Jouhet and Konan considered the particular case where in the definition above, $(f_i)_{i\in\mathbb{Z}}$ is the *frame sequence* associated with $\boldsymbol{\lambda}$, namely:

$$\mathrm{fs}(\boldsymbol{\lambda}) = (\dots,0,\underbrace{f_0=k,f_1=0,\dots,k,0}_{s_k \text{ pairs}},\dots,\underbrace{i,0,\dots,i,0}_{s_i-s_{i+1}\text{ pairs}},\dots,\underbrace{1,0,\dots,1,0}_{s_1-s_2\text{ pairs}},0,\dots),$$

which has weight $s_1^2+\cdots+s_k^2-s_1-\cdots-s_k$, where for all i, $\lambda^{(i)}$ has s_i-s_{i+1} parts.

Then they showed the following.

Theorem 2.3 ([DJK24])**.** *The map Λ induces a bijection between the sets*

$$\mathcal{P}_k = \{(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) : \boldsymbol{\lambda} \text{ is a } k\text{-tuple of partitions}\},$$

and

$$\mathcal{A}_k = \{(f_i)_{i\in\mathbb{Z}} : f_i + f_{i+1} \le k \text{ for all } i \ge 0 \text{ and } f_i = 0 \text{ for all } i < 0\}.$$

Considering well-chosen subsets of $\mathcal{A}_k$ and $\mathcal{P}_k$, they deduced bijections for the sum-side of several identities such as Bressoud's identity [Bre80, (3.4)]

$$\sum_{s_1\ge\cdots\ge s_k\ge 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(q^{k+1-r},q^{k+1+r},q^{2k+2};q^{2k+2})_\infty}{(q)_\infty}, \tag{2.1}$$

which is an even moduli counterpart to the Andrews–Gordon identities, or another identity of Bressoud [Bre80, (3.3)]

$$\sum_{s_1\ge\cdots\ge s_k\ge 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{k+2-j+2s},q^{k+1+j-2s},q^{2k+3};q^{2k+3})_\infty}{(q)_\infty}. \tag{2.2}$$

Using this bijection and identities (1.3) and (2.1) as ingredients, this allowed them to prove several new and known q-series identities. The 0 frequencies in the frame sequences were key in obtaining the minus signs in the powers in the numerator of the left-hand side of (2.2), for example.

Then, in [DJJ25] the two authors and Jouhet reformulated the bijection Λ in its form presented here, and used it on subsets of $\mathcal{A}_k$ and $\mathcal{P}_k$ with more intricate conditions, to prove other q-series identities, known and new, among which Theorem 1.4.

In this paper, our goal is to use particle motion to prove identities involving partitions with parity restrictions. To prove some of these identities, we will need some partitions where the parts -1 are allowed. Therefore, we extend Theorem 2.3 by allowing any offset of the frame sequence to the left or to the right.

Definition 2.4. Let $u \in \mathbb{Z}$ and let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ be a k-tuple of partitions (for some $k \geq 0$) into non-negative parts where for all i, $\lambda^{(i)}$ has $s_i - s_{i+1}$ parts. The u-*frame sequence* associated with $\boldsymbol{\lambda}$ is the generalised frequency sequence

$$\mathrm{fs}_u(\boldsymbol{\lambda}) = (\cdots, 0, \underbrace{f_u = k, f_{u+1} = 0, \dots, k, 0}_{s_k \text{ pairs}}, \dots, \underbrace{i, 0, \dots, i, 0}_{s_i - s_{i+1} \text{ pairs}}, \dots, \underbrace{1, 0, \dots, 1, 0}_{s_1 - s_2 \text{ pairs}}, 0, \dots),$$

which has weight $s_1^2 + \cdots + s_k^2 + (u-1)(s_1 + \cdots + s_k)$.

Theorem 2.5. *Let $u \in \mathbb{Z}$, $k \in \mathbb{Z}_{>0}$. Define*

$$\mathcal{P}_{k,u} = \{(\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda})) : \boldsymbol{\lambda} \text{ is a } k\text{-tuple of partitions}\},$$

and

$$\mathcal{A}_{k,u} = \{(f_i)_{i \in \mathbb{Z}} : f_i + f_{i+1} \leq k \text{ for all } i \in \mathbb{Z}, \text{ and } f_i = 0 \text{ for all } i < u\}.$$

Then the map Λ induces a bijection between the sets $\mathcal{P}_{k,u}$ and $\mathcal{A}_{k,u}$.

Proof. The proof of Theorem 2.3 in [DJK24] (and reformulated in [DJJ25]) actually never used the fact that the frame sequence starts at 0, only its shape. So the exact same proof as in these papers also proves this stronger result. □

Clearly, Theorem 2.5 generalises Theorem 2.3 (which is the case $u = 0$). But Theorem 2.1 also follows from it. Indeed, for any k-tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$,

$$\tilde{\Lambda}(\boldsymbol{\lambda}, \tilde{\mathrm{fs}}_{r,k}(\boldsymbol{\lambda})) = \Lambda(\boldsymbol{\lambda}', \mathrm{fs}_1(\boldsymbol{\lambda}')),$$

where $\boldsymbol{\lambda}'$ is obtained from $\boldsymbol{\lambda}$ by adding $\max(0, m-k+r)$ to each part of $\lambda^{(m)}$, for all $m \in \{1, \dots, k\}$. The result follows by seeing that Λ induces a bijection between

$$\left\{(\boldsymbol{\lambda}, \mathrm{fs}_1(\boldsymbol{\lambda})) : \boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)}) \text{and the parts of } \lambda^{(m)} \text{ are at least } \max(0, m-k+r) \text{ for all } m\right\} \subset \mathcal{P}_{k,1},$$

and

$$\{(f_i)_{i \in \mathbb{Z}} : f_i + f_{i+1} \leq k \text{ for all } i \in \mathbb{Z},\ f_i = 0 \text{ for all } i < 1, \text{ and } f_1 \leq k - r\} \subset \mathcal{A}_{k,1}.$$

To prove some properties about Λ, it is convenient to split it as a composition of intermediate steps of particle motion.

Definition 2.6. For any k-tuple of partitions $\boldsymbol{\lambda}$ of total length s_1, define

$$\mathrm{fs}_u(\boldsymbol{\lambda}) =: \theta^{(s_1)}, \theta^{(s_1-1)}, \dots, \theta^{(1)}, \theta^{(0)} = \Lambda(\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda})), \tag{2.3}$$

where each $\theta^{(i)}$ is obtained from $\theta^{(i+1)}$ by

$$\theta^{(i)} = \mathsf{pm}_{u+2i}^{(\lambda_i)}\left(\theta^{(i+1)}\right) \quad \text{for } i \in \{s_1 - 1, \dots, 1, 0\}. \tag{2.4}$$

Let us start by showing the following simple property.

Proposition 2.7. *For any k-tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ of total length s_1, let $\boldsymbol{\lambda}^+$ be the k-tuple of partitions obtained from $\boldsymbol{\lambda}$ by adding m to every part of $\lambda^{(m)}$ for all $m \in \{1, \dots, k\}$. Note that for all $u \in \mathbb{Z}$, $\mathrm{fs}_u(\boldsymbol{\lambda}) = \mathrm{fs}_u(\boldsymbol{\lambda}^+)$. Then for all $u \in \mathbb{Z}$,*

$$\Lambda(\boldsymbol{\lambda}^+, \mathrm{fs}_{u-1}(\boldsymbol{\lambda}^+)) = \Lambda(\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda})).$$

Proof. Recall that

$$\Lambda(\boldsymbol{\lambda}^+, \mathrm{fs}_{u-1}(\boldsymbol{\lambda}^+)) = \left(\mathsf{pm}_{u-1}^{(\lambda_0^+)} \circ \mathsf{pm}_{u+1}^{(\lambda_1^+)} \circ \cdots \circ \mathsf{pm}_{u+2\ell-3}^{(\lambda_{\ell-1}^+)}\right)(\mathrm{fs}_{u-1}(\boldsymbol{\lambda}^+)),$$

$$\Lambda(\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda})) = \left(\mathsf{pm}_u^{(\lambda_0)} \circ \mathsf{pm}_{u+2}^{(\lambda_1)} \circ \cdots \circ \mathsf{pm}_{u+2\ell-2}^{(\lambda_{\ell-1})}\right)(\mathrm{fs}_u(\boldsymbol{\lambda})).$$

Mimicking Definition 2.6, let us define $\theta^{(s_1),+} = \mathrm{fs}_{u-1}(\boldsymbol{\lambda}^+)$, and

$$\theta^{(i),+} = \mathsf{pm}_{u+2i-1}^{(\lambda_i^+)}\left(\theta^{(i+1),+}\right) \quad \text{for } i \in \{s_1 - 1, \dots, 1, 0\}.$$

We show by downward induction that for all $0 \leq i \leq s_1$, $\theta_j^{(i),+}$ and $\theta_j^{(i)}$ coincide for all $j \geq u+2i-1$, and that $\theta_{u+2i-2}^{(i),+} = \theta_{u+2i-1}^{(i),+} = 0$.

This is clearly true for $i = s_1$, as all frequencies of $j \geq u+2s_1-1$ are zero in both partitions. Moreover, by definition, the frequency of $u+2i-2$ is 0 in $\mathrm{fs}_{u-1}(\boldsymbol{\lambda}^+)$.

Now assume it is true for $i+1$ and prove it for i. Let m be such that λ_i is a part of $\lambda^{(m)}$ in $\boldsymbol{\lambda}$. Then the process $\mathsf{pm}_{u+2i-1}^{(\lambda_i+m)}\left(\theta^{(i+1),+}\right)$ starts with the pair $(m,0)$ at positions $(u+2i-1, u+2i)$. Through the first m particle motions, the pair $(m,0)$ at positions $(u+2i-1,u+2i)$ moves to the pair $(m,0)$ at positions $(u+2i, u+2i+1)$ (indeed, by the induction hypothesis, before applying the particle motion, the frequencies of $u+2i$ and $u+2i+1$ were 0). Then the remaining λ_i particle motions are applied from this position. This is the same as applying directly $\mathsf{pm}_{u+2i}^{(\lambda_i)}\left(\theta^{(i+1)}\right)$, as by the induction hypothesis, all the frequencies $j \geq u+2i+1$ coincided in $\theta^{(i+1),+}$ and $\theta^{(i+1)}$. Hence, the frequencies of all $j \geq u+2i-1$ coincide in $\theta^{(i),+}$ and $\theta^{(i)}$. Finally, because the first m particle motions in $\mathsf{pm}_{u+2i-1}^{(\lambda_i+m)}\left(\theta^{(i+1),+}\right)$ moved the pair $(m,0)$ at positions $(u+2i-1,u+2i)$ to the pair $(m,0)$ at positions $(u+2i,u+2i+1)$, the frequency $\theta_{u+2i-1}^{(i),+}$ becomes zero. Moreover, the frequency $\theta_{u+2i-2}^{(i),+}$ is zero by definition of $\mathrm{fs}_{u-1}(\boldsymbol{\lambda}^+)$. Hence the property is proved by induction, and $\theta^{(0),+} = \Lambda(\boldsymbol{\lambda}^+, \mathrm{fs}_{u-1}(\boldsymbol{\lambda}^+))$ is the same as $\theta^{(0)} = \Lambda(\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda}))$. ☐

We now rewrite some of the results of [DJJ25] but extended to the context of generalised frequency sequences and u-frame sequences. None of the proofs used the fact that the frame sequence started at index 0, so they work in the exact same way and we do not repeat them here. However we write the statement of the results in the more general context.

The first proposition is a restatement of [DJJ25, Proposition 5.4] in this context, and with v replaced by $v+2$.

Proposition 2.8 ([DJJ25, Proposition 5.4])**.** *Let $f = (f_i)_{i\in\mathbb{Z}}$ be a generalised frequency sequence. Suppose that u is an integer such that there exists $h \geq 1$ with $(f_u, f_{u+1}) = (h, 0)$ and $f_i + f_{i+1} \leq h$ for all $i \geq u$. For a non-negative integer m, let $(\overline{f}_i)_{i\in\mathbb{Z}} = \mathsf{pm}_u^{(m)}(f)$. Define*

$$v := \min\left\{ t \geq u : \sum_{i=u+2}^{t+2} (h - (f_{i-1} + f_i)) \geq m \right\}. \tag{2.5}$$

Then (f_u, f_{u+1}) moves to $(\overline{f}_v, \overline{f}_{v+1})$. Moreover $(\overline{f}_i)_{i\in\mathbb{Z}}$ is given explicitly by:

$$\overline{f}_i = \begin{cases} f_i & \text{if } i < u, \\ f_{i+2} & \text{if } u \leq i < v, \\ f_{v+2} + \sum_{j=u+2}^{v+2} (h - (f_{j-1} + f_j)) - m & \text{if } i = v, \\ f_{v+1} + m - \sum_{j=u+2}^{v+1} (h - (f_{j-1} + f_j)) & \text{if } i = v+1, \\ f_i & \text{if } i \geq v+2. \end{cases} \tag{2.6}$$

We define three sets which play a key role in all the proofs of this paper.

Definition 2.9. Let j, r, and k be non-negative integers such that $j + r \leq k$. Define

- $\mathcal{X}_{j,r,k,u}$ to be the set of all pairs $(\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ is a k-tuple of partitions and $\mathrm{fs}_u(\boldsymbol{\lambda})$ is the u-frame sequence corresponding to $\boldsymbol{\lambda}$, subject to the condition that each part of $\lambda^{(m)}$ is at least $m - j + \max\{m-(k-r), 0\}$ for each $m = 1, \ldots, k$,
- $\mathcal{Y}_{j,r,k,u}$ to be the set of all generalised frequency sequences $(f_i)_{i\in\mathbb{Z}}$ such that $f_i + f_{i+1} \leq k$ for all $i \geq u$, $f_i = 0$ for all $i < u$, and
$$f_u \in \{\ell + \max\{\ell - (j-r), 0\} : 0 \leq \ell \leq j\},$$
- $\mathcal{Z}_{j,r,k,u}$ to be the set of all generalised frequency sequences $(f_i)_{i\in\mathbb{Z}}$ such that $f_i + f_{i+1} \leq k$ for all $i \geq u$, $f_i = 0$ for all $i < u$, and
$$f_u \leq j - \max\{f_u + f_{u+1} - (k-r), 0\}.$$

The map Λ restricts to a bijection between two of these sets.

Proposition 2.10 ([DJJ25], Proposition 6.8)**.** *The map Λ is a weight-preserving bijection from $\mathcal{X}_{j,r,k,u}$ to $\mathcal{Z}_{j,r,k,u}$.*

Taking the notation of Definition 2.6, we prove a few results concerning parity conditions on $\{\theta^{(s_1)}, \ldots, \theta^{(0)}\}$, in order to show that Λ preserves parity conditions.

Lemma 2.11. *Let $u \in 2\mathbb{Z}$. Let $\boldsymbol{\lambda}$ be a k-tuple of partitions. For all $i \in \{s, \ldots, 0\}$, every odd part of $\theta^{(i+1)}$ appears an even number of times and λ_i is even if and only if every odd part of $\theta^{(i)}$ appears an even number of times.*

Proof. First show that if every odd part of $\theta^{(i+1)}$ appears an even number of times and λ_i is even, then every odd part of $\theta^{(i)}$ appears an even number of times. Let $(\theta^{(i+1)}_{u+2i}, \theta^{(i+1)}_{u+2i+1}) = (h, 0)$ for some $h > 0$. Suppose that, applying $\mathsf{pm}^{(\lambda_i)}_{u+2i}$, this pair $(\theta^{(i+1)}_{u+2i}, \theta^{(i+1)}_{u+2i+1})$ moves to $(\theta^{(i)}_v, \theta^{(i)}_{v+1})$. By Proposition 2.8, all entries $\theta^{(i)}_j$ for $j \notin \{v, v+1\}$ either remain at the same position or are shifted by two from their original position in $\theta^{(i+1)}$. Hence the parity of all $\theta^{(i)}_j$ for $j \notin \{v, v+1\}$ is preserved, and it suffices to show the result for $\theta^{(i)}_v$ and $\theta^{(i)}_{v+1}$.

From (2.4), we have $|\theta^{(i+1)}| + \lambda_i = |\theta^{(i)}|$. Using (2.6), we obtain

$$\sum_{t\in\mathbb{Z}}(t\,\theta^{(i+1)}_t) + \lambda_i = \sum_{t<u+2i}(t\,\theta^{(i+1)}_t) + \sum_{u+2i\le t<v}(t\,\theta^{(i+1)}_{t+2}) + v\cdot\theta^{(i)}_v + (v+1)\cdot\theta^{(i)}_{v+1} + \sum_{v+2\le t}(t\,\theta^{(i+1)}_t).$$

Taking modulo 2, we obtain

$$(u+2i)\theta^{(i+1)}_{u+2i} + (u+2i+1)\theta^{(i+1)}_{u+2i+1} + \lambda_i \equiv \lambda_i \equiv v\cdot\theta^{(i)}_v + (v+1)\cdot\theta^{(i)}_{v+1} \pmod 2. \tag{2.7}$$

Since λ_i is even by assumption, and exactly one of v and $v+1$ is odd, then this odd part must appear an even number of times, which completes the proof.

Now we show the reverse implication. We recover $\theta^{(i+1)}$ from $\theta^{(i)}$ as follows. Among all indices $j \geq 2i$, let v be the smallest index such that $\theta^{(i)}_j + \theta^{(i)}_{j+1}$ has the maximum value. We then apply reverse particle motion (see [DJJ25, Definition 5.8]) to $\theta^{(i)}$ at the pair $(\theta^{(i)}_v, \theta^{(i)}_{v+1})$ until it moves at positions $2i$ and $2i+1$ and the second entry becomes 0. The number of such inverse moves is exactly λ_i. As in the forward direction, all other entries except the pair $(\theta^{(i+1)}_{u+2i}, \theta^{(i+1)}_{u+2i+1})$ either remain at the same position or are shifted by two positions, so the parity of all other odd parts is preserved. Since $\theta^{(i+1)}_{u+2i+1} = 0$, it follows that all odd parts of $\theta^{(i+1)}$ appear an even number of times.

Finally, for λ_i, the claim follows directly from (2.7). By the assumption on $\theta^{(i)}$,

$$v\cdot\theta^{(i)}_v + (v+1)\cdot\theta^{(i)}_{v+1} \equiv 0 \pmod 2.$$

Hence by (2.7), λ_i is even, completing the proof. □

By a simple shifting argument, a similar lemma holds for u odd.

Lemma 2.12. *Let $u \in 2\mathbb{Z}+1$. Let $\boldsymbol{\lambda}$ be a k-tuple of partitions. For all $i \in \{s, \dots, 0\}$, every even part of $\theta^{(i+1)}$ appears an even number of times and λ_i is even if and only if every even part of $\theta^{(i)}$ appears an even number of times.*

Proof. Let $\tilde\theta^{(i+1)}$ be the frequency sequence obtained by shifting $\theta^{(i+1)}$ one step to the right, i.e. replacing $\theta^{(i+1)}_j$ by $\theta^{(i+1)}_{j-1}$ for all $j \in \mathbb{Z}$. Then by the definition of particle motion, $\theta^{(i)}$ is exactly the generalised frequency sequence obtained by shifting $\tilde\theta^{(i)} = \mathsf{pm}^{(\lambda_i)}_{u+2i+1}\left(\tilde\theta^{(i+1)}\right)$ one step to the left. The result follows by applying Lemma 2.11 to the $\tilde\theta^{(i)}$'s, as the odd parts of $\tilde\theta^{(i)}$ become the even parts of $\theta^{(i)}$. □

Let $\mathrm{shift}(\mathcal{X}_{j,r,k,u})$ be the set of all pairs $(\boldsymbol{\lambda}, \mathrm{fs}_{u-1}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ is a k-tuple of partitions such that each part of $\lambda^{(m)}$ is at least $m + \max\{m-j, 0\} + \max\{m-(k-r), 0\}$ for each $m = 1, \dots, k$. In other words,

$$\mathrm{shift}(\mathcal{X}_{j,r,k,u}) = \left\{(\boldsymbol{\lambda}^+, \mathrm{fs}_{u-1}(\boldsymbol{\lambda})) : (\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda})) \in \mathcal{X}_{j,r,k,u}\right\}.$$

The following lemma shows the shift invariance property.

Proposition 2.13. *The map Λ induces a bijection between $\mathrm{shift}(\mathcal{X}_{j,r,k,u})$ and $\mathcal{Z}_{j,r,k,u}$.*

Proof. By definition of $\mathrm{shift}(\mathcal{X}_{j,r,k,u})$, we have $(\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda})) \in \mathcal{X}_{j,r,k,u}$ if and only if $(\boldsymbol{\lambda}^+, \mathrm{fs}_{u-1}(\boldsymbol{\lambda})) \in \mathrm{shift}(\mathcal{X}_{j,r,k,u})$. By Proposition 2.7, for any $(\boldsymbol{\lambda}^+, \mathrm{fs}_{u-1}(\boldsymbol{\lambda})) \in \mathrm{shift}(\mathcal{X}_{j,r,k,u})$, we have $\Lambda(\boldsymbol{\lambda}^+, \mathrm{fs}_{u-1}(\boldsymbol{\lambda})) = \Lambda(\boldsymbol{\lambda}, \mathrm{fs}_u(\boldsymbol{\lambda}))$. And by Proposition 2.10, Λ induces a bijection between $\mathcal{X}_{j,r,k,u}$ and $\mathcal{Z}_{j,r,k,u}$. Hence it also induces a bijection between $\mathrm{shift}(\mathcal{X}_{j,r,k,u})$ and $\mathcal{Z}_{j,r,k,u}$. □

The next lemma follows directly from Lemma 2.12.

Lemma 2.14. *For $(\boldsymbol{\lambda}, \mathrm{fs}_{u-1}(\boldsymbol{\lambda})) \in \mathrm{shift}(\mathcal{X}_{j,r,k,u})$, let*

$$\mathrm{fs}_{u-1}(\boldsymbol{\lambda}) =: \theta^{(s_1)}, \theta^{(s_1-1)}, \dots, \theta^{(1)}, \theta^{(0)} = \Lambda(\boldsymbol{\lambda}, \mathrm{fs}_{u-1}(\boldsymbol{\lambda})),$$

where each $\theta^{(i)}$ is obtained from $\theta^{(i+1)}$ by $\theta^{(i)} = \mathsf{pm}^{(\lambda_i)}_{u+2i-1}\left(\theta^{(i+1)}\right)$ for $i = s_1-1, \dots, 1, 0$. Then every even part of $\theta^{(i+1)}$ appears an even number of times and λ_i is even if and only if every even part of $\theta^{(i)}$ appears an even number of times.

We saw in Proposition 2.10 that Λ induces a weight-preserving bijection between $\mathcal{X}_{j,r,k,u}$ and $\mathcal{Z}_{j,r,k,u}$. However $\mathcal{Z}_{j,r,k,u}$ is not immediately a set of frequency sequences whose generating function is well-known. Hence we do one last simple bijection to relate $\mathcal{Z}_{j,r,k,0}$ it to $\mathcal{Y}_{j,r,k,0}$, whose generating function can be nicely expressed as sums of products thanks to the Andrews–Gordon identities.

Proposition 2.15 (Generalisation of [DJJ25], Proposition 6.7)**.** *There exists a weight-preserving bijection from $\mathcal{Y}_{j,r,k,0}$ to $\mathcal{Z}_{j,r,k,0}$ via the map ϕ defined by*

$$(\dots,0,f_0,f_1,f_2,\dots)\mapsto(\dots,0,f_0',f_1,f_2,\dots),$$

where if $j\geq r$, then

$$f_0'=\begin{cases}f_0 & \text{if } f_0\leq j-r,\\ j-r+\ell & \text{if } f_0=j-r+2\ell \text{ for some } \ell\in\{1,\dots,r\},\end{cases}$$

and if $j<r$, then

$$f_0'=\ell,\quad \text{for } f_0=r-j+2\ell \text{ with } \ell\in\{0,\dots,j\}.$$

3. Proof of Theorem 1.11 and Corollary 1.14

In this section, we prove Theorem 1.11. First, we define a subset of $\mathcal{X}_{j,r,k,0}$ with additional parity conditions.

Definition 3.1. Let $j,r,$ and k be non-negative integers with $j+r\leq k$. Define $\mathcal{X}^o_{j,r,k}$ to be the set of all pairs $(\boldsymbol{\lambda},\mathrm{fs}_0(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(k)})$ is a k-tuple of partitions, such that for all $m\in\{1,\dots,k\}$,

- each part of $\lambda^{(m)}$ is at least $m-j+\max\{m-(k-r),0\}$ for each $m=1,\dots,k$,
- $\lambda^{(m)}$ is a partition into even parts.

Let us start by computing the generating function for $\mathcal{X}^o_{j,r,k}$.

Proposition 3.2. *Let $j,r,$ and k be non-negative integers with $j+r\leq k$. Then*

$$\sum_{(\boldsymbol{\lambda},\mathrm{fs}_0(\boldsymbol{\lambda}))\in\mathcal{X}^o_{j,r,k}} q^{|(\boldsymbol{\lambda},\mathrm{fs}_0(\boldsymbol{\lambda}))|}=\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)+(s_{j+1}-s_{j+2}+s_{j+3}-\cdots\pm s_{k-r})+(s_{k-r+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}.$$

Proof. Note that the generating function for partitions of length ℓ into even parts of size at least b is

$$\begin{cases}\frac{q^{b\ell}}{(q^2;q^2)_\ell} & \text{if } b \text{ is even},\\ \frac{q^{(b+1)\ell}}{(q^2;q^2)_\ell} & \text{if } b \text{ is odd}.\end{cases}$$

For non-negative integers $s_1,\dots,s_k$ with $s_1\geq\cdots\geq s_k\geq 0$, define $X^o_{j,r}(s_1,\dots,s_k)$ to be the set of k-tuples of partitions $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(k)})$ such that $(\boldsymbol{\lambda},\mathrm{fs}_0(\boldsymbol{\lambda}))\in\mathcal{X}^o_{j,r,k}$ and for each m, the length of $\lambda^{(m)}$ is s_m-s_{m+1}. Then $\mathrm{fs}_0(\boldsymbol{\lambda})$ is the same for all $\boldsymbol{\lambda}\in X^o_{j,r}(s_1,\dots,s_k)$; let us denote it by $\mathrm{fs}_0(s_1,\dots,s_k)$.

Recall that

$$|\,\mathrm{fs}_0(s_1,\dots,s_k)|=s_1^2+\cdots+s_k^2-(s_1+\cdots+s_k). \tag{3.1}$$

We can express $\mathcal{X}^o_{j,r,k}$ as

$$\mathcal{X}^o_{j,r,k}=\bigsqcup_{s_1\geq\cdots\geq s_k\geq 0} X^o_{j,r}(s_1,\dots,s_k)\times\{\mathrm{fs}_0(s_1,\dots,s_k)\}. \tag{3.2}$$

To compute the generating function for $X^o_{j,r}(s_1,\dots,s_k)$, we distinguish the different cases in $\max\{m-(k-r),0\}$. We obtain

$$\sum_{\boldsymbol{\lambda}\in X^o_{j,r}(s_1,\dots,s_k)} q^{|\boldsymbol{\lambda}|}=A_1\cdot A_2\cdot A_3,$$

with

$$\begin{aligned}A_1&=\sum q^{|\lambda^{(1)}|+\cdots+|\lambda^{(j)}|},\\ A_2&=\sum q^{|\lambda^{(j+1)}|+\cdots+|\lambda^{(k-r)}|},\\ A_3&=\sum q^{|\lambda^{(k-r+1)}|+\cdots+|\lambda^{(k)}|},\end{aligned}$$

where the sums run respectively on all tuples of partitions $(\lambda^{(1)},\dots,\lambda^{(j)})$, $(\lambda^{(j+1)},\dots,\lambda^{(k-r)})$, and $(\lambda^{(k-r+1)},\dots,\lambda^{(k)})$ such that $(\lambda^{(1)},\dots,\lambda^{(k)})\in X^o_{j,r}(s_1,\dots,s_k)$.

For all $m \in \{1, \dots, j\}$, $\lambda^{(m)}$ is a partition into $s_m - s_{m+1}$ even parts at least 0. Thus

$$A_1 = \prod_{m=1}^{j} \frac{1}{(q^2;q^2)_{s_m - s_{m+1}}}. \tag{3.3}$$

For the computation of A_2 and A_3, we need to distinguish two cases. First assume that $k-r-j$ is even (the case $k-r-j$ odd will follow).

For all $m \in \{j+1, \dots, k-r\}$, $\lambda^{(m)}$ is a partition into $s_m - s_{m+1}$ even parts at least $m-j$, hence

$$\begin{aligned} A_2 &= \frac{q^{2(s_{j+1}-s_{j+2})+2(s_{j+2}-s_{j+3})+4(s_{j+3}-s_{j+4})+4(s_{j+4}-s_{j+5})+\cdots+(k-r-j)(s_{k-r-1}-s_{k-r})+(k-r-j)(s_{k-r}-s_{k-r+1})}}{(q^2;q^2)_{s_{j+1}-s_{j+2}}(q^2;q^2)_{s_{j+2}-s_{j+3}}\cdots(q^2;q^2)_{s_{k-r}-s_{k-r+1}}} \\ &= \frac{q^{2s_{j+1}+2s_{j+3}+\cdots+2s_{k-r-1}-(k-r-j)s_{k-r+1}}}{(q^2;q^2)_{s_{j+1}-s_{j+2}}(q^2;q^2)_{s_{j+2}-s_{j+3}}\cdots(q^2;q^2)_{s_{k-r}-s_{k-r+1}}}. \end{aligned} \tag{3.4}$$

For all $m \in \{k-r+1, \dots, k\}$, $\lambda^{(m)}$ is a partition into even parts at least $2m-k+r-j$, hence

$$A_3 = \prod_{m=k-r+1}^{k} \frac{q^{(2m-k+r-j)(s_m-s_{m+1})}}{(q^2;q^2)_{s_m-s_{m+1}}} = \frac{q^{(k-r-j+2)s_{k-r+1}}q^{2s_{k-r+2}+2s_{k-r+3}+\cdots+2s_k}}{(q^2;q^2)_{s_{k-r+1}-s_{k-r+2}}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}. \tag{3.5}$$

By combining (3.3), (3.4), and (3.5), we obtain

$$\sum_{\boldsymbol{\lambda} \in X^o_{j,r}(s_1,\dots,s_k)} q^{|\boldsymbol{\lambda}|} = \frac{q^{2s_{j+1}+2s_{j+3}+\cdots+2s_{k-r-1}+2s_{k-r+1}+2s_{k-r+2}+\cdots+2s_k}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}.$$

Combining this with (3.1) and (3.2), we obtain the desired formula.

Now we treat the case where $k-r-j$ is odd. All arguments remain the same, except that equations (3.4) and (3.5) are replaced by

$$\begin{aligned} &\frac{q^{2(s_{j+1}-s_{j+2})+2(s_{j+2}-s_{j+3})+4(s_{j+3}-s_{j+4})+4(s_{j+4}-s_{j+5})+\cdots+(k-r-j-1)(s_{k-r-1}-s_{k-r})+(k-r-j+1)(s_{k-r}-s_{k-r+1})}}{(q^2;q^2)_{s_{j+1}-s_{j+2}}(q^2;q^2)_{s_{j+2}-s_{j+3}}\cdots(q^2;q^2)_{s_{k-r}-s_{k-r+1}}} \\ &\qquad = \frac{q^{2s_{j+1}+2s_{j+3}+\cdots+2s_{k-r}}q^{-(k-r-j+1)s_{k-r+1}}}{(q^2;q^2)_{s_{j+1}-s_{j+2}}(q^2;q^2)_{s_{j+2}-s_{j+3}}\cdots(q^2;q^2)_{s_{k-r}-s_{k-r+1}}}, \end{aligned}$$

and

$$\prod_{m=k-r+1}^{k} \frac{q^{(2m-k+r-j+1)(s_m-s_{m+1})}}{(q^2;q^2)_{s_m-s_{m+1}}} = \frac{q^{(k-r-j+3)s_{k-r+1}}q^{2s_{k-r+2}+2s_{k-r+3}+\cdots+2s_k}}{(q^2;q^2)_{s_{k-r+1}-s_{k-r+2}}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}},$$

respectively. Hence, we obtain again

$$\sum_{\boldsymbol{\lambda} \in X^o_{j,r}(s_1,\dots,s_k)} q^{|\boldsymbol{\lambda}|} = \frac{q^{2s_{j+1}+2s_{j+3}+\cdots+2s_{k-r}+2s_{k-r+1}+2s_{k-r+2}+\cdots+2s_k}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}},$$

and the result follows.

□

Now we define a set $\mathcal{Y}^o_{j,r,k}$ which will be in bijection with $\mathcal{Z}^o_{j,r,k}$ and $\mathcal{X}^o_{j,r,k}$, and whose generating function is expressed nicely as a sum of products.

Definition 3.3. For non-negative integers j, r, and k with $j + r \leq k$, define $\mathcal{Y}^o_{j,r,k}$ to be the set of all frequency sequences $(f_i)_{i\in\mathbb{Z}}$ such that

- $f_i + f_{i+1} \leq k$ for all $i \geq 0$,
- $f_i = 0$ for all $i < 0$,
- $f_0 \in \{\ell + \max\{\ell - (j-r), 0\} : 0 \leq \ell \leq j\}$, and
- f_i is even if i is odd. That is, odd parts appear an even number of times.

Now we prove that these three sets are indeed in bijection.

Proposition 3.4. *There exists a weight-preserving bijection between $\mathcal{X}^o_{j,r,k}$ and $\mathcal{Z}^o_{j,r,k}$.*

Proof. Since $\mathcal{X}^o_{j,r,k} \subseteq \mathcal{X}_{j,r,k,0}$ and $\mathcal{Z}^o_{j,r,k} \subseteq \mathcal{Z}_{j,r,k,0}$, we will show that the bijection $\Lambda : \mathcal{X}_{j,r,k,0} \to \mathcal{Z}_{j,r,k,0}$ (see Proposition 2.10) restricts to a bijection $\Lambda : \mathcal{X}^o_{j,r,k} \to \mathcal{Z}^o_{j,r,k}$.

Let $(\boldsymbol{\lambda}, \mathrm{fs}_0(\boldsymbol{\lambda})) \in \mathcal{X}^o_{j,r,k}$. We use the notation of Definition 2.6. By the definition of $\mathcal{X}^o_{j,r,k}$, every λ_i is even. Thus, by iterating Lemma 2.11, it follows that $\theta^{(s_1-1)}, \ldots, \theta^{(1)}$, and $\theta^{(0)} = \Lambda(\boldsymbol{\lambda}, \mathrm{fs}_0(\boldsymbol{\lambda}))$ all satisfy the condition that every odd part appears an even number of times. Therefore, the image of $\mathcal{X}^o_{j,r,k}$ under Λ satisfies the parity condition of $\mathcal{Z}^o_{j,r,k}$, and hence $\Lambda(\mathcal{X}^o_{j,r,k}) \subseteq \mathcal{Z}^o_{j,r,k}$. For the reverse inclusion, Lemma 2.11 also shows that the inverse map Λ^{-1} sends $\mathcal{Z}^o_{j,r,k}$ into $\mathcal{X}^o_{j,r,k}$, that is, $\Lambda^{-1}(\mathcal{Z}^o_{j,r,k}) \subseteq \mathcal{X}^o_{j,r,k}$, which completes the proof. □

Proposition 3.5. *There exists a weight-preserving bijection between $\mathcal{Y}^o_{j,r,k}$ and $\mathcal{Z}^o_{j,r,k}$.*

Proof. The bijection ϕ in Proposition 2.15 for $u = 0$ only modifies the frequency of 0, and hence has no effect on the number of appearances of the odd parts. Therefore, ϕ restricts to a bijection between $\mathcal{Y}^o_{j,r,k}$ and $\mathcal{Z}^o_{j,r,k}$. □

Finally, we use (1.6) to express the generating function for $\mathcal{Y}^o_{j,r,k}$ as a sum of products.

Proposition 3.6. *For non-negative integers j, r, and k with $j + r \leq k$, we have*

$$\sum_{f \in \mathcal{Y}^o_{j,r,k}} q^{|f|} = \sum_{s=0}^{j} \frac{(-q^2;q^2)_\infty (q^{2\lfloor \frac{k+2-r+j-2s}{2} \rfloor}, q^{2k+4-2\lfloor \frac{k+2-r+j-2s}{2} \rfloor}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}. \tag{3.6}$$

Proof. Let us start with the case $j \leq r$. Then, $\ell + \max\{\ell - (j-r), 0\} = r - j + 2\ell$ for $0 \leq \ell \leq j$. The set of all possible values for f_0 is

$$F = \{\ell + \max\{\ell - (j-r), 0\} : 0 \leq \ell \leq j\} = \{r - j + 2s : 0 \leq s \leq j\}.$$

Thus

$$\mathcal{Y}^o_{j,r,k} = \bigsqcup_{s=0}^{j} \{f \in \mathcal{Y}^o_{j,r,k} : f_0 = r - j + 2s\}.$$

Moreover, the set $\{f \in \mathcal{Y}^o_{j,r,k} : f_0 = a\}$ is in bijection with $\overline{\mathcal{W}}_{k,k-a}$ (by removing the parts 0). Hence

$$\sum_{f \in \mathcal{Y}^o_{j,r,k}} q^{|f|} = \sum_{s=0}^{j} \sum_{n \geq 0} \overline{W}_{k,k-r+j-2s}(n) q^n,$$

and the result follows using (1.6).

Now suppose $j > r$. Then the set

$$F = \{\ell + \max\{\ell - (j-r), 0\} : 0 \leq \ell \leq j\} = \{0, 1, \ldots, j-r, j-r+2, \ldots, j+r\}$$

can be expressed as the disjoint union $F = F_1 \sqcup F_2$, where

$$\begin{aligned} F_1 &= \{j+r, j+r-2, j+r-4, \cdots\} \\ &= \{r - j + 2s : s = \lfloor (j-r)/2 \rfloor, \lfloor (j-r)/2 \rfloor + 1, \ldots, j\}, \end{aligned}$$

and

$$\begin{aligned} F_2 &= \{j-r-1, j-r-3, j-r-5, \cdots\} \\ &= \{j - r - 1 - 2s : s = 0, 1, \ldots, \lfloor (j-r)/2 \rfloor - 1\}. \end{aligned}$$

By (1.6), the corresponding generating functions are, respectively,

$$\begin{aligned} \sum_{f \in \mathcal{Y}^o_{j,r,k} : f_0 \in F_1} q^{|f|} &= \sum_{s=\lfloor (j-r)/2 \rfloor}^{j} \sum_{n \geq 0} \overline{W}_{k,k-(r-j+2s)}(n) q^n \\ &= \sum_{s=\lfloor (j-r)/2 \rfloor}^{j} \frac{(-q^2;q^2)_\infty (q^{2\lfloor \frac{k-(r-j+2s)+2}{2} \rfloor}, q^{2k+4-2\lfloor \frac{k-(r-j+2s)+2}{2} \rfloor}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}, \end{aligned} \tag{3.7}$$

and

$$\sum_{f\in\mathcal{Y}^o_{j,r,k}:f_0\in F_2} q^{|f|} = \sum_{s=0}^{\lfloor (j-r)/2\rfloor-1}\sum_{n\geq 0}\overline{W}_{k,k-(j-r-1-2s)}(n)q^n \tag{3.8}$$
$$= \sum_{s=0}^{\lfloor (j-r)/2\rfloor-1}\frac{(-q^2;q^2)_\infty (q^{2\lfloor\frac{k-(j-r-1-2s)+2}{2}\rfloor}, q^{2k+4-2\lfloor\frac{k-(j-r-1-2s)+2}{2}\rfloor}, q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

We claim that

$$2k+4-2\left\lfloor\frac{k-(j-r-1-2s)+2}{2}\right\rfloor = 2\left\lfloor\frac{k-(r-j+2s)+2}{2}\right\rfloor.$$

If $k-j+r$ is odd, then

$$2k+4-2\left\lfloor\frac{k-(j-r-1-2s)+2}{2}\right\rfloor = 2\left(\frac{k+j-r-2s+1}{2}\right) = 2\left\lfloor\frac{k+j-r-2s+2}{2}\right\rfloor,$$

and hence the claim holds. The even case follows similarly. The summands in the right-hand sides of (3.8) and (3.7) are the same. Adding (3.7) and (3.8) together completes the proof. □

The proof of Theorem 1.11 follows directly from combining Propositions 3.2, 3.4, 3.5 and 3.6.

Let us conclude this section with the proof of Corollary 1.14.

Proof of Corollary 1.14. First, substituting $j=a$ and $r=0$ into Theorem 1.11 yields

$$\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_a)+(s_{a+1}-s_{a+2}+\cdots+(-1)^{k-a-1}s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} \tag{3.9}$$
$$= \sum_{s=0}^{a}\frac{(-q^2;q^2)_\infty(q^{2\lfloor\frac{k+a-2s+2}{2}\rfloor}, q^{2k+4-2\lfloor\frac{k+a-2s+2}{2}\rfloor}, q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

The left-hand side of the desired identity is obtained immediately. On the other hand, the right-hand side is written in a slightly different form. Note that $k-i+2\leq k+2$ for $i=0,\dots,a$, while $k+a-2s+2\geq k+2$ for $s=0,\dots,\lfloor a/2\rfloor$, and $k+a-2s+2<k+2$ for $s=\lfloor a/2\rfloor+1,\dots,a$. To obtain the desired right-hand side from (3.9), it suffices to show that

$$\left\{2\left\lfloor\frac{k-i+2}{2}\right\rfloor : i=0,\dots,a\right\} = \left\{2\left\lfloor\frac{k+a-2s+2}{2}\right\rfloor : s=\lfloor a/2\rfloor+1,\dots,a\right\}$$
$$\sqcup\left\{2k+4-2\left\lfloor\frac{k+a-2s+2}{2}\right\rfloor : s=0,\dots,\lfloor a/2\rfloor\right\}.$$

By an argument similar to that in the proof of Proposition 3.6, an elementary property of the floor function yields

$$2k+4-2\left\lfloor\frac{k+a-2s+2}{2}\right\rfloor = 2\left\lfloor\frac{k-a+2s+3}{2}\right\rfloor.$$

The proof follows from the fact that the set $\{k-i+2 : i=0,\dots,a\}$ can be written as a disjoint union

$$\{k-i+2 : i=0,\dots,a\} = (\{k-a+2,k-a+4,k-a+6,\dots\}\cap\{1,\dots,k+2\})$$
$$\sqcup(\{k-a+3,k-a+5,k-a+7,\dots\}\cap\{1,\dots,k+2\}),$$

where the first and second sets are equal to, respectively,

$$\{k+a-2s+2 : s=\lfloor a/2\rfloor+1,\dots,a\},\quad\text{and}\quad\{k-a+2s+3 : s=0,\dots,\lfloor a/2\rfloor\}.$$

□

4. Proof of Theorem 1.12

We now turn to the proof of Theorem 1.12, which involves different parity conditions. In this section, the (-1)-frame sequences will be useful.

First define a subset of $\mathrm{shift}(\mathcal{X}_{j,r,k,0})$ with parity conditions.

Definition 4.1. Let a,b, and k be non-negative integers with $2a+2b\leq k$. Define $\mathcal{X}^e_{a,b,k}$ to be the set of all pairs $(\boldsymbol{\lambda},\mathrm{fs}_{-1}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(k)})$ is a k-tuple of partitions, subject to the conditions that for all $m\in\{1,\dots,k\}$,

- each part of $\lambda^{(m)}$ is at least $m+\max\{m-2a,0\}+\max\{m-(k-2b),0\}$ for each $m=1,\dots,k$,

- $\lambda^{(m)}$ is a partition into even parts.

We first compute the generating function for $\mathcal{X}^e_{a,b,k}$.

Proposition 4.2. *Let $a, b,$ and k be non-negative integers with $2a + 2b \leq k$. Then*

$$\sum_{(\boldsymbol{\lambda},\mathrm{fs}_{-1}(\boldsymbol{\lambda}))\in\mathcal{X}^e_{a,b,k}} q^{|(\boldsymbol{\lambda},\mathrm{fs}_{-1}(\boldsymbol{\lambda}))|} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2-2(s_2+s_4+\cdots+s_{2a})+2(s_{k-2b+1}+s_{k-2b+3}+\cdots+s_{k-1})}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}.$$

Proof. This proof is analogous to that of Proposition 3.2.

For non-negative integers $s_1,\dots,s_k$ with $s_1 \geq \cdots \geq s_k \geq 0$, define $X^e_{a,b}(s_1,\dots,s_k)$ to be the set of k-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(k)})$ such that $(\boldsymbol{\lambda}, \mathrm{fs}_0(\boldsymbol{\lambda})) \in \mathcal{X}^e_{a,b,k}$ and for each m, the length of $\lambda^{(m)}$ is $s_m - s_{m+1}$. Then $\mathrm{fs}_{-1}(\boldsymbol{\lambda})$ is the same for all $\boldsymbol{\lambda} \in X^e_{a,b}(s_1,\dots,s_k)$; let us denote it by $\mathrm{fs}_{-1}(s_1,\dots,s_k)$.

$$|\,\mathrm{fs}_{-1}(s_1,\dots,s_k)| = s_1^2+\cdots+s_k^2 - 2(s_1+\cdots+s_k). \tag{4.1}$$

We can express $\mathcal{X}^e_{a,b,k}$ as the disjoint union

$$\mathcal{X}^e_{a,b,k} = \bigsqcup_{s_1\geq\cdots\geq s_k\geq 0} X^e_{a,b}(s_1,\dots,s_k)\times\{\mathrm{fs}_{-1}(s_1,\dots,s_k)\}, \tag{4.2}$$

To compute the generating function for $X^e_{a,b}(s_1,\dots,s_k)$, we distinguish the different cases in $\max\{m-2a,0\}+\max\{m-(k-2b),0\}$. We obtain

$$\sum_{\boldsymbol{\lambda}\in X^e_{a,b}(s_1,\dots,s_k)} q^{|\boldsymbol{\lambda}|} = A_1\cdot A_2\cdot A_3,$$

with

$$A_1 = \sum q^{|\lambda^{(1)}|+\cdots+|\lambda^{(2a)}|},$$
$$A_2 = \sum q^{|\lambda^{(2a+1)}|+\cdots+|\lambda^{(k-2b)}|},$$
$$A_3 = \sum q^{|\lambda^{(k-2b+1)}|+\cdots+|\lambda^{(k)}|},$$

where the sums run respectively on all tuples of partitions $(\lambda^{(1)},\dots,\lambda^{(2a)})$, $(\lambda^{(2a+1)},\dots,\lambda^{(k-2b)})$, and $(\lambda^{(k-2b+1)},\dots,\lambda^{(k)})$ such that $(\lambda^{(1)},\dots,\lambda^{(k)}) \in X^e_{a,b}(s_1,\dots,s_k)$.

For all $m \in \{1,\dots,2a\}$, $\lambda^{(m)}$ is a partition into $s_m - s_{m+1}$ even parts at least m. Thus

$$A_1 = \frac{q^{2s_1+2s_3+\cdots+2s_{2a-1}}q^{-(2a)s_{2a+1}}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{2a}-s_{2a+1}}}. \tag{4.3}$$

For all $m \in \{2a+1,\dots,k-2b\}$, $\lambda^{(m)}$ is a partition into $s_m - s_{m+1}$ even parts at least $2m-2a$, hence

$$A_2 = \frac{q^{(2a+2)s_{2a+1}}q^{2s_{2a+2}+2s_{2a+3}+\cdots+2s_{k-2b}}q^{-(2k-2a-4b)s_{k-2b+1}}}{(q^2;q^2)_{s_{2a+1}-s_{2a+2}}\cdots(q^2;q^2)_{s_{k-2b}-s_{k-2b+1}}}. \tag{4.4}$$

For all $m \in \{k-2b+1,\dots,k\}$, $\lambda^{(m)}$ is a partition into even parts at least $3m-k-2a+2b$, hence

$$A_3 = \frac{q^{(2k-2a-4b+4)s_{k-2b+1}}q^{2s_{k-2b+2}+4s_{k-2b+3}+2s_{k-2b+4}+4s_{k-2b+5}+\cdots+4s_{k-1}+2s_k}}{(q^2;q^2)_{s_{k-2b+1}-s_{k-2b+2}}\cdots(q^2;q^2)_{s_k}}. \tag{4.5}$$

Thus, by (4.3), (4.4), and (4.5), we obtain

$$\sum_{\boldsymbol{\lambda}\in X^e_{a,b}(s_1,\dots,s_k)} q^{|\boldsymbol{\lambda}|} = \frac{q^{2(s_1+s_3+\cdots+s_{2a-1})+2(s_{2a+1}+s_{2a+2}+\cdots+s_{k-2b})+(4s_{k-2b+1}+2s_{k-2b+2}+4s_{k-2b+3}+\cdots+4s_{k-1}+2s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}.$$

Combining this with (4.1) and (4.2) yields the desired formula. □

Following the same structure of proof as for Theorem 1.11, we define a set $\mathcal{Y}^e_{a,b,k}$ which will be in bijection with $\mathcal{X}^e_{a,b,k}$ and $\mathcal{Z}^e_{a,b,k}$.

Definition 4.3. For non-negative integers $a, b,$ and k with $2a + 2b \leq k$, we define $\mathcal{Y}^e_{a,b,k}$ to be the set of all frequency sequences $(f_i)_{i\in\mathbb{Z}}$ such that

- $f_i + f_{i+1} \leq k$ for all $i \geq 0$,

- $f_i = 0$ for all $i < 0$,
- $f_0 \in \{2(\ell + \max\{\ell - (a-b), 0\}) : 0 \le \ell \le a\}$, and
- f_i is even if i is even. That is, even parts appear an even number of times.

Now we prove that these three sets are indeed in bijection.

Proposition 4.4. *There exists a weight-preserving bijection between $\mathcal{X}^e_{a,b,k}$ and $\mathcal{Z}^e_{a,b,k}$.*

Proof. The argument parallels that of Proposition 3.4. Here we set $j = 2a$ and $r = 2b$. In that proof, replace j by $2a$, r by $2b$, Proposition 2.10 by Proposition 2.13 , $\mathcal{X}^o_{j,r,k}$ by$\mathcal{X}^e_{a,b,k}$, $\mathcal{Z}^o_{j,r,k}$ by $\mathcal{Z}^e_{a,b,k}$, and fs_0 by fs_{-1}, respectively. Then substitute Lemmas 2.11 with Lemma 2.14. With these replacements, the desired conclusion follows. □

Proposition 4.5. *There exists a weight-preserving bijection between $\mathcal{Y}^e_{a,b,k}$ and $\mathcal{Z}^e_{a,b,k}$.*

Proof. From the definitions of the subsets $\mathcal{Y}^e_{a,b,k} \subseteq \mathcal{Y}_{2a,2b,k,0}$ and $\mathcal{Z}^e_{a,b,k} \subseteq \mathcal{Z}_{2a,2b,k,0}$, every element of $\mathcal{Y}^e_{a,b,k}$ and $\mathcal{Z}^e_{a,b,k}$ has an even 0th frequency, and each even part appears an even number of times. We now show that the bijection ϕ constructed in Proposition 2.15 restricts to a bijection from $\mathcal{Y}^e_{a,b,k}$ to $\mathcal{Z}^e_{a,b,k}$.

Unlike Proposition 3.5, because of the additional parity conditions defining $\mathcal{Y}^e_{a,b,k}$ and $\mathcal{Z}^e_{a,b,k}$, every element in these sets must have an even 0th frequency. Hence, we need to verify explicitly that the map ϕ preserves this property.

If we take the variables j, r, and the parameter ℓ in Proposition 2.15 to be even, that is,

$$j = 2a, \quad r = 2b, \quad \ell = 2m,$$

then the restriction of ϕ to $\mathcal{Y}^e_{a,b,k}$ can be described explicitly as follows:

$$(\dots, 0, f_0, f_1, f_2, \dots) \mapsto (\dots, 0, f'_0, f_1, f_2, \dots),$$

where if $a \ge b$, then

$$f'_0 = \begin{cases} f_0 & \text{if } f_0 \le 2(a-b), \\ 2(a-b+m) & \text{if } f_0 = 2(a-b+2m) \text{ for some } 2m \in \{1, \dots, 2b\}, \end{cases}$$

and if $a < b$, then

$$f'_0 = 2m, \quad \text{for } f_0 = 2(b-a+2m) \text{ with } 2m \in \{0, \dots, 2a\}.$$

By the explicit description of ϕ, if f_0 is even then f'_0 is even. Hence, $\phi(\mathcal{Y}^e_{a,b,k}) \subseteq \mathcal{Z}^e_{a,b,k}$. Moreover, by the explicit formula again, it is easy to verify that the value f_0 obtained from an even f'_0 always satisfies the defining condition of f_0 in $\mathcal{Y}^e_{a,b,k}$. This shows $\phi(\mathcal{Z}^e_{a,b,k}) \subseteq \mathcal{Y}^e_{a,b,k}$ and completes the proof. □

Finally, we use Theorem 1.6 to express the generating function for $\mathcal{Y}^o_{j,r,k}$ as a sum of products.

Proposition 4.6. *For non-negative integers a and b with $2a + 2b \le k$, we have*

$$\sum_{f \in \mathcal{Y}^e_{a,b,k}} q^{|f|} = \sum_{s=0}^{a} \frac{(-q;q^2)_\infty (q^{k+1+2a-2b-4s}, q^{k+3-2a+2b+4s}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Proof. This proof is similar to that of Proposition 3.6. From Theorem 1.6, the generating function for the frequency sequences in $\mathcal{Y}^e_{a,b,k}$ satisfying $f_0 = 2s$ is

$$\frac{(-q;q^2)_\infty (q^{k+1-2s}, q^{k+3+2s}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}. \tag{4.6}$$

We use this to compute the generating function for $\mathcal{Y}^e_{a,b,k}$. If $a \le b$, then the set of all possible values for f_0 is

$$\{2(\ell + \max\{\ell - (a-b), 0\}) : 0 \le \ell \le a\} = \{-2a + 2b + 4\ell : 0 \le \ell \le a\},$$

and the proof follows immediately from (4.6).

If $a > b$, then the set

$$F = \{2(\ell + \max\{\ell - (a-b), 0\}) : 0 \le \ell \le a\} = \{0, 2, 4, \dots, 2a-2b-2, 2a-2b, 2a-2b+4, \dots, 2a+2b-4, 2a+2b\}$$

can be expressed as the disjoint union $F = F_1 \sqcup F_2$, where

$$F_1 = \{2a+2b, 2a+2b-4, 2a+2b-8, \dots\}$$

and

$$F_2 = \{2a-2b-2, 2a-2b-6, 2a-2b-10, \dots\}.$$

The first set is given by

$$F_1 = \{2b - 2a + 4s : s = \lfloor (a-b)/2 \rfloor, \lfloor (a-b)/2 \rfloor + 1, \dots, a\},$$

and the second by

$$F_2 = \{2a - 2b - 2 - 4s : s = 0, 1, \dots, \lfloor (a-b)/2 \rfloor - 1\}.$$

By (4.6), the corresponding generating functions are, respectively,

$$\sum_{f \in \mathcal{Y}^e_{a,b,k} : f_0 \in F_1} q^{|f|} = \sum_{s=\lfloor (a-b)/2 \rfloor}^{a} \frac{(-q;q^2)_\infty (q^{k+1-(2b-2a+4s)}, q^{k+3+(2b-2a+4s)}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty},$$

and

$$\sum_{f \in \mathcal{Y}^e_{a,b,k} : f_0 \in F_2} q^{|f|} = \sum_{s=0}^{\lfloor (a-b)/2 \rfloor - 1} \frac{(-q;q^2)_\infty (q^{k+1-(2a-2b-2-4s)}, q^{k+3+(2a-2b-2-4s)}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty} =$$

$$\sum_{s=0}^{\lfloor (a-b)/2 \rfloor - 1} \frac{(-q;q^2)_\infty (q^{k+1+2a-2b-4s}, q^{k+3-2a+2b+4s}, q^{2k+4}; q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Adding the two expressions completes the proof. □

5. Proof of Theorem 1.13 and Corollary 1.15

The beginning of this section is very similar to the previous one, and so we give less detail here. However, the end of the proof of Theorem 1.13 requires a new argument, which we detail in Proposition 5.4.

Definition 5.1. Let a, b, and k be non-negative integers with $2a + 2b - 1 \leq k$. Define $\widetilde{\mathcal{X}}^e_{a,b,k}$ to be the set of all pairs $(\boldsymbol{\lambda}, \mathrm{fs}_{-1}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ is a k-tuple of partitions, subject to the conditions that for all $m \in \{1, \dots, k\}$,

- each part of $\lambda^{(m)}$ is at least $m + \max\{m - 2a, 0\} + \max\{m - (k - 2b + 1), 0\}$ for each $m = 1, \dots, k$, and
- $\lambda^{(m)}$ is a partition into even parts.

We first compute the generating function for $\widetilde{\mathcal{X}}^e_{a,b,k}$.

Proposition 5.2. *Let a, b, and k be non-negative integers with $2a + 2b - 1 \leq k$. Then*

$$\sum_{(\boldsymbol{\lambda}, \mathrm{fs}_{-1}(\boldsymbol{\lambda})) \in \widetilde{\mathcal{X}}^e_{a,b,k}} q^{|(\boldsymbol{\lambda}, \mathrm{fs}_{-1}(\boldsymbol{\lambda}))|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 - 2(s_2 + s_4 + \cdots + s_{2a}) + 2(s_{k-2b+2} + s_{k-2b+4} + \cdots + s_k)}}{(q^2;q^2)_{s_1 - s_2} \cdots (q^2;q^2)_{s_{k-1} - s_k} (q^2;q^2)_{s_k}}.$$

Proof. This proof follows the same argument as Proposition 4.2, with the following modifications: $\mathcal{X}^e_{a,b,k}$ and $X^e_{a,b}(s_1, \dots, s_k)$ are replaced by $\widetilde{\mathcal{X}}^e_{a,b,k}$ and $\widetilde{X}^e_{a,b}(s_1, \dots, s_k)$, respectively, and the part leading to (4.4) and (4.5) is replaced by the following.

For $2a < m \leq k - 2b + 1$, parts of $\lambda^{(m)}$ are at least $2m - 2a$, hence

$$A_2 = \frac{q^{(2a+2)s_{2a+1}} q^{2s_{2a+2} + 2s_{2a+3} + \cdots + 2s_{k-2b+1}} q^{-(2k-2a-4b+2)s_{k-2b+2}}}{(q^2;q^2)_{s_{2a+1} - s_{2a+2}} \cdots (q^2;q^2)_{s_{k-2b+1} - s_{k-2b+2}}}.$$

For $k - 2b + 1 < m \leq k$, parts of $\lambda^{(m)}$ are at least $3m - k - 2a + 2b - 1$, hence

$$A_3 = \frac{q^{(2k-2a-4b+6)s_{k-2b+2}} q^{2s_{k-2b+3} + 4s_{k-2b+4} + 2s_{k-2b+5} + 4s_{k-2b+6} + \cdots + 2s_{k-1} + 4s_k}}{(q^2;q^2)_{s_{k-2b+1} - s_{k-2b+2}} \cdots (q^2;q^2)_{s_k}}.$$

This yields

$$\sum_{\boldsymbol{\lambda} \in \widetilde{X}^e_{a,b}(s_1, \dots, s_k)} q^{|\boldsymbol{\lambda}|} = \frac{q^{2(s_1 + s_3 + \cdots + s_{2a-1}) + 2(s_{2a+1} + s_{2a+2} + \cdots + s_{k-2b} + s_{k-2b+1}) + (4s_{k-2b+2} + 2s_{k-2b+3} + 4s_{k-2b+4} + \cdots + 2s_{k-1} + 4s_k)}}{(q^2;q^2)_{s_1 - s_2} \cdots (q^2;q^2)_{s_{k-1} - s_k} (q^2;q^2)_{s_k}}.$$

Therefore, we obtain the desired formula. □

We show that $\widetilde{\mathcal{X}}^e_{a,b,k}$ and $\widetilde{\mathcal{Z}}^e_{a,b,k}$ are in bijection.

Proposition 5.3. *There exists a weight-preserving bijection between $\widetilde{\mathcal{X}}^e_{a,b,k}$ and $\widetilde{\mathcal{Z}}^e_{a,b,k}$.*

Proof. This is exactly the same as the proof of Proposition 4.4 but with $r = 2b - 1$ instead of $r = 2b$. □

Finally, we need one last proposition in order to use Theorem 1.12 to finish the proof of Theorem 1.13.

Proposition 5.4. *The following equality holds:*

$$(1+q)\sum_{f\in\widetilde{\mathcal{Z}}^e_{a,b,k}} q^{|f|} = \sum_{f\in\mathcal{Z}^e_{a,b-1,k}} q^{|f|} + q\sum_{f\in\mathcal{Z}^e_{a,b,k}} q^{|f|}.$$

Proof. We have the inclusions $\mathcal{Z}^e_{a,b,k} \subseteq \widetilde{\mathcal{Z}}^e_{a,b,k} \subseteq \mathcal{Z}^e_{a,b-1,k}$. Moreover, the map $(f_0,f_1,f_2,\dots)\mapsto(f_0,f_1-1,f_2,\cdots)$ is a bijection from $\widetilde{\mathcal{Z}}^e_{a,b,k}\setminus\mathcal{Z}^e_{a,b,k}$ to $\mathcal{Z}^e_{a,b-1,k}\setminus\widetilde{\mathcal{Z}}^e_{a,b,k}$. Thus

$$\begin{aligned}(1+q)\sum_{f\in\widetilde{\mathcal{Z}}^e_{a,b,k}} q^{|f|} &= \sum_{f\in\widetilde{\mathcal{Z}}^e_{a,b,k}} q^{|f|} + \sum_{f\in\widetilde{\mathcal{Z}}^e_{a,b,k}\setminus\mathcal{Z}^e_{a,b,k}} q^{|f|+1} + q\sum_{f\in\mathcal{Z}^e_{a,b,k}} q^{|f|}\\ &= \sum_{f\in\widetilde{\mathcal{Z}}^e_{a,b,k}} q^{|f|} + \sum_{f\in\mathcal{Z}^e_{a,b-1,k}\setminus\widetilde{\mathcal{Z}}^e_{a,b,k}} q^{|f|} + q\sum_{f\in\mathcal{Z}^e_{a,b,k}} q^{|f|}\\ &= \sum_{f\in\mathcal{Z}^e_{a,b-1,k}} q^{|f|} + q\sum_{f\in\mathcal{Z}^e_{a,b,k}} q^{|f|}.\end{aligned}$$

□

Using Theorem 1.12 to evaluate the two sums in Proposition 5.4 completes the proof of Theorem 1.13.

We conclude with the proof of Corollary 1.15.

Proof of Corollary 1.15. By Theorem 1.12 and Theorem 1.13 with $b=0$, we obtain

(5.1)
$$\sum_{s_1\ge\cdots\ge s_k\ge 0}\frac{q^{s_1^2+\cdots+s_k^2-2(s_2+s_4+\cdots+s_{2a})}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s=0}^{a}\frac{(-q;q^2)_\infty(q^{k+1+2a-4s},q^{k+3-2a+4s},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty},$$

and

(5.2)
$$\sum_{s_1\ge\cdots\ge s_k\ge 0}\frac{q^{s_1^2+\cdots+s_k^2-2(s_2+s_4+\cdots+s_{2a})}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$=\frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{s=0}^{a}\left[(q^{k+3+2a-4s},q^{k+1-2a+4s},q^{2k+4};q^{2k+4})_\infty + q(q^{k+1+2a-4s},q^{k+3-2a+4s},q^{2k+4};q^{2k+4})_\infty\right],$$

respectively. We first verify that, under this specialisation, the right-hand sides of (5.1) and (5.2) coincide. Indeed, in the right-hand side of (5.2), we apply the change of variables $s\mapsto a-s$ in the second q-Pochhammer symbol appearing inside the summation. Under this reindexing, the second q-Pochhammer symbol is equal to the first one, and hence two right-hand sides are the same.

We next show that the desired right-hand side can be obtained from the right-hand side of (5.1). The argument is similar to that used in the proof of Corollary 1.14. We split the sum in the right-hand side of (5.1) into the two ranges $s=0,\dots,\lfloor a/2\rfloor$ and $s=\lfloor a/2\rfloor+1,\dots,a$. For $s=\lfloor a/2\rfloor+1,\dots,a$, we have $k+3-2a+4s<k+1$. Thus, the right-hand side of (5.1) can be rewritten as

(5.3)
$$\sum_{s=0}^{\lfloor a/2\rfloor}\frac{(-q;q^2)_\infty(q^{k+1+2a-4s},q^{k+3-2a+4s},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty} + \sum_{s=\lfloor a/2\rfloor+1}^{a}\frac{(-q;q^2)_\infty(q^{k+3-2a+4s},q^{k+1+2a-4s},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

Note that the set $\{k+1-2i : i=0,\dots,a\}$ can be decomposed as a disjoint union

$$(\{k+1-2a,k+1-2a+4,k+1-2a+8,\dots\}\cap\{1,\dots,k+1\}) \sqcup (\{k+3-2a,k+3-2a+4,k+3-2a+8,\dots\}\cap\{1,\dots,k+1\}),$$

and the first and second sets are equal to

$$\{k+1+2a-4s : s=\lfloor a/2\rfloor+1,\dots,a\},\quad\text{and}\quad\{k+3-2a+4s : s=0,\dots,\lfloor a/2\rfloor\},$$

respectively. Combining this decomposition with (5.3) yields the desired result. □

6. Open problems

We propose two directions for further research.

First, one may seek combinatorial proofs of the identities that follow directly from our results. From Theorem 1.11, we obtain two different infinite summation expressions of the same product expression. More precisely, specializing Theorem 1.11 with $j = k - a$ and $r = a$, we obtain $\mathrm{AK}_{a,k}(q)$ appearing in Corollary 1.17. Taking the difference $\mathrm{AK}_{a,k}(q) - \mathrm{AK}_{a+1,k}(q)$ yields

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_{k-a}+s_{k-a+1}+\cdots+s_k}(1-q^{2s_{k-a}})}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(-q^2;q^2)_\infty(q^{2a+2},q^{2k+2-2a},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

On the other hand, specializing Theorem 1.11 with $j = 0$ and $r = k - 2a$ gives another identity

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+(s_1-s_2+s_3-\cdots+s_{2a-1}-s_{2a})+(s_{2a+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(-q^2;q^2)_\infty(q^{2a+2},q^{2k+2-2a},q^{2k+4};q^{2k+4})_\infty}{(q^2;q^2)_\infty}.$$

By the two identities above, we obtain

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_{k-a}+s_{k-a+1}+\cdots+s_k}(1-q^{2s_{k-a}})}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+(s_1-s_2+s_3-\cdots+s_{2a-1}-s_{2a})+(s_{2a+1}+\cdots+s_k)}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}. \tag{6.1}$$

Problem 6.1. Find a combinatorial proof of (6.1).

Similarly, comparing product expressions leads to identities between two summation expressions having different numbers of summation variables. By [DJJ25, Theorem 1.12] and Theorem 1.13, we obtain

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{2(s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k)}(-q^{1+2s_k};q^2)_\infty}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s_1\geq\cdots\geq s_{2k}\geq 0} \frac{q^{s_1^2+\cdots+s_{2k}^2-2(s_2+s_4+\cdots+s_{2j})+2(s_{2k-2r+2}+s_{2k-2r+4}+\cdots+s_{2k})}}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{2k-1}-s_{2k}}(q^2;q^2)_{s_{2k}}}. \tag{6.2}$$

Problem 6.2. Find a combinatorial proof of (6.2) (for instance, via particle motion, or by any other combinatorial method).

Second, we propose generalisations in other directions and alternative proofs of our results. Stanton [Sta18] proved not only the identity (1.4), but also a binomial extension [Sta18, Theorem 3.1] of the Andrews–Gordon identities: Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k} \cdot \frac{q^{-s_1-\cdots-s_j}(1+q^{s_1+s_2})(1+q^{s_2+s_3})\cdots(1+q^{s_{j-1}+s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \sum_{s=0}^{j}\binom{j}{s}\frac{(q^{2k+3},q^{k+1-r+j-2s},q^{k+2+r-j+2s};q^{2k+3})_\infty}{(q)_\infty}. \tag{6.3}$$

Moreover, the j factors q^{-s_1} and $q^{-s_i}(1+q^{s_{i-1}+s_i})$, $2 \leq i \leq j$ may be replaced by any j-element subset of $\{q^{-s_1}\} \cup \{q^{-s_i}(1+q^{s_{i-1}+s_i}) : 2 \leq i \leq k-r\}$. It would be interesting to develop analogous binomial extensions of our results.

Problem 6.3. Find binomial extensions of Theorem 1.11, Theorem 1.12, and Theorem 1.13 in the spirit of (6.3).

In joint work [DJJ25] of the authors with Jouhet, Stanton-type generalisations with binomial coefficients were proved using Bailey pairs. If explicit formulas answering the previous problem can be found, it would be natural to seek Bailey pair proofs of those identities. Likewise, it would be interesting to obtain Bailey pair proofs of our main results.

Problem 6.4. Prove the q-series identities in Theorem 1.11, Theorem 1.12, and Theorem 1.13 using Bailey pairs.

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