GENERALISATIONS OF CAPPARELLI'S AND PRIMC'S IDENTITIES, II: PERFECT $A_{n-1}^{(1)}$ CRYSTALS AND EXPLICIT CHARACTER FORMULAS

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Abstract. In the first paper of this series, we gave infinite families of coloured partition identities which generalise Primc's and Capparelli's classical identities.

In this second paper, we study the representation theoretic consequences of our combinatorial results. First, we show that the difference conditions we defined in our n^2-coloured generalisation of Primc's identity, which have a very simple expression, are actually the energy function with values in $\{0,1,2\}$ for the perfect crystal of the tensor product of the vector representation and its dual in $A_{n-1}^{(1)}$.

Then we introduce a new type of partitions, grounded partitions, which allows us to retrieve connections between character formulas and partition generating functions without having to perform a specialisation.

Finally, using the formulas for the generating functions of our generalised partitions, we recover the Kac-Peterson character formula for the characters of all the irreducible highest weight $A_{n-1}^{(1)}$-modules of level 1, and give a new character formula as a sum of infinite products with obviously positive coefficients in the generators $e^{-\alpha_i}$ $(i \in \{0,\dots,n-1\})$, where the α_i's are the simple roots.

1. Introduction and statement of results

Both our papers ([5] and this one) aim to be self-contained and accessible to both combinatorialists and representation theorists (and hopefully others), even though this one is more representation theoretic and [5] more combinatorial. Therefore, we recall the background and definitions necessary to understand our results (here mostly about crystal base theory, and in [5] mostly about integer partitions and q-series).

1.1. **Background.** A *partition* λ of a positive integer n is a non-increasing sequence of natural numbers $(\lambda_1,\dots,\lambda_s)$ whose sum is n, written as the sum $\lambda_1+\cdots+\lambda_s$. The numbers $\lambda_1,\dots,\lambda_s$ are called the *parts* of λ, and $|\lambda| = n$ is the *weight* of λ. For example, the partitions of 4 are $4, 3+1, 2+2, 2+1+1$, and $1+1+1+1$.

The Rogers-Ramanujan identities [26] state that for $a = 0$ or 1, the number of partitions of n such that the difference between two consecutive parts is at least 2 and the part 1 appears at most $1-a$ times is equal to the number of partitions of n into parts congruent to $\pm(1+a) \mod 5$. In the 1980's, Lepowsky and Wilson [15, 16] gave an interpretation and proof of these identities in terms of characters for level 3 standard modules of the affine Lie algebra $A_1^{(1)}$ by using vertex operators. Since then, a very fruitful interaction between partition identities and representation theory has been developed, see for example [4, 18, 19, 20, 21, 22, 24, 27]. More detail on the history of this field can be found in the first paper of this series [5].

In the present paper, we focus on the interaction between partition identities and crystal base theory. Crystal bases were introduced independently by Kashiwara [14] and Lusztig [17] to study representations of quantum algebras, which are q-deformations of universal enveloping algebras of classical Lie algebras. They have a nice combinatorial structure, and admit particularly simple tensor products.

One of the most important questions in representation theory is finding nice explicit formulas for characters of representations. If $\hat{\mathfrak{g}}$ is an affine Lie algebra, and V an irreducible module of $\hat{\mathfrak{g}}$ with highest weight Λ, then by definition, the character $\mathrm{ch}(V)$ of V multiplied by $e^{-\Lambda}$ can be expressed as a power series in $e^{-\alpha_0},\dots,e^{-\alpha_{n-1}}$ with positive coefficients, where $\alpha_0,\dots,\alpha_{n-1}$ are the simple roots of $\hat{\mathfrak{g}}$. However, finding explicit expressions for characters is not easy. The most famous example, the *Weyl-Kac character formula* [9], gives a beautiful factorized expression for the character, but the coefficients of the monomials in $e^{-\alpha_i}$ in this expression are not obviously positive.

Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki [11, 12] introduced the theory of perfect crystals to find such nice expressions for characters via the so-called *(KMN)^2 crystal base character formula*.

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It allows one to construct crystals of irreducible highest weight modules for all classical weights of the same level. Then the crystal base character formula allows one to identify these perfect crystals with partitions satisfying certain difference conditions, which in certain cases gives rise to character formulas as partition generating functions. However, these formulas are in general obtained after doing a specialisation, for example replacing all the $e^{-\alpha_i}$'s by q (which is the principal specialisation). In this paper, we will prove a non-specialised character formula, with obviously positive coefficients, for all the irreducible highest weight $A_{n-1}^{(1)}$-modules of level 1.

But first, let us present our starting point, Primc's partition identity (again, more detail can be found in our first paper [5]). In [23], Primc used the $(\mathrm{KMN})^2$ crystal base character formula to study level 1 standard modules of $A_1^{(1)}$ and $A_2^{(1)}$. He computed an energy function for the perfect crystal of the tensor product of the vector representation and its dual in $A_1^{(1)}$ and $A_2^{(1)}$, and through the crystal base character formula, he gave the principal specialisation of the character formula in terms of partitions with difference conditions.

In the $A_1^{(1)}$ case, the energy matrix of the perfect crystal associated to the tensor product of the vector representation and its dual is the following:

$$P_2 = \begin{array}{c} \\ a_1b_0 \\ a_0b_0 \\ a_1b_1 \\ a_0b_1 \end{array} \begin{array}{c} \begin{array}{cccc} a_1b_0 & a_0b_0 & a_1b_1 & a_0b_1 \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}, \tag{1.1}$$

and in $A_2^{(1)}$, the energy matrix is given by:

$$P_3 = \begin{array}{c} \\ a_2b_0 \\ a_2b_1 \\ a_1b_0 \\ a_0b_0 \\ a_2b_2 \\ a_1b_1 \\ a_0b_1 \\ a_1b_2 \\ a_0b_2 \end{array} \begin{array}{c} \begin{array}{ccccccccc} a_2b_0 & a_2b_1 & a_1b_0 & a_0b_0 & a_2b_2 & a_1b_1 & a_0b_1 & a_1b_2 & a_0b_2 \end{array} \\ \begin{pmatrix} 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 \end{pmatrix} \end{array}. \tag{1.2}$$

Consider coloured partitions satisfying the difference conditions of (1.1) (resp. (1.2)), where the coefficient (i,j) in the matrix gives the minimal difference between consecutive parts coloured i and j. Using the Weyl-Kac charater formula [9], Primc proved that in both cases, when performing the principal specialisation (corresponding to some dilations on the variables in the generating function), the generating function for such partitions reduces to $\frac{1}{(q;q)_\infty}$, which is simply the generating function for partitions. Here we used, for $n \in \mathbb{N} \cup \{\infty\}$, the standard q-series notation

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

In the first paper of this series [5], we gave a large family of coloured partition identities which generalise and refine Primc's identities. To do so, we gave difference conditions which generalise both (1.1) and (1.2). Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of colour symbols. For all $i,k,i',k' \in \mathbb{N}$, we defined the minimal difference Δ in the following way:

$$\Delta(a_ib_k, a_{i'}b_{k'}) = \chi(i \geq i') - \chi(i = k = i') + \chi(k \leq k') - \chi(k = i' = k'), \tag{1.3}$$

where $\chi(prop)$ equals 1 if the proposition *prop* is true and 0 otherwise.

Restricting Δ to colours a_ib_j for $i,j \in \{0,1\}$ gives (1.1), and restricting it to colours a_ib_j for $i,j \in \{0,1,2\}$ gives (1.2).

Our general theorem in [5] gives the generating function for partitions $\lambda_1 + \cdots + \lambda_s$ into parts coloured a_ib_j for all $i,j \in \{0,\dots,n-1\}$, satisfying the difference conditions

$$\lambda_j - \lambda_{j+1} \geq \Delta(c(\lambda_j), c(\lambda_{j+1})),$$

where for all j, $c(\lambda_j)$ denotes the colour of the part λ_j. Such partitions are called generalised Primc partitions and their set is denoted by $\mathcal{P}_n$. Let $P_n(m; u_0, \dots, u_{n-1}; v_0, \dots, v_{n-1})$ be the number of partitions of m in $\mathcal{P}_n$, such that for $i \in \{0, \dots, n-1\}$, the symbol a_i (resp. b_i) appears u_i (resp. v_i) times in the colour sequence. Defining the generating function

$$G_n^P(q; b_0, \cdots, b_{n-1}) := \sum_{m,u_0,\dots,u_{n-1},v_0,\dots,v_{n-1} \geq 0} P_n(n; u_0, \dots, u_{n-1}; v_0, \dots, v_{n-1}) q^m b_0^{v_0-u_0} \cdots b_{n-1}^{v_{n-1}-u_{n-1}},$$

we showed the following.

Theorem 1.1. [5, Theorem 1.27] *Let n be a positive integer. We have:*

$$\begin{aligned}
G_n^P(q; b_0, \cdots, b_{n-1}) &= [x^0] \prod_{i=0}^{n-1} (-b_i^{-1} xq; q)_\infty (-b_i x^{-1}; q)_\infty \\
&= \frac{1}{(q;q)_\infty^n} \sum_{\substack{s_1,\dots,s_{n-1} \in \mathbb{Z} \\ s_n = 0}} b_0^{s_1} \prod_{i=1}^{n-1} b_i^{-s_i+s_{i+1}} q^{s_i(s_i - s_{i+1})} \qquad (1.4) \\
&= \frac{1}{(q;q)_\infty} \left(\prod_{i=1}^{n-1} \frac{\left(q^{i(i+1)}; q^{i(i+1)}\right)_\infty}{(q;q)_\infty} \right) \sum_{\substack{r_1,\dots,r_{n-1}: \\ 0 \leq r_j \leq j-1 \\ r_n = 0}} \prod_{i=1}^{n-1} b_i^{-r_i + r_{i+1}} q^{r_i(r_i - r_{i+1})} \\
&\qquad \times \left(- \left(\prod_{\ell=0}^{i-1} b_\ell b_i^{-1} \right) q^{\frac{i(i+1)}{2} + (i+1)r_i - i r_{i+1}}; q^{i(i+1)} \right)_\infty \qquad (1.5) \\
&\qquad \times \left(- \left(\prod_{\ell=0}^{i-1} b_i b_\ell^{-1} \right) q^{\frac{i(i+1)}{2} - (i+1)r_i + i r_{i+1}}; q^{i(i+1)} \right)_\infty .
\end{aligned}$$

We can obtain a product formula for our generating function by doing the following dilations, which correspond to the principal specialisation that Primc considered in his paper:

$$\begin{cases} q \mapsto q^n \\ b_i \mapsto q^i \quad \text{for all } i \in \{0, \dots, n-1\} . \end{cases} \qquad (1.6)$$

Corollary 1.2. [5, Corollary 1.26] *By doing the transformations described in* (1.6)*, we obtain the generating function for classical integer partitions:*

$$\begin{aligned}
G_n^P(q^n; 1, \cdots, q^{n-1}) &= [x^0] \prod_{i=0}^{n-1} (-q^{n-i} x; q^n)_\infty (-q^i x^{-1}; q^n)_\infty \\
&= [x^0] (-qx; q)_\infty (-x^{-1}; q)_\infty \\
&= \frac{1}{(q;q)_\infty} .
\end{aligned}$$

The cases $n = 2$ and $n = 3$ in the corollary above recover Primc's original results.

In [5], we also gave a multi-parameter family of generalisations of Capparelli's identity [4], another partition identity which originally arose from representation theory via the theory of vertex operators. Let us also state this generalisation, as it gives a different (but related) expression for the character formula.

Definition 1.3. Let $\pi = \pi_1 + \cdots + \pi_r$ be a partition. We say that another partition $\lambda = \lambda_1 + \cdots + \lambda_s$ *contains the pattern* π if there is some index i such that

$$\lambda_i = \pi_1, \quad \lambda_{i+1} = \pi_2, \quad \dots, \quad \lambda_{i+r-1} = \pi_r.$$

If λ does not contain the pattern π, we say that λ *avoids* π.

Let us recall from [5] some conditions that the parameters in our generalisation need to satisfy.

Definition 1.4. A function δ is said to satisfy Condition 1 if it is defined on the set of colours $\{a_k b_\ell : k \neq \ell\}$, has integer values, and for all k, ℓ,

$$\min\{k, \ell\} < \delta(a_k b_\ell) \leq \max\{k, \ell\}.$$

Definition 1.5. A function γ is said to satisfy Condition 2 if it is defined on the set of pairs of colours $\{(a_{k_1} b_{\ell_1}, a_{k_2} b_{\ell_2}) : k_1 \neq \ell_1, k_2 \neq \ell_2\}$, has integer values, and if for all k_1, k_2, ℓ_1, ℓ_2, it satisfies the following:

- If $\max\{k_1, \ell_2\} < \min\{k_2, \ell_1\}$, we have
$$\max\{k_1, \ell_2\} < \gamma(a_{k_1} b_{\ell_1}, a_{k_2} b_{\ell_2}) \leq \min\{k_2, \ell_1\}.$$
- If $k_1 > \ell_1$, $k_2 > \ell_2$, and $\{\ell_2 + 1, \ldots, k_2\} \setminus \{\ell_1 + 1, \ldots, k_1\} \neq \emptyset$, we have
$$\gamma(a_{k_1} b_{\ell_1}, a_{k_2} b_{\ell_2}) \in \{\ell_2 + 1, \ldots, k_2\} \setminus \{\ell_1 + 1, \ldots, k_1\}.$$
- If $k_1 < \ell_1$, $k_2 < \ell_2$, and $\{k_1 + 1, \ldots, \ell_1\} \setminus \{k_2 + 1, \ldots, \ell_2\} \neq \emptyset$, we have
$$\gamma(a_{k_1} b_{\ell_1}, a_{k_2} b_{\ell_2}) \in \{k_1 + 1, \ldots, \ell_1\} \setminus \{k_2 + 1, \ldots, \ell_2\}.$$

These functions now allow us to define forbidden patterns and generalised Capparelli partitions.

Definition 1.6. Let n be a positive integer, and let δ and γ be functions satisfying Conditions 1 and 2, respectively. We define $\mathcal{C}_n(\delta, \gamma)$, the set of generalised Capparelli partitions related to δ and γ, to be the set of partitions λ such that

- $\lambda \in \mathcal{P}_n$,
- λ has no part coloured $a_0 b_0$,
- λ does not contain any of the following patterns, where p is any positive integer:
  - for any $i \in \{1, \ldots, n-1\}$,
  $$p_{a_i b_i} + p_{a_i b_i},$$
  (i.e. free colours cannot repeat)
  - for any k_1, k_2, ℓ_1, ℓ_2 such that $\max\{k_1, \ell_2\} < \min\{k_2, \ell_1\}$ and $i = \gamma(a_{k_1} b_{\ell_1}, a_{k_2} b_{\ell_2})$,
  $$p_{a_{k_1} b_{\ell_1}} + p_{a_i b_i} + p_{a_{k_2} b_{\ell_2}},$$
  - for any $k_2 > \ell_2$,
    * for any $2 \leq u \leq \infty$, any k_1, ℓ_1, and $i = \delta(a_{k_2} b_{\ell_2})$,
    $$(p+u)_{a_{k_1} b_{\ell_1}} + p_{a_i b_i} + p_{a_{k_2} b_{\ell_2}},$$
    (here we take the convention that $u = \infty$ if the pattern $p_{a_i b_i} + p_{a_{k_2} b_{\ell_2}}$ is at the beginning of the partition)
    * for any $k_1 \leq \ell_1$, and $i = \delta(a_{k_2} b_{\ell_2})$,
    $$(p+1)_{a_{k_1} b_{\ell_1}} + p_{a_i b_i} + p_{a_{k_2} b_{\ell_2}},$$
    * for any $k_1 > \ell_1$ such that $\{\ell_2 + 1, \ldots, k_2\} \setminus \{\ell_1 + 1, \ldots, k_1\} \neq \emptyset$, and for $i = \gamma(a_{k_1} b_{\ell_1}, a_{k_2} b_{\ell_2})$,
    $$(p+1)_{a_{k_1} b_{\ell_1}} + p_{a_i b_i} + p_{a_{k_2} b_{\ell_2}},$$
  - for any $k_1 < \ell_1$,
    * for any $2 \leq u \leq \infty$, any k_2, ℓ_2, and $i = \delta(a_{k_1} b_{\ell_1})$,
    $$p_{a_{k_1} b_{\ell_1}} + p_{a_i b_i} + (p-u)_{a_{k_2} b_{\ell_2}},$$
    (here we take the convention that $u = \infty$ if the pattern $p_{a_{k_1} b_{\ell_1}} + p_{a_i b_i}$ is at the end of the partition)
    * for any $k_2 \geq \ell_2$, and $i = \delta(a_{k_1} b_{\ell_1})$,
    $$(p+1)_{a_{k_1} b_{\ell_1}} + (p+1)_{a_i b_i} + p_{a_{k_2} b_{\ell_2}},$$
    * for any $k_2 < l_2$ such that $\{k_1 + 1, \ldots, \ell_1\} \setminus \{k_2 + 1, \ldots, \ell_2\} \neq \emptyset$, and for $i = \gamma(a_{k_1} b_{\ell_1}, a_{k_2} b_{\ell_2})$,
    $$(p+1)_{a_{k_1} b_{\ell_1}} + (p+1)_{a_i b_i} + p_{a_{k_2} b_{\ell_2}}.$$

Let $C_n(m;\delta,\gamma;u_0,\dots,u_{n-1};v_0,\dots,v_{n-1})$ be the number of partitions of m in $\mathcal{C}_n(\gamma,\delta)$, such that for $i\in\{0,\dots,n-1\}$, the symbol a_i (resp. b_i) appears u_i (resp. v_i) times in the colour sequence. We define the generating function

$$G_n^C(\delta,\gamma;q;b_0,\cdots,b_{n-1}) := \sum_{m,u_0,\dots,u_{n-1},v_0,\dots,v_{n-1}\geq 0} C_n(m;\delta,\gamma;u_0,\dots,u_{n-1};v_0,\dots,v_{n-1})q^m b_0^{v_0-u_0}\cdots b_{n-1}^{v_{n-1}-u_{n-1}}.$$

Through a bijection, we showed the following relation between our generalisations of Capparelli's and Primc's partitions with difference conditions.

Theorem 1.7. [5, Theorem 1.24] *For all positive integers n and all functions δ and γ satisfying Conditions 1 and 2 respectively, we have*

$$G_n^C(\delta,\gamma;q;b_0,\cdots,b_{n-1}) = (q;q)_\infty G_n^P(q;b_0,\cdots,b_{n-1}).$$

Through Theorem 1.1, the generating function for generalised Capparelli partitions can also be written as a sum of infinite products.

In this paper, we use our results above to give new character formulas.

1.2. **Statement of Results.** We will define all the necessary notions from crystal base theory in Section 2. For now, let us define a few notations which will allow us to state our main theorems.

Let n be a positive integer, and consider the Cartan datum for the generalised Cartan matrix of affine type $A_{n-1}^{(1)}$. We denote by $\bar{P}=\mathbb{Z}\Lambda_0\oplus\cdots\oplus\mathbb{Z}\Lambda_{n-1}$ the lattice of the classical weights, where the elements Λ_ℓ ($\ell\in\{0,\dots,n-1\}$) are the fundamental weights. The set of all the level 1 classical weights is given by $\bar{P}_1^+=\{\Lambda_\ell:\ell\in\{0,\cdots,n-1\}\}$. The null root is denoted by δ, and the simple roots by α_i, $i\in\{0,\cdots,n-1\}$. Let $\mathcal{B}=\{v_i:i\in\{0,\cdots,n-1\}\}$ be the crystal of the vector representation of $A_{n-1}^{(1)}$ and let $\mathcal{B}^\vee=\{v_i^\vee:i\in\{0,\cdots,n-1\}\}$ be its dual. For all $v_i\in\mathcal{B}$, we denote by $\overline{\mathrm{wt}}v_i\in\bar{P}$ the classical weight of v_i. We finally set $\mathbb{B}$ to be the tensor product $\mathcal{B}\otimes\mathcal{B}^\vee$.

Given that (1.1) and (1.2) are energy matrices for perfect crystals coming from the tensor product of the vector representation and its dual in $A_1^{(1)}$ and $A_2^{(1)}$, respectively, it is natural to wonder whether our generalised difference conditions Δ defined in (1.3) are also energy functions for certain perfect crystals. We answer this question in the affirmative by showing the following.

Theorem 1.8. *Let n be a positive integer, and let $\mathcal{B}$ denote the crystal of the vector representation of $A_{n-1}^{(1)}$. The crystal $\mathbb{B}=\mathcal{B}\otimes\mathcal{B}^\vee$ is a perfect crystal of level 1. Furthermore, the energy function on $\mathbb{B}\otimes\mathbb{B}$ such that $H((v_0\otimes v_0^\vee)\otimes(v_0\otimes v_0^\vee))=0$ satisfies for all $k,\ell,k',\ell'\in\{0,\dots,n-1\}$,*

$$H((v_{\ell'}\otimes v_{k'}^\vee)\otimes(v_\ell\otimes v_k^\vee)) = \Delta(a_kb_\ell;a_{k'}b_{\ell'}), \tag{1.7}$$

where Δ is the minimal difference for generalised Primc partitions defined in (1.3).

Primc showed Theorem 1.8 in the cases $n=2$ and $n=3$. The theorem is still true when $n=1$, in which case the crystal $\mathcal{B}$ has a single vertex and a loop 0, and the partitions corresponding to its crystal are simply the classical partitions.

In [2], Benkart, Frenkel, Kang, and Lee gave another formulation of the energy function of certain level 1 perfect crystals of classical types, including the $A_{n-1}^{(1)}$-crystal studied in Theorem 1.8. However, they did not give a closed expression valid for all $k,\ell,k',\ell'\in\{0,\dots,n-1\}$ as we did in Theorem 1.8 and (1.3). They used the fact that, when removing the 0-arrows from the crystal graph on Figure 4.4 (see more details on crystals and energy functions in Sections 2 and 4), the energy function H is constant on each connected component, and gave a table with the value of H for a representative of each connected component. The value of H for the other vertices can then be obtained by determining to which connected component they belong. Both their and our energy functions satisfy $H((v_0\otimes v_0^\vee)\otimes(v_0\otimes v_0^\vee))=0$, so they must be the same, even though their formulations differ. In this sense, Theorem 1.8 gives a simpler, more explicit and unified formula for the $A_{n-1}^{(1)}$ energy function in [2].

Our proof of Theorem 1.8 in Sections 6 and 7 relies on explicitly building paths in the crystal graph. We only treat the case $n\geq 3$, as $n=1$ and $n=2$ give crystals with a slightly different shape, and we already know that the theorem is true in these cases.

Theorem 1.8 gives a simple explicit expression for the energy function. Using the $(\mathrm{KMN})^2$ crystal base character formula [11], it allows us to relate the generating function $G_n^P(q;b_0,\cdots,b_{n-1})$ of generalised Primc partitions and the generating function $G_n^C(\delta,\gamma;q;b_0,\cdots,b_{n-1})$ of generalised Capparelli partitions with the character of the irreducible highest weight module $L(\Lambda_0)$.

Unlike previous connections between character formulas and partition generating functions, where a specific specialisation (often the principal specialisation) was needed, here we give a **non-dilated character formula**. This is obtained through a new combinatorial method relying on so-called "grounded partitions", which we introduce in Section 3. This leads to the following combinatorial character formula, where all the definitions will be made clear in Sections 2 and 3.

Theorem 1.9. *Let $\widehat{g}$ be an affine Kac–Moody Lie algebra , let $\mathcal{B}$ be a perfect crystal of level ℓ, let λ be a dominant integral weight of level ℓ with constant ground state path $\cdots\otimes g\otimes g$, and assume that $H(g\otimes g)=0$. Setting $q=e^{-\frac{\delta}{d_0}}$ and $c_b=e^{\overline{\mathrm{wt}}b}$ for all $b\in\mathcal{B}$, we have $c_g=1$, and the character of the irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-module $L(\lambda)$ is given by the following expressions:*

$$\sum_{\pi\in\mathcal{P}_{c_g}^{\gg}} C(\pi)q^{|\pi|}=e^{-\lambda}\mathrm{ch}(\mathrm{L}(\lambda)),$$

$$\sum_{\pi\in\mathcal{P}_{c_g}^{\ggg}} C(\pi)q^{|\pi|}=\frac{e^{-\lambda}\mathrm{ch}(\mathrm{L}(\lambda))}{(q;q)_\infty},$$

where $\mathcal{P}_{c_g}^{\gg}$ and $\mathcal{P}_{c_g}^{\ggg}$ are sets of grounded partitions defined in Section 3.

In other words, the characters of irreducible highest weight modules whose weights have constant ground state paths can be computed as generating functions for some types of partitions.

Applying this theorem to $A_{n-1}^{(1)}$ at level 1 leads to the following character formulas.

Theorem 1.10. *Let n be a positive integer, and let $\Lambda_0,\ldots,\Lambda_{n-1}$ be the fundamental weights of $A_{n-1}^{(1)}$. By setting $e^{\overline{\mathrm{wt}}v_i}=b_i$ and $e^{-\delta}=q$, we have the following identities:*

$$G_n^P(q;b_0,\cdots,b_{n-1})=\frac{e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0))}{(q;q)_\infty},$$

$$G_n^C(\delta,\gamma;q;b_0,\cdots,b_{n-1})=e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0)).$$

This result gives an evaluation of the character of the irreducible highest weight module for the particular weight Λ_0, but we can extend our techniques to retrieve the characters for the other level 1 weights of $\bar{P}_1^+$.

Theorem 1.11. *Let n be a positive integer, and let $\Lambda_0,\ldots,\Lambda_{n-1}$ be the fundamental weights of $A_{n-1}^{(1)}$. By setting $e^{\overline{\mathrm{wt}}v_i}=b_i$ and $e^{-\delta}=q$, we have the following identities for any $\ell\in\{0,\ldots,n-1\}$:*

$$G_n^P(q;b_0q,\cdots,b_{\ell-1}q,b_\ell,\ldots,b_{n-1})=\frac{e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell))}{(q;q)_\infty},$$

$$G_n^C(\delta,\gamma;q;b_0q,\cdots,b_{\ell-1}q,b_\ell,\ldots,b_{n-1})=e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell)).$$

The case $\ell=0$ of Theorem 1.11 gives Theorem 1.10.

As mentioned earlier, finding an expression of the character as a series with positive coefficients is an important problem. In [10], using modular forms and string functions, Kac and Peterson gave a formula for $e^{-\Lambda}\mathrm{ch}(L(\Lambda))$ for all the irreducible highest weight level 1 modules $L(\Lambda)$ of most classical types as a series in $\mathbb{Z}[[e^{-\alpha_0},e^{-\alpha_1},\cdots,e^{-\alpha_{n-1}}]]$ with obviously positive coefficients. This built on earlier work of Kac [8], in which he proved the particular case where $M=L(\Lambda_0)$ in $A_n^{(1)}$, $D_n^{(1)}$, and $E_n^{(1)}$.

In [1], Bartlett and Warnaar used Hall-Littlewood polynomials to give explicitly positive formulas for the characters of certain highest weight modules of the affine Lie algebras $C_n^{(1)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$, which also led to generalisations for the Macdonald identities in types $B_n^{(1)}$, $C_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$. However their approach failed to give a formula for the case $A_n^{(1)}$. Using Macdonald-Koornwinder theory, Rains and Warnaar [25] later found additional character formulas for these types, together with new Rogers-Ramanujan type identities.

In [6], Griffin, Ono, and Warnaar obtained a limiting Rogers-Ramanujan type identity for the principal specialisation of the character of some particular weights $(m-k)\Lambda_0 + k\Lambda_1$ in $A_n^{(1)}$. On the other hand, Meurman and Primc [19] treated the case of all levels of $A_1^{(1)}$ via vertex operator algebras.

Here, using our non-dilated character formula from Theorem 1.11, we recover, for all $\ell \in \{0,\dots,n-1\}$, the character formula of Kac-Peterson. Moreover, we give a new explicit expression for $e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell))$ as a sum of $(n-1)!$ series with positive coefficients $\mathbb{Z}[[e^{-\alpha_0}, e^{-\alpha_1}, \cdots, e^{-\alpha_{n-1}}]]$, each of which are infinite product generating functions for partitions into distinct parts with congruence conditions.

Theorem 1.12. *Let n be a positive integer, and let $\Lambda_0,\dots,\Lambda_{n-1}$ be the fundamental weights of $A_{n-1}^{(1)}$. For all $\ell \in \{0,\dots,n-1\}$, we have*

$$
\begin{aligned}
&e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell))\\
&= \frac{1}{(e^{-\delta};e^{-\delta})_\infty^{n-1}} \sum_{\substack{s_1,\dots,s_{n-1}\in\mathbb{Z}\\ s_0=s_n=0}} e^{-s_\ell\delta} \prod_{i=1}^{n-1} e^{s_i\alpha_i} e^{s_i(s_{i+1}-s_i)\delta} \qquad (1.8)\\
&= \left(\prod_{i=1}^{n-1} \frac{\left(e^{-i(i+1)\delta};e^{-i(i+1)\delta}\right)_\infty}{(e^{-\delta};e^{-\delta})_\infty}\right) \sum_{\substack{r_1,\dots,r_{n-1}:\\ 0\le r_j\le j-1\\ r_n=0}} e^{-r_\ell\delta} \prod_{i=1}^{n-1} e^{r_i\alpha_i} e^{r_i(r_{i+1}-r_i)\delta}\\
&\qquad \times \left(-e^{(ir_{i+1}-(i+1)r_i-\frac{i(i+1)}{2}-\ell\chi(i\ge l>0))\delta+\sum_{j=1}^i j\alpha_j}; e^{-i(i+1)\delta}\right)_\infty \qquad (1.9)\\
&\qquad \times \left(-e^{((i+1)r_i-ir_{i+1}-\frac{i(i+1)}{2}+\ell\chi(i\ge l>0))\delta-\sum_{j=1}^i j\alpha_j}; e^{-i(i+1)\delta}\right)_\infty,
\end{aligned}
$$

where $\delta = \alpha_0 + \cdots + \alpha_{n-1}$ is the null root and r_0 is taken to be 0.

The character formula (1.8) is, up to a change of variables, a reformulation of the Kac-Peterson character formula for the type $A_{n-1}^{(1)}$ given in [10, p.217]. Thus, our partition identity Theorem 1.1, combined with Theorem 1.11, makes the connection between the KMN[2] crystal base character formula and the Kac-Peterson character formula.

The *principal specialisation* [9, Chapter 10] for the affine type $A_{n-1}^{(1)}$ consists in transforming the generators with

$$e^{-\alpha_i} \mapsto q \quad \text{for all } i \in \{0,\dots,n-1\}.$$

In that case, we have a natural transformation $b_i := q^i b_0$ and a dilated version of the character formula can be deduced from Theorems 1.1 and 1.11.

Corollary 1.13. *Let n be a positive integer, and let $\Lambda_0,\dots,\Lambda_{n-1}$ be the fundamental weights of $A_{n-1}^{(1)}$. For all $\ell \in \{0,\cdots,n-1\}$, the principal specialisation of $e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell))$, denoted by $\mathbb{F}_{\mathbb{1}}(e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell)))$, is the generating function of the classical integer partitions with no parts divisible by n :*

$$
\begin{aligned}
\mathbb{F}_{\mathbb{1}}(e^{-\Lambda_\ell}\mathrm{ch}(L(\Lambda_\ell))) &= (q^n;q^n)_\infty \times G_n^P(q^n; q^n b_0,\cdots,q^{n+\ell-1}b_0, q^\ell,\cdots,q^{n-1}b_0)\\
&= (q^n;q^n)_\infty \times [x^0]\left(\prod_{i=0}^{\ell-1}(-q^{-i}b_0^{-1}x;q^n)_\infty(-q^{n+i}b_0x^{-1};q^n)_\infty\right.\\
&\qquad\qquad \left.\times \prod_{i=\ell}^{n-1}(-q^{n-i}b_0^{-1}x;q^n)_\infty(-q^ib_0x^{-1};q^n)_\infty\right)\\
&= (q^n;q^n)_\infty \times [x^0](-q^{1-\ell}b_0^{-1}x;q)_\infty(q^\ell b_0x^{-1};q)_\infty\\
&= \frac{(q^n;q^n)_\infty}{(q;q)_\infty}.
\end{aligned}
$$

In this particular case, we recover the principal specialisation of the Weyl-Kac character formula [9].

The remainder of this paper is organised as follows. In Section 2, we recall the necessary definitions and theorems about representation theory and crystal bases. In Section 3, we define *grounded partitions*, which will play a key role in obtaining a non-specialised character formula. In Section 4, we define the $A_{n-1}^{(1)}$

crystals related to our difference condition/energy function Δ. In Section 5, we prove our character formulas (Theorems 1.10, 1.11, and 1.12) assuming that Δ is an energy function for our crystal. Finally, in Sections 6 and 7, we prove that this is indeed the case, by constructing some paths on the crystal graph.

2. Basics on Crystals

In this section, we recall the definitions and basic theorems from crystal base theory which are necessary for our purpose. We refer to the book [7], which we consider to be a good summary of the basic theory of Kac-Moody algebras [7, Chapter 2], quantum groups [7, Chapter 3] and crystal bases [7, Chapters 4, 10]. For a more combinatorial approach and more emphasis on the finite dimensional case, we refer the reader to [3].

Throughout this section, n is a fixed positive integer.

2.1. **Cartan datum and quantum affine algebras.** A square matrix $A = \left(a_{i,j}\right)_{i,j\in\{0,\dots,n-1\}}$ is said to be a generalised Cartan matrix if A has the following properties:

- for all $i \in \{0,\dots,n-1\}$, $a_{i,i} = 2$,
- for all $i \neq j$ in $\{0,\dots,n-1\}$, $a_{i,j} \in \mathbb{Z}_{\leq 0}$,
- $a_{i,j} = 0$ if and only if $a_{j,i} = 0$,

Moreover, if there exists a diagonal matrix D with positive integer coefficients such that DA is symmetric, then A is said to be symmetrisable. In addition, if the rank of the matrix A is $n-1$, then A is said to be of affine type. In this paper, we always assume that this is the case.

Let us consider such a matrix A. Let $P^\vee$ be a free abelian group of rank $n+1$ with $\mathbb{Z}$-basis $\{h_0,\dots,h_{n-1},d\}$:

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_{n-1} \oplus \mathbb{Z}d.$$

We call $P^\vee$ the *dual weight lattice*. The complexification $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ is called the *Cartan subalgebra*. The linear functionals α_i and Λ_i $(i \in \{0,\dots,n-1\})$ on $\mathfrak{h}$ given by

$$\begin{array}{lll} \langle h_j,\alpha_i\rangle := \alpha_i(h_j) = a_{j,i} & \langle d,\alpha_i\rangle := \alpha_i(d) = \delta_{i,0} & \\ \langle h_j,\Lambda_i\rangle := \Lambda_i(h_j) = \delta_{i,j} & \langle d,\Lambda_i\rangle := \Lambda_i(d) = 0 & (i,j \in \{0,\dots,n-1\}) \end{array} \tag{2.1}$$

are respectively the *simple roots* and *fundamental weights*. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. We denote by $\Pi = \{\alpha_i \mid i \in \{0,\dots,n-1\}\} \subset \mathfrak{h}^*$ the set of simple roots, and define $\Pi^\vee = \{h_i \mid i \in \{0,\dots,n-1\}\} \subset \mathfrak{h}$ to be the set of *simple coroots*. We also set

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$$

to be the *weight lattice*. It contains the set of *dominant integral weights*

$$P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all i } \in \{0,\dots,n-1\}\}.$$

The quintuple $(A,\Pi,\Pi^\vee,P,P^\vee)$ is said to be a *Cartan datum* for the Cartan matrix A. The *Kac-Moody affine Lie algebra* $\widehat{\mathfrak{g}}$ attached to this datum is the Lie algebra with generators e_i, f_i $(i \in \{0,\dots,n-1\})$ and $h \in P^\vee$, with the following defining relations ([7, Definition 2.1.3]):

(1) $[h,h'] = 0$ for all $h,h' \in P^\vee$,
(2) $[e_i,f_j] = \delta_{ij}h_j$,
(3) $[h,e_i] = \alpha_i(h)e_i$ for all $h \in P^\vee$,
(4) $[h,f_i] = -\alpha_i(h)f_i$ for all $h \in P^\vee$,
(5) $(\mathrm{ad}e_i)^{1-a_{i,j}}e_j = (\mathrm{ad}f_i)^{1-a_{i,j}}f_j = 0$ for $i \neq j$,

where $\mathrm{ad}x : y \mapsto [x,y]$.

We also define the *coroot lattice*

$$\bar{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_{n-1},$$

and its complexification $\bar{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \bar{P}^\vee$. The $\mathbb{Z}$-submodule

$$\bar{P} := \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_{n-1}$$

of P is called the lattice of *classical weights*.

Remark. By (2.1), for all $j \neq 0$, we have

$$\alpha_j = \sum_{i=0}^{n-1} a_{i,j}\Lambda_i \ \in \bar{P}.$$

We will denote by $\overline{\alpha}_0$ the restriction of α_0 to $\bar{P}$.

Let $\bar{P}^+ := \sum_{i=0}^{n} \mathbb{Z}_{\geq 0}\Lambda_i$ denote the corresponding set of dominant weights.

The center

$$\mathbb{Z}c = \{h \in P^\vee : \langle h, \alpha_i\rangle = 0 \text{ for all i } \in \{0, \dots, n-1\}\}$$

of the affine Lie algebra $\widehat{\mathfrak{g}}$ is one-dimensional and generated by the *canonical central element* c, where

$$c = c_0 h_0 + \cdots + c_{n-1}h_{n-1}.$$

The space of imaginary roots

$$\mathbb{Z}\delta = \{\lambda \in P : \langle h_i, \lambda\rangle = 0 \text{ for all } i \in \{0, \dots, n-1\}\}$$

of $\widehat{\mathfrak{g}}$ is also one-dimensional, generated by the *null root* δ, where

$$\delta = d_0\alpha_0 + d_1\alpha_1 + \cdots + d_{n-1}\alpha_{n-1}.$$

The vector ${}^t(d_0, d_1, \dots, d_{n-1}) \in \mathbb{C}^n$ spans the kernel of the Cartan matrix A. The *level* ℓ of a dominant weight $\lambda \in P^+$ is given by the expression $\langle c, \lambda\rangle := \lambda(c) = \ell$.

For any $k \in \mathbb{Z}$ and an indeterminate q, let us set

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

We also set $[0]_q\,! = 1$ and for $k \geq 1$, $[k]_q\,! = [k]_q[k-1]_q \cdots [1]_q$. For $m \geq k \geq 0$, define

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_q = \frac{[m]_q\,!}{[k]_q\,!\,[m-k]_q\,!}.$$

We now have all the definitions necessary to introduce quantum affine Lie algebras.

Definition 2.1. [7, Definition 3.1.1] The *quantum affine algebra* $U_q(\widehat{\mathfrak{g}})$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra with unit element over $\mathbb{C}(q)$ (where q is an indeterminate) with generators e_i, f_i $(i \in \{0, \dots, n-1\})$, and q^h $(h \in P^\vee)$, satisfying the defining relations:

(1) $q^0 = 1,\ q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,
(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in P^\vee$, $i \in \{0, \dots, n-1\}$,
(3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee$, $i \in \{0, \dots, n-1\}$,
(4) $e_i f_j - f_j e_i = \delta_{i,j} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in \{0, \dots, n-1\}$,
(5) $\displaystyle\sum_{k=0}^{1-a_{i,j}} \left\langle \begin{matrix} 1-a_{i,j} \\ k \end{matrix} \right\rangle_{q_i} e_i^{1-a_{i,j}-k} e_j e_i^k = 0$ for $i \neq j$,
(6) $\displaystyle\sum_{k=0}^{1-a_{i,j}} \left\langle \begin{matrix} 1-a_{i,j} \\ k \end{matrix} \right\rangle_{q_i} f_i^{1-a_{i,j}-k} f_j f_i^k = 0$ for $i \neq j$.

Here $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$, where $D = diag(s_i : i \in \{0, \dots, n-1\})$ is a symmetrising matrix of A.

Definition 2.2. The *quantum affine algebra* $U'_q(\widehat{\mathfrak{g}})$ is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $e_i, f_i, K_i^{\pm 1}$ $(i \in \{0, \dots, n-1\})$.

Contrarily to $U_q(\widehat{\mathfrak{g}})$, the quantum affine algebra $U'_q(\widehat{\mathfrak{g}})$ admits some non-trivial finite-dimensional irreducible modules.

2.2. **Integrable modules, highest weight modules and character formula.** We are now ready to define irreducible highest weight modules and characters.

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra with bracket $[\cdot,\cdot]$, and let V be a vector space. Then V is a $\mathfrak{g}$-*module* if there is a bilinear map $\mathfrak{g}\times V\to V$, denoted by $(x,v)\mapsto x\cdot v$, satisfying for all $x,y\in\mathfrak{g}$ and all $v\in V$:

$$[x,y]\cdot v = x\cdot(y\cdot v)-y\cdot(x\cdot v).$$

A subspace W of a $\mathfrak{g}$-module V is called a *submodule* of V if for all $x\in\mathfrak{g}$, $x\cdot W\subseteq W$.

A $\mathfrak{g}$-module V is said to be *irreducible* if its only submodules are V and 0.

The notion of modules extends naturally from an affine Lie algebra $\widehat{\mathfrak{g}}$ to its quantum affine algebra $U_q(\widehat{\mathfrak{g}})$.

Definition 2.4. A $U_q(\widehat{\mathfrak{g}})$-module M is said to be *integrable* if it satisfies the following properties:

(a) M has a weight space decomposition: $M=\bigoplus_{\lambda\in P}M_\lambda$, where $M_\lambda=\{v\in M\mid q^h\cdot v=q^{\lambda(h)}v$ for all $h\in P^\vee\}$;

(b) there are finitely many $\lambda_1,\dots,\lambda_k\in P$ such that $\mathrm{wt}(M)\subseteq\Omega(\lambda_1)\cup\dots\cup\Omega(\lambda_k)$, where $\mathrm{wt}(M)=\{\lambda\in P\mid M_\lambda\neq 0\}$ and $\Omega(\lambda_j)=\{\mu\in P\mid \mu\in\lambda_j+\sum_{i\in\{0,\dots,n-1\}}\mathbb{Z}_{\leq 0}\alpha_i\}$;

(c) the elements e_i and f_i act locally nilpotently on M for all $i\in\{0,\dots,n-1\}$.

We denote by $\mathcal{O}^q_{\text{int}}$ the category of integrable $U_q(\widehat{\mathfrak{g}})$-modules.

For all $\lambda\in P$, a module of *highest weight* λ is an integrable module such that:

(a) $\mathrm{wt}(M)\subseteq\Omega(\lambda)$;

(b) $\dim M_\lambda=1$;

(b) $M=U_q(\widehat{\mathfrak{g}})M_\lambda$.

For all $\lambda\in P$, up to isomorphism, there exists a unique highest weight module which is *irreducible*. We denote by $L(\lambda)$ the *irreducible highest weight* $U_q(\widehat{\mathfrak{g}})$-module of highest weight λ.

Definition 2.5. Let M be an integrable module such that $\dim M_\lambda<\infty$ for all $\lambda\in\mathrm{wt}(M)$. The *character* of M is defined by

$$\mathrm{ch}(M)=\sum_{\lambda\in\mathrm{wt}(M)}\dim M_\lambda\cdot e^\lambda, \tag{2.2}$$

where the e^λ's are formal basis elements of the group algebra $\mathbb{C}[\mathfrak{h}^*]$, with the multiplication defined by $e^\lambda e^\mu=e^{\lambda+\mu}$.

When M is a highest weight module of highest weight λ, its character satisfies

$$e^{-\lambda}\mathrm{ch}(M)=\sum_{\mu\in\mathrm{wt}(M)}\dim M_\mu\cdot e^{\mu-\lambda}\quad\in\quad\mathbb{Z}_{\geq 0}[[e^{-\alpha_i},i\in\{0,\dots,n-1\}]].$$

All these definitions on modules also hold in the case of the $\widehat{\mathfrak{g}}$-modules M', where the weight spaces are given by $M'_\lambda=\{v\in M'\mid h\cdot v=\lambda(h)v$ for all $h\in P^\vee\}$. Thus, looking at the generators of the weight spaces, for a fixed weight $\lambda\in P$, the *irreducible highest weight* $\widehat{\mathfrak{g}}$-module can be identified with the *irreducible highest weight* $U_q(\widehat{\mathfrak{g}})$-module, and we have equality of characters.

2.3. **Crystal bases.** The crystal base theory was developed independently by Kashiwara [14] and Lusztig [17] to study the category $\mathcal{O}^q_{\text{int}}$ of integrable $U_q(\widehat{g})$-modules. If M is a module in the category $\mathcal{O}^q_{\text{int}}$, then for each $i\in\{0,\dots,n-1\}$, a weight vector $u\in M_\lambda$ can be written uniquely in the form $u=\sum_{k=0}^N f_i^{(k)}u_k$, for some $N\geq 0$ and $u_k\in M_{\lambda+k\alpha_i}\cap\ker e_i$ for all $k=0,1,\dots,N$, with $f_i^{(k)}=f_i^k/([k]_{q_i}!)$. The *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$, for $i\in\{0,\dots,n-1\}$, are then defined as follows:

$$\tilde{e}_iu=\sum_{k=1}^N f_i^{(k-1)}u_k,\qquad \tilde{f}_iu=\sum_{k=0}^N f_i^{(k+1)}u_k. \tag{2.3}$$

Crystal bases will be seen as bases at $q=0$. To do so, let us define the *localisation* of $\mathbb{C}[q]$ at $q=0$ by $\mathbb{A}_0=\{f=g/h\mid g,h\in\mathbb{C}[q],\ h(0)\neq 0\}$.

Definition 2.6. [7, Definition 4.2.2] Assume that M is a $U_q(\widehat{\mathfrak{g}})$-module in the category $\mathcal{O}^q_{\text{int}}$. A free $\mathbb{A}_0$-submodule $\mathcal{L}$ of M is a *crystal lattice* if

(i) $\mathcal{L}$ generates M as a vector space over $\mathbb{C}(q)$;
(ii) $\mathcal{L} = \bigoplus_{\lambda\in P} \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = M_\lambda \cap \mathcal{L}$;
(iii) $\tilde{e}_i\mathcal{L} \subset \mathcal{L}$ and $\tilde{f}_i\mathcal{L} \subset \mathcal{L}$, for all $i \in \{0,\dots,n-1\}$.

Since the operators $\tilde{e}_i$ and $\tilde{f}_i$ preserve the lattice $\mathcal{L}$, they also define operators on the quotient $\mathcal{L}/q\mathcal{L}$.

Definition 2.7. [7, Definition 4.2.3] A *crystal base* for a $U_q(\widehat{\mathfrak{g}})$-module $M \in \mathcal{O}^q_{\text{int}}$ is a pair $(\mathcal{L},\mathcal{B})$ such that

(1) $\mathcal{L}$ is a crystal lattice of M;
(2) $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{C} \otimes_{\mathbb{A}_0} \mathcal{L}$;
(3) $\mathcal{B} = \sqcup_{\lambda\in P}\mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda/q\mathcal{L}_\lambda)$;
(4) $\tilde{e}_i\mathcal{B}_\lambda \subset \mathcal{B}_{\lambda+\alpha_i} \cup \{0\}$ and $\tilde{f}_i\mathcal{B}_\lambda \subset \mathcal{B}_{\lambda-\alpha_i} \cup \{0\}$ for all $i \in \{0,\dots,n-1\}$;
(5) $\tilde{f}_ib = b'$ if and only if $b = \tilde{e}_ib'$, for $b,b' \in \mathcal{B}$ and $i \in \{0,\dots,n-1\}$.

To each module $M \in \mathcal{O}^q_{\text{int}}$, one can associate a corresponding crystal base $(\mathcal{L},\mathcal{B})$, which is unique up to isomorphism [7, Chapter 5]. Therefore, from now on, we will refer to "the" crystal base of M.

Furthermore, the *crystal graph* associated to $(\mathcal{L},\mathcal{B})$ can be defined as follows. The set of vertices is $\mathcal{B}$, and the oriented edges are built as follows:

$$b \xrightarrow{i} b' \quad \text{if and only if} \quad \tilde{f}_ib = b' \text{ (or equivalently } \tilde{e}_ib' = b).$$

Remark 2.8. *When $\tilde{f}_ib = 0$ (resp. $\tilde{e}_ib = 0$), then there is no edge labelled i coming out of b (resp. arriving in b).*

The crystal graph can be viewed as a combinatorial data of the module M.

For $i \in \{0,\dots,n-1\}$, let us define functions $\varepsilon_i,\varphi_i : \mathcal{B} \to \mathbb{Z}$ as follows:

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^kb \in \mathcal{B}\},$$
$$\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^kb \in \mathcal{B}\}.$$

Thus $\varepsilon_i(b)$ is the length of the longest chain of i-arrows ending at b in the crystal graph, and $\varphi_i(b)$ is the length of the longest chain of i-arrows starting from b. Furthermore, we have $\varphi_i(b) - \varepsilon_i(b) = \lambda(h_i)$ for all $b \in \mathcal{B}_\lambda$. Thus, by setting $\mathrm{wt}b = \lambda$,

$$\varepsilon(b) = \sum_{i=0}^{n-1} \varepsilon_i(b)\Lambda_i, \qquad \text{and} \qquad \varphi(b) = \sum_{i=0}^{n-1} \varphi_i(b)\Lambda_i, \tag{2.4}$$

we then have $\overline{\mathrm{wt}}b = \varphi(b)-\varepsilon(b)$ for all $b \in \mathcal{B}_\lambda$, where $\overline{\mathrm{wt}}b$ is the projection of $\mathrm{wt}b$ on $\overline{P}$. Also, by the definition of the weight vectors u_k in the Kashiwara operators (2.3), we have for all $b \in \mathcal{B}$ such that $\tilde{e}_ib \neq 0$,

$$\mathrm{wt}\tilde{e}_ib - \mathrm{wt}b = \alpha_i. \tag{2.5}$$

Let us now introduce the notion of crystal.

Definition 2.9. [7, Definition 4.5.1] Let $A = (a_{i,j})_{0\leq i,j\leq n-1}$ be a Cartan matrix with associated *Cartan datum* $(A,\Pi,\Pi^\vee,P,P^\vee)$. A *crystal* associated with $(A,\Pi,\Pi^\vee,P,P^\vee)$ is a set $\mathcal{B}$ together with maps

$$\begin{aligned} &\mathrm{wt} : \mathcal{B} \longrightarrow P, \\ &\tilde{e}_i,\tilde{f}_i : \mathcal{B} \longrightarrow \mathcal{B}\cup\{0\} \qquad (i \in \{0,\dots,n-1\}), \\ &\varepsilon_i,\varphi_i : \mathcal{B} \longrightarrow \mathbb{Z}\cup\{-\infty\} \qquad (i \in \{0,\dots,n-1\}), \end{aligned}$$

satisfying the following properties for all $i \in \{0,\dots,n-1\}$:

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$,
(2) $\mathrm{wt}(\tilde{e}_ib) = \mathrm{wt}b + \alpha_i$ if $\tilde{e}_ib \in \mathcal{B}$,
(3) $\mathrm{wt}(\tilde{f}_ib) = \mathrm{wt}b - \alpha_i$ if $\tilde{f}_ib \in \mathcal{B}$,
(4) $\varepsilon_i(\tilde{e}_ib) = \varepsilon_i(b) - 1$ if $\tilde{e}_ib \in \mathcal{B}$,
(5) $\varphi_i(\tilde{e}_ib) = \varphi_i(b) + 1$ if $\tilde{e}_ib \in \mathcal{B}$,
(6) $\varepsilon_i(\tilde{f}_ib) = \varepsilon_i(b) + 1$ if $\tilde{f}_ib \in \mathcal{B}$,
(7) $\varphi_i(\tilde{f}_ib) = \varphi_i(b) - 1$ if $\tilde{f}_ib \in \mathcal{B}$,
(8) $\tilde{f}_ib = b'$ if and only if $b = \tilde{e}_ib'$ for $b,b' \in B$,

(9) if $\varphi_i(b) = -\infty$ for $b \in \mathcal{B}$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

In particular, if $(\mathcal{L}, \mathcal{B})$ is a crystal base, then $\mathcal{B}$ is a crystal.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two crystals. A *crystal morphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$ is a map $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ such that

- $\Psi(0) = 0$;
- Ψ commutes with $\mathrm{wt}, \varepsilon_i, \varphi_i$ for all $i \in \{0, \dots, n-1\}$;
- for $b, b' \in \mathcal{B}_1$ such that $\tilde{f}_i b = b'$ and $\Psi(b), \Psi(b') \in \mathcal{B}_2$, we have $\tilde{f}_i \Psi(b) = \Psi(b')$, $\tilde{e}_i \Psi(b') = \Psi(b)$.

A morphism Ψ is said to be *strict* if it commutes with $\tilde{e}_i$, $\tilde{f}_i$ for all $i \in \{0, \dots, n-1\}$.

The theory of crystal bases behaves very nicely with respect to the tensor product of $\mathcal{O}^q_{\text{int}}$-modules, as can be seen in the next theorem.

Theorem 2.10. [7, Theorem 4.4.1] *Let $M_1, M_2 \in \mathcal{O}_{\text{int}}$, and let $(\mathcal{L}_1, \mathcal{B}_1), (\mathcal{L}_2, \mathcal{B}_2)$ be the corresponding crystal bases. We set $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathbb{A}_0} \mathcal{L}_2$ and $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \equiv \mathcal{B}_1 \times \mathcal{B}_2$. Then $(\mathcal{L}, \mathcal{B})$ is a crystal base of $M_1 \otimes_{\mathbb{C}(q)} M_2$, with*

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \ \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \ \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \ \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \ \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \end{aligned} \tag{2.6}$$

where $b_1 \otimes 0 = 0 \otimes b_2 = 0$ for all $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$. Furthermore, we have

$$\begin{aligned} \mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt} b_1 + \mathrm{wt} b_2, \\ \varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)\}, \\ \varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)\}. \end{aligned}$$

The last tool we need in this paper is the notion of energy function, defined as follows.

Definition 2.11. [7, Definition 10.2.1] Let $M \in \mathcal{O}^q_{\text{int}}$ be a module, and $(\mathcal{L}, \mathcal{B})$ be the corresponding crystal base. An *energy function* on $\mathcal{B} \otimes \mathcal{B}$ is a map $H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$ satisfying

$$H\left(\tilde{e}_i(b_1 \otimes b_2)\right) = \begin{cases} H(b_1 \otimes b_2) & \text{if } \ i \neq 0, \\ H(b_1 \otimes b_2) + 1 & \text{if } \ i = 0 \text{ and } \varphi_0(b_1) \geq \varepsilon_0(b_2) \\ H(b_1 \otimes b_2) - 1 & \text{if } \ i = 0 \text{ and } \varphi_0(b_1) < \varepsilon_0(b_2), \end{cases} \tag{2.7}$$

for all $i \in \{0, \dots, n-1\}$ and b_1, b_2 with $\tilde{e}(b_1 \otimes b_2) \neq 0$.

By definition, in the crystal graph of $\mathcal{B} \otimes \mathcal{B}$, the value of $H(b_1 \otimes b_2)$, when it exists, determines all the values $H(b'_1 \otimes b'_2)$ for vertices $b'_1 \otimes b'_2$ in the same connected component as $b_1 \otimes b_2$. Note that the conditions (2.7) are equivalent to the following:

$$\begin{aligned} H\left(\tilde{e}_i(b_1 \otimes b_2)\right) &= \begin{cases} H(b_1 \otimes b_2) + \chi(i=0) & \text{if } \ \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ H(b_1 \otimes b_2) - \chi(i=0) & \text{if } \ \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ H(\tilde{f}_i(b_1 \otimes b_2)) &= \begin{cases} H(b_1 \otimes b_2) - \chi(i=0) & \text{if } \ \varphi_i(b_1) > \varepsilon_i(b_2) \\ H(b_1 \otimes b_2) + \chi(i=0) & \text{if } \ \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned} \tag{2.8}$$

Figure 2.1 gives the crystal graph $\mathcal{B}$ of the vector representation of $A_1^{(1)}$ [7, 10.5.2], the tensor product $\mathcal{B} \otimes \mathcal{B}$, and an energy function H on $\mathcal{B} \otimes \mathcal{B}$.

Figure 2.1.

$$\mathcal{B}: \quad \begin{tikzcd} \boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[l, bend left, "0"] \end{tikzcd}$$

$$\mathcal{B}\otimes\mathcal{B}: \quad \begin{tikzcd} \boxed{0}\otimes\boxed{0} \arrow[r, "1"] & \boxed{1}\otimes\boxed{0} \arrow[d, "1"] \\ \boxed{0}\otimes\boxed{1} \arrow[u, "0"] & \boxed{1}\otimes\boxed{1} \arrow[l, "0"] \end{tikzcd} \qquad H: \quad \boxed{i}\otimes\boxed{j} \ \mapsto\ \chi(i \geq j)$$

2.4. **Perfect crystals.** The theory of perfect crystals was developed by Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki [11, 12] to study the irreducible highest weight modules over quantum affine algebras. Indeed, perfect crystals provide a construction of the crystal base $\mathcal{B}(\lambda)$ of any irreducible $U_q(\widehat{\mathfrak{g}})$-module $L(\lambda)$ corresponding to a classical weight $\lambda \in \bar{P}^+$. We call *affine crystal* an abstract crystal associated with an affine Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ (quantum algebra $U_q(\widehat{\mathfrak{g}})$), while the term *classical crystal* is used for an abstract crystal associated to the classical Cartan datum $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$ (quantum algebra $U'_q(\widehat{\mathfrak{g}})$ defined in Definition 2.2).

All the theorems in this section are due to Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki, but we give references to the book [7] for reader's convenience.

Let us start by defining perfect crystals.

Definition 2.12. [7, Definition 10.5.1] For a positive integer ℓ, a finite classical crystal $\mathcal{B}$ is said to be a *perfect crystal of level* ℓ for the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ if

(1) there is a finite-dimensional $U'_q(\widehat{\mathfrak{g}})$-module with a crystal base whose crystal graph is isomorphic to $\mathcal{B}$;
(2) $\mathcal{B}\otimes\mathcal{B}$ is connected;
(3) there exists a classical weight λ_0 such that

$$\mathrm{wt}(\mathcal{B}) \subset \lambda_0 + \frac{1}{d_0}\sum_{i\neq 0} \mathbb{Z}_{\leq 0}\alpha_i \quad \text{and} \quad |\mathcal{B}_{\lambda_0}| = 1;$$

(4) for any $b \in \mathcal{B}$, we have

$$\langle c, \varepsilon(b)\rangle = \sum_{i=0}^{n-1} \varepsilon_i(b)\Lambda_i(c) \geq \ell;$$

(5) for each $\lambda \in \bar{P}^+_\ell := \{\mu \in \bar{P}^+ \mid \langle c, \mu\rangle = \ell\}$, there exist unique vectors b^λ and b_λ in $\mathcal{B}$ such that $\varepsilon(b^\lambda) = \lambda$ and $\varphi(b_\lambda) = \lambda$.

In the remainder of this section, we fix a perfect crystal $\mathcal{B}$.

The maps $\lambda \mapsto \varepsilon(b_\lambda)$ and $\lambda \mapsto \varphi(b^\lambda)$ then define two bijections on $\bar{P}^+_\ell$.

As a consequence of the last condition, for any $\lambda \in \bar{P}^+_\ell$, the vertex operator theory [7, (10.4.4)] leads to a natural crystal isomorphism

$$\begin{aligned} \mathcal{B}(\lambda) &\overset{\sim}{\to} \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B} \\ u_\lambda &\mapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda. \end{aligned} \tag{2.9}$$

Definition 2.13. For $\lambda \in \bar{P}^+_\ell$, the *ground state path of weight* λ is the tensor product

$$\mathfrak{p}_\lambda = (g_k)_{k=0}^\infty = \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0,$$

where the elements $g_k \in \mathcal{B}$ are such that

$$\begin{array}{lll} \lambda_0 = \lambda & g_0 = b_\lambda & \\ \lambda_{k+1} = \varepsilon(b_{\lambda_k}) & g_{k+1} = b_{\lambda_{k+1}} & \text{for all } \ k \geq 0 \cdot \end{array} \tag{2.10}$$

A tensor product $\mathfrak{p} = (p_k)_{k=0}^\infty = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0$ of elements $p_k \in \mathcal{B}$ is said to be a λ-*path* if $p_k = g_k$ for k large enough.

Iterating the isomorphism (2.9), we obtain

$$\begin{array}{ccccccc} \mathcal{B}(\lambda) & \overset{\sim}{\to} & \mathcal{B}(\lambda_1)\otimes\mathcal{B} & \overset{\sim}{\to} & \mathcal{B}(\lambda_2)\otimes\mathcal{B}\otimes\mathcal{B} & \overset{\sim}{\to} & \cdots \\ u_\lambda & \mapsto & u_{\lambda_1}\otimes g_0 & \mapsto & u_{\lambda_2}\otimes g_1 \otimes g_0 & \mapsto & \cdots, \end{array}$$

which gives a natural bijection stated in the next theorem.

Theorem 2.14. [7, Theorem 10.6.4] *Let $\lambda \in \bar{P}_\ell^+$. Then there is a crystal isomorphism*

$$\begin{aligned}\mathcal{B}(\lambda) &\xrightarrow{\sim} \mathcal{P}(\lambda)\\ u_\lambda &\mapsto \mathfrak{p}_\lambda\end{aligned}$$

between the crystal base $\mathcal{B}(\lambda)$ of $L(\lambda)$ and the set $\mathcal{P}(\lambda)$ of λ-paths.

We describe the crystal structure of $\mathcal{P}(\lambda)$ as follows [7, (10.48)]. For any $\mathfrak{p} = (p_k)_{k=0}^\infty \in \mathcal{P}(\lambda)$, let $N \geq 0$ be the smallest integer such that $p_k = g_k$ for all $k \geq N$. We then set

$$\begin{aligned}
\overline{\mathrm{wt}}\mathfrak{p} &= \lambda_N + \sum_{k=0}^{N-1} \overline{\mathrm{wt}} p_k,\\
\tilde{e}_i\mathfrak{p} &= \cdots \otimes g_{N+1} \otimes \tilde{e}_i\left(g_N \otimes \cdots \otimes p_0\right),\\
\tilde{f}_i\mathfrak{p} &= \cdots \otimes g_{N+1} \otimes \tilde{f}_i\left(g_N \otimes \cdots \otimes p_0\right),\\
\varepsilon_i(\mathfrak{p}) &= \max\left(\varepsilon_i(\mathfrak{p}') - \varphi_i(g_N), 0\right),\\
\varphi_i(\mathfrak{p}) &= \varphi_i(\mathfrak{p}') + \max\left(\varphi_i(g_N) - \varepsilon_i(\mathfrak{p}'), 0\right),
\end{aligned}$$

where $\mathfrak{p}' := p_{N-1} \otimes \cdots \otimes p_1 \otimes p_0$, and $\overline{\mathrm{wt}}$ is viewed as the classical weight of an element of $\mathcal{B}$ or $\mathcal{P}(\lambda)$.

The explicit expression for the affine weight $\mathrm{wt}\mathfrak{p}$ in P is given in the following theorem, which is known as the (KMN)^2 crystal base character formula, and plays a key role in connecting characters with partition generating functions.

Theorem 2.15. [7, Theorem 10.6.7] *Let $\lambda \in \bar{P}_\ell^+$, let H be an energy function on $\mathcal{B} \otimes \mathcal{B}$, and let $\mathfrak{p} = (p_k)_{k=0}^\infty \in \mathcal{P}(\lambda)$. Then the weight of $\mathfrak{p}$ and the character of the irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-module $L(\lambda)$ are given by the following expressions:*

$$\begin{aligned}
\mathrm{wt}\mathfrak{p} &= \lambda + \sum_{k=0}^{\infty}\left(\overline{\mathrm{wt}}p_k - \overline{\mathrm{wt}}g_k\right) - \left(\sum_{k=0}^{\infty}(k+1)\Big(H(p_{k+1}\otimes p_k) - H(g_{k+1}\otimes g_k)\Big)\right)\frac{\delta}{d_0},\\
&= \lambda + \sum_{k=0}^{\infty}\left(\overline{\mathrm{wt}}p_k - \overline{\mathrm{wt}}g_k\right) - \left(\sum_{l=k}^{\infty}(H(p_{l+1}\otimes p_l) - H(g_{l+1}\otimes g_l))\right)\frac{\delta}{d_0}, && (2.11)\\
\mathrm{ch}(\mathrm{L}(\lambda)) &= \sum_{\mathfrak{p}\in\mathcal{P}(\lambda)} e^{\mathrm{wt}\mathfrak{p}}. && (2.12)
\end{aligned}$$

A specialisation of Theorem 2.15 gives the following corollary.

Corollary 2.16. *Suppose that λ is such that $b_\lambda = b^\lambda = g$, and set $H(g \otimes g) = 0$. Then $\overline{\mathrm{wt}}g = 0$, $g_k = g$ for all $k \in \mathbb{Z}_{\geq 0}$, and we have*

$$\mathrm{wt}\mathfrak{p} = \lambda + \sum_{k=0}^{\infty}\overline{\mathrm{wt}\mathfrak{p}}_k - \left(\sum_{l=k}^{\infty} H(p_{l+1}\otimes p_l)\right)\frac{\delta}{d_0}.$$

This is the main result which we will use in the next section to connect crystal base theory to integer partitions.

Note that the d_0 in Theorem 2.15 and Corollary 2.16 did not appear in [7] and the original work [11], but this was a typo which was fixed later in [13] for example. However, it did not affect their results, as for most classical types (including $A_{n-1}^{(1)}$), we have $d_0 = 1$.

3. Perfect crystals and grounded partitions

To make the connection between our combinatorial partition identities and character formulas, we introduce in this section a new type of coloured partitions: grounded partitions.

Let $\mathcal{C}$ be a set of colours, and let $\mathbb{Z}_{\mathcal{C}} = \{k_c : k \in \mathbb{Z}, c \in \mathcal{C}\}$ be the set of coloured integers. First, we relax the condition that parts of (coloured) partitions have to be in non-increasing order.

Definition 3.1. Let $\gg$ be a binary relation defined on $\mathbb{Z}_{\mathcal{C}}$. A *generalised coloured partition* with relation $\gg$ is a finite sequence $(\pi_0, \dots, \pi_s)$ of coloured integers, where for all $i \in \{0, \dots, s-1\}$, $\pi_i \gg \pi_{i+1}$.

In the following, $c(\pi_i) \in \mathcal{C}$ denotes the colour of the part π_i. The quantity $|\pi| = \pi_0 + \cdots + \pi_s$ is the weight of π, and $C(\pi) = c(\pi_0) \cdots c(\pi_s)$ is its colour sequence.

Remark. The binary relation is not necessarily an order. When $\gg$ is a strict total order, we can easily check that every finite set of coloured parts defines a classical coloured partition, by ordering the parts. In the same way, for a large total order, the generalised coloured partitions are finite multi-sets of coloured integers.

Let us choose a particular colour c_g. We now define grounded partitions, which are directly related to ground state paths.

Definition 3.2. A *grounded partition* with ground c_g and relation $\gg$ is a non-empty generalised coloured partition $\pi = (\pi_0, \dots, \pi_s)$ with relation $\gg$, such that $\pi_s = 0_{c_g}$, and when $s > 0$, $\pi_{s-1} \neq 0_{c_g}$.
Let $\mathcal{P}_{c_g}^{\gg}$ denote the set of such partitions.

In the following, we explicitly write $\pi = (\pi_0, \dots, \pi_{s-1}, 0_{c_g})$. The trivial partition in $\mathcal{P}_{c_g}^{\gg}$ is then (0_{c_g}).

Example 3.3. *For the set of classical integer partitions $\pi = (\pi_1, \dots, \pi_s)$, where parts satisfy $\pi_1 \geq \cdots \geq \pi_s > 0$, the empty partition is such that $s = 0$. This set is in bijection with the set $\mathcal{P}_c$ of grounded coloured partitions with only one colour c, defined by the relation*

$$k_c \gg l_c \text{ if and only if } k - l \geq 0.$$

The bijection is given by

$$(\pi_1, \dots, \pi_s) \mapsto ((\pi_1)_c, \dots, (\pi_s)_c, 0_c),$$

where the empty partition $\emptyset$ corresponds to the grounded partition (0_c).

We now make the connection between grounded partitions and crystal base theory. **For the remainder of this section, we fix an affine Kac–Moody Lie algebra $\widehat{g}$, a perfect crystal $\mathcal{B}$ of level ℓ, a weight $\lambda \in \bar{P}_\ell^+$ with constant ground state path $\cdots \otimes g \otimes g$, and assume that** $H(g \otimes g) = 0$**.** Let $\mathcal{C}_{\mathcal{B}} = \{c_b : b \in \mathcal{B}\}$ be the set of colours indexed by $\mathcal{B}$. We define the binary relation $\gtrdot$ on $\mathbb{Z}_{\mathcal{C}_{\mathcal{B}}}$ by

$$k_{c_b} \gtrdot k'_{c_{b'}} \text{ if and only if } k - k' = H(b' \otimes b). \tag{3.1}$$

This relation leads to the following.

Proposition 3.4. *Let ϕ be the map between λ-paths and grounded partitions defined as follows:*

$$\phi : \quad \mathfrak{p} \mapsto (\pi_0, \dots, \pi_{s-1}, 0_{c_g}),$$

where $\mathfrak{p} = (p_k)_{k \geq 0}$ is a λ-path in $\mathcal{P}(\lambda)$, $s \geq 0$ is the unique non-negative integer such that $p_{s-1} \neq g$ and $p_k = g$ for all $k \geq s$, and for all $k \in \{0, \dots, s-1\}$, the part π_k has colour c_{p_k} and size

$$\sum_{l=k}^{s-1} H(p_{k+1} \otimes p_k).$$

Then ϕ is a bijection between $\mathcal{P}(\lambda)$ and $\mathcal{P}_{c_g}^{\gtrdot}$. Furthermore, by taking $c_b = e^{\overline{\mathrm{wt}} b}$, we have for all $\pi \in \mathcal{P}_{c_g}^{\gtrdot}$,

$$e^{-\lambda + \mathrm{wt} \phi^{-1}(\pi)} = C(\pi) e^{-\frac{\delta}{d_0}|\pi|}. \tag{3.2}$$

Proof. It is easy to see that $\phi(\mathfrak{p})$ belongs to $\mathcal{P}_{c_g}^{\gtrdot}$, since by (3.1) we have $\pi_k \gtrdot \pi_{k+1}$ for all $k \in \{0, \dots, s-1\}$, and $p_{s-1} \neq g$ implies that $\pi_{s-1} \neq 0_{c_g}$. Note that the ground state path $\cdots \otimes g \otimes \otimes g \otimes g$ is associated to (0_{c_g}).

Let us now give the inverse bijection. Start with $\pi \in (\pi_0, \dots, \pi_{s-1}, 0_{c_g}) \in \mathcal{P}_{c_g}^{\gtrdot}$, different from (0_{c_g}), with colour sequence $c_{p'_0} \cdots c_{p'_{s-1}} c_g$. Recall that $\pi_s = 0_{c_g}$. We set $\phi^{-1}(\pi) = (p_k)_{k \geq 0}$, where $p_k = g$ for all $k \geq s$ and $p_k = p'_k$ for all $k \in \{0, \dots, s-1\}$.

- We first show that $p_{s-1} \neq g$. Assume for the purpose of contradiction that $p_{s-1} = g$. By (3.1), we know that $\pi_{s-1} \gtrdot 0_{c_g}$ if and only if

  $$\pi_{s-1} - 0_{c_g} = H(p_s \otimes p_{s-1}) = H(g \otimes g) = 0,$$

  i.e. if and only if $\pi_{s-1} = 0_{c_g}$. This contradicts the fact that $\pi_{s-1} \neq 0_{c_g}$.

- By (3.1), we also have, for all $k \in \{0, \dots, s-1\}$, $\pi_k - \pi_{k+1} = H(p_{k+1} \otimes p_k)$. Therefore

$$\pi_k = \pi_k - 0_{c_g} = \sum_{l=k}^{s-1} \pi_l - \pi_{l+1} = \sum_{l=k}^{s-1} H(p_{l+1} \otimes p_l).$$

With what precedes, we have $\phi(\phi^{-1}(\pi)) = \pi$ and $\phi^{-1}(\phi(\mathfrak{p})) = \mathfrak{p}$. We obtain (3.2) by Corollary 2.16 and by observing that

$$\pi_k = \sum_{l=k}^{s-1} H(p_{l+1} \otimes p_l) = \sum_{l=k}^{\infty} H(p_{l+1} \otimes p_l),$$

since $H(p_{l+1} \otimes p_l) = H(g \otimes g) = 0$ for all $l \geq s$. $\square$

Example 3.5. *Let us consider the energy matrix* (1.1) *given by Primc for the case $A_1^{(1)}$, with the correspondence $c_{v_j \otimes v_i^\vee} = a_i b_j$ for all $i, j \in \{0, 1\}$. Let us set the ground $g = v_0 \otimes v_0^\vee$ corresponding to the classical weight Λ_0, so that $c_g = a_0 b_0$.*

- *The ground state path $\mathfrak{p}_{\Lambda_0} = \cdots \otimes (v_0 \otimes v_0^\vee) \otimes (v_0 \otimes v_0^\vee)$ corresponds to the partition $\phi(\mathfrak{p}_{\Lambda_0}) = (0_{a_0 b_0})$.*
- *For $\mathfrak{p} = \cdots \otimes (v_0 \otimes v_0^\vee) \otimes (v_0 \otimes v_0^\vee) \otimes (v_1 \otimes v_0^\vee) \otimes (v_0 \otimes v_1^\vee) \otimes (v_1 \otimes v_1^\vee)$, we have*

$$\phi(\mathfrak{p}) = (3_{a_1 b_1}, 3_{a_1 b_0}, 1_{a_0 b_1}, 0_{a_0 b_0}).$$

The next proposition allows us to describe the set $\mathcal{P}_{c_g}^{\gg}$ of grounded partitions for the relation $\gg$ defined by

$$k_{c_b} \gg k'_{c_{b'}} \text{ if and only if } k - k' \geq H(b' \otimes b). \tag{3.3}$$

We refer to this relation as *minimal difference conditions*. One can view the partitions of $\mathcal{P}_{c_g}^{\gtrdot}$ as the partitions of $\mathcal{P}_{c_g}^{\gg}$ such that the differences between consecutive parts are minimal. Note that contrarily to $\mathcal{P}_{c_g}^{\gtrdot}$, the set $\mathcal{P}_{c_g}^{\gg}$ has some partitions $\pi = (\pi_0, \dots, \pi_{s-1}, 0_{c_g})$ such that $c(\pi_{s-1}) = c_g$. For this reason, the set $\mathcal{P}_{c_g}^{\gtrdot}$ is not exactly the set of all minimal partitions of $\mathcal{P}_{c_g}^{\gg}$, but is still related to it.

Proposition 3.6. *Recall that $\mathcal{P}_{c_g}$ is the set of grounded partitions where all parts have colour c_g. There is a bijection Φ between $\mathcal{P}_{c_g}^{\gg}$ and $\mathcal{P}_{c_g}^{\gtrdot} \times \mathcal{P}_{c_g}$, such that if $\Phi(\pi) = (\mu, \nu)$, then $|\pi| = |\mu| + |\nu|$, and by setting $c_g = 1$, we have $C(\pi) = C(\mu)$.*

Proof. We set $\Phi(0_{c_g}) = ((0_{c_g}), (0_{c_g}))$. Let us now consider any $\pi = (\pi_0, \dots, \pi_{s-1}, 0_{c_g}) \in \mathcal{P}_{c_g}^{\gg}$, different from (0_{c_g}), with colour sequence $c_{p'_0} \cdots c_{p'_{s-1}} c_g$, and build $\Phi(\pi) = (\mu, \nu)$. Recall that $\pi_{s-1} \neq \pi_s = 0_{c_g}$. Let us set $\mathfrak{p} = (p_k)_{k \geq 0}$, with $p_k = g$ for all $k \geq s$ and $p_k = p'_k$ for all $k \in \{0, \dots, s-1\}$, and set

$$r = \max\{k \in \{0, \dots, s\} : p_{k-1} \neq g\}.$$

Since $p_k = g$ for all $k \geq r$, with the convention $c_g = 1$, we obtain that $C(\pi) = c_{p_0} \cdots c_{p_{s-1}} = c_{p_0} \cdots c_{p_{r-1}}$. Note that $r = 0$ if and only if all the parts of π have colour c_g. We set $\mu = (\mu_0, \dots, \mu_{r-1}, 0_{c_g}) = \phi(\mathfrak{p})$. By Proposition 3.4, for all $k \in \{0, \dots, r-1\}$, the part μ_k is coloured by c_{p_k} and has size

$$\sum_{l=k}^{r-1} H(p_{l+1} \otimes p_l).$$

Let us now build $\nu = (\nu_0, \dots, \nu_{t-1}, 0_{c_g}) \in \mathcal{P}_{c_g}$, where $c(\nu_k) = c_g$ and $\nu_k > 0$ for all $k \in \{0, \cdots, t-1\}$. We distinguish two different cases.

- If $r < s$, then we set $t = s$ and $\nu = (\nu_0, \dots, \nu_{s-1}, 0_{c_g})$, where

$$\begin{cases} \nu_k &= \pi_k - \mu_k \quad \text{for } k \in \{0, \dots, r-1\}, \\ \nu_k &= \pi_k \quad \text{for } k \in \{r, \dots, s-1\}. \end{cases}$$

  By (3.3), the sequence $(\nu_0, \dots, \nu_{r-1})$ is non-increasing. Moreover the fact that $H(g \otimes g) = 0$ and $\pi_{s-1} \neq 0_{c_g}$ implies that $\nu_{s-1} > 0$, and $(\nu_0, \dots, \nu_{s-1})$ is a non-increasing sequence of positive integers.

Finally, let us check that $\nu_{r-1} \geq \nu_r$. We have

$$\begin{aligned}\nu_{r-1} - \nu_r &= \pi_{r-1} - \pi_r - \mu_{r-1} \\ &\geq H(p_r \otimes p_{r-1}) - H(p_r \otimes p_{r-1}) && \text{by (3.3)} \\ &\geq 0.\end{aligned}$$

Thus $(\nu_0, \dots, \nu_{s-1})$ is indeed a non-increasing sequence of positive integers.

- By definition we have $r \leq s$, so the only other possible case is $r = s$. As before, $(\pi_0 - \mu_0, \dots, \pi_s - \mu_s)$ is a non-increasing sequence of non-negative integers, now with $\pi_s - \mu_s = 0 - 0 = 0$. We then set
$$t = \min\{k \in \{0, \dots, s\} : \pi_k = \mu_k\},$$
and $\nu_k = \pi_k - \mu_k$ for all $k \in \{0, \dots, t-1\}$.

Observe that for $\Phi(\pi) = (\mu, \nu)$, with $\pi = (\pi_0, \dots, \pi_{s-1}, 0_{c_g})$, $\mu = (\mu_0, \dots, \mu_{r-1}, 0_{c_g})$ and $\nu = (\nu_0, \dots, \nu_{t-1}, 0_{c_g})$, we always have $s = \max\{r, t\}$, and by adding $s - \min\{r, t\}$ parts 0_{c_g} at the end of the shorter partition, we have $\pi_k = \mu_k + \nu_k$ and $c(\pi_k) = c(\mu_k)$ for all $k \in \{0, \dots, s-1\}$.

The map Φ^{-1} from $\mathcal{P}^{\gg}_{c_g} \times \mathcal{P}_{c_g}$ to $\mathcal{P}^{\ggg}_{c_g}$ simply consists in adding the parts of $\mu = (\mu_0, \dots, \mu_{r-1}, 0_{c_g}) \in \mathcal{P}^{\gg}_{c_g}$ to those of $\nu = (\nu_0, \cdots, \nu_{t-1}, 0_{c_g}) \in \mathcal{P}_{c_g}$ to obtain a grounded partition $\pi \in \mathcal{P}^{\ggg}_{c_g}$ in the following way:

- if $t \leq r$, then π_k has size $\mu_k + \nu_k$ and colour $c(\mu_k)$, where we set $\nu_k = 0$ for all $k \in \{t, \cdots, r-1\}$, and we obtain the partition
$$\pi = (\pi_0, \cdots, \pi_{r-1}, 0_{c_g}),$$
- if $t > r$, the first r parts are defined as in the case $t \leq r$, and the remaining parts are $\pi_k = \nu_k$ for all $k \in \{r, \dots, t-1\}$ with colour c_g, and we obtain the partition
$$\pi = (\pi_0, \cdots, \pi_{t-1}, 0_{c_g}).$$

□

Examples 3.7. *Let us consider the energy matrix* (1.1) *given Primc for the case* $A_1^{(1)}$*, and set* $c_g = a_0b_0$*. We give three examples of the previous bijection, for the different cases* $t < r$*,* $t = r$*, and* $t > r$*.*

- *Case* $t < r$*: Let* $\pi = (10_{a_0b_0}, 7_{a_1b_0}, 5_{a_0b_1}, 3_{a_1b_1}, 2_{a_0b_0}, 1_{a_1b_0}, 0_{a_0b_0})$*. Given the minimal difference conditions in* (1.1)*, our bijection gives*
$$\mu = (6_{a_0b_0}, 5_{a_1b_0}, 3_{a_0b_1}, 3_{a_1b_1}, 2_{a_0b_0}, 1_{a_1b_0}, 0_{a_0b_0}) \quad \text{and} \quad \nu = (4_{a_0b_0}, 2_{a_0b_0}, 2_{a_0b_0}, 0_{a_0b_0}).$$
- *Case* $t = r$*: Let* $\pi = (10_{a_0b_0}, 7_{a_1b_0}, 5_{a_0b_1}, 4_{a_1b_1}, 3_{a_0b_0}, 2_{a_1b_0}, 0_{a_0b_0})$*. We have*
$$\mu = (6_{a_0b_0}, 5_{a_1b_0}, 3_{a_0b_1}, 3_{a_1b_1}, 2_{a_0b_0}, 1_{a_1b_0}, 0_{a_0b_0}) \quad \text{and} \quad \nu = (4_{a_0b_0}, 2_{a_0b_0}, 2_{a_0b_0}, 1_{a_0b_0}, 1_{a_0b_0}, 1_{a_0b_0}, 0_{a_0b_0}).$$
- *Case* $t > r$*: Let* $\pi = (8_{a_0b_0}, 5_{a_1b_0}, 3_{a_0b_1}, 2_{a_1b_1}, 1_{a_0b_0}, 1_{a_0b_0}, 0_{a_0b_0})$*. We obtain*
$$\mu = (4_{a_0b_0}, 3_{a_1b_0}, 1_{a_0b_1}, 1_{a_1b_1}, 0_{a_0b_0}) \quad \text{and} \quad \nu = (4_{a_0b_0}, 2_{a_0b_0}, 2_{a_0b_0}, 1_{a_0b_0}, 1_{a_0b_0}, 1_{a_0b_0}, 0_{a_0b_0}).$$

We are now able to prove Theorem 1.9.

Proof of Theorem 1.9. By Propositon 3.4 and (2.12),
$$\sum_{\pi \in \mathcal{P}^{\gg}_{c_g}} C(\pi) q^{|\pi|} = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{-\lambda} e^{\mathrm{wt}\,\mathfrak{p}} = e^{-\lambda} \mathrm{ch}(\mathrm{L}(\lambda)).$$
By Corollary 2.16, $\overline{\mathrm{wt}} g = 0$. Thus $c_g = e^0 = 1$, and Proposition 3.6 yields
$$\sum_{\pi \in \mathcal{P}^{\ggg}_{c_g}} C(\pi) q^{|\pi|} = \frac{1}{(q;q)_\infty} \sum_{\pi \in \mathcal{P}^{\gg}_{c_g}} C(\pi) q^{|\pi|} = \frac{e^{-\lambda} \mathrm{ch}(\mathrm{L}(\lambda))}{(q;q)_\infty}.$$

□

By this theorem, the characters of irreducible highest weight modules of level ℓ can be computed as generating functions for some grounded partitions. It is the key that connects generalised Primc partitions to the characters of irreducible highest weight modules of level 1 for the affine Lie algebra $A_{n-1}^{(1)}$.

4. Perfect crystal of type $A_{n-1}^{(1)}$: tensor product of the vector representation and its dual

We now describe the perfect crystal $\mathbb{B}$ used in Theorem 1.8. Throughout this section, we fix an integer $n \geq 3$.

Consider the Cartan datum for the matrice $A = (a_{ij})_{i,j \in \{0,\dots,n-1\}}$ where for all $i, j \in \{0, \dots, n-1\}$,

$$a_{ij} = 2\delta_{i,j} - \chi(i - j \equiv \pm 1 \mod n). \tag{4.1}$$

It corresponds to the affine type $A_{n-1}^{(1)}$ [7, 10.1.1]. We then have the corresponding canonical central element c and null root δ, which are expressed in the following way:

$$\begin{aligned} c &= h_0 + h_1 + \dots + h_{n-1}, \\ \delta &= \alpha_0 + \alpha_1 + \dots + \alpha_{n-1}. \end{aligned} \tag{4.2}$$

Any dominant integral weight $\lambda = k_0\Lambda_0 + \dots + k_{n-1}\Lambda_{n-1} \in \bar{P}^+$ has level

$$\langle c, \lambda \rangle = k_0 + \dots + k_{n-1}.$$

Thus, the set of classical weights of level 1 is exactly $\bar{P}_1^+ = \{\Lambda_i : i \in \{0, \dots, n-1\}\}$, the set of fundamental weights.

A perfect crystal of level 1 is given by the crystal graph in Figure 4.1 [7, 11.1.1].

Figure 4.1.

$$\mathcal{B}: \quad \begin{tikzcd} \boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & \boxed{n-2} \arrow[r, "n-1"] & \boxed{n-1} \arrow[llll, bend left, "0"] \end{tikzcd}$$

The $U_q'(\widehat{\mathfrak{g}})$-module corresponding to this crystal is called the *vector representation* of $A_{n-1}^{(1)}$. The most important property of this crystal is the order in which the arrows appear. The only purpose of labelling the vertices is to ease the calculations in the remainder of this paper. Noting that this crystal graph is cyclic, we identify $\{0, \dots, n-1\}$ with the group $(\mathbb{Z}/n\mathbb{Z}, +)$. In this way, the crystal graph of $\mathcal{B}$ can be defined locally around each arrow i as shown in Figure 4.2.

Figure 4.2.

$$\mathcal{B}(\xrightarrow{i}): \quad \boxed{i-1} \xrightarrow{\;i\;} \boxed{i}$$

Remark. For the type $A_1^{(1)}$, the Cartan matrice A is defined differently and is given by

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Nonetheless, the crystal graph of the vector representation behaves in the same way as in the case $n \geq 3$.

For all $i \in \{0, \dots, n-1\}$, let v_i be the element of $\mathcal{B}$ corresponding to the vertex labelled i. The functions of this crystal are given by the following relations:

$$\overline{\mathrm{wt}} v_i = \Lambda_{i+1} - \Lambda_i \quad \text{for all } i \in \{0, \dots, n-1\}, \tag{4.3}$$

$$\begin{cases} \tilde{f}_i v_{i-1} &= v_i \\ \varphi_i v_{i-1} &= 1 \\ \tilde{f}_i v_j = \varphi_i v_j &= 0 \quad \text{if} \quad j \neq i-1, \end{cases} \tag{4.4}$$

$$\begin{cases} \tilde{e}_i v_i &= v_{i-1} \\ \varepsilon_i v_i &= 1 \\ \tilde{e}_i v_j = \varepsilon_i v_j &= 0 \quad \text{if} \quad j \neq i. \end{cases} \tag{4.5}$$

We note that for this crystal, the unique *maximal weight* λ_0, as defined in Condition (3) of Definition 2.12, is attained in v_0 (i.e. $\lambda_0 = \overline{\mathrm{wt}} v_0$). For all $i \in \{0, \dots, n-1\}$, we have

$$\begin{aligned} \overline{\mathrm{wt}} v_0 - \overline{\mathrm{wt}} v_i &= \sum_{j=1}^{i} \overline{\mathrm{wt}} v_{j-1} - \overline{\mathrm{wt}} v_j \\ &= \sum_{j=1}^{i} \alpha_j \quad \text{by (2.5)}. \end{aligned}$$

The fact that the null root vanishes on $\bar{\mathfrak{h}}$ implies that in $\bar{P}$, $\overline{\alpha}_0 = -(\alpha_1 + \cdots + \alpha_{n-1})$. We also remark that the crystal $\mathcal{B}$ has a unique *minimal weight*, attained in v_{n-1} :

$$\begin{aligned} \overline{\mathrm{wt}} v_i - \overline{\mathrm{wt}} v_{n-1} &= \sum_{j=i+1}^{n-1} \overline{\mathrm{wt}} v_{j-1} - \overline{\mathrm{wt}} v_j \\ &= \sum_{j=i+1}^{n-1} \alpha_j \quad \text{by (2.5)}. \end{aligned}$$

Let us consider the dual $\mathcal{B}^\vee$ of $\mathcal{B}$, which is the crystal obtained from $\mathcal{B}$ by reversing the edges in its graph, as shown on Figure 4.3.

Figure 4.3.

$$\begin{tikzcd} \mathcal{B}^\vee : & \boxed{0} \arrow[rrrrr, bend right, "0"'] & \boxed{1} \arrow[l, "1"'] & \cdots \arrow[l, "2"'] & \boxed{n-2} \arrow[l, "n-2"'] & \boxed{n-1} \arrow[l, "n-1"'] \end{tikzcd}$$

Let $v^\vee$ denote the element of $\mathcal{B}^\vee$ corresponding to v in $\mathcal{B}$. We then have the relations

$$\overline{\mathrm{wt}} v^\vee = -\overline{\mathrm{wt}} v\,, \quad \tilde{f}_i v^\vee = (\tilde{e}_i v)^\vee\,, \quad \varphi_i v^\vee = \varepsilon_i v\,, \quad \tilde{e}_i v^\vee = (\tilde{f}_i v)^\vee \quad \text{and} \quad \varepsilon_i v^\vee = \varphi_i v. \tag{4.6}$$

Recall that the duality is an involution, since by the previous equalities, we have

$$(\tilde{f}_i[(v^\vee)^\vee], \tilde{e}_i[(v^\vee)^\vee], \varphi_i[(v^\vee)^\vee], \varepsilon_i[(v^\vee)^\vee]) = (\tilde{f}_i[(v^\vee)^\vee], \tilde{e}_i[(v^\vee)^\vee], \varphi_i v, \varepsilon_i v), \tag{4.7}$$

and the map $v \mapsto (v^\vee)^\vee$ is an isomorphism between $\mathcal{B}$ and $(\mathcal{B}^\vee)^\vee$. Thus $(\mathcal{B}^\vee)^\vee$ can be identified with $\mathcal{B}$.

The dual $\mathcal{B}^\vee$ is also a perfect crystal of level 1, as his maximal weight is attained in the dual $v_{n-1}^\vee$ of the minimal vertex v_{n-1} of $\mathcal{B}$.

Moreover, for two crystals $\mathcal{B}_1$ and $\mathcal{B}_2$, we have

$$(\mathcal{B}_1 \otimes \mathcal{B}_2)^\vee = \mathcal{B}_2^\vee \otimes \mathcal{B}_1^\vee. \tag{4.8}$$

By Theorem 2.10, $\mathcal{B} \otimes \mathcal{B}^\vee$ is a crystal for the tensor product of the vector representation of $A_{n-1}^{(1)}$ and its dual, and the tensor rules (2.6) on $\mathcal{B} \otimes \mathcal{B}^\vee$ become

$$\tilde{e}_i(v_k \otimes v_l^\vee) = \begin{cases} \tilde{e}_i v_k \otimes v_l^\vee & \text{if } \varphi_i(v_k) \geq \varphi_i(v_l) \\ v_k \otimes \tilde{e}_i v_l^\vee & \text{if } \varphi_i(v_k) < \varphi_i(v_l) \end{cases},$$

$$\tilde{f}_i(v_k \otimes v_l^\vee) = \begin{cases} \tilde{f}_i v_k \otimes v_l^\vee & \text{if } \varphi_i(v_k) > \varphi_i(v_l) \\ v_k \otimes \tilde{f}_i v_l^\vee & \text{if } \varphi_i(v_k) \leq \varphi_i(v_l) \end{cases}.$$

Using (4.4) and (4.5), we can draw the corresponding crystal graph, given in Figure 4.4.

Figure 4.4.

$$
\mathcal{B}\otimes\mathcal{B}^\vee :\quad
\begin{tikzcd}
v_0\otimes v_{n-1}^\vee \arrow[r, "1"] \arrow[d, "n-1"'] & v_1\otimes v_{n-1}^\vee \arrow[r, "2"] \arrow[d, "n-1"'] & v_2\otimes v_{n-1}^\vee \arrow[r, dashed] \arrow[d, "n-1"'] & v_{n-2}\otimes v_{n-1}^\vee \arrow[r, "n-1"] & v_{n-1}\otimes v_{n-1}^\vee \arrow[d, "n-1"'] \\
v_0\otimes v_{n-2}^\vee \arrow[r, "1"] \arrow[d, dashed] & v_1\otimes v_{n-2}^\vee \arrow[r, "2"] \arrow[d, dashed] & v_2\otimes v_{n-2}^\vee \arrow[r, dashed] & v_{n-2}\otimes v_{n-2}^\vee \arrow[d, dashed] & v_{n-1}\otimes v_{n-2}^\vee \arrow[llll, bend left, "0"] \arrow[d, dashed] \\
v_0\otimes v_2^\vee \arrow[r, "1"] \arrow[d, "2"'] & v_1\otimes v_2^\vee \arrow[r, "2"] & v_2\otimes v_2^\vee \arrow[d, "2"'] & v_{n-2}\otimes v_2^\vee \arrow[r, "n-1"] \arrow[d, "2"'] & v_{n-1}\otimes v_2^\vee \arrow[llll, bend left, "0"] \arrow[d, "2"'] \\
v_0\otimes v_1^\vee \arrow[r, "1"] & v_1\otimes v_1^\vee \arrow[d, "1"'] & v_2\otimes v_1^\vee \arrow[r, dashed] \arrow[d, "1"'] & v_{n-2}\otimes v_1^\vee \arrow[r, "n-1"] \arrow[d, "1"'] & v_{n-1}\otimes v_1^\vee \arrow[llll, bend left, "0"] \arrow[d, "1"'] \\
v_0\otimes v_0^\vee \arrow[uuuu, bend right, "0"'] & v_1\otimes v_0^\vee \arrow[r, "2"] \arrow[uuuu, bend right, "0"'] & v_2\otimes v_0^\vee \arrow[r, dashed] \arrow[uuuu, bend right, "0"'] & v_{n-2}\otimes v_0^\vee \arrow[r, "n-1"] \arrow[uuuu, bend right, "0"'] & v_{n-1}\otimes v_0^\vee \arrow[llll, bend left, "0"]
\end{tikzcd}
$$

Again, the crystal graph of $\mathcal{B}\otimes\mathcal{B}^\vee$ can be defined locally by giving the vertices adjacent to the edges labelled i, as shown on Figure 4.5.

Figure 4.5.

$$
\begin{array}{ccc}
\mathcal{B}\otimes\mathcal{B}^\vee(\xrightarrow{i}) : & \begin{tikzcd} v_k\otimes v_i^\vee \arrow[d, "i"'] \\ v_k\otimes v_{i-1}^\vee \end{tikzcd} & \begin{tikzcd} v_{i-1}\otimes v_i^\vee \arrow[r, "i"] & v_i\otimes v_i^\vee \arrow[d, "i"'] \\ & v_i\otimes v_{i-1}^\vee \end{tikzcd} \\
k\notin\{i-1,i\} & & \begin{tikzcd} v_{i-1}\otimes v_k^\vee \arrow[r, "i"] & v_i\otimes v_k^\vee \end{tikzcd}
\end{array}
$$

We obtain, for all $i\in\{0,\dots,n-1\}$, the relations

$$
\begin{cases}
\varphi_i(v_{i-1}\otimes v_i^\vee)=\varepsilon_i(v_i\otimes v_{i-1}^\vee)=2\\
\varphi_i(v_i\otimes v_{i-1}^\vee)=\varepsilon_i(v_{i-1}\otimes v_i^\vee)=0\\
\varphi_i(v_i\otimes v_i^\vee)=\varepsilon_i(v_i\otimes v_i^\vee)=1\\
\varphi_i(v_{i-1}\otimes v_{i-1}^\vee)=\varepsilon_i(v_{i-1}\otimes v_{i-1}^\vee)=0
\end{cases},
\qquad
\begin{cases}
\varphi_i(v_k\otimes v_i^\vee)=\varepsilon_i(v_i\otimes v_k^\vee)=1\\
\varphi_i(v_{i-1}\otimes v_k^\vee)=\varepsilon_i(v_k\otimes v_{i-1}^\vee)=1\\
\varphi_i(v_k\otimes v_l^\vee)=\varepsilon_i(v_l\otimes v_k^\vee)=0
\end{cases},\quad \forall\, l,k\notin\{i,i-1\}. \tag{4.9}
$$

The local configurations for the vertices are given in Figure 4.6.

Figure 4.6.

$$
k-l\notin\{\pm1\}:\qquad
\begin{tikzcd}
 & {} \arrow[d, "l+1"'] & \\
{} \arrow[r, "k"] & v_k\otimes v_l^\vee \arrow[r, "k+1"] \arrow[d, "l"'] & {} \\
 & {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {} \arrow[d, "i+1"'] & & \\
{} \arrow[r, "i-1"] & v_{i-1}\otimes v_i^\vee \arrow[r, "i"] & v_i\otimes v_i^\vee \arrow[d, "i"'] & \\
 & & v_i\otimes v_{i-1}^\vee \arrow[r, "i+1"] \arrow[d, "i-1"'] & {} \\
 & & {} &
\end{tikzcd}
$$

The values of the functions ε, φ defined in (2.4) are

$$\begin{cases} \varphi(v_{i-1}\otimes v_i^\vee) = \varepsilon(v_i\otimes v_{i-1}^\vee) = 2\Lambda_i \\ \varepsilon(v_{i-1}\otimes v_i^\vee) = \varphi(v_i\otimes v_{i-1}^\vee) = \Lambda_{i-1}+\Lambda_{i+1} \\ \varphi(v_i\otimes v_i^\vee) = \varepsilon(v_i\otimes v_i^\vee) = \Lambda_i \end{cases}, \qquad \begin{cases} \varphi(v_k\otimes v_l^\vee) = \Lambda_{k+1}+\Lambda_l \\ \varepsilon(v_k\otimes v_l^\vee) = \Lambda_{l+1}+\Lambda_k \end{cases}, \tag{4.10}$$

where $k-l\notin\{0,\pm1\}$. For all $k,l\in\{0,\dots,n-1\}$, the weight of $v_k\otimes v_l^\vee$ is given by

$$\overline{\mathrm{wt}}(v_k\otimes v_l^\vee) = \Lambda_{k+1}-\Lambda_k+\Lambda_l-\Lambda_{l+1}. \tag{4.11}$$

We then observe that

$$\langle c, \varepsilon(v_k\otimes v_l^\vee)\rangle = 1+\chi(k\neq l). \tag{4.12}$$

By [11, Lemma 4.6.2], since $\mathcal{B}$ and $\mathcal{B}^\vee$ are perfect crystals of level 1, their tensor product $\mathbb{B}$ is also a perfect crystal of level 1. We observe that the potential grounds of $\mathbb{B}$ are the vertices $v_i\otimes v_i^\vee$, since by (4.10), for all $i\in\{0,\dots,n-1\}$, we have that

$$\varepsilon(b^{\Lambda_i}) = \Lambda_i \text{ if and only if } b^{\Lambda_i} = v_i\otimes v_i^\vee \quad\text{and}\quad \varphi(b_{\Lambda_i}) = \Lambda_i \text{ if and only if } b_{\Lambda_i} = v_i\otimes v_i^\vee.$$

5. Proof of the character formulas

In this section, we prove our character formulas given in Theorems 1.10, 1.11, and 1.12, under the assumption that Theorem 1.8 is true. We will then prove Theorem 1.8 in the last two sections.

5.1. **Proof of Theorem 1.10.** We show that the set of grounded partitions $\mathcal{P}_{c_g}^{\gg}$, with $\gg$ defined in (3.3), grounded at c_g for $g=(v_0\otimes v_0^\vee)$, is in bijection with the set of generalised Primc partitions.

Let $(\pi_0,\dots,\pi_{s-1},0_{c_g})\in\mathcal{P}_{c_g}^{\gg}$, and let b be the vertex in $\mathbb{B}$ corresponding to the colour of π_{s-1}. Since $\pi_{s-1}\neq 0_{c_g}$, the minimal size of π_{s-1} is $H((v_0\otimes v_0^\vee)\otimes(v_i\otimes v_j^\vee))$ if $(i,j)\neq(0,0)$, and 1 if $(i,j)=(0,0)$. This corresponds to the size of minimal parts in generalised Primc partitions. By Theorem 1.8, for all $(i,j)\neq(0,0)$,

$$H((v_0\otimes v_0^\vee)\otimes(v_i\otimes v_j^\vee)) = \Delta(a_jb_i, a_0b_0) = 1.$$

Thus the generalised Primc partitions and the grounded partitions in $\mathcal{P}_{c_g}^{\gg}$ coincide in terms of minimal difference conditions and minimal part sizes, with the colour correspondence $c_{v_l\otimes v_k^\vee}\leftrightarrow a_kb_l$. Thus their generating functions are the same with the correspondence $e^{\overline{\mathrm{wt}}v_i}=b_i$, since by (4.11),

$$e^{\overline{\mathrm{wt}}(v_l\otimes v_k^\vee)} = e^{\overline{\mathrm{wt}}(l)-\overline{\mathrm{wt}}(k)} = b_k^{-1}b_l.$$

Using the character formula of Theorem 1.9, this gives the desired result. □

5.2. **Proof of Theorem 1.11.** Let us now turn to the proof of Theorem 1.11. It uses some notions defined in our first paper [5], such as bound and free colours, reduced colour sequences, kernel, insertions, types. As they are only needed for this proof, we do not redefine them here, and refer the reader to Sections 1 and 2 of [5].

Let us fix $\ell\in\{0,\dots,n-1\}$ and recall that in the perfect crystal $\mathbb{B}$, we have $b^{\Lambda_\ell}=b_{\Lambda_\ell}=v_\ell\otimes v_\ell^\vee$. Assuming that Theorem 1.8 is true, we also have that $H[(v_\ell\otimes v_\ell^\vee)\otimes(v_\ell\otimes v_\ell^\vee)]=\Delta(a_\ell b_\ell;a_\ell b_\ell)=0$. Let us set $g=(v_\ell\otimes v_\ell^\vee)$ to be the ground in $\mathbb{B}$, and consider the set $\mathcal{P}_{c_g}^{\gg}$ of grounded partitions with ground c_g. For $\pi=(\pi_0,\dots,\pi_{s-1},0_{c_g})\in\mathcal{P}_{c_g}^{\gg}$, we write $c(\pi_k)=c_{(v_{j_k}\otimes v_{i_k}^\vee)}$.

By (3.3), for all $k\in\{0,\cdots,s-2\}$, the parts of π satisfy the difference conditions

$$\pi_k-\pi_{k+1}\geq H((v_{j_{k+1}}\otimes v_{i_{k+1}}^\vee)\otimes(v_{j_k}\otimes v_{i_k}^\vee)) = \Delta(a_{i_k}b_{j_k}, a_{i_{k+1}}b_{j_{k+1}}).$$

Let us now study the size of the smallest part.

- If $(j_{s-1},i_{s-1})\neq(\ell,\ell)$, the minimal size of π_{s-1} is

$$\Delta(a_{i_{s-1}}b_{j_{s-1}}, a_\ell b_\ell) = \begin{cases} \chi(i_{s-1}>\ell) = \chi(i_{s-1}\geq\ell)+\chi(\ell>j_{s-1}) & \text{if} \quad j_{s-1}=\ell \quad (i_{s-1}\neq\ell) \\ \chi(i_{s-1}\geq\ell)+\chi(\ell>j_{s-1}) & \text{if} \quad j_{s-1}\neq\ell \end{cases}.$$

- Otherwise, we have that $j_{s-1} = i_{s-1} = \ell$, and then $c(\pi_{s-1}) = c_g$. In this case, $\Delta(a_\ell b_\ell; a_\ell b_\ell) = 0$ implies that the size of π_{s-1} must be at least 1 in order to have $\pi_{s-1} \neq 0_{c_g}$. We observe that, in that case, we still have $1 = \chi(i_{s-1} \geq \ell) + \chi(\ell > j_{s-1})$.

In both cases, our grounded partition π, without the part 0_{c_g}, is a partition satisfying the difference condition Δ of generalised Primc partitions, but such that the minimal size for the last part, denoted by $\Delta(a_{i_{s-1}} b_{j_{s-1}}, a_\infty b_\infty)$ with our conventions from [5], is given by the expression

$$\Delta(a_{i_{s-1}} b_{j_{s-1}}, a_\infty b_\infty) = \chi(i_{s-1} \geq \ell) + \chi(\ell > j_{s-1}). \tag{5.1}$$

Moreover we observe that this is always equal to 1 when $a_{i_{s-1}} b_{j_{s-1}}$ is a free colour. Thus in the case $\ell = 0$, the minimal part always has size 1, independently of its colour. For larger ℓ, the minimal part may have size $0, 1$, or 2 according to (5.1). Besides, we keep the convention $\Delta(a_\infty b_\infty, c) = 1$, as it our first paper.

The proof of Theorem 1.1 in [5] relies on a correspondence between generalised Primc partitions and coloured Frobenius partitions having the same kernel. In the case where the kernel ends with a free colour $a_k b_k$, the generalised Primc partition is also a partition grounded in c_g by adding 0_{c_g}, and the type of the insertions inside the secondary pairs remain the same.

When the kernel ends with a bound colour $a_k b_{k'}$, $k \neq k'$, we modify the type of the insertion of $a_{k'} b_{k'}$ to the right of $a_k b_{k'}$, and it becomes

$$\begin{aligned} T_\Delta(a_k b_{k'}) &= \Delta(a_k b_{k'}, a_{k'} b_{k'}) + \Delta(a_k b_{k'}, a_\infty b_\infty) - \Delta(a_k b_{k'}, a_\infty b_\infty) \\ &= 1 + \chi(k > k') - (\chi(k \geq \ell) + \chi(\ell > k')). \end{aligned} \tag{5.2}$$

Note that this value is still in $\{0, 1\}$, since it can be rewritten as $\chi(\ell > k) + \chi(k > k') - \chi(\ell > k')$. The types of the others insertions are the same as the types for the generalised Primc partitions in [5].

Recall from [5] that a *n^2-coloured Frobenius partition* is a pair of coloured partitions

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{s-1} \\ \mu_0 & \mu_1 & \cdots & \mu_{s-1} \end{pmatrix},$$

where $\lambda = \lambda_0 + \lambda_1 + \cdots + \lambda_{s-1}$ is a partition into s distinct non-negative parts, each coloured with some a_i, $i \in \{0, \dots, n-1\}$, with the following order

$$0_{a_{n-1}} < 0_{a_{n-2}} < \cdots < 0_{a_0} < 1_{a_{n-1}} < 1_{a_{n-2}} < \cdots < 1_{a_0} < \cdots, \tag{5.3}$$

and $\mu = \mu_0 + \mu_1 + \cdots + \mu_{s-1}$ is a partition into s distinct non-negative parts, each coloured with some b_i, $i \in \{0, \dots, n-1\}$, with the order

$$0_{b_0} < 0_{b_1} < \cdots < 0_{b_{n-1}} < 1_{b_0} < 1_{b_1} < \cdots < 1_{b_{n-1}} < \cdots. \tag{5.4}$$

The colour sequence of such a partition is defined to be $c(\lambda_0)c(\mu_0), \dots, c(\lambda_{s-1})c(\mu_{s-1})$. Here the size corresponding to the colour $c(\lambda_i)c(\mu_i)$ is $\lambda_i + \mu_i$.

We consider coloured Frobenius partitions such that the minimal size for $\lambda_{s-1} + \mu_{s-1}$ is given by $\Delta'(a_k b_{k'}, a_\infty b_\infty) = \Delta(a_k b_{k'}, a_\infty b_\infty)$, where $c(\lambda_{s-1}) = a_k$, $c(\mu_{s-1}) = b_{k'}$, and $\Delta(a_k b_{k'}, a_\infty b_\infty)$ was defined in (5.1). We say that such coloured Frobenius partitions are *grounded* at c_g. We have $\Delta'(a_k b_k, a_\infty b_\infty) = 1$ for any free colour $a_k b_k$. Note that the differences are the same as those defined in [5]:

$$\Delta'(a_i b_j, a_{i'} b_{j'}) = \chi(i \geq i') + \chi(j \leq j').$$

Here we keep the convention $\Delta'(a_\infty b_\infty, c) = 1$. When the kernel of the coloured Frobenius partition ends with a bound colour $a_k b_{k'}$, the type of the insertion of the free colour $a_{k'} b_{k'}$ to its right, according to the differences $\Delta'' := 2 - \Delta'$, is given by

$$\begin{aligned} T_{\Delta''}(a_k b_{k'}) &= \Delta''(a_k b_{k'}, a_{k'} b_{k'}) + \Delta''(a_{k'} b_{k'}, a_\infty b_\infty) - \Delta''(a_k b_{k'}, a_\infty b_\infty) \\ &= 2 - [\Delta'(a_k b_{k'}, a_{k'} b_{k'}) + \Delta'(a_{k'} b_{k'}, a_\infty b_\infty) - \Delta'(a_k b_{k'}, a_\infty b_\infty)] \\ &= 2 - [1 + \chi(k > k') + 1 - (\chi(k \geq \ell) + \chi(\ell > k'))] \\ &= \chi(k \geq \ell) + \chi(\ell > k') - \chi(k > k'). \end{aligned} \tag{5.5}$$

The types of all the insertions which are not at the right end of the kernel are the same as the types for Δ'' of the coloured Frobenius partitions in [5]. Thus, (5.2) yields the relation

$$T_\Delta(a_k b_{k'}) + T_{\Delta''}(a_k b_{k'}) = 1.$$

This means that an insertion has Δ-type 1 if and only if it has Δ''-type 0. Thus, by Theorem 3.1 of [5], the generating function for our *grounded* generalised Primc partitions with a fixed kernel is the same as the generating function for grounded coloured Frobenius partitions with the same kernel. Therefore, the generating function for generalised Primc partitions with minimal part size $\Delta(a_k b_{k'}, a_\infty b_\infty)$ is the same as the generating function for coloured Frobenius partitions with minimal part size $\Delta'(a_k b_{k'}, a_\infty b_\infty) = \chi(k \geq \ell) + \chi(\ell > k')$. The generating function for the latter, where for all $i \in \{0, \dots, n-1\}$, the power of b_i counts the number of colours b_i minus the number of colours a_i in the colour sequence, is given by

$$[x^0] \prod_{i=0}^{\ell-1} (-b_i^{-1}x; q)_\infty (-b_i q x^{-1}; q)_\infty \times \prod_{i=\ell}^{n-1} (-b_i^{-1} x q; q)_\infty (-b_i x^{-1}; q)_\infty.$$

In this product, the minimal size for λ_{s-1} with colour a_k is $\chi(k \geq \ell)$, while the minimal size for μ_{s-1} with colour $b_{k'}$ is $\chi(k' < \ell)$, so that the minimal size for $\lambda_{s-1} + \mu_{s-1}$ is indeed $\chi(k \geq \ell) + \chi(\ell > k')$. We conclude by noting that, by Theorem 1.1, this generating function is obtained by doing the changes of variables $b_i \mapsto b_i q^{\chi(i<\ell)}$ in

$$G_n^P(q; b_0, \cdots, b_{n-1}) = [x^0] \prod_{i=0}^{n-1} (-b_i^{-1} x q; q)_\infty (-b_i x^{-1}; q)_\infty,$$

which gives Theorem 1.11. $\square$

5.3. **Proof of Theorem 1.12.** Finally, we turn to the proof of Theorem 1.12, which gives the expression of the character for $L(\Lambda_\ell)$ as a sum of series with positive coefficients.

By the definition of characters, the function $e^{-\Lambda_\ell}\mathrm{ch}(\mathrm{L}(\Lambda_\ell))$ can be expressed as a power series in $e^{-\alpha_i}$ for $i \in \{0, \dots, n-1\}$, or, by a change of variables, as a power series in $e^{-\delta}$ and e^{α_i} for $i \neq 0$. By definition of the crystal graph $\mathcal{B}$, we have $\tilde{f}_i v_{i-1} = v_i$, so that by (2.5), we have $\overline{\mathrm{wt}} v_{i-1} - \overline{\mathrm{wt}} v_i = \alpha_i$ for $i \in \{1, \dots, n-1\}$ and $\overline{\mathrm{wt}} v_{n-1} - \overline{\mathrm{wt}} v_0 = \overline{\alpha}_0$. The change of variables $e^{\overline{\mathrm{wt}} v_i} = b_i$ then gives $e^{\alpha_i} = b_{i-1} b_i^{-1}$ for $i \in \{1, \dots, n-1\}$. The changes of variables are natural, since for all $i \neq 0$, the weight α_i in P is indeed a classical weight in $\bar{P}$. In addition, the series $G_n^P(b_0 q, \cdots, b_{\ell-1} q, b_\ell, \cdots, b_{n-1})$ can be expressed in terms of summands of the form

$$\left(\prod_{i=0}^{n-1} b_i^{r_i}\right) q^m \quad \text{with} \quad \sum_{i=0}^{n-1} r_i = 0,$$

so that we can always retrieve the exponent of $b_{i-1} b_i^{-1}$, for all $i \in \{1, \dots, n-1\}$, which corresponds to $\sum_{j=0}^{i-1} r_j$. Thus the identification

$$\begin{aligned} e^{-\delta} &\longleftrightarrow q \\ e^{\alpha_i} &\longleftrightarrow b_{i-1} b_i^{-1} \end{aligned}$$

is unique, and our generalisation of Primc's identity allows us to retrieve the non-dilated version of the characters for all the irreducible highest weight modules with classical weight of level 1 for the type $A_{n-1}^{(1)}$.

Looking at Formula (1.5), we obtain the following correspondences (recall that $r_1 = 0 = r_n$)

$$\begin{aligned} \prod_{i=1}^{n-1} b_i^{-r_i + r_{i+1}} &= \prod_{i=1}^{n-1} (b_{i-1} b_i^{-1})^{r_i} = \prod_{i=1}^{n-1} e^{r_i \alpha_i}, \\ \prod_{j=0}^{i-1} b_j b_i^{-1} &= \prod_{j=1}^{i} (b_{j-1} b_j^{-1})^j = e^{\sum_{j=1}^{i} j \alpha_j}. \end{aligned}$$

By doing these transformations in (1.5), we then obtain by Theorem 1.10 and Theorem 1.1 that

$$
\begin{aligned}
e^{-\Lambda_0}\mathrm{ch}(L(\Lambda_0)) &= \frac{1}{(e^{-\delta};e^{-\delta})_\infty^{n-1}} \sum_{\substack{s_1,\dots,s_{n-1}\in\mathbb{Z}\\ s_n=0}} \prod_{i=1}^{n-1} e^{s_i\alpha_i} e^{s_i(s_{i+1}-s_i)\delta} \\
&= \frac{1}{(e^{-\delta};e^{-\delta})_\infty^{n-1}} \sum_{\substack{r_1,\dots,r_{n-1}:\\ 0\le r_j\le j-1\\ r_n=0}} \prod_{i=1}^{n-1} e^{r_i\alpha_i} e^{r_i(r_{i+1}-r_i)\delta} \left(e^{-i(i+1)\delta};e^{-i(i+1)\delta}\right)_\infty \\
&\qquad\times \left(-e^{(ir_{i+1}-(i+1)r_i-\frac{i(i+1)}{2})\delta+\sum_{j=1}^i j\alpha_j};e^{-i(i+1)\delta}\right)_\infty \\
&\qquad\times \left(-e^{((i+1)r_i-ir_{i+1}-\frac{i(i+1)}{2})\delta-\sum_{j=1}^i j\alpha_j};e^{-i(i+1)\delta}\right)_\infty .
\end{aligned}
$$

Note that for all $\ell \in \{0,\dots,n-1\}$ and $j \in \{1,\dots,n-1\}$, the transformation $b_j \mapsto b_j q^{\chi(j<\ell)}$ is equivalent to $b_{j-1}b_j^{-1} \mapsto q^{\chi(j=\ell)}b_{j-1}b_j^{-1}$. This corresponds to the transformations $e^{\alpha_j} \mapsto e^{-\chi(j=\ell)\delta+\alpha_j}$ for all $j \in \{1,\dots,n-1\}$, and Theorem 1.12 follows. $\square$

6. Tools for the proof of Theorem 1.8

We already know that the crystal graph of $\mathbb{B}\otimes\mathbb{B}$ is connected, as $\mathbb{B}$ is a perfect crystal. However, here we reprove this by building the paths in this graph, as these paths will allow us to compute the energy function. First, let us define some tools that will help us simplify the construction of the paths.

6.1. **Symmetry in the crystal graph of $\mathbb{B}\otimes\mathbb{B}$.** We use some properties of duality to exhibit symmetries in the crystal graph of $\mathbb{B}\otimes\mathbb{B}$. Let us start with a general proposition about energy and duality.

Proposition 6.1. *Let $\mathcal{B}$ be a self-dual crystal and H be an energy function on $\mathcal{B}\otimes\mathcal{B}$. Then for all $b_1,b_2\in\mathcal{B}$, we have*

$$H[b_1\otimes b_2]-H[\tilde{f}_i(b_1\otimes b_2)] = H[b_2^\vee\otimes b_1^\vee]-H[\tilde{e}_i(b_2^\vee\otimes b_1^\vee)]. \tag{6.1}$$

Proof. By (2.8), we have

$$
\begin{aligned}
H[b_1\otimes b_2]-H[\tilde{f}_i(b_1\otimes b_2)] &= \begin{cases} -\chi(i=0) & \text{if } \varphi_i(b_1)\le\varepsilon_i(b_2)\\ +\chi(i=0) & \text{if } \varphi_i(b_1)>\varepsilon_i(b_2),\end{cases}\\
H[b_2^\vee\otimes b_1^\vee]-H[\tilde{e}_i(b_2^\vee\otimes b_1^\vee)] &= \begin{cases} -\chi(i=0) & \text{if } \varphi_i(b_2^\vee)\ge\varepsilon_i(b_1^\vee)\\ +\chi(i=0) & \text{if } \varphi_i(b_2^\vee)<\varepsilon_i(b_1^\vee).\end{cases}
\end{aligned}
$$

On the other hand, we have by duality :

$$
\begin{aligned}
\varphi_i(b_2^\vee) &= \varepsilon_i(b_2),\\
\varepsilon_i(b_1^\vee) &= \varphi_i(b_1).
\end{aligned}
$$

The equality (6.1) follows. $\square$

We now use Proposition 6.1 to give the symmetry in $\mathbb{B}\otimes\mathbb{B}$.

Proposition 6.2. *Let $\mathcal{B}$ be a crystal, let $\mathcal{B}^\vee$ be its dual, and let $\mathbb{B}=\mathcal{B}\otimes\mathcal{B}^\vee$. Let H be an energy function on $\mathbb{B}\otimes\mathbb{B}$.*

For all $\sigma_1,\sigma_2,\sigma_3,\sigma_4,\tau_1,\tau_2,\tau_3,\tau_4\in\mathcal{B}$, there exists a path between $(\sigma_1\otimes\sigma_2^\vee)\otimes(\sigma_3\otimes\sigma_4^\vee)$ and $(\tau_1\otimes\tau_2^\vee)\otimes(\tau_3\otimes\tau_4^\vee)$ in $\mathbb{B}\otimes\mathbb{B}$ if and only if there exists a path between $(\sigma_4\otimes\sigma_3^\vee)\otimes(\sigma_2\otimes\sigma_1^\vee)$ and $(\tau_4\otimes\tau_3^\vee)\otimes(\tau_2\otimes\tau_1^\vee)$ in $\mathbb{B}\otimes\mathbb{B}$. These paths are dual, i.e. we can obtain one from the other by reversing the edges and replacing each vertex by its dual.

Moreover, if there exists a path between $(\sigma_1\otimes\sigma_2^\vee)\otimes(\sigma_3\otimes\sigma_4^\vee)$ and $(\tau_1\otimes\tau_2^\vee)\otimes(\tau_2\otimes\tau_1^\vee)$, we have

$$H[(\sigma_1\otimes\sigma_2^\vee)\otimes(\sigma_3\otimes\sigma_4^\vee)] = H[(\sigma_4\otimes\sigma_3^\vee)\otimes(\sigma_2\otimes\sigma_1^\vee)]. \tag{6.2}$$

Proof. Note that $\mathbb{B}\otimes\mathbb{B}$ is self-dual.

The first claim about the paths follows directly from the definition of duality.

We prove (6.2) by induction on the length of the path between $(\sigma_1\otimes\sigma_2^\vee)\otimes(\sigma_3\otimes\sigma_4^\vee)$ and $(\tau_1\otimes\tau_2^\vee)\otimes(\tau_2\otimes\tau_1^\vee)$.

- If the path is of length 0, i.e. we consider the vertex $(\tau_1 \otimes \tau_2^\vee) \otimes (\tau_2 \otimes \tau_1^\vee)$, (6.2) is simply the identity.
- Now assume that (6.2) is true for all vertices $(\sigma_1 \otimes \sigma_2^\vee) \otimes (\sigma_3 \otimes \sigma_4^\vee)$ having a path of length n to $(\tau_1 \otimes \tau_2^\vee) \otimes (\tau_2 \otimes \tau_1^\vee)$, and show it for paths of length $n+1$. Let $(\sigma_1 \otimes \sigma_2^\vee) \otimes (\sigma_3 \otimes \sigma_4^\vee)$ be such that there is a path of length $n+1$ between $(\sigma_1 \otimes \sigma_2^\vee) \otimes (\sigma_3 \otimes \sigma_4^\vee)$ and $(\tau_1 \otimes \tau_2^\vee) \otimes (\tau_2 \otimes \tau_1^\vee)$.

  Then there exists i such that
$$(\sigma_1 \otimes \sigma_2^\vee) \otimes (\sigma_3 \otimes \sigma_4^\vee) = g_i\big((\rho_1 \otimes \rho_2^\vee) \otimes (\rho_3 \otimes \rho_4^\vee)\big),$$
  where g_i is equal to either $\tilde{f}_i$ or $\tilde{e}_i$, and such that there is a path of length n between $(\rho_1 \otimes \rho_2^\vee) \otimes (\rho_3 \otimes \rho_4^\vee)$ and $(\tau_1 \otimes \tau_2^\vee) \otimes (\tau_2 \otimes \tau_1^\vee)$.

  By the induction hypothesis, we have
$$H[(\rho_1 \otimes \rho_2^\vee) \otimes (\rho_3 \otimes \rho_4^\vee)] = H[(\rho_4 \otimes \rho_3^\vee) \otimes (\rho_2 \otimes \rho_1^\vee)].$$
  Moreover, by Proposition 6.1, we have
$$H[(\sigma_1 \otimes \sigma_2^\vee) \otimes (\sigma_3 \otimes \sigma_4^\vee)] - H[(\rho_1 \otimes \rho_2^\vee) \otimes (\rho_3 \otimes \rho_4^\vee)] = H[(\sigma_4 \otimes \sigma_3^\vee) \otimes (\sigma_2 \otimes \sigma_1^\vee)] - H[(\rho_4 \otimes \rho_3^\vee) \otimes (\rho_2 \otimes \rho_1^\vee)].$$
  Combining the two equalities completes the proof.

□

In particular, by Proposition 6.2, if we find a path from $(v_0 \otimes v_0^\vee) \otimes (v_0 \otimes v_0^\vee)$ to $(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)$, then we immediately have a path from $(v_0 \otimes v_0^\vee) \otimes (v_0 \otimes v_0^\vee)$ to $(v_k \otimes v_l^\vee) \otimes (v_{k'} \otimes v_{l'}^\vee)$ as well, and by (6.2), this yields the following symmetry on the energy function:
$$H[(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)] = H[(v_k \otimes v_l^\vee) \otimes (v_{k'} \otimes v_{l'}^\vee)].$$
Besides, by (1.3), we have
$$\begin{aligned}\Delta(a_k b_l; a_{k'} b_{l'}) &= \chi(k \geq k') - \chi(k = l = k') + \chi(l \leq l') - \chi(l = k' = l') \\ &= \begin{cases} \chi(k > k') + \chi(l < l') & \text{if} \quad l = k' \\ \chi(k \geq k') + \chi(l \leq l') & \text{if} \quad l \neq k' \end{cases}, \end{aligned} \tag{6.3}$$
and then
$$\Delta(a_k b_l; a_{k'} b_{l'}) = \Delta(a_{l'} b_{k'}; a_l b_k).$$
Therefore, if we prove that $H[(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)] = \Delta(a_k b_l; a_{k'} b_{l'})$, we equivalently have $H[(v_k \otimes v_l^\vee) \otimes (v_{k'} \otimes v_{l'}^\vee)] = \Delta(a_{l'} b_{k'}; a_l b_k)$. Thus, to prove Theorem 1.8 in Section 7, we will distinguish several cases according to some relations between k, k', l, l', and by interchanging $k \equiv l'$ and $k' \equiv l$, the symmetry will imply the remaining cases.

6.2. **Redefining the minimal differences Δ.** To build a path from $(v_0 \otimes v_0^\vee) \otimes (v_0 \otimes v_0^\vee)$ to $(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)$ and show that
$$H[(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)] = \Delta(a_k b_l; a_{k'} b_{l'}),$$
we will distinguish the cases $k' = l$ and $k' \neq l$. But first, let us define a tool which will make our proofs simpler.

Definition 6.3. Let us identify $\{0, \ldots, n-1\}$ with $\mathbb{Z}/n\mathbb{Z}$, and consider the natural order on $\{0, \ldots, n-1\}$,
$$0 < 1 < \cdots < n-2 < n-1.$$
We also define, for all $i, j \in \{0, \ldots, n-1\}$, the intervals
$$\operatorname{int}(i,j) := \{i+1, i+2, \ldots, j-1, j\}.$$

Lemma 6.4. *For all $i \in \{0, \ldots, n-1\}$, we have the following:*
$$\begin{array}{ccc} i < i-1 & \iff & i = 0, \\ \operatorname{int}(i,i) & = & \{0, \ldots, n-1\}, \\ \{0, \ldots, n-1\} \setminus \operatorname{int}(i,j) = \operatorname{int}(j,i) & \iff & i \neq j, \\ 0 \notin \operatorname{int}(j,i) & \iff & j < i, \\ 0 \in \operatorname{int}(i,j) & \iff & j \leq i. \end{array}$$

The aim of this lemma is to rewrite the difference conditions Δ according to the fact that 0 belongs to some interval or not. By (6.3), Δ can be reformulated as follows:

$$\Delta(a_k b_l; a_{k'} b_{l'}) = \begin{cases} \chi(0 \notin \mathrm{int}(k', k)) + \chi(0 \notin \mathrm{int}(l, l')) & \text{if} \quad l = k' \\ \chi(0 \in \mathrm{int}(k, k')) + \chi(0 \in \mathrm{int}(l', l)) & \text{if} \quad l \neq k' \end{cases} . \tag{6.4}$$

Proof of Lemma 6.4. The first equivalence is straightforward, since $i > i - 1$ if and only if $i \neq 0$, and $0 < n - 1 = -1$.

The second equality follows from the definition of int, since we go around $\{0, \dots, n-1\}$.

Note that

$$\mathrm{int}(i, j) = \{i + 1, i + 2, \dots, j - 1, j\},$$

while

$$\mathrm{int}(j, i) = \{j + 1, j + 2, \dots, i - 1, i\},$$

and if $i \neq j$, these two sets are complementary in $\{0, \dots, n-1\}$. Moreover, when $i \neq j$, we have $i \in \mathrm{int}(j, i)$ and $j \in \mathrm{int}(i, j)$, so that both sets never equal $\emptyset$ nor $\{0, \dots, n-1\}$. Otherwise, when $i = j$, they would both be equal to $\{0, \dots, n-1\}$. This gives the third equivalence.

For the fourth equivalence, the fact that $0 \in \{0, \dots, n-1\}$ gives

$$\begin{aligned} 0 \notin \mathrm{int}(j, i) &\Longleftrightarrow 0 \notin \{j + 1, i + 2, \dots, j - 1, i\}, \\ &\Longleftrightarrow i \neq j \text{ and } \emptyset \neq \{j + 1, j + 2, \dots, i - 1, i\} \subseteq \{1, \dots, n - 1\} \\ &\Longleftrightarrow j < j + 1 \leq i. \end{aligned}$$

Finally, for the last equivalence, we note that

$$\begin{aligned} \chi(j \leq i) &= \chi(j < i) + \chi(j = i) \\ &= \chi(j < i)\chi(j \neq i) + \chi(j = i) \\ &= \chi(0 \notin \mathrm{int}(j, i))\chi(i \neq j) + \chi(i = j) \\ &= \chi(0 \in \mathrm{int}(i, j))\chi(i \neq j) + \chi(i = j)\chi(0 \in \mathrm{int}(i, i)). \end{aligned}$$

This concludes the proof. □

7. Proof of Theorem 1.8

We are now ready to build the paths in $\mathbb{B} \otimes \mathbb{B}$, and use them to compute the energy function $H[(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)]$. We will use the relations in (4.9) and the local configurations of the vertices as defined in (4.6). The symmetric of $(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)$ is $(v_k \otimes v_l^\vee) \otimes (v_{k'} \otimes v_{l'}^\vee)$, obtained by interchanging $k' \equiv l, l' \equiv k$. We distinguish several cases:

1. $k' = l'$ and $l = k$,
2. $k' = l \neq k = l'$,
3. $k' = l$ and $k \neq l'$,
4. $k' \neq k = l = l'$ (Symmetric: $l \neq k = k' = l'$),
5. $l' \neq k' = k \neq l$ (Symmetric: $k \neq l = l' \neq k'$),
6. $k \neq k'$, $k' \neq l$ and $l \neq l'$
   (a) $k + 1, k' \notin \mathrm{int}(l, l')$ (Symmetric: $l' + 1, l \notin \mathrm{int}(k', k)$),
   (b) $k + 1 \in int(l, l')$ and $k' \notin \mathrm{int}(l, l')$ (Symmetric: $l' + 1 \in \mathrm{int}(k', k)$ and $l \notin \mathrm{int}(k', k)$)
   (c) $k + 1 \notin \mathrm{int}(l, l')$ and $k' \in \mathrm{int}(l, l')$ (Symmetric: $l' + 1 \notin \mathrm{int}(k', k)$ and $l \in \mathrm{int}(k', k)$)
   (d) $k + 1, k' \in \mathrm{int}(l, l')$ and $l' + 1, l \in \mathrm{int}(k', k)$.

7.1. **The case** $k' = l'$ **and** $l = k$**.** We want to compute the energy $H[(v_{k'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_l^\vee)]$ for all k', l. To do so, we build a path from $(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee)$ to $(v_{k'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_l^\vee)$. We consider the case $k' \neq l$, as otherwise the two elements are the same. By (4.9), we have

$$\varphi_i(v_{k'} \otimes v_{k'}^\vee) = \varepsilon_i(v_{k'} \otimes v_{k'}^\vee) = \chi(i = k').$$

By the tensor rules (2.6), we then obtain the path

$$\begin{array}{r}
(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee) \xrightarrow{k'} \underbrace{(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'-1}^\vee) \xrightarrow{k'-1} \cdots \xrightarrow{l+1}}_{\text{empty if } k'=l+1} (v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_l^\vee) \\
\Big\downarrow {\scriptstyle k'+1} \qquad\qquad \\
(v_{k'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_l^\vee) \xleftarrow{l} \underbrace{(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{l-1} \otimes v_l^\vee) \xleftarrow{l-1} \cdots \xleftarrow{k'+2} (v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'+1} \otimes v_l^\vee)}_{\text{empty if } k'+1=l}
\end{array}$$

This path is only made of forward moves $\tilde{f}_i$, with $i \in \mathrm{int}(l, k') \sqcup \mathrm{int}(k', l)$ appearing once, where we change the right side of the tensor products. By (2.8), we then have

$$H[(v_{k'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_l^\vee)] - H[(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee)] = \chi(0 \in \mathrm{int}(l, k')) + \chi(0 \in \mathrm{int}(k', l)) = 1. \tag{7.1}$$

By Proposition 6.2, we have the dual path from $(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee)$ to $(v_l \otimes v_l^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee)$, and

$$H[(v_l \otimes v_l^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee)] - H[(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee)] = 1. \tag{7.2}$$

Here we need to compute $H((v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee))$. By interchanging k' and l, we obtain

$$H[(v_{k'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_l^\vee)] - H[(v_l \otimes v_l^\vee) \otimes (v_l \otimes v_l^\vee)] = 1. \tag{7.3}$$

Subtracting (7.1) to (7.3) yields

$$H[(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee)] = H[(v_l \otimes v_l^\vee) \otimes (v_l \otimes v_l^\vee)],$$

and we have an explicit path from $(v_l \otimes v_l^\vee) \otimes (v_l \otimes v_l^\vee)$ to $(v_{k'} \otimes v_{k'}^\vee) \otimes (v_{k'} \otimes v_{k'}^\vee)$ by combining the previous paths.

Recall that by definition, $H[(v_0 \otimes v_0^\vee) \otimes (v_0 \otimes v_0^\vee)] = 0$. Thus setting $k' = 0$ yields by (6.4) that for all $l \in \{0, \ldots, n-1\}$,

$$\begin{aligned} H[(v_l \otimes v_l^\vee) \otimes (v_l \otimes v_l^\vee)] &= 0 \\ &= 2\chi(0 \notin \mathrm{int}(l, l)) \\ &= \Delta(a_l b_l; a_l b_l). \end{aligned} \tag{7.4}$$

Plugging this into (7.1) gives, for all $k' \neq l$,

$$\begin{aligned} H[(v_{k'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_l^\vee)] &= 1 \\ &= \chi(0 \in \mathrm{int}(l, k')) + \chi(0 \in \mathrm{int}(k', l)) \\ &= \Delta(a_l b_l; a_{k'} b_{k'}). \end{aligned} \tag{7.5}$$

7.2. **The case $k' = l \neq k = l'$.** We now build a path from $(v_l \otimes v_l^\vee) \otimes (v_k \otimes v_k^\vee)$ to $(v_l \otimes v_k^\vee) \otimes (v_k \otimes v_l^\vee)$. By (4.10), we know that $\varepsilon_i(v_k \otimes v_k^\vee) = \chi(i = k)$ and $\varepsilon_i(v_k \otimes v_l^\vee) = 0$ if $i \notin \{l+1, k\}$. Since $k \neq l$, we have for all $i \in \mathrm{int}(k, l)$ that $(v_l \otimes v_i^\vee) \neq (v_l \otimes v_{l+1}^\vee)$, and then $(v_l \otimes v_i^\vee) \xrightarrow{i} (v_l \otimes v_{i-1}^\vee)$. We obtain the path

$$\begin{array}{r}
(v_l \otimes v_l^\vee) \otimes (v_k \otimes v_k^\vee) \xrightarrow{k} \underbrace{(v_l \otimes v_l^\vee) \otimes (v_k \otimes v_{k-1}^\vee) \xrightarrow{k-1} \cdots \xrightarrow{l+1}}_{\text{empty if } l+1=k} (v_l \otimes v_l^\vee) \otimes (v_k \otimes v_l^\vee) \\
\Big\downarrow {\scriptstyle l} \qquad\qquad\quad \\
(v_l \otimes v_k^\vee) \otimes (v_k \otimes v_l^\vee) \xleftarrow{k+1} (v_l \otimes v_{k+1}^\vee) \otimes (v_k \otimes v_l^\vee) \underbrace{\xleftarrow{k+2} \cdots \xleftarrow{l-1} (v_l \otimes v_{l-1}^\vee) \otimes (v_k \otimes v_l^\vee)}_{\text{empty if } l=k+1}
\end{array}$$

In the upper part of the path, we move forward (i.e. with some $\tilde{f}_i$) by modifying the right side of the tensor product with arrows in $\mathrm{int}(l, k)$ appearing once. Then, in the lower part of the path, we move forward by modifying the left side of the tensor product with arrows in $\mathrm{int}(k,l)$ appearing once. Using that $k \neq l$, the energy function satisfies:

$$\begin{aligned}
H[(v_l \otimes v_k^\vee) \otimes (v_k \otimes v_l^\vee)] &= H[(v_l \otimes v_l^\vee) \otimes (v_k \otimes v_k^\vee)] + \chi(0 \in \mathrm{int}(l,k)) - \chi(0 \in \mathrm{int}(k,l)) && \text{by (2.8)}\\
&= 1 + 2\chi(0 \in \mathrm{int}(l,k)) - 1 && \text{by (7.5)}\\
&= \Delta(a_l b_k; a_k b_l) && \text{by (6.4).}
\end{aligned}$$

7.3. **The case $k' = l$ and $k \neq l'$.** The vertices $(v_{l'} \otimes v_l^\vee) \otimes (v_l \otimes v_k^\vee)$ and $(v_k \otimes v_l^\vee) \otimes (v_l \otimes v_{l'}^\vee)$ are symmetric.

Since $k \neq l'$, we have that $\mathrm{int}(k,l) \neq \mathrm{int}(l',l)$. By symmetry, we can assume that $\mathrm{int}(l',l) \not\subset \mathrm{int}(k,l) \subset \mathrm{int}(l',l)$, so that $l'+1 \notin \mathrm{int}(k,l)$. In that case, we necessarily have $k \neq l$. Then, $\varphi_l(v_{l'} \otimes v_l^\vee) = 1 = \varepsilon_l(v_l \otimes v_l^\vee)$ and $\varphi_i(v_{l'} \otimes v_l^\vee) = 0$ for all $i \in \mathrm{int}(k,l) \setminus \{l\}$. Thus we have the path

$$\begin{array}{r}
\underbrace{(v_l \otimes v_l^\vee) \otimes (v_l \otimes v_l^\vee) \xleftarrow{l} (v_{l-1} \otimes v_l^\vee) \otimes (v_l \otimes v_l^\vee) \xleftarrow{l-1} \cdots \xleftarrow{l'+1}}_{\text{empty if } l=l'} (v_{l'} \otimes v_l^\vee) \otimes (v_l \otimes v_l^\vee)\\
\Big\downarrow l \qquad\qquad\\
(v_{l'} \otimes v_l^\vee) \otimes (v_l \otimes v_k^\vee) \xleftarrow{k+1} \cdots \xleftarrow{l+1} (v_{l'} \otimes v_l^\vee) \otimes (v_l \otimes v_{l-1}^\vee)
\end{array}$$

and the energy function is given by

$$\begin{aligned}
H[(v_{l'} \otimes v_l^\vee) \otimes (v_l \otimes v_k^\vee)] &= \chi(l' \neq l)\chi(0 \in \mathrm{int}(l',l)) + \chi(0 \in \mathrm{int}(k,l)) && \text{by (2.8)}\\
&= \chi(0 \notin \mathrm{int}(l,l')) + \chi(0 \notin \mathrm{int}(l,k)) && \text{by Lemma 6.4}\\
&= \Delta(a_k b_l; a_l b_{l'}) && \text{by (6.4).}
\end{aligned}$$

This was the last case where $k' = l$. Also, we have already studied a special case where $k' \neq l$, which was the case $l' = k' \neq l = k$. We now study the other cases where $k' \neq l$.

7.4. **The case $k' \neq k = l = l'$ (Symmetric case: $l \neq k = k' = l'$).** Since $l \notin \mathrm{int}(l,k')$, we have the path

$$(v_{l+1} \otimes v_{l+1}^\vee) \otimes (v_l \otimes v_l^\vee) \xleftarrow{l+1} \underbrace{(v_l \otimes v_{l+1}^\vee) \otimes (v_l \otimes v_l^\vee) \xleftarrow{l+2} \cdots \xleftarrow{k'}}_{\text{empty if } k'=l+1} (v_l \otimes v_{k'}^\vee) \otimes (v_l \otimes v_l^\vee).$$

Thus the energy function satisfies

$$\begin{aligned}
H[(v_l \otimes v_{k'}^\vee) \otimes (v_l \otimes v_l^\vee)] &= 1 + \chi(0 \in \mathrm{int}(l,k')) && \text{by (2.8) and (7.5)}\\
&= \chi(0 \in \mathrm{int}(l,l)) + \chi(0 \in \mathrm{int}(l,k')) && \text{by Lemma 6.4}\\
&= \Delta(a_l b_l; a_{k'} b_l) && \text{by (6.4).}
\end{aligned}$$

7.5. **The case $l' \neq k' = k \neq l$ (Symmetric case: $k \neq l = l' \neq k'$).** We first assume that $l'+1 \notin \mathrm{int}(k',l)$. Since $l' \neq k'$, it means that

$$\mathrm{int}(l',k') \sqcup \mathrm{int}(k',l) = \mathrm{int}(l',l).$$

Since $l'+1$ and k' do not belong to $\mathrm{int}(k',l)$, we have by (4.10) that $\varphi_i(v_{l'} \otimes v_{k'}^\vee) = 0$ for all $i \in \mathrm{int}(k',l)$. This gives the path

$$\begin{array}{r}
(v_{l'+1} \otimes v_{l'+1}^\vee) \otimes (v_{k'+1} \otimes v_{k'+1}^\vee) \xleftarrow{l'+1} \underbrace{(v_{l'} \otimes v_{l'+1}^\vee) \otimes (v_{k'+1} \otimes v_{k'+1}^\vee) \xleftarrow{l'+2} \cdots \xleftarrow{k'}}_{\text{empty if } k'=l'+1} (v_{l'} \otimes v_{k'}^\vee) \otimes (v_{k'+1} \otimes v_{k'+1}^\vee)\\
\Big\downarrow k'+1 \qquad\qquad\\
(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_{k'}^\vee) \underbrace{\xleftarrow{l} \cdots \xleftarrow{k'+2} (v_{l'} \otimes v_{k'}^\vee) \otimes (v_{k'+1} \otimes v_{k'}^\vee)}_{\text{empty if } k'+1=l}.
\end{array}$$

We deduce the following formula for the energy function:

$$\begin{aligned}
H[(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_{k'}^\vee)] &= 1 + \chi(0 \in \mathrm{int}(l',k')) + \chi(0 \in \mathrm{int}(k',l)) && \text{by (2.8) and (7.5)}\\
&= \chi(0 \in \mathrm{int}(k',k')) + \chi(0 \in \mathrm{int}(l',l)) && \text{by Lemma 6.4}\\
&= \Delta(a_{k'} b_l; a_{k'} b_{l'}) && \text{by (6.4).}
\end{aligned}$$

Let us now assume that $l'+1 \in \mathrm{int}(k',l)$. Since $\mathrm{int}(k',l) \neq \emptyset$ and $l' \neq k'$, we necessarily have that $k'+1 \neq l$ and $\mathrm{int}(k',l') \subset \mathrm{int}(k',l-1)$, so that $l' \neq l$. Note also that, by (4.10),

$$\varphi_{k'}(v_{l'} \otimes v^\vee_{k'-1}) = 0 = \varepsilon_{k'}(v_{k'-1} \otimes v^\vee_{k'}),$$

since $k' \neq l'+1$, and $\varphi_i(v_{l'} \otimes v^\vee_{k'}) = 0$ for all $i \in \mathrm{int}(l,k') \setminus \{k'\}$. We then have the path

$$(v_{k'} \otimes v^\vee_{k'-1}) \otimes (v_{k'} \otimes v^\vee_{k'-1}) \underbrace{\xleftarrow{k'}}_{\bullet} \underbrace{(v_{k'} \otimes v^\vee_{k'-1}) \otimes (v_{k'} \otimes v^\vee_{k'}) \xrightarrow{k'+1} \cdots \xrightarrow{l'}}_{\text{nonempty since } k' \neq l'+1} (v_{l'} \otimes v^\vee_{k'-1}) \otimes (v_{k'} \otimes v^\vee_{k'})$$

$$\underbrace{\Big\uparrow{\scriptstyle k'}}_{\star}$$

$$\underbrace{(v_{l'} \otimes v^\vee_{k'}) \otimes (v_l \otimes v^\vee_{k'}) \xrightarrow{l+1} \cdots \xrightarrow{k'-1}}_{\bullet} (v_{l'} \otimes v^\vee_{k'}) \otimes (v_{k'-1} \otimes v^\vee_{k'}) \underbrace{\xrightarrow{k'}}_{\star} (v_{l'} \otimes v^\vee_{k'-1}) \otimes (v_{k'-1} \otimes v^\vee_{k'}) \cdot$$

By the previous case ($l' \neq k' = k \neq l$), we obtain the energy function

$$\begin{aligned} H[(v_{k'} \otimes v^\vee_{k'-1}) \otimes (v_{k'} \otimes v^\vee_{k'-1})] &= \chi(0 \in \mathrm{int}(k',k')) + \chi(0 \in \mathrm{int}(k'-1,k'-1)) \\ &= 2\chi(0 \in \mathrm{int}(k',k')). \end{aligned} \tag{7.6}$$

In the computation of H, by (2.8), the moves marked by $\star$ cancel each other, since it is the same arrow that operates backward (i.e. by some $\tilde{e}_i$) consecutively on the right and on the left side of the tensor product. Besides, the moves marked by $\bullet$ give $\mathrm{int}(l,k')$ and operate backward on the right side of the tensor product. As a consequence,

$$\begin{aligned} H[(v_{l'} \otimes v^\vee_{k'}) \otimes (v_l \otimes v^\vee_{k'})] &= H[(v_{k'} \otimes v^\vee_{k'-1}) \otimes (v_{k'} \otimes v^\vee_{k'-1})] - \chi(0 \in \mathrm{int}(k',l')) - \chi(0 \in \mathrm{int}(l,k')) && \text{by (2.8)} \\ &= 2\chi(0 \in \mathrm{int}(k',k')) - \chi(0 \in \mathrm{int}(k',l')) - \chi(0 \in \mathrm{int}(l,k')) && \text{by (7.6)} \\ &= \chi(0 \in \mathrm{int}(k',k')) + \chi(0 \in \mathrm{int}(k',l)) - \chi(0 \in \mathrm{int}(k',l')) && \\ &= \chi(0 \in \mathrm{int}(k',k')) + \chi(0 \in \mathrm{int}(l',l)) && \text{by Lemma 6.4} \\ &= \Delta(a_{k'}b_l; a_{k'}b_{l'}) && \text{by (6.4).} \end{aligned}$$

7.6. **The case $k \neq k'$, $k' \neq l$ and $l \neq l'$.**

7.6.1. *The sub-case $k+1, k' \notin \mathrm{int}(l,l')$ (Symmetric case : $l'+1, l \notin \mathrm{int}(k',k)$).* We have $l'+1, k' \notin \mathrm{int}(l,l')$, so that $\varphi_i(v_{l'} \otimes v^\vee_{k'}) = 0$ for all $i \in \mathrm{int}(l,l')$. Besides, $k+1 \notin \mathrm{int}(l,l')$, so that $\tilde{e}_i(v_i \otimes v^\vee_k) = (v_{i-1} \otimes v^\vee_k)$. We obtain the path

$$(v_{l'} \otimes v^\vee_{k'}) \otimes (v_{l'} \otimes v^\vee_k) \xleftarrow{l'} \cdots \xleftarrow{l+1} (v_{l'} \otimes v^\vee_{k'}) \otimes (v_l \otimes v^\vee_k).$$

By Case 7.4 and the symmetric of Case 7.5, we have

$$H[(v_{l'} \otimes v^\vee_{k'}) \otimes (v_{l'} \otimes v^\vee_k)] = \chi(0 \in \mathrm{int}(k,k')) + \chi(0 \in \mathrm{int}(l',l')), \tag{7.7}$$

and the energy function becomes

$$\begin{aligned} H[(v_{l'} \otimes v^\vee_{k'}) \otimes (v_l \otimes v^\vee_k)] &= H[(v_{l'} \otimes v^\vee_{k'}) \otimes (v_{l'} \otimes v^\vee_k)] - \chi(0 \in \mathrm{int}(l,l')) && \text{by (2.8)} \\ &= \chi(0 \in \mathrm{int}(k,k')) + \chi(0 \in \mathrm{int}(l',l')) - \chi(0 \in \mathrm{int}(l,l')) && \text{by (7.7)} \\ &= \chi(0 \in \mathrm{int}(k,k')) + \chi(0 \in \mathrm{int}(l',l)) && \text{by Lemma 6.4} \\ &= \Delta(a_k b_l; a_{k'} b_{l'}) && \text{by (6.4).} \end{aligned}$$

7.6.2. *The sub-case $k+1 \in \mathrm{int}(l,l')$ and $k' \notin \mathrm{int}(l,l')$ (Symmetric case: $l'+1 \in \mathrm{int}(k',k)$ and $l \notin \mathrm{int}(k',k)$).* This case is very similar to the previous one. We use the following path:

$$\underbrace{(v_{l'} \otimes v^\vee_{k'}) \otimes (v_{l'} \otimes v^\vee_k) \xleftarrow{l'} \cdots \xleftarrow{k+2} (v_{l'} \otimes v^\vee_{k'}) \otimes (v_{k+1} \otimes v^\vee_k)}_{\star} \underbrace{\xrightarrow{k}}_{\bullet} (v_{l'} \otimes v^\vee_{k'}) \otimes (v_{k+1} \otimes v^\vee_{k-1})$$

$$\underbrace{\Big\uparrow{\scriptstyle k+1}}_{\star}$$

$$\underbrace{(v_{l'} \otimes v^\vee_{k'}) \otimes (v_{l'} \otimes v^\vee_k) \xrightarrow{l+1} \cdots \xrightarrow{k} (v_{l'} \otimes v^\vee_{k'}) \otimes (v_k \otimes v^\vee_k)}_{\star} \underbrace{\xrightarrow{k}}_{\bullet} (v_{l'} \otimes v^\vee_{k'}) \otimes (v_k \otimes v^\vee_{k-1})$$

Note that the moves marked by $\bullet$ cancel each other, and the moves marked by $\star$ give $\operatorname{int}(l,l')$, so that the calculation is the same as in the previous case.

7.6.3. *The sub-case $k+1 \notin \operatorname{int}(l,l')$ and $k' \in \operatorname{int}(l,l')$ (Symmetric case: $l'+1 \notin \operatorname{int}(k',k)$ and $l \in \operatorname{int}(k',k)$).* We have $l, k+1 \notin \operatorname{int}(l,l')$, so that $\varepsilon_i(v_l \otimes v_k^\vee) = 0$ for all $i \in \operatorname{int}(l,l')$. Note that $k'+1 \in \operatorname{int}(l,l')$, since $k' \in \operatorname{int}(l,l')$ and $k' \neq l'$. This gives the path

$$\underbrace{(v_l \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee) \xrightarrow{l+1} \cdots \xrightarrow{k'} (v_{k'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)}_{\star} \underbrace{\xrightarrow{k'}}_{\bullet} (v_{k'} \otimes v_{k'-1}^\vee) \otimes (v_l \otimes v_k^\vee)$$

$$\underbrace{\Big\downarrow{\scriptstyle k'+1}}_{\star}$$

$$\underbrace{(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee) \xleftarrow{l'} \cdots \xleftarrow{k'+2} (v_{k'+1} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)}_{\star} \underbrace{\xrightarrow{k'}}_{\bullet} (v_{k'+1} \otimes v_{k'-1}^\vee) \otimes (v_l \otimes v_k^\vee)$$

As before, the moves marked by $\bullet$ cancel each other, and the moves $\star$ give $\operatorname{int}(l,l')$. We move with the $\tilde{f}_i$'s by changing the left side of the tensor product, and we get

$$\begin{aligned} H[(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)] &= H[(v_l \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)] - \chi(0 \in \operatorname{int}(l,l')) && \text{by (2.8)} \\ &= \chi(0 \in \operatorname{int}(k,k')) + \chi(0 \in \operatorname{int}(l,l)) - \chi(0 \in \operatorname{int}(l,l')) && \text{by (7.7)} \\ &= \chi(0 \in \operatorname{int}(k,k')) + \chi(0 \in \operatorname{int}(l',l)) && \text{by Lemma 6.4} \\ &= \Delta(a_k b_l; a_{k'} b_{l'}) && \text{by (6.4).} \end{aligned}$$

7.6.4. *The sub-case $k+1, k' \in \operatorname{int}(l,l')$ and $l'+1, l \in \operatorname{int}(k',k)$.* Note that this case overlaps with the case $k' = l' \neq k = l$ that we already checked in the first part. Omitting that case, we can assume by symmetry that $k \neq l$. We obtain the path

$$\underbrace{(v_{l'} \otimes v_{l'}^\vee) \otimes (v_k \otimes v_k^\vee) \xrightarrow{l'} \cdots \xrightarrow{k'+1}}_{\text{empty if } k'=l'} (v_{l'} \otimes v_{k'}^\vee) \otimes (v_k \otimes v_k^\vee) \xleftarrow{k} \cdots \xleftarrow{l+1} (v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)\,.$$

Since $k \neq l$, the fact that $l \in \operatorname{int}(k',k)$ implies that $\operatorname{int}(k',k) = \operatorname{int}(k',l) \sqcup \operatorname{int}(l,k)$, and the fact that $k+1 \in \operatorname{int}(l,l')$ implies that $\operatorname{int}(l,l') = \operatorname{int}(l,k) \sqcup \operatorname{int}(k,l')$, so that $k', l'+1 \notin \operatorname{int}(l,k)$. Also, if $k' \neq l'$, then $l'+1 \in int(k',k)$ implies that $\operatorname{int}(k',k) = \operatorname{int}(k',l') \sqcup \operatorname{int}(l',k)$, so that $k \notin \operatorname{int}(k',l')$. Since $l \neq l'$ and $k' \neq l$, the fact that $k' \in \operatorname{int}(l,l')$ implies that

$$\operatorname{int}(l',k') = \operatorname{int}(l',l) \sqcup \operatorname{int}(l,k'),$$

and the fact that $l \in \operatorname{int}(k',k)$ and $l \neq k$ implies that

$$\operatorname{int}(l,k') = \operatorname{int}(l,k) \sqcup \operatorname{int}(k,k').$$

Thus the computation of H gives

$$\begin{aligned} H[(v_{l'} \otimes v_{k'}^\vee) \otimes (v_l \otimes v_k^\vee)] &= 1 - \chi(k' \neq l')\chi(0 \in \operatorname{int}(k',l')) - \chi(0 \in \operatorname{int}(l,k)) && \text{by (2.8) and (7.5)} \\ &= 1 - \chi(0 \notin \operatorname{int}(l',k')) - \chi(0 \in \operatorname{int}(l,k)) && \text{by Lemma 6.4} \\ &= \chi(0 \in \operatorname{int}(l',k')) - \chi(0 \in \operatorname{int}(l,k)) \\ &= \chi(0 \in \operatorname{int}(l',l)) + \chi(0 \in \operatorname{int}(l,k')) - \chi(0 \in \operatorname{int}(l,k)) \\ &= \chi(0 \in \operatorname{int}(l',l)) + \chi(0 \in \operatorname{int}(k,k')) \\ &= \Delta(a_k b_l; a_{k'} b_{l'}) && \text{by (6.4).} \end{aligned}$$

We have checked all the possible choices of k, l, k', l'. Our proof of Theorem 1.8 is thus complete.

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