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INTEGER PARTITIONS : ROGERS-RAMANUJAN TYPE IDENTITIES AND ASYMPTOTICS

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Résumé

Les partitions d'entiers, un domaine à l'interface de la combinatoire et de la théorie des nombres, se trouvent au centre de cette thèse qui s'articule autour de trois parties.

Dans la première, nous étudions des identités de partitions du type Rogers-Ramanujan. Nous commençons par donner trois nouvelles preuves du théorème de Schur pour les surpartitions. Puis nous démontrons deux nouvelles généralisations d'identités de partitions d'Andrews aux surpartitions. Enfin nous donnons une preuve combinatoire et un raffinement du théorème de Siladic, une identité de partitions provenant de la théorie des algèbres de Lie.

Dans la deuxième partie, nous explorons des aspects asymptotiques de la théorie des partitions. Nous expliquons d'abord la méthode du cercle de Hardy-Ramanujan-Rademacher et sa variante dûe à Wright. Nous utilisons ensuite cette dernière pour donner un équivalent asymptotique de certaines quantités liées aux surpartitions. Enfin nous introduisons une nouvelle méthode, la méthode du cercle à deux variables, qui permet de calculer l'asymptotique bivariée des coefficients des formes de Jacobi et mock Jacobi, et nous l'utilisons pour résoudre la conjecture de Dyson sur l'asymptotique du crank et montrer que le même résultat est aussi vrai pour le rang.

La troisième et dernière partie concerne un analogue pour les surpartitions des coefficients q-binomiaux. Nous prouvons différents résultats, parmi lesquels une formule exacte, un analogue du triangle de Pascal et une identité du type Rogers-Ramanujan.

Abstract

Integer partitions, a field lying at the interface between combinatorics and number theory, is at the heart of this thesis, which is comprised of three parts.

In the first part, we study partition identities of the Rogers-Ramanujan type. We start by giving three new proofs of Schur's theorem for overpartitions. Then we prove two new generalisations of partition identities due to Andrews to overpartitions. Finally we give a combinatorial proof and refinement of Siladić's theorem, a partition identity which was originally derived by Lie algebraic methods.

In the second part, we investigate asymptotic aspects of the theory of partitions. First we explain the Hardy-Ramanujan-Rademacher circle method and its variant due to Wright. Then we use the latter to give an asymptotic formula for certain quantities related to overpartitions. Finally we introduce a new method, the two-variable circle method, which allows one to compute the bivariate asymptotics for coefficients of Jacobi and mock Jacobi forms, and we use it to solve Dyson's conjecture on the crank asymptotics and to show that the same result also holds for the rank.

The third and last part concerns an overpartition analogue of q-binomial coefficients. We prove several results, among which are an exact formula, an analogue of the Pascal triangle and a Rogers-Ramanujan type identity.

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Part I.

Introduction and preliminaries

1. Introduction (français)

1.1. État de l'art

1.1.1. Les débuts de la théorie des partitions

Leibniz fut le premier mathématicien à étudier les partitions d'entiers. Dans une lettre datant de 1674, il demanda à Bernoulli le nombre de manières de décomposer un entier positif n comme somme d'entiers positifs plus petits. En d'autres termes, il lui demanda le nombre de partitions de n. L'ordre des termes de la somme, appelés parts, n'importe pas. Nous adopterons donc la convention de les écrire en ordre décroissant. Par exemple, il existe trois partitions de 3 : 3, $2+1$ et $1+1+1$.

Leibniz voulait connaître, pour tout entier positif n, le nombre de partitions de n, que nous noterons $p(n)$. Nous remarquons que $p(n) = 0$ lorsque n est négatif, et nous adoptons la convention que $p(0) = 1$ (nous considérons que la partition vide est l'unique partition de 0). Le tableau 1.1 montre les partitions de n et $p(n)$ pour $n = 1, \dots, 7$.

Face à une suite d'entiers comme $(p(n))_{n\in\mathbb{N}}$, nous pouvons nous poser plusieurs questions.

D'abord, nous pouvons nous demander quelle est la proportion d'entiers positifs n pour lesquels $p(n)$ est premier. Leibniz avait suggéré que $p(n)$ pourrait toujours être premier, car il l'avait observé pour $n = 1, \dots, 6$. Mais $p(7) = 15 = 3 \times 5$ n'est pas premier donc la conjecture de Leibniz est fausse. Cependant, une légère modification de sa conjecture aboutit à une question majeure de la théorie des partitions, encore ouverte à ce jour : y a-t-il une infinité d'entiers n pour lesquels $p(n)$ est premier ? En 2000, Ono [Ono00] a fait un premier pas vers la résolution de ce problème en montrant que tout nombre premier divise au moins une valeur de $p(n)$.

Nous pouvons aussi nous interroger sur la proportion d'entiers positifs n pour lesquels $p(n)$ est pair ou impair. Des expériences réalisées sur ordinateur ont mené à la conjecture, toujours ouverte elle aussi, que $p(n)$ est aussi souvent impair que pair, c'est à dire que $p(n)$ est pair pour $\sim \frac{x}{2}$ entiers $n \leq x$ lorsque x tend vers l'infini (voir Parkin et Shanks [PS67]).

Après ces questions ayant trait à la théorie des nombres, nous pouvons aussi

TABLE 1.1.: Partitions des sept premiers entiers positifs

n	$p(n)$	partitions de n
1	1	1
2	2	2, 1+1
3	3	3, 2+1, 1+1+1
4	5	4, 3+1, 2+2, 2+1+1, 1+1+1+1
5	7	5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1
6	11	6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1
7	15	7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1

poser une question plus analytique. Si nous calculons quelques autres valeurs de $p(n)$, nous remarquons que cette fonction croît très rapidement. Par exemple

$$p(10) = 42, p(20) = 627, p(50) = 204226, p(100) = 190569292,$$
$$p(200) = 3972999029388 \text{ et } p(1000) = 24061467864032622473692149727991.$$

Il était donc intéressant de comprendre à quelle vitesse la fonction $p(n)$ grandit asymptotiquement et de chercher une formule exacte pour $p(n)$. Ces questions ont été résolues au début du vingtième siècle par Hardy, Ramanujan et Rademacher grâce à leur "méthode du cercle", comme nous l'expliquons dans la suite de cette introduction (section 1.1.5) et dans la partie III.

1.1.2. Séries génératrices

Après que Leibniz ait posé les bases de la théorie des partitions, il fallut attendre environ soixante-dix ans avant qu'Euler ne trouve les premiers résultats profonds du domaine. Il commença à étudier les partitions d'entiers en 1740, lorsqu'il reçut une lettre de Naudé lui demandant de calculer le nombre de partitions de 50 en 7 parts distinctes. Il fut intéressé par la question et donna une première solution lors d'une présentation à l'Académie de Saint Petersbourg en 1741 [Eul51]. Puis il prouva son résultat une deuxième fois, avec une méthode différente, dans son célèbre livre *Introductio in Analysin Infinitorum* [Eul48], paru en 1748. Présentons maintenant l'idée de cette preuve. Ce n'est pas facile - et cela prendrait un temps considérable - d'écrire à la main toutes les manières de décomposer 50 comme somme de 7 entiers distincts puis de les compter. De plus, même si quelqu'un avait la patience de le faire, cela ne donnerait que peu d'intuition sur la manière de traiter la question générale "de combien de façons peut-on écrire un entier positif n comme somme de m entiers positifs distincts ?". Afin de répondre à ce problème, Euler introduisit un outil encore fondamental de nos jours : les séries génératrices. Pour comprendre son idée, citons Euler lui-même [Eul48] :

> 297. Soit l'expression suivante :
>
> $$(1 + x^{\alpha}z)(1 + x^{\beta}z)(1 + x^{\gamma}z)(1 + x^{\delta}z)(1 + x^{\epsilon}z) \cdots .$$
>
> Nous demandons sa forme si les facteurs sont effectivement multipliés. Nous supposons qu'il a la forme $1 + Pz + Qz^2 + Rz^3 + Sz^4 + \cdots$, où il est clair que P est égal à la somme des puissances $x^{\alpha} + x^{\beta} + x^{\gamma} + x^{\delta} + x^{\epsilon} + \cdots$. Ensuite Q est la somme des produits de puissances prises deux par deux, c'est à dire que Q est la somme des différentes puissances de x dont les exposants sont la somme de deux des différents termes de la suite $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$, etc. De manière semblable R est la somme des puissances de x dont les exposants sont la somme de trois des différents termes. Enfin, S est la somme des puissances de x dont les exposants sont la somme de quatre des différents termes de la même suite $\alpha, \beta, \gamma, \delta, \epsilon$, etc., et ainsi de suite.
>
> 298. Les puissances individuelles de x qui constituent les valeurs des lettres P, Q, R, S, etc. ont un coefficient de 1 si leurs exposants peuvent être formés d'une seule manière à partir de $\alpha, \beta, \gamma, \delta$, etc. Si le même exposant d'une puissance de x peut être obtenu de plusieurs manières en tant que somme de deux, trois, ou plus de termes de la suite $\alpha, \beta, \gamma, \delta$, etc., alors cette puissance a un coefficient égal au nombre de façons dont l'exposant peut être obtenu. Ainsi, si dans la valeur de Q se trouve Nx^n, c'est parce que n a N différentes façons d'être exprimé comme la somme de deux termes de la suite α, β, γ, etc. De plus, si dans l'expression du [*produit*] donné, le terme Nx^nz^m apparaît, c'est parce qu'il existe N différentes façons dans [*le produit d'exprimer N*] comme la somme de m termes de la suite $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, etc.
>
> 299. Si le produit $(1 + x^{\alpha}z)\left(1 + x^{\beta}z\right)(1 + x^{\gamma}z)\left(1 + x^{\delta}z\right)\cdots$ est effectivement multiplié, alors à partir de l'expression obtenue, le nombre de façons différentes dont un nombre donné peut être écrit comme la somme du nombre désiré de termes de la suite $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$,

> etc. devient immédiatement apparent. Par exemple si nous désirons savoir de combien de manière différentes le nombre n peut être la somme de m termes de la suite donnée, alors nous trouvons le terme $x^n z^m$ et son coefficient est le nombre désiré.

Ainsi, d'après le raisonnement d'Euler, le coefficient de $z^m q^n$ dans le produit infini

$$(1+zq)(1+zq^2)(1+zq^3)(1+zq^4)\cdots$$

est égal au nombre de partitions de n en m parts distinctes, que nous noterons $Q(m,n)$. Ainsi la série génératrice de $Q(m,n)$ est

$$\sum_{m,n\geq 0} Q(m,n) z^m q^n = \prod_{k\geq 1}(1+zq^k).$$

Cela nous permet de trouver une relation de récurrence simple pour $Q(m,n)$ qui nous aidera à calculer $Q(7,50)$. En remarquant que

$$\prod_{k\geq 1}(1+zq^k) = (1+zq)\prod_{k\geq 1}(1+zq^{k+1}) = (1+zq)\prod_{k\geq 1}(1+(zq)q^k),$$

nous déduisons que

$$\begin{aligned}\sum_{m,n\geq 0} Q(m,n) z^m q^n &= (1+zq)\sum_{m,n\geq 0} Q(m,n)(zq)^m q^n \\ &= \sum_{m\geq 0}\sum_{n\geq m} Q(m,n-m) z^m q^n + \sum_{m\geq 1}\sum_{n\geq m} Q(m-1,n-m) z^m q^n,\end{aligned}$$

où la deuxième ligne est obtenue par un changement de variables. Ainsi les coefficient de $z^m q^n$ des deux côtés de l'égalité sont égaux, soit

$$Q(m,n) = Q(m,n-m) + Q(m-1,n-m).$$

Avec cette formule, il est aisé de calculer $Q(7,50)$ (ou toute autre valeur) récursivement, et de trouver que cela vaut 522.

Dans le même livre, Euler remarqua que "la condition que les nombres doivent être différents est éliminée si le produit est placé au dénominateur". En d'autres termes, si l'on note $p(m,n)$ le nombre de partitions de n en m parts qui ne doivent pas nécessairement être distinctes, alors la série génératrice de $p(m,n)$ est

$$\begin{aligned}\sum_{m,n\geq 0} p(m,n) z^m q^n &= \prod_{k\geq 1}\left(1+zq^k+z^2q^{2k}+z^3q^{3k}+\cdots\right) \\ &= \prod_{k\geq 1}\frac{1}{1-zq^k}\end{aligned}$$

De la même manière que précédemment nous pouvons déduire que

$$p(m,n) = p(m-1,n-1) + p(m,n-m),$$

et utiliser cette formule pour montrer qu'il y a 8496 partitions de 50 en 7 parts.

Euler parvint même à utiliser les séries génératrices pour trouver une relation de récurrence très efficace pour calculer $p(n)$. D'abord, il remarqua que la série génératrice des partitions est

$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{k\geq 1} \frac{1}{1-q^k}.$$

Ensuite il étudia le produit infini

$$\prod_{k\geq 1} \left(1-q^k\right) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + \cdots$$

et conjectura qu'il est égal à

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

Il réussit à prouver cette assertion quelques années plus tard. Cette formule, connue actuellement sous le nom de Théorème des Nombres Pentagonaux d'Euler, fut le point de départ de la théorie des fonctions theta et des formes modulaires, l'un des domaines les plus importants de la théorie des nombres aujourd'hui [Kob93]. En combinant la série génératrice des partitions et le théorème des nombres pentagonaux, il montra que

$$\left(\sum_{n\geq 0} p(n)q^n\right)\left(\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}\right) = 1.$$

En comparant les coefficients de q^n de chaque côté de l'égalité, il obtint que $p(0) = 1$ et pour $n \geq 1$,

$$p(n) = p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots.$$

C'est encore à ce jour l'algorithme le plus efficace pour calculer $p(n)$, puisqu'il permet de calculer les valeurs de $p(1), \ldots, p(n)$ en temps $O\left(n^{\frac{3}{2}}\right)$.

Les séries génératrices introduites par Euler sont encore l'outil le plus utile de la théorie des partitions, et presque tous les articles sur le sujet les utilisent, que ce soit pour prouver des identités de partitions, des congruences ou des équivalents asymptotiques, pour n'en citer que quelques-uns.

Table 1.2.: L'identité d'Euler pour les sept premiers entiers

n	partitions de n en parts impaires	partitions de n en parts distinctes
1	1	1
2	1+1	2
3	3, 1+1+1	3, 2+1
4	3+1, 1+1+1+1	4, 3+1
5	5, 3+1+1, 1+1+1+1+1	5, 4+1, 3+2
6	5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1	6, 5+1, 4+2, 3+2+1
7	7, 5+1+1, 3+3+1, 3+1+1+1+1, 1+1+1+1+1+1+1	7, 6+1, 5+2, 4+3, 4+2+1

1.1.3. Identités de partitions

Euler fut aussi le premier à découvrir une identité de partitions [Eul48] en 1748.

Théorème 1.1 (Euler)**.** *Pour tout entier positif n, le nombre de partitions de n en parts distinctes est égal au nombre de partitions de n en parts impaires.*

Le tableau 1.2 illustre l'identité d'Euler sur les sept premiers entiers.

Plus généralement, une identité de partitions est un énoncé de la forme "Pour tout entier n, le nombre de partitions de n satisfaisant certaines conditions est égal au nombre de partitions de n satisfaisant d'autres conditions." Une identité de partitions peut être prouvée de plusieurs façons. Nous pouvons trouver une formule exacte pour le nombre de partitions de chaque type pour tout n, et montrer qu'elles sont égales. Mais souvent ce n'est pas facile à réaliser en pratique. Il est préférable de considérer la série génératrice de chaque type de partitions. Notons $a(n)$ le nombre de partitions de n de type A et $b(n)$ le nombre de partitions de n de type B. Même si nous ne pouvons pas trouver directement une formule générale pour $a(n)$ et $b(n)$, si nous pouvons calculer leurs séries génératrices et montrer qu'elles sont égales, c'est à dire que $\sum_{n\geq 0} a(n)q^n = \sum_{n\geq 0} b(n)q^n$, alors en comparant les coefficients de q^n des deux côtés nous déduisons que $a(n) = b(n)$ pour tout n. Une autre manière de prouver une identité de partitions est d'associer chaque partition de type A avec exactement une partition de type B et vice versa. Il s'agit d'une preuve bijective. Le plus souvent, il est plus facile de prouver une identité de partitions en utilisant des séries génératrices que des bijections, mais les bijections donnent plus d'information combinatoire car elles indiquent exactement quelle partition de type A correspond à quelle partition de type B.

Comparons maintenant les trois approches sus-mentionnées sur l'exemple de l'identité d'Euler.

Nous avons vu dans la section précédente qu'il est difficile de trouver une formule exacte pour le nombre de partitions de n en m parts distinctes. Même si nous enlevons la condition sur le nombre de parts, cela ne s'avère pas beaucoup plus facile. De même il n'y a pas de manière élémentaire de trouver le nombre de partitions de n en parts impaires pour tout n. En fait, il a fallu attendre 1937 et la méthode du cercle de Hardy-Ramanujan-Rademacher pour arriver à une formule exacte pour ces quantités (et pour $p(n)$). Il va sans dire qu'Euler n'a donc pas démontré son identité avec cette méthode. En revanche, les séries génératrices de ces types de partitions ne sont pas trop difficiles à calculer. Comme nous l'avons vu dans la section précédente, la série génératrice des partitions en parts distinctes est égale à

$$\prod_{k\geq 1}(1+q^k),$$

et la série génératrice des partitions en parts impaires est égale à

$$(1+q+q^2+\cdots)(1+q^3+q^6+\cdots)(1+q^5+q^{10}+\cdots)\cdots = \prod_{k\geq 0}\frac{1}{1-q^{2k+1}}.$$

En utilisant le fait que

$$1+q^k = \frac{1-q^{2k}}{1-q^k},$$

nous obtenons

$$\begin{aligned}\prod_{k\geq 1}(1+q^k) &= \prod_{k\geq 1}\frac{1-q^{2k}}{1-q^k}\\ &= \frac{\cancel{(1-q^2)}\cancel{(1-q^4)}\cdots}{(1-q)\cancel{(1-q^2)}(1-q^3)\cancel{(1-q^4)}(1-q^5)\cdots}\\ &= \prod_{k\geq 0}\frac{1}{1-q^{2k+1}},\end{aligned}$$

et l'identité d'Euler est prouvée grâce à une simple manipulation des séries génératrices.

Expliquons maintenant l'idée d'une preuve bijective de l'identité d'Euler. Nous devons trouver une manière d'associer chaque partition de n en parts impaires avec une partition de n en parts distinctes, et étant donnée une partition en parts distinctes, être capable de retrouver de quelle partition en parts impaires elle provient. Commençons avec une partition en parts impaires et essayons de la transformer en partition en parts distinctes. Nous voulons que toutes les parts soient distinctes donc s'il y a deux occurrences d'une même part dans la partition, nous les fusionnons en une part de taille double. Nous répétons ce procédé jusqu'à ce que les parts soient toutes distinctes. Illustrons cette méthode sur un exemple :

$$\begin{aligned} 7+5+5+5+3+1+1+1+1 &\mapsto 7+(5+5)+5+3+(1+1)+(1+1) \\ &\mapsto 10+7+5+3+2+2 \\ &\mapsto 10+7+5+3+(2+2) \\ &\mapsto 10+7+5+4+3. \end{aligned}$$

Maintenant nous devons trouver la transformation réciproque. Partant d'une partition en parts distinctes, nous séparons chaque part paire en deux parts de taille moitié et répétons cette procédure jusqu'à ce que toutes les parts soient impaires. Sur l'exemple précédent, nous voyons que ce procédé permet de retrouver la partition dont nous sommes partis.

$$\begin{aligned} 10+7+5+4+3 &\mapsto (5+5)+7+5+(2+2)+3 \\ &\mapsto 7+5+5+5+3+2+2 \\ &\mapsto 7+5+5+5+3+(1+1)+(1+1) \\ &\mapsto 7+5+5+5+3+1+1+1+1. \end{aligned}$$

L'ordre dans lequel nous fusionnons ou séparons les parts n'importe pas. Cette procédure fonctionne pour toute partition en parts impaires ou distinctes ; elle donne donc une preuve bijective de l'identité d'Euler. En 1969, Andrews [And69c] étendit cette preuve pour généraliser l'identité d'Euler à d'autres types de partitions que l'on peut aussi associer par une procédure de fusion/séparation.

Après Euler, Sylvester [Syl73] fut le suivant à faire des découvertes dans le domaine des identités de partitions à la fin du dix-neuvième siècle. Parmi d'autres résultats, il introduisit une représentation graphique des partitions appelée diagramme de Ferrers et l'utilisa pour prouver que pour tous les entiers positifs k et n, le nombre de partitions de n dont la plus grande part est égale à k est égal au nombre de partitions de n en k parts.

L'avancée majeure suivante dans le domaine des identités de partitions a été faite plusieurs années plus tard, avec la découverte des identités de Rogers-Ramanujan.

Théorème 1.2 (Première identité de Rogers-Ramanujan)**.** *Pour tout entier positif n, le nombre de partitions de n telles que la différence entre deux parts consécutives est d'au moins 2 est égal au nombre de partitions de n en parts congrues à 1 ou 4 modulo 5.*

Théorème 1.3 (Deuxième identité de Rogers-Ramanujan)**.** *Pour tout entier positif n, le nombre de partitions de n telles que la différence entre deux parts consécutives est d'au moins 2 et la plus petite part est supérieure ou égale à 2 est égal au nombre de partitions de n en parts congrues à 2 ou 3 modulo 5.*

En termes de séries génératrices, ces deux identités peuvent être écrites de la manière suivante :

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})},$$

et

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}.$$

Ces deux identités ont été publiées par Rogers [Rog94] en 1894, mais sont passées quasiment inaperçues à l'époque. Cependant, en 1913, Ramanujan a redécouvert ces identités séries-produits et les a envoyées à Hardy dans une lettre. Hardy était incapable de les démontrer et les envoya à Littlewood, MacMahon et Perron, qui ne réussirent pas non plus, même si MacMahon vit le lien entre les identités séries-produits et les identités de partitions et les vérifia jusqu'à $n = 89$. Le mystère fut résolu en 1917 lorsque Ramanujan, à la lecture d'anciens volumes de *Proceedings of the London Mathematical Society*, tomba par hasard le papier de Rogers. Il fut très impressionné par son travail, et ils commencèrent à échanger des lettres qui conduisirent à une simplification de l'argument original de Rogers. Ils publièrent leur nouvelle preuve dans un papier commun [RR19] en 1919. Tenu à l'écart des développements des mathématiques britanniques par la première guerre mondiale, le mathématicien allemand Schur prouva à son tour ces identités, indépendamment, en 1917 [Sch17]. Au fil du temps, les identités de Rogers-Ramanujan ont acquis le statut d'identités les plus célèbres du domaine, et des dizaines de preuves utilisant différentes techniques ont été publiées, par exemple [Bre83, GM81, Wat29]. Cependant nous ne disposons encore d'aucune preuve bijective simple à ce jour.

Plus généralement, une identité de partitions du type "pour tout n, le nombre de partitions de n avec certaines conditions de différence est égal au nombre de partitions de n avec certaines conditions de congruence" est appelée une identité de partitions du type Rogers-Ramanujan. Ces identités ont été largement étudiées, comme nous allons le voir maintenant.

Si nous regardons les énoncés de l'identité d'Euler et la première identité de Rogers-Ramanujan, nous remarquons qu'elles sont toutes deux de la forme "pour tout n, le nombre de partitions de n telles que deux parts consécutives diffèrent d'au moins k égale le nombre de partitions de n en parts congrues à 1 ou -1 modulo $k+3$". L'identité d'Euler correspond à $k=1$ et la première identité de Rogers-Ramanujan à $k=2$. Ainsi nous pouvons nous demander si cette identité est vraie aussi pour $k=3$. Il se trouve que bien qu'elle soit vraie pour $n=1,\dots,8$, elle est fausse pour $n=9$, car il y a trois partitions en parts congrues à $\pm 1 \mod 6$ ($7+1$, $5+1+1$ et $1+\cdots+1$) et quatre partitions telles que deux parts consécutives diffèrent d'au moins 3 (9, $8+1$, $7+2$ et $6+3$). En 1946, Lehmer [Leh46] élimina tout espoir de trouver une généralisation de ce type en prouvant que pour tout $k \geq 3$, il est impossible de trouver un ensemble $N \subseteq \mathbb{N}$ tel que pour tout n, le nombre de partitions de n telles que deux parts consécutives diffèrent d'au moins k soit égal au nombre de partitions de n en parts appartenant à N. Mais en 1956, Alder [Ald56] remarqua qu'au lieu d'une égalité, une inégalité pourrait être vraie pour tout k, et il conjectura que pour tous $k, n \geq 0$, le nombre de partitions de n telles que deux parts consécutives diffèrent d'au moins k est supérieur ou égal au nombre de partitions de n en parts congrues à 1 ou -1 modulo $k+3$.

Bien que Lehmer ait prouvé que cette généralisation est impossible, Schur [Sch26] avait prouvé en 1926 un théorème similaire pour $k=3$ en modifiant les conditions de différence plutôt que celles de congruence.

Théorème 1.4 (Schur)**.** *Pour tout entier n, soit $A(n)$ le nombre de partitions de n en parts congrues à 1 ou -1 modulo 6, $B(n)$ le nombre de partitions de n en parts distinctes congrues à 1 ou 2 modulo 3, et $C(n)$ le nombre de partitions de n telles que deux parts consécutives diffèrent d'au moins 3 et deux multiples de 3 consécutifs ne peuvent pas être des parts. Alors pour tout n,*

$$A(n) = B(n) = C(n).$$

Cela explique pourquoi la conjecture était fausse pour $n=9$, puisque l'on comptait la partition $6+3$.

Le théorème de Schur est devenu lui aussi l'une des identités de partitions les plus importantes, et plusieurs preuves ont été données, utilisant diverses techniques comme les bijections [Bes91, Bre80], la méthode des mots pondérés [AG93], et les récurrences et équations aux q-différences [And67b, And68b, And71b].

Il est naturel de se demander si la généralisation “pour tout entier n, le nombre de partitions de n telles que deux parts consécutives diffèrent d'au moins k et deux multiples de k consécutifs ne peuvent pas être des parts est égal au nombre de partitions de n en parts satisfaisant certaines conditions de congruence” est vraie pour d'autres valeurs de k. Dans les années 1960, Göllnitz [G67̈] et Gordon [Gor65] ont montré indépendamment qu'un tel théorème existe pour $k = 2$ en prouvant que le nombre de partitions de n telles que deux parts consécutives diffèrent d'au moins 2 et deux multiples de 2 consécutifs ne peuvent pas être des parts est égal au nombre de partitions de n en parts congrues à $1, 4$ ou 7 modulo 8. Cependant, une telle généralisation n'est pas possible pour $k \geq 4$, comme l'a prouvé Alder [Ald48] en 1948.

En 1961, Gordon [Gor61] trouva la première généralisation des identités de Rogers-Ramanujan.

Théorème 1.5 (Gordon)**.** *Soit $A_{k,a}(n)$ le nombre de partitions de n en parts congrues à $0, \pm a \mod 2k+1$. Soit $B_{k,a}(n)$ le nombre de partitions de n de la forme $\lambda_1 + \lambda_2 + \cdots + \lambda_j$ telles que $\lambda_i \geq \lambda_{i+1}$, $\lambda_i - \lambda_{i+k-1} \geq 2$ et au plus $a-1$ des λ_i peuvent être égaux à 1. Alors pour tous $1 \leq a \leq k$ et $n \geq 0$,*

$$A_{k,a}(n) = B_{k,a}(n).$$

Ce théorème mena à deux généralisations majeures d'Andrews [And69b, And74].

Le théorème de Schur a également été généralisé de deux manières différentes par Andrews [And68a, And69a]. Dans ces généralisations, il considère les partitions en parts distinctes congrues à 2^k modulo $2^n - 1$ ou en parts distinctes congrues à -2^k modulo $2^n - 1$ pour $0 \leq k \leq n-1$. Le théorème de Schur correspond à $n = 2$. Les cas $n = 3$ de ces théorèmes sont les suivants.

Théorème 1.6 (Andrews)**.** *Soit $A(n)$ le nombre de partitions de n en parts distinctes congrues à 1,2 ou 4 modulo 7. Soit $B(n)$ le nombre de partitions de n de la forme $n = \lambda_1 + \cdots + \lambda_s$ telles que $\lambda_i - \lambda_{i+1} \geq 7$ si $\lambda_{i+1} \equiv 1, 2, 4 \ (mod\ 7)$, $\lambda_i - \lambda_{i+1} \geq 12$ si $\lambda_{i+1} \equiv 3 \ (mod\ 7)$, $\lambda_i - \lambda_{i+1} \geq 10$ si $\lambda_{i+1} \equiv 5, 6 \ (mod\ 7)$ et $\lambda_i - \lambda_{i+1} \geq 15$ si $\lambda_{i+1} \equiv 0 \ (mod\ 7)$. Alors pour tout entier n, $A(n) = B(n)$.*

Théorème 1.7 (Andrews)**.** *Soit $C(n)$ le nombre de partitions de n en parts distinctes congrues à 3,5 ou 6 modulo 7. Soit $D(n)$ le nombre de partitions de n de la forme $n = \lambda_1 + \cdots + \lambda_s$ telles que $\lambda_i - \lambda_{i+1} \geq 7$ si $\lambda_i \equiv 3, 5, 6 \ (mod\ 7)$, $\lambda_i - \lambda_{i+1} \geq 12$ si $\lambda_i \equiv 4 \ (mod\ 7)$, $\lambda_i - \lambda_{i+1} \geq 10$ si $\lambda_i \equiv 1, 2 \ (mod\ 7)$, $\lambda_i - \lambda_{i+1} \geq 15$ si $\lambda_i \equiv 0 \ (mod\ 7)$ et $\lambda_s \neq 1, 2, 4, 7$. Alors pour tout entier n, $C(n) = D(n)$.*

Les identités d'Andrews ont depuis lors été généralisées et raffinées [All97, CL06], et été utilisées par Yee pour résoudre la plupart des cas de la conjecture d'Alder [1] [And71a, Yee08]. Elles jouent aussi un rôle naturel en théorie des représentations [AO91] et en algèbre quantique [Oh15].

Depuis les années 1980, de nombreuses connexions entre les représentations des algèbres de Lie, les équations aux q-différences et les identités de partitions du type Rogers-Ramanujan ont été révélées. Pour les équations aux q-différences, voir [CLM06], [FFJ$^+$09] et [Jer12]. Concernant les partitions, Lepowsky et Wilson [LW84] ont été les premiers à établir le lien avec les algèbres de Lie en donnant une interprétation des identités de Rogers-Ramanujan en termes de représentations de l'algèbre de Lie affine $sl_2(\mathbb{C})^\sim$. Réciproquement, l'étude des identités de Rogers-Ramanujan les a aussi aidés dans leur compréhension des algèbres de Lie, car ils ont cherché à comprendre la signification de la condition de différence au moins 2 entre deux parts consécutives. Des méthodes similaires à celles de Lepowsky et Wilson ont par la suite été appliquées à d'autres représentations d'algèbres de Lie affines, ce qui mena à de nouvelles identités de partitions du type Rogers-Ramanujan, découvertes par Capparelli [Cap93], Primc [Pri99], Meurman-Primc [MP87] et Siladić [Sil] pour en citer quelques-uns. Par exemple, l'identité de Capparelli est la suivante.

Théorème 1.8 (Capparelli)**.** *Soit $C(n)$ le nombre de partitions de n en parts congrues à ± 2 ou ± 3 modulo 12. Soit $D(n)$ le nombre de partitions de n de la forme $n = \lambda_1 + \cdots + \lambda_s$ telles que $\lambda_s > 1$, $\lambda_i - \lambda_{i+1} \geq 2$, et si $\lambda_i - \lambda_{i+1} < 4$ alors soit λ_i et λ_{i+1} sont tous deux multiples de 3, soit $\lambda_i \equiv 1 \mod 3$, soit $\lambda_{i+1} \equiv -1 \mod 3$. Alors pour tout entier n, $C(n) = D(n)$.*

Cette identité, présentée par Capparelli comme conjecture à une conférence, a d'abord été prouvée combinatoirement par Andrews [And92] et Alladi, Andrews et Gordon [AAG95], puis avec des techniques d'algèbres de Lie par Capparelli lui-même [Cap96]. Simultanément, Tamba et Xie la prouvèrent en utilisant la théorie des opérateurs vertex [TX95]. Cependant, la plupart des identités de partitions du type Rogers-Ramanujan provenant de l'étude des algèbres de Lie ne sont pas encore bien comprises combinatoirement.

1.1.4. Congruences

Les congruences sont aussi un sujet important de la théorie des partitions. C'est à nouveau Ramanujan qui initia la recherche dans ce domaine. Se basant sur le tableau des valeurs de $p(n)$ pour $n = 0, \ldots, 200$ calculées par MacMahon,

1. La conjecture d'Alder a été complètement démontrée en 2011 by Alfes, Jameson et Lemke Oliver [AJO11].

il annonça en 1919 qu'il avait découvert trois congruences simples vérifiées par $p(n)$, plus précisément que pour tout $n \geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \mod 5, \\ p(7n+5) &\equiv 0 \mod 7, \\ p(11n+6) &\equiv 0 \mod 11. \end{aligned}$$

Il eut l'idée des ces congruences parce que le tableau des valeurs de $p(n)$ de MacMahon était présenté sous la forme de 5 colonnes, et il remarqua que les nombres de la dernière colonne étaient toujours divisibles par 5. Si Hardy et MacMahon n'avaient pas remarqué cette propriété intéressante, c'était peut-être parce qu'ils pensaient que les partitions, des objets additifs par nature, n'avaient aucune raison d'avoir des propriétés de divisibilité. Ramanujan prouva les deux premières congruences dans [Ram19] et annonça dans une courte note [Ram20] qu'il avait aussi trouvé une preuve de la dernière. Après la mort de Ramanujan en 1920, Hardy [Ram21] réussit à extraire une preuve des trois congruences à partir d'un manuscrit de Ramanujan.

En voyant ces congruences, un combinatoricien se pose naturellement la question d'une interprétation combinatoire. Par exemple, pour la première congruence, il voudrait diviser les partitions de n en 5 ensembles de même taille en fonction d'une condition combinatoire. Cependant les preuves originales de Ramanujan reposent sur des identités entre q-séries et ne donnent pas d'intuition combinatoire. En 1944, Dyson [Dys44], encore étudiant à Cambridge à l'époque, définit le rang d'une partition comme sa plus grande part moins son nombre de parts, et conjectura que cette fonction permettait d'expliquer les deux premières congruences de Ramanujan. Plus précisément, il dit que pour tout n, les partitions de $5n+4$ (resp. $7n+5$) peuvent être divisées en 5 (resp. 7) classes de même taille selon la valeur de leur rang modulo 5 (resp. 7). Cette conjecture fut prouvée dix ans plus tard par Atkin et Swinnerton-Dyer [ASD54].

Cependant le rang ne permet pas d'expliquer la congruence modulo 11. C'est pourquoi Dyson [Dys44] conjectura l'existence d'une autre fonction qu'il appela le "crank", qui donnerait une interprétation combinatoire des trois congruences. Le crank a finalement été découvert par Andrews et Garvan [AG88, Gar88] en 1988. Si pour une partition λ, $o(\lambda)$ désigne le nombre de 1 dans λ et $\mu(\lambda)$ désigne le nombre de parts strictement plus grandes que $o(\lambda)$, alors le crank de λ est défini par

$$\operatorname{crank}(\lambda) := \begin{cases} \text{plus grande part de } \lambda & \text{si } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{si } o(\lambda) > 0. \end{cases}$$

Par exemple le crank de la partition $5+3+3+2$ est égal à 5 car 1 n'est pas une part, et celui de $5+2+1+1$ est égal à -1. Soit $M(m,n)$ le nombre de partitions de n avec crank m, et $N(m,n)$ le nombre de partitions de n avec rang m. Des congruences similaires à celles de Ramanujan existent aussi pour $M(m,n)$ [Mah05] et $N(m,n)$ [BO10].

Dans [Ram19], Ramanujan énonça aussi une conjecture plus générale : soit $\delta = 5^a 7^b 11^c$ et soit λ un entier tel que $24\lambda \equiv 1 \mod \delta$, alors pour tout $n \geq 0$,

$$p(n\delta + \lambda) \equiv 0 \mod \delta.$$

Dans un manuscrit non publié [BO99], il donna une preuve de cette conjecture pour un a arbitraire et $b = c = 0$. Il commença une preuve pour b arbitraire et $a = c = 0$ mais ne la termina jamais. S'il l'avait fait, il aurait remarqué que la conjecture devait être modifiée. En effet, si sa conjecture est vraie pour toutes les valeurs de $p(n)$ dont il disposait à l'époque, Chowla [Cho34] trouva en 1934 que $p(243)$ n'est pas divisible par 7^3 bien que $24 \times 243 \equiv 1 \mod 7^3$. Cependant en 1938 Watson [Wat38] parvint à prouver une version modifiée de la conjecture pour toutes les puissances de 5 et 7, et Atkin [Atk67] prouva la totalité de la conjecture modifiée en 1967 : si $\delta = 5^a 7^b 11^c$ et $24\lambda \equiv 1 \mod \delta$, alors pour tout $n \geq 0$,

$$p(n\delta + \lambda) \equiv 0 \mod 5^a 7^{[(b+2)/2]} 11^c.$$

De nombreuses autres congruences ont été prouvées pour des fonctions liées aux partitions [AO01, Gar10, Gar12, Lov00, Lov01, LO02, Ono11] et c'est encore aujourd'hui un domaine de recherche très actif.

1.1.5. Asymptotique et la méthode du cercle de Hardy-Ramanujan

Depuis Leibniz et les débuts de la théorie des partitions, les mathématiciens ont cherché à trouver une formule exacte pour la fonction $p(n)$.

Hardy et Ramanujan [HR18b] furent les premiers à étudier $p(n)$ analytiquement en 1918. Ils prouvèrent en particulier l'équivalent asymptotique suivant lorsque n tend vers l'infini :

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Ils furent surpris de voir à quel point la valeur obtenue avec cette formule était proche de la valeur exacte de $p(200)$ calculée par MacMahon. Cela leur donna l'intuition qu'une formule exacte pour $p(n)$ pouvait être obtenue par une méthode similaire, et ils prouvèrent la formule suivante :

$$p(n) = \frac{1}{2\sqrt{2}} \sum_{k=1}^{a\sqrt{n}} \sqrt{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}} \frac{\mathrm{d}}{\mathrm{d}n} \left(\exp \left(\frac{\pi \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}}{k} \right) \right),$$

où $\omega_{h,k}$ est une racine 24-ème de l'unité, (h, k) désigne le plus grand diviseur commun de h et k, et a est une constante arbitraire, avec pour seule contrainte que n soit plus grand qu'une certaine valeur $n_0(a)$ qui dépend de a. Cette formule est extrêmement précise. Par exemple, il suffit de calculer les huit premiers termes de la série pour obtenir $p(200) = 3972999029388$, qui est la valeur exacte.

Cependant il y a une relation entre n et a, donc il ne s'agit pas d'une formule exacte pour $p(n)$, dans le sens où on ne peut pas simplement substituer n dans la formule et obtenir le résultat. Quelques années plus tard, en 1937, Rademacher [Rad37] améliora la méthode de Hardy et Ramanujan pour trouver une expression de $p(n)$ en tant que série convergente.

Théorème 1.9. *Pour tout entier positif n, nous avons*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{1/2}\right)}{\left(x - \frac{1}{24}\right)^{1/2}} \right]_{x=n},$$

où

$$A_k(n) = \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}}.$$

La technique qu'ils ont utilisée pour prouver ces formules est appelée la Méthode du Cercle de Rogers-Ramanujan-Rademacher. Son principe est le suivant. D'abord nous écrivons $p(n)$ sous forme d'intégrale :

$$p(n) = \frac{1}{2i\pi} \oint_{\gamma} \frac{P(x)}{x^{n+1}} \,\mathrm{d}x,$$

où P est la série génératrice des partitions et γ est un cercle centré à l'origine de rayon strictement inférieur à 1. Ensuite nous utilisons le fait que les singularités de la fonction intégrée sont les racines de l'unité. Nous divisons le cercle en petits arcs qui déterminent quelle singularité est la plus proche, et nous estimons l'intégrale sur chacun de ces arcs en utilisant le fait que P est (presque) une forme modulaire.

Dans les années 1920, peu après la mort de Ramanujan, Hardy et Littlewood publièrent une série de papiers, intitulés *Some problems of Partitio Numerorum* [HL20, HL23, HL25] en référence au chapitre sur les partitions d'entiers dans le livre d'Euler [Eul48], dans lesquels ils utilisèrent la méthode du cercle pour résoudre plusieurs problèmes importants en théorie additive des nombres comme le problème de Waring. Cette méthode s'est révélée très utile en combinatoire additive et a été utilisée pour prouver des résultats majeurs tels que le théorème de Roth [Rot53] ou le fait que chaque entier impair est la somme d'au plus cinq nombres premiers [Tao], pour n'en citer que quelques-uns.

En 1933, Wright [Wri33] montra que si nous voulons seulement trouver un équivalent asymptotique pour le nombre de partitions de n et n'avons pas besoin d'une formule exacte, alors la méthode du cercle de Hardy-Ramanujan peut être simplifiée en considérant seulement un arc autour du pôle dominant et en montrant que la contribution asymptotique de l'intégrale sur le reste du cercle est négligeable. Récemment, cette méthode a été utilisée à nouveau pour trouver des équivalents asymptotiques pour plusieurs fonctions liées aux partitions, dans [BM13, BM14a, BM14b] pour en citer quelques-uns. Cependant, il y a une erreur dans les papiers sus-mentionnés, car les auteurs utilisent la même formule asymptotique proche et loin du pôle dominant, ce qui n'est pas valide loin de ce dernier. Heureusement la preuve peut être corrigée et les résultats principaux restent vrais. Hormis les papiers originaux de Wright, la méthode apparaît sous sa forme correcte pour la première fois dans [BDar].

Le crank, introduit pour expliquer les congruences de Ramanujan, a aussi été étudié asymptotiquement. En 1989, Dyson conjectura l'équivalent asymptotique suivant [Dys89] :

Conjecture 1.10 (Dyson)**.** *Lorsque $n \to \infty$, nous avons*

$$M\left(m,n\right) \sim \frac{1}{4}\beta \operatorname{sech}^2\left(\frac{1}{2}\beta m\right) p(n)$$

avec $\beta := \frac{\pi}{\sqrt{6n}}$.

Dyson savait que la formule était vraie pour βm fixé, mais il demanda sur quel intervalle (pouvant dépendre de n) cette formule pouvait être vraie et quel était le terme d'erreur. Pour m fixé, il est possible d'obtenir un équivalent directement, car dans ce cas la série génératrice de $M(m,n)$ est le produit d'une forme modulaire et d'une fonction theta partielle [BM13]. Cependant, la source de difficulté de la conjecture de Dyson est qu'il s'agit d'un problème d'asymptotique à deux variables. Bien qu'il existe plusieurs variantes de la méthode du cercle qui ont été appliquées avec succès à de nombreux problèmes, les techniques existantes n'étaient pas suffisantes pour résoudre cette conjecture.

1.1.6. Les surpartitions

Les séries basiques hypergéométriques (ou q-séries) sont des séries construites en utilisant les q-factorielles

$$(a)_n = (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

pour $n \in \mathbb{N} \cup \{\infty\}$. Des identités telles que les identités de Rogers-Ramanujan et le théorème des nombres pentagonaux d'Euler sont appelées identités de q-séries. Elles jouent un rôle important en combinatoire, théorie des nombres, théorie des représentations et physique mathématique, pour ne citer que quelques domaines. Alors que les identités mentionnées jusqu'à présent dans cette introduction ont une interprétation combinatoire en termes de partitions, certaines autres identités importantes, comme le théorème q-binomial [GR04]

$$\sum_{n \geq 0} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty},$$

un q-analogue du théorème binomial, n'en ont pas. Pour pouvoir donner une interprétation combinatoire de cette identité et d'autres, il est utile de considérer une généralisation des partitions : les surpartitions. Une surpartition est une partition dans laquelle la dernière occurrence d'un nombre peut être surlignée. Il est équivalent de considérer les partitions dans lesquelles la première occurrence d'un nombre peut être surlignée (selon le contexte, l'une ou l'autre des définitions peut être plus pratique à utiliser). Par exemple, il y a 8 surpartitions de 3 :

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, 1+1+\overline{1}.$$

Bien qu'elles ne portaient pas le nom de surpartitions à l'époque, elles ont été utilisées par Andrews [And67a] dès 1967 pour donner des interprétations combinatoires du théorème q-binomial, de la transformation de Heine et de l'identité de Lebesgue. Ensuite elles ont été utilisées en 1987 par Joichi et Stanton [JS87] dans leur théorie algorithmique des preuves bijectives d'identités de q-séries. Elles apparaissent aussi dans des preuves bijectives de la sommation ${}_1\psi_1$ de Ramanujan et de la sommation de q-Gauss [Cor03, CL02]. Ce sont Corteel et Lovejoy [CL04] qui leur ont donné leur nom en 2004 et ont révélé leur généralité en donnant des interprétations combinatoires de plusieurs identités de q-séries.

Au-delà des identités de q-séries, les surpartitions sont une généralisation très intéressante des partitions. Plusieurs identités de partitions ont des analogues pour les surpartitions ou des généralisations. Par exemple, Lovejoy a prouvé un analogue pour les surpartitions des identités de Gordon [Lov03], Andrews-Santos and Gordon-Göllnitz [Lov04], et la généralisation suivante du théorème de Schur [Lov05b].

Théorème 1.11 (Lovejoy). *Soit $A(k,n)$ le nombre de surpartitions de n en parts congrues à 1 ou 2 modulo 3 ayant k parts non surlignées. Soit $B(k,n)$ le nombre de surpartitions $\lambda_1 + \cdots + \lambda_s$ de n, ayant k parts non surlignées et satisfaisant les conditions de différence*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 3\chi(\overline{\lambda_{i+1}}) \text{ si } \lambda_{i+1} \equiv 1,2 \mod 3, \\ 1 + 3\chi(\overline{\lambda_{i+1}}) \text{ si } \lambda_{i+1} \equiv 0 \mod 3, \end{cases}$$

où $\chi(\overline{\lambda_{i+1}}) = 1$ si λ_{i+1} est surligné et 0 sinon. Alors pour tous $k, n \geq 0$, $A(k,n) = B(k,n)$.

Le travail de Lovejoy a conduit à d'autres recherches dans le domaine et plusieurs nouvelles identités de partitions ont été découvertes [CSS13, CM07, LM08].

Les surpartitions ont aussi des propriétés arithmétiques intéressantes [ACKOar, BL08, Mah04, Tre06] et sont liées aux domaines des algèbres de Lie [KK04], de la physique mathématique [DLM03, FJM05a, FJM05b] et des fonctions supersymétriques [DLM03].

1.2. Contributions de cette thèse

Après avoir décrit plusieurs aspects de la théorie des partitions, présentons maintenant les nouveaux résultats de cette thèse.

1.2.1. Identités de partitions

La partie II traite des identités de partitions. Dans le chapitre 4, nous donnons trois nouvelles preuves du théorème de Schur pour les surpartitions (théorème 1.11). Dans les chapitres 5 et 6, nous généralisons les deux identités d'Andrews aux surpartitions. Enfin dans le chapitre 7, nous donnons une preuve combinatoire et un raffinement du théorème de Siladić, une identité de partitions provenant de la théorie des algèbres d'opérateurs vertex. Toutes ces preuves font intervenir des équations aux q-différences et des récurrences, mais des techniques très différentes sont utilisées dans chacune d'entre elles, ce qui montre la diversité de cette méthode.

1.2.1.1. Le théorème de Schur

Rappelons le théorème de Schur.

Théorème 1.12 (Schur). *Pour tout entier n, soit $A(n)$ le nombre de partitions de n en parts congrues à 1 ou -1 modulo 6, $B(n)$ le nombre de partitions de n en parts distinctes congrues à 1 ou 2 modulo 3, et $C(n)$ le nombre de partitions de n telles que deux parts consécutives diffèrent d'au moins 3 et deux multiples de 3 consécutifs ne peuvent pas être des parts. Alors pour tout n,*

$$A(n) = B(n) = C(n).$$

Outre la preuve originale de Schur [Sch26], le théorème de Schur a été largement étudié et toutes sortes de preuves ont été données. Bessenrodt [Bes91] et Bressoud [Bre80] l'ont prouvé bijectivement, Alladi et Gordon l'ont prouvé en utilisant la technique des mots pondérés, et Andrews en a donné trois preuves utilisant des récurrences [And67b, And68b, And71b].

Comme mentionné précédemment, le théorème de Schur a été généralisé aux surpartitions, le théorème de Schur correspondant au cas $k = 0$ dans le théorème 1.11. Cependant le théorème de Lovejoy n'avait que deux preuves. Il a été découvert en utilisant la méthode des mots pondérés [Lov05b] et ensuite prouvé bijectivement [RP09]. Il n'était pas clair si les preuves d'Andrews du théorème de Schur utilisant des récurrences pouvaient être adaptées pour prouver le théorème 1.11.

Dans le chapitre 4, nous répondons à cette question en donnant trois nouvelles preuves du théorème 1.11 utilisant des récurrences, basées sur les trois preuves d'Andrews [And67b, And68b, And71b]. Cependant les équations et les techniques utilisées pour les résoudre sont différentes et plus complexes. Ces preuves sont présentées dans le papier [Dou14b]. Donnons cependant d'ores et déjà l'idée de ces preuves.

Première preuve La première preuve utilise des récurrences obtenues par un raisonnement combinatoire basé sur la plus petite part des surpartitions. La série génératrice des surpartitions comptées par $A(k,n)$ est facile à calculer et égale

$$\sum_{k,n\geq 0} A(k,n)d^k q^n = \prod_{n=0}^{\infty} \frac{(1+q^{3n+1})(1+q^{3n+2})}{(1-dq^{3n+1})(1-dq^{3n+2})}.$$

Notre but est de prouver que la série génératrice des surpartitions comptées par $B(k,n)$ est la même, mais elle n'est pas aussi facile à calculer et nous devons utiliser des récurrences et des équations aux q-différences.

Soit $b_j(k,m,n)$ le nombre de surpartitions comptées par $B(k,n)$ ayant m parts, telles que la plus petite part est strictement supérieure à j. Nous obtenons d'abord des équations de récurrence telles que

$$b_0(k,m,n) - b_1(k,m,n) = b_0(k,m-1,n-3m+2) + b_0(k-1,m-1,n-1).$$

Pour ce faire, nous remarquons que $b_0(k,m,n) - b_1(k,m,n)$ est le nombre de surpartitions comptées par $b_0(k,m,n)$ ayant leur plus petite part égale à 1. Ensuite, nous enlevons la plus petite part, et nous en déduisons une condition sur la deuxième plus petite part en utilisant les conditions de différence et le fait que la plus petite part était surlignée ou non.

Ensuite, nous définissons

$$f_i(x) = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} b_i(k,m,n)x^m d^k q^n,$$

et nous traduisons ces récurrences en équations aux q-différences pour les fonctions f_i, et après quelques substitutions, nous obtenons une équation aux q-différences pour f_0 uniquement :

$$\begin{aligned}(1-dxq)(1-dxq^2)f_0(x) &= (1+xq+xq^2+dxq^3-dx^2q^3-dx^2q^6)f_0(xq^3)\\ &\quad + xq^3(1-xq^3)f_0(xq^6).\end{aligned}$$

Notre but est de trouver $f_0(1)$, qui est exactement la série génératrice des surpartitions comptées par $B(k,n)$. Si $d=0$, ce qui correspond au cas des partitions, l'équation peut être résolue assez facilement. Ici la présence de la variable d rend l'équation beaucoup plus compliquée. Pour obtenir $f_0(1)$, nous définissons $F(x) := f_0(x)\prod_{k=0}^{\infty}\frac{(1-dxq^{3k+1})}{(1-xq^{3k})}$, déduisons l'équation aux q-différences vérifiée par F, et la traduisons en une équation de récurrence pour (A_n), où $F(x) = \sum_{n=0}^{\infty} A_n x^n$. Plus précisément,

$$\begin{aligned}(1-q^{3n})A_n &= (1+dq^2+q^{3n-1})(1+q^{3n-2})A_{n-1}\\ &\quad - dq^2(1+q^{3n-2})(1+q^{3n-5})A_{n-2}.\end{aligned}$$

Cela nous conduit à définir $a_n = A_n\prod_{k=0}^{n-1}\frac{1}{1+q^{3k+1}}$ pour simplifier l'équation, et en définissant $f(x) = \sum_{n=0}^{\infty} a_n x^n$, nous obtenons

$$(1-x)(1-dxq^2)f(x) = (1+xq^2)f(xq^3).$$

Une itération de l'équation précédente conduit à

$$f(x) = \prod_{k=0}^{\infty}\frac{(1+xq^{3k+2})}{(1-xq^{3k})(1-dxq^{3k+2})}.$$

Finalement, nous utilisons le théorème de comparaison d'Appell [Die57, p. 101], stipulant que

$$\lim_{x\to 1^-}(1-x)\sum_{n\geq 0}u_nx^n = \lim_{n\to\infty} u_n,$$

pour retrouver $f_0(1)$. Nous avons

$$\begin{aligned}\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\frac{A_nx^n}{\prod_{k=0}^{n-1}(1+q^{3k+1})} &= \frac{\lim_{n\to\infty}A_n}{\prod_{k=0}^{\infty}(1+q^{3k+1})}\\ &= \prod_{k=0}^{\infty}\frac{(1+q^{3k+2})}{(1-q^{3k+3})(1-dq^{3k+2})}.\end{aligned}$$

Ainsi

$$\lim_{n\to\infty}A_n = \prod_{k=0}^{\infty}\frac{(1+q^{3k+2})(1+q^{3k+1})}{(1-q^{3k+3})(1-dq^{3k+2})},$$

et

$$f_0(x) = (1-x)\prod_{k=0}^{\infty}\frac{(1-xq^{3k+3})}{(1-dxq^{3k+1})}\sum_{n=0}^{\infty}A_nx^n.$$

Nous appliquons le théorème de comparaison d'Appell une nouvelle fois et obtenons

$$f_0(1) = \prod_{k=0}^{\infty}\frac{(1+q^{3k+1})(1+q^{3k+2})}{(1-dq^{3k+1})(1-dq^{3k+2})},$$

ce qui conclut la preuve.

Deuxième preuve La deuxième preuve est assez similaire à la première, hormis le fait qu'elle repose sur des récurrences basées sur la plus grande part des surpartitions.

Soit $\psi_m(k,n)$ le nombre de surpartitions comptées par $B(k,n)$ telles que la plus grande part est inférieure ou égale à m. Par un raisonnement combinatoire dans lequel nous enlevons la plus grande part de la surpartition, nous obtenons des récurrences telles que

$$\psi_{3m+1}(k,n) = \psi_{3m}(k,n) + \psi_{3m+1}(k-1,n-3m-1) + \psi_{3m-2}(k,n-3m-1).$$

Ensuite nous définissons

$$a_m(q,d) = 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\psi_m(k,n)q^nd^k,$$

et en faisant des allers-retours entre récurrences et équations aux q-différences d'une manière similaire à la première preuve et en finissant aussi avec le théorème de comparaison d'Appell, nous montrons que $\lim_{m\to\infty} a_m(q,d)$, qui est la série génératrice de toutes les surpartitions comptées par $B(k,n)$, est bien égale à

$$\prod_{n=0}^{\infty} \frac{(1+q^{3n+1})(1+q^{3n+2})}{(1-dq^{3n+1})(1-dq^{3n+2})}.$$

Troisième preuve Cette preuve utilise aussi des récurrences basées sur la plus grande part des surpartitions, mais nous ajoutons une variable supplémentaire qui compte le nombre de parts congrues à 1 ou 2 modulo 3 plus deux fois le nombre de parts congrues à 0 modulo 3. Cela permet de prouver le théorème sans faire des allers-retours entre récurrences et équations aux q-différences ou utiliser le théorème de comparaison d'Appell, mais cela rend les équations légèrement plus compliquées. On peut cependant dire qu'il s'agit de la preuve la plus élémentaire du théorème de Schur pour les surpartitions.

Soit $\psi_m(M,n,k)$ le nombre de surpartitions de n avec k parts non surlignées, telles que M égale le nombre de parts congrues à 1 ou 2 modulo 3 plus deux fois le nombre de parts congrues à 0 modulo 3, vérifiant les conditions de différences, et telles que la plus grande part est inférieure ou égale à m. En utilisant un raisonnement similaire à celui de la deuxième preuve, nous obtenons des équations de récurrence telles que

$$\begin{aligned}\psi_{3m+1}(M,k,n) = \psi_{3m}(M,k,n) &+ \psi_{3m+1}(M-1,k-1,n-3m-1)\\ &+ \psi_{3m-2}(M-1,k,n-3m-1).\end{aligned}$$

Définissons

$$a_m(q,d,t) := 1 + \sum_{M=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \psi_m(M,n,k) q^n d^k t^M.$$

Nous cherchons à déterminer $\lim_{m\to\infty} a_m(q,d,t)$, la série génératrice des surpartitions comptées par $B(k,n)$ ayant M parts (les parts divisibles par 3 étant comptées deux fois). Nous remarquons que $(a_{3m+3}(q,d,t))_{m\in\mathbb{N}}$ vérifie la même équation de récurrence que $(a_{3m-1}(q,d,tq^3))_{m\in\mathbb{N}}$. En utilisant les conditions initiales nous déduisons que

$$a_{3m+3}(q,d,t) = \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} a_{3m-1}(q,d,tq^3).$$

En faisant tendre m vers l'infini et en itérant, nous obtenons

$$\lim_{m\to\infty} a_m(q,d,t) = \prod_{n=0}^{\infty} \frac{(1+tq^{3n+1})(1+tq^{3n+2})}{(1-dtq^{3n+1})(1-dtq^{3n+2})}, \tag{1.1}$$

et le théorème est prouvé.

1.2.1.2. Les théorèmes d'Andrews

Dans la première partie de cette introduction, nous avons mentionné deux identités de partitions dues à Andrews [And69a, And68a] qui généralisent le théorème de Schur, mais nous avons énoncé seulement deux cas particuliers (théorèmes 1.6 et 1.7). Introduisons maintenant quelques notations nécessaires pour énoncer les théorèmes en toute généralité.

Soit $A = \{a(1), \dots, a(r)\}$ un ensemble de r entiers distincts tel que pour tout $1 \leq k \leq r$,

$$\sum_{i=1}^{k-1} a(i) < a(k),$$

et tel quel les $2^r - 1$ sommes possibles d'éléments de A soient toutes distinctes. Nous appelons cette ensemble de sommes $A' = \{\alpha(1), \dots, \alpha(2^r-1)\}$, où $\alpha(1) < \cdots < \alpha(2^r-1)$. Remarquons que $\alpha(2^k) = a(k+1)$ pour tout $0 \leq k \leq r-1$ et que chaque α entre $a(k)$ et $a(k+1)$ a $a(k)$ comme plus grand terme. Soit N un entier positif tel que $N \geq \alpha(2^r-1) = a(1) + \cdots + a(r)$. Définissons aussi $\alpha(2^r) = a(r+1) = N + a(1)$. Soit A_N l'ensemble des entiers positifs congrus à l'un des $a(i)$ modulo N, $-A_N$ l'ensemble des entiers positifs congrus à un $-a(i)$ modulo N, A'_N l'ensemble des entiers positifs congrus à l'un des $\alpha(i)$ modulo N, et $-A'_N$ l'ensemble des entiers positifs congrus à un des $-\alpha(i)$ modulo N. Soit $\beta_N(m)$ le reste de la division euclidienne de m par N. Pour $\alpha \in A'$, soit $w(\alpha)$ le nombre de termes apparaissant dans la somme qui définit α et $v(\alpha)$ le plus petit $a(i)$ apparaissant dans cette somme.

Pour mieux comprendre ces notations, nous invitons le lecteur à considérer l'exemple où $a(k) = 2^{k-1}$ pour $1 \leq k \leq r$ et $\alpha(k) = k$ pour $1 \leq k \leq 2^r - 1$.

Nous pouvons maintenant énoncer les généralisations du théorème de Schur dues à Andrews.

Théorème 1.13 (Andrews)**.** *Soit $D(A_N; n)$ le nombre de partitions de n en parts distinctes appartenant à A_N. Soit $E(A'_N; n)$ le nombre de partitions de n en parts appartenant à A'_N, de la forme $n = \lambda_1 + \cdots + \lambda_s$ telles que*

$$\lambda_i - \lambda_{i+1} \geq N w(\beta_N(\lambda_{i+1})) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Alors pour tout $n \geq 0$, $D(A_N; n) = E(A'_N; n)$.

Théorème 1.14 (Andrews)**.** *Soit $F(-A_N;n)$ le nombre de partitions de n en parts distinctes appartenant à $-A_N$. Soit $G(-A'_N;n)$ le nombre de partitions de n en parts appartenant à $-A'_N$, de la forme $n = \lambda_1 + \cdots + \lambda_s$, telles que*

$$\lambda_i - \lambda_{i+1} \geq Nw(\beta_N(-\lambda_i)) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

et $\lambda_s \geq N(w(\beta_N(-\lambda_s) - 1)$. Alors pour tout $n \geq 0$, $F(-A_N;n) = G(-A'_N;n)$.

Andrews a prouvé le théorème 1.13 [And69a] en étendant sa preuve du théorème de Schur basée sur la plus petite part des partitions [And68b] et le théorème 1.14 [And68a] en étendant sa preuve du théorème de Schur basée sur la plus grande part des partitions [And67b].

Comme le théorème de Schur a été généralisé avec succès aux surpartitions, il était naturel de se demander s'il est aussi possible d'étendre les théorèmes d'Andrews en toute généralité aux surpartitions. Nous répondons à cette question en prouvant les généralisations suivantes.

Théorème 1.15. *Soit $D(A_N;k,n)$ le nombre de surpartitions de n en parts appartenant à A_N, ayant k parts non surlignées. Soit $E(A'_N;k,n)$ le nombre de surpartitions de n en parts appartenant à A'_N de la forme $n = \lambda_1 + \cdots + \lambda_s$, ayant k parts non surlignées, telles que*

$$\lambda_i - \lambda_{i+1} \geq N\left(w\left(\beta_N(\lambda_{i+1})\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}),$$

où $\chi(\overline{\lambda_{i+1}}) = 1$ si λ_{i+1} est surligné et 0 sinon. Alors pour tous $k,n \geq 0$, $D(A_N;k,n) = E(A'_N;k,n)$.

Théorème 1.16. *Soit $F(-A_N;k,n)$ le nombre de surpartitions de n en parts appartenant à $-A_N$, ayant k parts non surlignées. Soit $G(-A'_N;k,n)$ le nombre de surpartitions de n en parts appartenant à $-A'_N$ de la forme $n = \lambda_1 + \cdots + \lambda_s$, ayant k parts non surlignées, telles que*

$$\lambda_i - \lambda_i \geq N\left(w\left(\beta_N(\lambda_i)\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + v(\beta_N(\lambda_i)) - \beta_N(\lambda_i),$$

$$\lambda_s \geq N(w(\beta_N(-\lambda_s)) - 1).$$

Alors pour tous $k,n \geq 0$, $F(-A_N;k,n) = G(-A'_N;k,n)$.

Le théorème de Schur (resp. le théorème de Schur pour les surpartitions) correspond à $N = 3$, $r = 2$, $a(1) = 1$, $a(2) = 2$ dans les théorèmes 1.13 et 1.14 (resp. les théorèmes 1.15 et 1.16). De nouveau, le cas $k = 0$ du théorème 1.15 (resp. du théorème 1.16) donne le théorème 1.13 (resp. le théorème 1.14).

Illustrons les théorèmes 1.15 et 1.16 sur des exemples dans le cas $N = 7$, $r = 3$, $a(1) = 1$, $a(2) = 2$, $a(3) = 4$. Pour le théorème 1.15, les surpartitions de

4 comptées par $E(A'_7;k,4)$ sont 4, $\overline{4}$, $3+1$, $\overline{3}+1$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $1+1+1+1$ et $\overline{1}+1+1+1$. Les surpartitions de 4 en parts congrues à 1,2 ou 4 modulo 7 (comptées par $D(A_7;k,4)$) sont 4, $\overline{4}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$ et $\overline{1}+1+1+1$. Dans les deux cas, nous avons une surpartition avec 0 parts non surlignées, trois surpartitions avec 1 part non surlignée, trois surpartitions avec 2 parts non surlignées, deux surpartitions avec 3 parts non surlignées et une surpartition avec 4 parts non surlignées. Pour le théorème 2.16, les surpartitions de 8 comptées par $G(-A'_7;k,8)$ sont 8, $\overline{8}$, $5+3$ et $\overline{5}+3$. Les surpartitions de 8 en parts congrues à 3, 5 ou 6 modulo 7 (comptées par $F(-A_7;k,8)$) sont $5+3$, $\overline{5}+3$, $5+\overline{3}$ et $\overline{5}+\overline{3}$. Dans les deux cas, nous avons une surpartition avec 0 parts non surlignées, deux surpartitions avec 1 part non surlignée et une surpartition avec 2 parts non surlignées.

Bien que les énoncés des théorèmes 1.15 et 1.16 ressemblent à ceux des théorèmes d'Andrews, les preuves sont beaucoup plus complexes et utilisent plusieurs nouvelles idées. Le théorème 1.15 a été prouvé dans le papier [Douar] et le théorème 1.16 dans [Dou15]. Nous présentons à la fois les preuves d'Andrews et les nôtres dans les chapitres 5 et 6, mais donnons déjà les idées générales de nos preuves ici.

Pour prouver le théorème 1.15, nous établissons d'abord l'équation aux q-différences satisfaite par la série génératrice des surpartitions comptées par $E(A'_N;k,n)$, grâce à un raisonnement combinatoire sur la plus petite part des surpartitions. Ensuite nous prouvons par induction sur r qu'une fonction satisfaisant cette équation aux q-différences est égale à

$$\prod_{k=1}^{r} \frac{(-q^{a(k)};q^N)_\infty}{(dq^{a(k)};q^N)_\infty},$$

qui est la série génératrice des surpartitions comptées par $D(A_N;k,n)$. Pour ce faire, nous faisons des allers-retours entre équations aux q-différences pour les séries génératrices et équations de récurrence pour leurs coefficients, afin de faire baisser le degré de l'équation aux q-différences d'un, un peu comme dans la première preuve du théorème de Schur pour les surpartitions.

La preuve du théorème 1.16 est relativement similaire. D'abord nous donnons l'équation de récurrence vérifiée par la série génératrice des surpartitions comptées par $G(-A'_N;k,n)$ dont la plus grande part est $\leq m$, en utilisant un raisonnement combinatoire sur la plus grande part des surpartitions. Puis nous prouvons par induction sur r que la limite lorsque m tend vers l'infini d'une fonction satisfaisant cette équation de récurrence est égale à

$$\prod_{j=1}^{r} \frac{(-q^{N-a(j)};q^N)_\infty}{(dq^{N-a(j)};q^N)_\infty},$$

qui est la série génératrice des surpartitions comptées par $F(-A_N; k, n)$.

1.2.1.3. L'identité de Siladić

Siladić [Sil] a prouvé le théorème suivant en étudiant les représentations de l'algèbre de Lie affine tordue $A_2^{(2)}$.

Théorème 1.17 (Siladić)**.** *Le nombre de partitions $\lambda_1 + \cdots + \lambda_s$ d'un entier n en parts différentes de 2 telles que la différence entre deux parts consécutives est au moins 5 (ie. $\lambda_i - \lambda_{i+1} \geq 5$) et*

$$\begin{aligned}
\lambda_i - \lambda_{i+1} = 5 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \mod 16,\\
\lambda_i - \lambda_{i+1} = 6 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 2, \pm 6 \mod 16,\\
\lambda_i - \lambda_{i+1} = 7 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 3 \mod 16,\\
\lambda_i - \lambda_{i+1} = 8 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 4 \mod 16,
\end{aligned}$$

est égal au nombre de partitions de n en parts distinctes impaires.

Dans le chapitre 7, nous donnons une preuve combinatoire et un raffinement de l'identité de Siladić. Elle a été présentée dans le papier [Dou14a]. Notre raffinement du théorème 1.17 est le suivant :

Théorème 1.18. *Pour $n \in \mathbb{N}$ et $k \in \mathbb{N}^*$, soit $A(k, n)$ le nombre de partitions $\lambda_1 + \cdots + \lambda_s$ de n telles que k égale le nombre de parts impaires plus deux fois le nombre de parts paires, satisfaisant les conditions suivantes :*

1. $\forall i \geq 1, \lambda_i \neq 2$,
2. $\forall i \geq 1, \lambda_i - \lambda_{i+1} \geq 5$,
3. $\forall i \geq 1$,

$$\begin{aligned}
\lambda_i - \lambda_{i+1} = 5 &\Rightarrow \lambda_i \equiv 1, 4 \mod 8,\\
\lambda_i - \lambda_{i+1} = 6 &\Rightarrow \lambda_i \equiv 1, 3, 5, 7 \mod 8,\\
\lambda_i - \lambda_{i+1} = 7 &\Rightarrow \lambda_i \equiv 0, 1, 3, 4, 6, 7 \mod 8,\\
\lambda_i - \lambda_{i+1} = 8 &\Rightarrow \lambda_i \equiv 0, 1, 3, 4, 5, 7 \mod 8.
\end{aligned}$$

Pour $n \in \mathbb{N}$ et $k \in \mathbb{N}^*$, soit $B(k, n)$ le nombre de partitions de n en k parts distinctes impaires. Alors pour tous $n \in \mathbb{N}$ et $k \in \mathbb{N}^*$, $A(k, n) = B(k, n)$.

Comme dans le théorème de Schur, la série génératrice des partitions comptées par $B(k, n)$ est facile à calculer et égale

$$\sum_{k,n \geq 0} B(k, n) t^k q^n = \prod_{k \geq 0} \left(1 + tq^{2k+1}\right),$$

mais la série génératrice des partitions comptées par $A(k,n)$ est plus difficile à déterminer.

Pour $N \in \mathbb{N}$, $k,n \in \mathbb{N}^*$, soit $a_N(k,n)$ le nombre de partitions comptées par $A(k,n)$ dont la plus grande part est inférieure ou égale à N. Définissons

$$G_N(t,q) := 1 + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_N(k,n) t^k q^n.$$

Ainsi $G_\infty(t,q) = \lim_{N\to\infty} G_N(t,q)$ est la série génératrice des partitions comptées par $A(k,n)$.

Notre stratégie est de prouver le théorème suivant

Théorème 1.19. *Pour tout* $m \in \mathbb{N}^*$,

$$G_{2m}(t,q) = (1+tq)G_{2m-3}(tq^2,q).$$

En effet nous pouvons ensuite faire tendre m vers l'infini et déduire que

$$\begin{aligned} G_\infty(t,q) =& (1+tq)G_\infty(tq^2,q) = (1+tq)(1+tq^3)G_\infty(tq^4,q) = \cdots \\ =& \prod_{k\geq 0}\left(1+tq^{2k+1}\right). \end{aligned}$$

Pour y parvenir, nous utilisons la méthode suivante. Dans la section 7.2.1, nous donnons d'abord une formulation équivalente du théorème 1.17 qui est plus facile à manipuler en termes de partitions. Puis, dans la section 7.2.2, nous établissons des équations aux q-différences vérifiées par les séries génératrices des partitions considérées dans le théorème 1.17. Enfin dans la section 7.2.3, nous utilisons ces équations aux q-différences pour prouver le théorème 1.19 par induction.

1.2.2. Asymptotique et méthode du cercle à deux variables

Dans la partie III, nous étudions des aspects asymptotiques de la théorie des partitions. D'abord, dans le chapitre 8, nous expliquons le principe de la méthode du cercle de Hardy-Ramanujan-Rademacher [HR18b, Rad37] pour calculer la formule exacte pour $p(n)$, et dans le chapitre 9 nous appliquons la méthode du cercle de Wright [Wri33], qui donne seulement un équivalent asymptotique mais est beaucoup plus simple, pour calculer un équivalent de $p(n)$ lorsque n tend vers l'infini. Ensuite, dans le chapitre 10, nous appliquons la méthode du cercle de Wright pour donner un équivalent asymptotique de deux quantités liées aux surpartitions avec différences impaires restreintes. Finalement, dans le chapitre 11, nous présentons une nouvelle généralisation de la méthode du cercle de Wright aux formes de Jacobi et aux formes mock Jacobi, que nous appelons la méthode du cercle à deux variables. Dans le chapitre 12, nous utilisons cette méthode pour calculer un équivalent asymptotique de $M(m,n)$, ce qui résout la conjecture de Dyson, et dans le chapitre 13 nous l'utilisons pour trouver un équivalent asymptotique de $N(m,n)$.

1.2.2.1. Surpartitions avec différences impaires restreintes

Dans le chapitre 10, nous étudions les surpartitions telles que la différence entre deux parts consécutives ne peut être impaire que si la plus grande de ces deux parts est surlignée, et utilisons des équations aux q-différences pour calculer leur série génératrice sous forme de q-série hypergéométrique à deux variables. Cette série génératrice se spécialise dans un cas en forme modulaire et dans un autre cas en forme mock modulaire mixte. Nous considérons aussi la série génératrice à deux variables des mêmes surpartitions avec plus petite part impaire, et à nouveau nous trouvons des spécialisations modulaires et mock modulaires mixtes. Ces propriétés de modularité nous permettent de calculer des équivalents asymptotiques en utilisant la méthode du cercle de Wright. Ces résultats sont présentés dans le papier [BDLM15].

Soit $\overline{t}(n)$ le nombre de surpartitions telles que la différence entre deux parts consécutives peut être impaire seulement si la plus grande des deux parts est surlignée, et si la plus petite part de la surpartition est impaire alors elle est surlignée. Soit $\overline{s}(n)$ le nombre de partitions comptées par $\overline{t}(n)$ avec plus petite part impaire. Ainsi $\overline{t}(4) = 8$ et $\overline{s}(4) = 4$, les huit surpartitions comptées par $\overline{t}(4)$ étant

$$4,\ \overline{4},\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ 2+\overline{2},\ \overline{2}+1+\overline{1},\ 1+1+1+\overline{1}$$

et les quatre surpartitions comptées par $\overline{s}(4)$ étant

$$3+\overline{1},\ \overline{3}+\overline{1},\ \overline{2}+1+\overline{1},\ 1+1+1+\overline{1}.$$

D'abord, nous déterminons les séries génératrices q-hypergéométriques de $\overline{t}(m,n)$ (resp. $\overline{s}(m,n)$), le nombre de surpartitions comptées par $\overline{t}(n)$ (resp. $\overline{s}(n)$) avec m parts.

Théorème 1.20. *Nous avons les identités suivantes :*

$$\sum_{m,n\geq 0} \bar{t}(m,n)x^m q^n = \frac{(-xq)_\infty}{(xq)_\infty}\left(1+\sum_{n\geq 1}\frac{(-q^3;q^3)_{n-1}(-x)^n q^n}{(-q)_{n-1}(q^2;q^2)_n}\right), \tag{1.2}$$

$$\sum_{m,n\geq 1} \bar{s}(m,n)x^m q^n = \sum_{n\geq 1}\frac{(q^3;q^3)_{n-1}x^n q^n}{(q)_{n-1}(q^2;q^2)_n}. \tag{1.3}$$

La seconde identité est obtenue directement par un argument combinatoire, mais la première est plus subtile et la preuve repose sur le fait que les deux membres de l'égalité satisfont une certaine équation aux q-différences.

Lorsque $x = 1$ dans (1.2) ou -1 dans (1.3), alors nous avons une forme modulaire, et lorsque $x = -1$ dans (1.2) ou 1 dans (1.3), alors nous avons le produit d'une forme modulaire et d'une fonction mock theta, appelé une forme mock modulaire mixte (voir [LO13]). On définit les fonctions mock theta [BL07] $\overline{\gamma}(q)$ et $\overline{\chi}(q)$ par

$$\overline{\gamma}(q) := \sum_{n\geq 0}\frac{(-1;q)_n(q;q)_n q^{\binom{n+1}{2}}}{(q^3;q^3)_n}$$

et

$$\overline{\chi}(q) := \sum_{n\geq 0}\frac{(-1;q)_n(-q;q)_n q^{\binom{n+1}{2}}}{(-q^3;q^3)_n}.$$

Soient $\bar{t}_+(n)$ (resp. $\bar{s}_+(n)$) le nombre de surpartitions comptées par $\bar{t}(n)$ (resp. $\bar{s}(n)$) avec plus grande part paire, et $\bar{t}_-(n)$ (resp. $\bar{s}_-(n)$) le nombre de surpartitions comptées par $\bar{t}(n)$ (resp. $\bar{s}(n)$) avec plus grande part impaire.

Corollaire 1.21. *Nous avons*

$$\begin{aligned}
\sum_{n\geq 0}\bar{t}(n)q^n &= \frac{\left(q^3;q^3\right)_\infty}{(q;q)_\infty\left(q^2;q^2\right)_\infty},\\
\sum_{n\geq 0}\left(\bar{t}_+(n)-\bar{t}_-(n)\right)q^n &= \frac{\left(-q^3;q^3\right)_\infty}{(-q;q)_\infty^3}\overline{\chi}(q),\\
1+3\sum_{n\geq 1}\left(\bar{s}_+(n)-\bar{s}_-(n)\right)q^n &= \frac{\left(-q^3;q^3\right)_\infty}{(-q;q)_\infty^3},\\
1+3\sum_{n\geq 1}\bar{s}(n)q^n &= \frac{\left(q^3;q^3\right)_\infty}{(q;q)_\infty\left(q^2;q^2\right)_\infty}\overline{\gamma}(q).
\end{aligned}$$

Ensuite nous utilisons ces séries génératrices et la méthode du cercle de Wright pour déduire des équivalents asymptotiques de $\overline{s}(n)$ and $\overline{t}_+(n) - \overline{t}_-(n)$, qui sont les cas du corollaire 1.21 dans lesquels les séries génératrices sont des formes mock modulaires mixtes.

Théorème 1.22. *Lorsque $n \to \infty$, nous avons*

$$\overline{s}(n) \sim \frac{\sqrt{21}}{36n} e^{\frac{\pi\sqrt{7n}}{3}}, \tag{1.4}$$

$$\overline{t}_+(n) - \overline{t}_-(n) \sim (-1)^n \frac{\sqrt{3}}{18n^{-\frac{3}{4}}} e^{\frac{2\pi\sqrt{n}}{3}}. \tag{1.5}$$

Remarquons que si dans la plupart des exemples dans la littérature, le pôle dominant dans la méthode du cercle est $q = 1$, dans le cas de (1.5) il s'agit de $q = -1$.

Le chapitre 10 est organisé de la manière suivante. Dans la section 10.2 nous prouvons le théorème 1.20 et le corollaire 1.21 en utilisant des arguments analytiques et combinatoires. Puis dans la section 10.3 nous prouvons le théorème 1.22 en utilisant la méthode du cercle de Wright.

1.2.2.2. La méthode du cercle à deux variables

Dans le chapitre 11, nous présentons une généralisation à deux variables de la méthode du cercle de Wright.

Les calculs d'asymptotique à deux variables sont généralement beaucoup plus difficiles que ceux à une variable. Il n'est donc pas surprenant que la conjecture de Dyson (conjecture 1.10) soit restée ouverte depuis 1989. Dans cette thèse, nous donnons une nouvelle méthode qui permet de calculer un équivalent asymptotique à deux variables des coefficients de Fourier des formes de Jacobi (et des formes mock Jacobi) et ainsi de résoudre la conjecture de Dyson : la méthode du cercle à deux variables.

La conjecture de Dyson Dans le chapitre 12, nous prouvons que la conjecture de Dyson est vraie. Cela fait l'objet du papier [BDar].

Théorème 1.23. *La conjecture de Dyson est vraie. Plus précisément, si $|m| \leq \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$, lorsque $n \to \infty$ nous avons*

$$M(m,n) = \frac{\beta}{4}\mathrm{sech}^2\left(\frac{\beta m}{2}\right) p(n) \left(1 + O\left(\beta^{\frac{1}{2}} |m|^{\frac{1}{3}}\right)\right),$$

avec $\beta := \frac{\pi}{\sqrt{6n}}$.

Un calcul direct montre le corollaire suivant.

Corollaire 1.24. *Presque toutes les partitions vérifient la conjecture de Dyson. Plus précisément*

$$\sharp\left\{\lambda \vdash n | \mathrm{crank}(\lambda)| \leq \frac{\sqrt{n}}{\pi\sqrt{6}} \log n\right\} \sim p(n).$$

En fait, la conjecture de Dyson découle d'un résultat plus général concernant les coefficients $M_k(m,n)$ définis pour $k \in \mathbb{N}$ par

$$\mathcal{C}_k(\zeta;q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_k(m,n)\,\zeta^m q^n := \frac{(q)_\infty^{2-k}}{(\zeta q)_\infty (\zeta^{-1}q)_\infty}.$$

Remarquons que $M(m,n) = M_1(m,n)$. En notant $p_k(n)$ le nombre de partitions de n en k couleurs, nous avons

Théorème 1.25. *Pour k fixé et $|m| \leq \frac{1}{6\beta_k} \log n$, nous avons lorsque $n \to \infty$*

$$M_k(m,n) = \frac{\beta_k}{4} \operatorname{sech}^2\left(\frac{\beta_k m}{2}\right) p_k(n) \left(1 + O\left(\beta_k^{\frac{1}{2}} |m|^{\frac{1}{3}}\right)\right),$$

avec $\beta_k := \pi\sqrt{\frac{k}{6n}}$.

Le chapitre 12 est organisé comme suit. Dans la section 12.2, nous rappelons des faits de base sur les formes modulaires et formes de Jacobi qui sont les composantes de base de $\mathcal{C}_k$ et énonçons des propriétés des polynômes d'Euler. Dans la section 12.3, nous déterminons le comportement asymptotique de $\mathcal{C}_k$. Dans la section 12.4, nous utilisons la méthode du cercle à deux variables pour finir la preuve du théorème 1.25. Dans la section 12.5, nous illustrons le théorème 1.23 numériquement.

Asymptotique du rang Dans le chapitre 13, nous montrons que la méthode du cercle à deux variables permet aussi de prouver que la même formule asymptotique que celle de la conjecture de Dyson est également vérifiée par le rang. Cela a donné lieu au papier [DMar]. La situation du rang est plus compliquée que celle du crank, car sa série génératrice n'est pas une forme de Jacobi mais une forme mock Jacobi, ce qui signifie qu'il existe une fonction non holomorphe telle que la somme de la série génératrice du rang et de cette fonction a des propriétés de modularité. Cependant la méthode du cercle à deux variables fonctionne quand même, bien que certains calculs deviennent plus compliqués, et nous prouvons le théorème suivant.

Théorème 1.26. *Si* $|m| \leq \frac{\sqrt{n}\log n}{\pi\sqrt{6}}$, *nous avons lorsque* $n \to \infty$

$$N(m,n) = \frac{\beta}{4}\mathrm{sech}^2\left(\frac{\beta m}{2}\right) p(n)\left(1 + O\left(\beta^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right).$$

Le chapitre 13 est organisé comme suit. Dans la section 13.2, nous prouvons quelques estimations préliminaires pour la série génératrice du rang en utilisant les propriétés de transformation des formes mock Jacobi. Dans la section 13.3, nous utilisons ces résultats pour estimer la série génératrice proche et loin du pôle dominant, et dans la section 13.4, nous appliquons la méthode du cercle à deux variables pour établir le théorème 1.26.

1.2.3. Une extension des coefficients q-binomiaux

Dans le chapitre 14, nous étudions un analogue pour les surpartitions des coefficients q-binomiaux

$$\begin{bmatrix} M+N \\ N \end{bmatrix}_q = \frac{(q)_{M+N}}{(q)_M(q)_N}.$$

Ces polynômes jouent un rôle important en combinatoire et en théorie des nombres. Ils sont la série génératrice du nombre d'inversions dans les permutations d'un multiensemble, du nombre de sous-espaces de dimension N des espaces vectoriels de dimension $M+N$ sur $\mathbb{F}_q$ et du nombre de partitions avec des restrictions sur la plus grande part et le nombre de parts. Ils sont aussi des q-analogues des coefficients binomiaux classiques, ce qui signifie que si l'on pose $q = 1$ dans la définition des coefficients q-binomiaux, nous obtenons les coefficients binomiaux classiques.

Comme $\begin{bmatrix} M+N \\ N \end{bmatrix}_q$ est la série génératrice du nombre de partitions qui rentrent dans un rectangle de taille $M \times N$, c'est à dire avec plus grande part $\leq M$ et nombre de parts $\leq N$ (voir par exemple [And84]), nous définissons notre analogue pour les surpartitions des coefficients q-binomiaux, que nous appellerons *coefficients sur-q-binomiaux*, comme la série génératrice du nombre de surpartitions qui rentrent dans un rectangle de taille $M \times N$.

Notre premier résultat est une formule exacte pour les coefficients sur-q-binomiaux $\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q$.

Théorème 1.27. *Pour tous* M *et* N *entiers positifs,*

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q = \sum_{k=0}^{\min\{M,N\}} q^{\frac{k(k+1)}{2}} \frac{(q)_{M+N-k}}{(q)_k(q)_{M-k}(q)_{N-k}}.$$

Par exemple, d'après le théorème 1.27, nous trouvons que

$$\overline{\begin{bmatrix}6\\3\end{bmatrix}} = 1 + 2q + 4q^2 + 8q^3 + 10q^4 + 12q^5 + 12q^6 + 8q^7 + 4q^8 + 2q^9,$$

et nous pouvons vérifier que les douze surpartitions de 5 qui rentrent dans un rectangle de taille 3×3 sont les suivantes

$$3+2, \overline{3}+2, 3+\overline{2}, \overline{3}+\overline{2}, 3+1+1, \overline{3}+1+1, 3+1+\overline{1}$$
$$\overline{3}+1+\overline{1}, 2+2+1, 2+\overline{2}+1, 2+2+\overline{1}, 2+\overline{2}+\overline{1}.$$

Tout comme les coefficients q-binomiaux vérifient des récurrences simples, qui sont des q-analogues du triangle de Pascal,

$$\begin{aligned}\begin{bmatrix}M+N\\N\end{bmatrix} &= \begin{bmatrix}M+N-1\\N\end{bmatrix} + q^M \begin{bmatrix}M+N-1\\N-1\end{bmatrix},\\ \begin{bmatrix}M+N\\N\end{bmatrix} &= \begin{bmatrix}M+N-1\\N-1\end{bmatrix} + q^N \begin{bmatrix}M+N-1\\N\end{bmatrix},\end{aligned}$$

les coefficients sur-q-binomiaux vérifient aussi des récurrences similaires.

Théorème 1.28. *Pour tous M et N entiers positifs, nous avons*

$$\overline{\begin{bmatrix}M+N\\N\end{bmatrix}} = \overline{\begin{bmatrix}M+N-1\\N-1\end{bmatrix}} + q^N \overline{\begin{bmatrix}M+N-1\\N\end{bmatrix}} + q^N \overline{\begin{bmatrix}M+N-2\\N-1\end{bmatrix}},$$

$$\overline{\begin{bmatrix}M+N\\N\end{bmatrix}} = \overline{\begin{bmatrix}M+N-1\\N\end{bmatrix}} + q^M \overline{\begin{bmatrix}M+N-1\\N-1\end{bmatrix}} + q^M \overline{\begin{bmatrix}M+N-2\\N-1\end{bmatrix}}.$$

En utilisant les coefficients sur-q-binomiaux, nous établissons plusieurs identités. Les identités de Rogers-Ramanujan, introduites dans la section 1.1.3, ont été prouvées par différentes méthodes. Parmi celles-ci, l'une des plus belles et élémentaires est une preuve due à Andrews qui utilise des relations de récurrence vérifiées par les coefficients q-binomiaux [And88, And89]. Inspirés par cette preuve, nous prouvons une identité du type Rogers-Ramanujan pour les surpartitions.

Théorème 1.29. *Soit $A(n)$ le nombre de surpartitions $\lambda_1 + \cdots + \lambda_\ell$ de n, sans part $\overline{1}$, et vérifiant les conditions de différence*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 1, & \textit{si } \lambda_i \textit{ n'est pas surlignée,} \\ 2, & \textit{si } \lambda_i \textit{ est surlignée.} \end{cases}$$

Soit $B(n)$ le nombre de surpartitions de n telles que les parts non surlignées sont congrues à 2 modulo 4, et $C(n)$ le nombre de partitions de n en parts non divisibles par 4. Alors pour tout entier positif n,

$$A(n) = B(n) = C(n).$$

Remarque. *Il s'agit d'un cas particulier de [Lov03, Théorème 1.2], qui a été généralisé par Chen, Sang, et Shi [CSS13]. Alors que les résultats précédents ont été obtenus en utilisant des équations aux q-différences, nous utilisons les récurrences des coefficients sur-q-binomiaux pour démontrer le théorème 1.29.*

L'égalité $B(n) = C(n)$ peut être prouvée facilement en examinant les séries génératrices. Ainsi l'identité importante est $A(n) = B(n)$. Illustrons maintenant le théorème 1.29 dans le cas $n = 8$. Il existe seize surpartitions vérifiant les conditions de différence :

$$8, \overline{8}, 7+1, \overline{7}+1, 6+2, \overline{6}+2, 6+\overline{2}, \overline{6}+\overline{2}, 5+3, \overline{5}+3,$$
$$5+\overline{3}, \overline{5}+\overline{3}, 5+2+1, \overline{5}+2+1, 4+3+1, 4+\overline{3}+1,$$

et aussi seize surpartitions qui vérifient les conditions de congruence :

$$\overline{8}, \overline{7}+\overline{1}, \overline{6}+\overline{2}, 6+2, \overline{6}+2, 6+\overline{2}, \overline{5}+\overline{3}, \overline{5}+\overline{2}+\overline{1}, \overline{5}+2+\overline{1}, \overline{4}+\overline{3}+\overline{1},$$
$$\overline{4}+2+2, \overline{4}+2+\overline{2}, \overline{3}+2+2+\overline{1}, \overline{3}+2+\overline{2}+\overline{1}, 2+2+2+2, 2+2+2+\overline{2}.$$

Le chapitre 14 est organisé comme suit. Dans la section 14.2, nous donnons une formule exacte pour les coefficients sur-q-binomiaux et des récurrences analogues au triangle de Pascal (Proposition 1.28). Dans la section 14.3, nous exposons plusieurs identités de q-séries qui peuvent être prouvées à l'aide des coefficients sur-q-binomiaux. Enfin dans la section 14.4, nous prouvons le théorème 1.29 en utilisant les coefficients sur-q-binomiaux.

2. Introduction (English)

2.1. State of the art

2.1.1. The beginnings of the theory of partitions

Leibniz was the first mathematician who studied integer partitions. In a letter from 1674 [Lei90], he asked Bernoulli about the number of ways to decompose a positive integer n as a sum of other positive integers. In other words, he asked about the number of partitions of n. The order of the summands (which are called parts) is not taken into consideration, so we will always write them in decreasing order. For example, there are three partitions of 3: 3, $2+1$ and $1+1+1$.

Leibniz was interested in knowing, for any positive integer n, the number of partitions of n, which we denote by $p(n)$. We notice that $p(n) = 0$ when n is negative, and we use the convention that $p(0) = 1$ (we consider the empty sequence to be the only partition of 0). Table 2.1 shows the partitions of n and $p(n)$ for $n = 1, \dots, 7$.

When one sees a sequence of integers as $(p(n))_{n\in\mathbb{N}}$, one can ask several questions.

First, one can wonder how often $p(n)$ is prime. Leibniz suggested that $p(n)$ might always be prime as it is the case for $n = 1, \dots, 6$. But $p(7) = 15 = 3\times 5$ is not prime. However, if we modify his conjecture a little, it leads us to a major question in the theory of partitions, still open today: Are there infinitely many integers n such that $p(n)$ is prime? In 2000, Ono [Ono00] made a first step towards solving this problem by showing that every prime divides at least one value of $p(n)$.

One can also ask how often $p(n)$ is odd or even. Computer experiments lead to the conjecture, yet unsolved too, that $p(n)$ is "equally often" even and odd, that is that $p(n)$ is even for $\sim \frac{x}{2}$ integers $n \leq x$ as x tends to infinity (see Parkin and Shanks [PS67]).

After those number theoretic questions, one can also ask a more analytical one. If one computes some larger values of $p(n)$, one sees that it grows quite

Table 2.1.: Partitions of the first seven integers

n	$p(n)$	partitions of n
1	1	1
2	2	2, 1+1
3	3	3, 2+1, 1+1+1
4	5	4, 3+1, 2+2, 2+1+1, 1+1+1+1
5	7	5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1
6	11	6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1
7	15	7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1

rapidly. For example

$$p(10) = 42, p(20) = 627, p(50) = 204226, p(100) = 190569292,$$
$$p(200) = 3972999029388 \text{ and } p(1000) = 24061467864032622473692149727991.$$

So it is also interesting to understand how fast the partition function grows asymptotically, and to see whether there is an exact formula for $p(n)$. These questions were solved by Hardy, Ramanujan and Rademacher with their "circle method", as we will explain later in this introduction (Section 2.1.5) and Part III.

2.1.2. Generating functions

After Leibniz set the basis of the theory of partitions, we must wait approximately seventy years for Euler to find the first really deep results. His study of integer partitions started in 1740 when he received a letter from Naudé asking about the number of partitions of 50 into 7 distinct parts. He was interested in this question and gave a first solution during a presentation he made at the St. Petersburg Academy in 1741 [Eul51]. He then proved it again with a different method in his very influential book *Introductio in Analysin Infinitorum* [Eul48] in 1748. Let us present the idea this proof. It is not easy (and would take a lot of time) to write down all the ways to write 50 as a sum of 7 integers and then to count them. Moreover, even if someone managed to do it for this particular example, it would give little insight about how to treat the general question "how many ways are there to write a positive integer n as a sum of m distinct positive integers?". To address these issues, Euler introduced a tool which is still fundamental now: generating functions. To understand his idea, let us quote Euler himself [Eul48]:

> 297. Let the following expression be given:
>
> $$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z)(1+x^{\epsilon}z)\cdots.$$
>
> We ask about the form if the factors are actually multiplied. We suppose that it has the form $1+Pz+Qz^2+Rz^3+Sz^4+\cdots$, where it is clear that P is equal to the sum of the powers $x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\epsilon}+\cdots$. Then Q is the sum of the products of the powers taken two at a time, that is Q is the sum of the different powers of x whose exponents are the sum of two of the different terms in the sequence $\alpha,\beta,\gamma,\delta,\epsilon,\zeta,\eta$, etc. In like manner R is the sum of powers of x whose exponents are the sum of three of the different terms. Further, S is the sum of powers of x whose exponents are the sum of four of the different terms in that same sequence $\alpha,\beta,\gamma,\delta,\epsilon$, etc., and so forth.
>
> 298. The individual powers of x which constitute the values of the letters P,Q,R,S, etc. have a coefficient of 1 if their exponents can be formed in only one way from $\alpha,\beta,\gamma,\delta$, etc. If the same exponent of a power of x can be obtained in several ways as the sum of two, three, or more terms of the sequence $\alpha,\beta,\gamma,\delta$, etc., then that power has a coefficient equal to the number of ways the exponent can be obtained. Thus, if in the value of Q there is found Nx^n, this is because n has N different ways of being expressed as a sum of two terms from the sequence α,β,γ, etc. Further, if in the expression of the given [*product*] the term Nx^nz^m occurs, this is because there are N different ways in [*the product that* N] can be a sum of m terms of the sequence $\alpha,\beta,\gamma,\delta,\epsilon,\zeta$, etc.
>
> 299. If the given product
>
> $$(1+x^{\alpha}z)\left(1+x^{\beta}z\right)(1+x^{\gamma}z)\left(1+x^{\delta}z\right)\cdots$$
>
> is actually multiplied, then from the resulting expression it becomes immediately apparent how many different ways a given number can be the sum of any desired number of terms from the sequence $\alpha,\beta,\gamma,\delta,\epsilon,\zeta$, etc. For example if it is desired to know how many different ways the number n can be the sum of m terms of the

given sequence, then we find the term $x^n z^m$ and its coefficient is the desired number.

Thus, following Euler's reasoning, the coefficient of $z^m q^n$ in the infinite product

$$(1+zq)(1+zq^2)(1+zq^3)(1+zq^4)\cdots$$

is equal to the number of partitions of n into m distinct parts, which we will write $Q(m,n)$. So the generating function of $Q(m,n)$ is

$$\sum_{m,n\geq 0} Q(m,n) z^m q^n = \prod_{k\geq 1}(1+zq^k).$$

Now this allows us to find a simple recurrence relation for $Q(m,n)$, which will help us compute $Q(7,50)$. Noting that

$$\prod_{k\geq 1}(1+zq^k) = (1+zq)\prod_{k\geq 1}(1+zq^{k+1}) = (1+zq)\prod_{k\geq 1}(1+(zq)q^k),$$

we deduce that

$$\begin{aligned}\sum_{m,n\geq 0} Q(m,n) z^m q^n &= (1+zq)\sum_{m,n\geq 0} Q(m,n)(zq)^m q^n \\ &= \sum_{m\geq 0}\sum_{n\geq m} Q(m,n-m) z^m q^n + \sum_{m\geq 1}\sum_{n\geq m} Q(m-1,n-m) z^m q^n,\end{aligned}$$

where the second line follows from a change of variables. Then the coefficients of $z^m q^n$ on both sides must be equal, thus

$$Q(m,n) = Q(m,n-m) + Q(m-1,n-m).$$

With this formula, one can easily compute $Q(7,50)$ (or any other value) recursively, and find that it equals 522.

In the same book, Euler noticed that "the condition that the numbers be different is eliminated if the product is put into the denominator." In other words, if we let $p(m,n)$ denote the number of partitions of n into m parts which do not need to be distinct, then the generating function for $p(m,n)$ is

$$\begin{aligned}\sum_{m,n\geq 0} p(m,n) z^m q^n &= \prod_{k\geq 1}\left(1+zq^k+z^2q^{2k}+z^3q^{3k}+\cdots\right) \\ &= \prod_{k\geq 1}\frac{1}{1-zq^k}\end{aligned}$$

In the same way as before, we can deduce that

$$p(m,n) = p(m-1,n-1) + p(m,n-m),$$

and use this formula to show that there are 8496 partitions of 50 into 7 parts.

Euler was even able to use generating functions to find a very efficient recurrence relation to compute $p(n)$. First, he noticed that the generating function for partitions is

$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{k\geq 1} \frac{1}{1-q^k}.$$

Then he studied the infinite product

$$\prod_{k\geq 1} \left(1-q^k\right) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + \cdots$$

and conjectured that this is equal to

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

He was able to prove this statement a few years later. This formula, now known as Euler's Pentagonal Number Theorem, was the starting point of the theory of theta functions and modular forms, one of the most important fields in number theory today [Kob93]. Combining the generating function for partitions and the pentagonal number theorem, he showed that

$$\left(\sum_{n\geq 0} p(n)q^n\right)\left(\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}\right) = 1.$$

Comparing the coefficients of q^n on each side, he obtained that $p(0) = 1$, and for $n \geq 1$,

$$p(n) = p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots .$$

This is still the most efficient algorithm to compute $p(n)$, as it gives the values of $p(1), \ldots, p(n)$ in time $O\left(n^{\frac{3}{2}}\right)$.

The generating functions introduced by Euler are still the most useful tool in the theory of partitions, and almost every paper on the subject uses them, whether it is to prove partition identities, congruences, or asymptotic formulas, to name only a few.

Table 2.2.: Euler's identity for the first seven integers

n	partitions of n into odd parts	partitions of n into distinct parts
1	1	1
2	1+1	2
3	3, 1+1+1	3, 2+1
4	3+1, 1+1+1+1	4, 3+1
5	5, 3+1+1, 1+1+1+1+1	5, 4+1, 3+2
6	5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1	6, 5+1, 4+2, 3+2+1
7	7, 5+1+1, 3+3+1, 3+1+1+1+1, 1+1+1+1+1+1+1	7, 6+1, 5+2, 4+3, 4+2+1

2.1.3. Partition identities

The first mathematician who discovered a partition identity was yet again Euler, who proved the following [Eul48] in 1748.

Theorem 2.1 (Euler)**.** *For every positive integer n, the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

Table 2.2 illustrates Euler's identity on the first seven integers.

More generally, a partition identity is a statement of the type "For every integer n, the number of partitions of n satisfying some conditions is equal to the number of partitions of n satisfying some other conditions". There are several ways to prove a partition identity. We can find an exact formula for the number of partitions of each type for every n and show that they are equal. But often this is not easy to do in practice. So instead we can consider the generating functions of each type of partitions. Let $a(n)$ denote the number of partitions of n of type A and $b(n)$ the number of partitions of n of type B. Even if we cannot find a nice formula for $a(n)$ and $b(n)$ directly, if we can compute their generating functions and show that they are equal, namely that $\sum_{n\geq 0} a(n)q^n = \sum_{n\geq 0} b(n)q^n$, then by comparing the coefficients of q^n on each side we deduce that $a(n) = b(n)$ for all n. Another way to prove a partition identity is to pair every partition of type A with a unique partition of type B and vice versa. This is called a bijective proof. It is often easier to prove a partition identity using generating functions than bijections, but bijections give more combinatorial insight as we know exactly which partition of type A corresponds to which partition of type B.

Let us now compare the three approaches aforementioned on the example of Euler's identity.

We saw in the previous section that it is not easy to find an exact formula for the number of partitions of n into m distinct parts. Even if we remove the condition on the number of parts, this is not much easier. In the same way there is no elementary way to find the number of partitions of n into odd parts for every n. Actually, we must wait until 1937 and the Hardy-Ramanujan-Rademacher circle method to find an exact formula for these quantities (and for $p(n)$). Obviously Euler did not prove his identity in this way. However the generating functions of those types of partitions are not too hard to compute. As seen in the previous section, the generating function for partitions into distinct parts equals

$$\prod_{k \geq 1} (1 + q^k),$$

and the generating function for partitions into odd parts equals

$$(1 + q + q^2 + \cdots)(1 + q^3 + q^6 + \cdots)(1 + q^5 + q^{10} + \cdots) \cdots = \prod_{k \geq 0} \frac{1}{1 - q^{2k+1}}.$$

Using the fact that

$$1 + q^k = \frac{1 - q^{2k}}{1 - q^k},$$

we find that

$$\begin{aligned} \prod_{k \geq 1} (1 + q^k) &= \prod_{k \geq 1} \frac{1 - q^{2k}}{1 - q^k} \\ &= \frac{\cancel{(1 - q^2)}\cancel{(1 - q^4)} \cdots}{(1 - q)\cancel{(1 - q^2)}(1 - q^3)\cancel{(1 - q^4)}(1 - q^5) \cdots} \\ &= \prod_{k \geq 0} \frac{1}{1 - q^{2k+1}}, \end{aligned}$$

and Euler's identity is proved, by a simple manipulation of the generating functions.

Now let us explain the idea of a bijective proof of Euler's identity. We need to find a way to associate each partition of n into odd parts with a partition of n into distinct parts, and when given a partition into distinct parts, to be able to find from which partition into odd parts it came. Let us start with a partition into odd parts and try to transform it into a partition into distinct parts. We want all the parts to be distinct, so if there are two copies of a part, we merge them into a part of double size. We repeat this procedure until the parts are all distinct. Let us illustrate this with an example:

$$\begin{aligned}7+5+5+5+3+1+1+1+1 &\mapsto 7+(5+5)+5+3+(1+1)+(1+1)\\ &\mapsto 10+7+5+3+2+2\\ &\mapsto 10+7+5+3+(2+2)\\ &\mapsto 10+7+5+4+3.\end{aligned}$$

Now we need to find the inverse of this transformation. Starting with a partition into distinct parts, we split every even part into two equal halves and repeat this procedure until all parts are odd. On the previous example, we see that this allows us to find the partition we started with.

$$\begin{aligned}10+7+5+4+3 &\mapsto (5+5)+7+5+(2+2)+3\\ &\mapsto 7+5+5+5+3+2+2\\ &\mapsto 7+5+5+5+3+(1+1)+(1+1)\\ &\mapsto 7+5+5+5+3+1+1+1+1.\end{aligned}$$

The order in which we split or merge parts does not matter. This procedure works for any partition into odd or distinct parts, so it gives a bijective proof of Euler's identity. In 1969, Andrews [And69c] extended this proof to generalise Euler's identity to other sets of partitions that we can also associate by a merging/splitting procedure.

After Euler, Sylvester [Syl73] was the next to make discoveries in the field of partition identities at the end of the nineteenth century. Among other results, he introduced a graphical representation of partitions called the Ferrers diagram and used it to prove that for every positive integers k and n, the number of partitions of n with largest part k is equal to the number of partitions of n into k parts.

The next major improvement in the field of partition identities has been made several years later, with the discovery of the famous Rogers-Ramanujan identities.

Theorem 2.2 (First Rogers-Ramanujan identity). *For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.*

Theorem 2.3 (Second Rogers-Ramanujan identity). *For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 and the smallest part is larger than 1 is equal to the number of partitions of n into parts congruent to 2 or 3 modulo 5.*

In terms of generating functions, these two identities can be written as

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})},$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}.$$

These two identities were published by Rogers [Rog94] in 1894, but went almost unnoticed at the time. However, in 1913, Ramanujan rediscovered these product-series identities and sent them to Hardy in a letter. Hardy was unable to prove them and sent them to Littlewood, MacMahon and Perron, who could not prove them either, although MacMahon saw the link between the product-series identities and the partition identities and was able to verify them up to $n = 89$. The mystery was solved in 1917 when Ramanujan, who was looking at old volumes of *Proceedings of the London Mathematical Society*, accidentally found Rogers' paper. He was very impressed by Rogers' work, and they started exchanging letters which led to a simplification of the original argument. They published their new proof in a joint paper [RR19] in 1919. Kept aside from the developments of British mathematics by the First World War, the German mathematician Schur also proved these identities independently in 1917 [Sch17]. Over the years, the Rogers-Ramanujan identities have acquired the status of the most celebrated identities in the field and dozens of proofs involving different techniques have been published, for example [Bre83, GM81, Wat29]. However, finding a simple bijective proof is still an open question now.

More generally, a partition identity of the type "for all n, the number of partitions of n with some difference conditions equals the number of partitions of n with some congruence conditions" is called a partition identity of the Rogers-Ramanujan type. Mathematicians have studied many such identities, as we will show now.

If we look at the statements of Euler's identity and the first Rogers-Ramanujan identity, we see that they are both of the form "for all n, the number of partitions of n such that consecutive parts differ by at least k equals the number of partitions of n into parts congruent to 1 or -1 modulo $k+3$". Euler's identity corresponds to $k = 1$ and the first Rogers-Ramanujan identity to $k = 2$. Thus we can wonder if this identity is also true for $k = 3$. It turns out that while it holds for $n = 1, \dots, 8$, it fails for $n = 9$, as there are three partitions into parts congruent to $\pm 1 \mod 6$ ($7+1$, $5+1+1$ and $1+\cdots+1$) and four partitions such that consecutive parts differ by at least 3 (9, $8+1$, $7+2$ and $6+3$). In

1946, Lehmer [Leh46] wiped out any hope for such a generalisation by proving that for any $k \geq 3$, one cannot even find a set $N \subseteq \mathbb{N}$ such that for all n, the number of partitions of n such that consecutive parts differ by at least k equals the number of partitions of n into parts in N. But in 1956, Alder [Ald56] noticed that instead of an equality, an inequality might still hold for all k and he conjectured that for all $k, n \geq 0$, the number of partitions of n into parts congruent to 1 or -1 modulo $k+3$ is no larger than the number of partitions of n such that consecutive parts differ by at least k.

Even if Lehmer proved that this generalisation is impossible, in 1926, Schur [Sch26] had proved a similar theorem for $k = 3$ by modifying the difference conditions instead of the congruence conditions.

Theorem 2.4 (Schur). *For any integer n, let $A(n)$ denote the number of partitions of n into parts congruent to 1 or -1 modulo 6, $B(n)$ denote the number of partitions of n into distinct parts congruent to 1 or 2 modulo 3, and $C(n)$ the number of partitions of n such that parts differ by at least 3 and no two consecutive multiples of 3 appear. Then for all n,*

$$A(n) = B(n) = C(n).$$

This explains why the conjecture failed for $n = 9$, as we counted the partition $6 + 3$.

Schur's theorem also became a very influential partition identity, and several proofs have been given using a variety of different techniques such as bijections [Bes91, Bre80], the method of weighted words [AG93], and recurrences [And67b, And68b, And71b].

One can wonder whether the possible generalisation "for any integer n, the number of partitions of n such that parts differ by at least k and no two consecutive multiples of k appear equals the number of partitions of n into parts satisfying some congruence conditions" holds for other values of k. During the 1960's Göllnitz [Gö67] and Gordon [Gor65] independently showed that such a theorem exists for $k = 2$ by proving that the number of partitions of n such that parts differ by at least 2 and no two consecutive multiples of 2 appear equals the number of partitions of n into parts congruent to $1, 4$ or 7 modulo 8. However, such a generalisation is not possible for $k \geq 4$, as shown by Alder [Ald48] in 1948.

In 1961, Gordon [Gor61] finally found the first generalisation of the Rogers-Ramanujan identities.

Theorem 2.5 (Gordon). *Let $A_{k,a}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm a \mod 2k+1$. Let $B_{k,a}(n)$ denote the number of* *partitions of n of the form* $\lambda_1+\lambda_2+\cdots+\lambda_j$*, where* $\lambda_i \geq \lambda_{i+1}$ *and* $\lambda_i-\lambda_{i+k-1} \geq 2$ *and at most* $a-1$ *of the* λ_i*'s are* 1*. Then for* $1 \leq a \leq k$*, for all* $n \geq 0$*,*

$$A_{k,a}(n) = B_{k,a}(n).$$

It led to two major generalisations of Andrews [And69b, And74].

Schur's theorem was also generalised in two different ways by Andrews [And68a, And69a]. In those generalizations, he considered partitions into distinct parts congruent to 2^k modulo 2^n-1 or into distinct parts congruent to -2^k modulo 2^n-1 for $0 \leq k \leq n-1$. Schur's theorem correspond to $n=2$. The cases $n=3$ of these theorems are the following.

Theorem 2.6 (Andrews)**.** *Let* $A(n)$ *denote the number of partitions of n into distinct parts congruent to* 1,2 *or* 4 *modulo* 7*. Let* $B(n)$ *denote the number of partitions of n of the form* $n = \lambda_1+\cdots+\lambda_s$*, where* $\lambda_i - \lambda_{i+1} \geq 7$ *if* $\lambda_{i+1} \equiv 1,2,4 \pmod 7$*,* $\lambda_i - \lambda_{i+1} \geq 12$ *if* $\lambda_{i+1} \equiv 3 \pmod 7$*,* $\lambda_i - \lambda_{i+1} \geq 10$ *if* $\lambda_{i+1} \equiv 5,6 \pmod 7$ *and* $\lambda_i - \lambda_{i+1} \geq 15$ *if* $\lambda_{i+1} \equiv 0 \pmod 7$*. Then for all n,* $A(n) = B(n)$*.*

Theorem 2.7 (Andrews)**.** *Let* $C(n)$ *denote the number of partitions of n into distinct parts congruent to* 3,5 *or* 6 *modulo* 7*. Let* $D(n)$ *denote the number of partitions of n of the form* $n = \lambda_1+\cdots+\lambda_s$*, where* $\lambda_i - \lambda_{i+1} \geq 7$ *if* $\lambda_i \equiv 3,5,6 \pmod 7$*,* $\lambda_i - \lambda_{i+1} \geq 12$ *if* $\lambda_i \equiv 4 \pmod 7$*,* $\lambda_i - \lambda_{i+1} \geq 10$ *if* $\lambda_i \equiv 1,2 \pmod 7$ *and* $\lambda_i - \lambda_{i+1} \geq 15$ *if* $\lambda_i \equiv 0 \pmod 7$ *and* $\lambda_s \neq 1,2,4,7$*. Then for all n,* $C(n) = D(n)$*.*

Andrews' identities have since then been generalised and refined [All97, CL06], they were used by Yee to solve most cases of Alder's conjecture [1] [And71a, Yee08] and they also play a natural role in group representation theory [AO91] and quantum algebra [Oh15].

Since the 1980's, many connections between representations of Lie algebras, q-difference equations and Rogers-Ramanujan type partition identities have emerged. For q-difference equations, see [CLM06], [FFJ+09] and [Jer12]. Regarding partitions, Lepowsky and Wilson [LW84] were the first ones to establish the link with Lie algebras by giving an interpretation of the Rogers-Ramanujan identities in terms of representations of the affine Lie algebra $sl_2(\mathbb{C})^\sim$. In turn, the study of the Rogers-Ramanujan identities also helped them understand more about Lie algebras, as they tried to understand the meaning of the difference 2 condition. Methods similar to Lepowsky and Wilson's were subsequently

1. Alder's conjecture was completely proved in 2011 by Alfes, Jameson and Lemke Oliver [AJO11].

applied to other representations of affine Lie algebras, yielding new partition identities of the Rogers-Ramanujan type discovered by Capparelli [Cap93], Primc [Pri99], Meurman-Primc [MP87] and Siladić [Sil] to name a few. For example, Capparelli's identity is the following.

Theorem 2.8 (Capparelli). *Let $C(n)$ denote the number of partitions of n into parts congruent to ± 2 or ± 3 modulo 12. Let $D(n)$ denote the number of partitions of n of the form $n = \lambda_1 + \cdots + \lambda_s$ such that $\lambda_s > 1$, $\lambda_i - \lambda_{i+1} \geq 2$, and if $\lambda_i - \lambda_{i+1} < 4$ then either λ_i and λ_{i+1} are both multiples of 3, or $\lambda_i \equiv 1 \mod 3$, or $\lambda_{i+1} \equiv -1 \mod 3$. Then $C(n) = D(n)$.*

This identity, presented as a conjecture by Capparelli at a conference, was first proved combinatorially by Andrews [And92] and Alladi, Andrews and Gordon [AAG95], and later with Lie-algebraic techniques by Capparelli himself [Cap96]. Simultaneously, Tamba and Xie also proved Capparelli's conjecture using vertex operator theory [TX95]. However, many of the Rogers-Ramanujan type partition identities arising from the study of Lie algebras have yet to be understood combinatorially.

2.1.4. Congruences

Congruences are also an important subject in the theory of partitions. Ramanujan is the one who initiated research in this field. Basing his work on the table of values of $p(n)$ for $n = 0, \ldots, 200$ computed by MacMahon, he announced in 1919 [Ram19] that he had found three simple congruences satisfied by $p(n)$, namely that for all $n \geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \mod 5, \\ p(7n+5) &\equiv 0 \mod 7, \\ p(11n+6) &\equiv 0 \mod 11. \end{aligned}$$

He actually first had the idea of these congruences because MacMahon's table of values of $p(n)$ was written in the form of five columns, and he noticed that the numbers in the last column were always divisible by 5. Surprisingly, both Hardy and MacMahon didn't notice this interesting property, perhaps because they thought that partitions, additive objects by nature, had no reason to have any divisibility property. Ramanujan proved the first two congruences in [Ram19] and announced in a short note [Ram20] that he had also found a proof of the last one. After Ramanujan's death in 1920, Hardy [Ram21] was able to extract a proof of the three congruences from a manuscript of Ramanujan.

Seeing these congruences, a combinatorialist would like to find a combinatorial interpretation. For example, for the first congruence, he would want to divide the partitions of n into 5 sets of equal size according to some combinatorial condition. However Ramanujan's original proofs of these congruences rely on q-series identities and give little combinatorial insight into why they are true. In 1944, Dyson [Dys44], an undergraduate student at Cambridge at the time, defined the rank of a partition as its largest part minus its number of parts, and conjectured that it can explain the first two of Ramanujan's congruences. Namely, he said that for all n, the partitions of $5n + 4$ (resp. $7n + 5$) can be divided in 5 (resp. 7) different classes of same size according to their rank modulo 5 (resp. 7). This was proved ten years later by Atkin and Swinnerton-Dyer [ASD54].

However the rank fails to explain the modulo 11 congruence. Therefore Dyson [Dys44] conjectured the existence of another statistic which he called the "crank" which would give a combinatorial explanation for all the Ramanujan congruences. The crank was finally found by Andrews and Garvan [AG88, Gar88] in 1988. If for a partition λ, $o(\lambda)$ denotes the number of ones in λ, and $\mu(\lambda)$ is the number of parts strictly larger than $o(\lambda)$, then the crank of λ is defined as

$$\mathrm{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

For example the partition $5 + 3 + 3 + 2$ has crank 5 because it does not have 1 as a part, and $5 + 2 + 1 + 1$ has crank -1. Let $M(m, n)$ denote the number of partitions of n with crank m, and $N(m, n)$ denote the number of partitions of n with rank m. Congruences of the same type as those of Ramanujan also exist for $M(m, n)$ [Mah05] and $N(m, n)$ [BO10].

In [Ram19], Ramanujan also stated a more general conjecture: let $\delta = 5^a 7^b 11^c$ and let λ be an integer such that $24\lambda \equiv 1 \mod \delta$, then for all $n \geq 0$,

$$p(n\delta + \lambda) \equiv 0 \mod \delta.$$

In his unpublished manuscript [BO99], he gave a proof of this conjecture for arbitrary a and $b = c = 0$. He began a proof for arbitrary b and $a = c = 0$ but never finished it. If he would have done so, he would have seen that the statement actually needed to be modified. Indeed, even if the conjecture is true for all the values he had at the time ($n \leq 200$), Chowla [Cho34] found in 1934 that $p(243)$ is not divisible by 7^3 even though $24 \times 243 \equiv 1 \mod 7^3$. However in 1938 Watson [Wat38] was able to prove a modified conjecture for all powers of 5 and 7, and Atkin [Atk67] finally proved the full modified conjecture in

1967: If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \mod \delta$, then for all $n \geq 0$,

$$p(n\delta + \lambda) \equiv 0 \mod 5^a 7^{[(b+2)/2]} 11^c.$$

Many other congruences for functions related to partitions have been proved [AO01, Gar10, Gar12, Lov00, Lov01, LO02, Ono11] and it is still to this day a very active research area.

2.1.5. Asymptotics and the Hardy-Ramanujan circle method

Since Leibniz and the beginning of the theory of partitions, mathematicians have been interested in finding an exact formula for the partition function $p(n)$.

Hardy and Ramanujan [HR18b] were the first to study $p(n)$ analytically in 1918. In particular they proved the following asymptotic formula when n tends to infinity:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

They were surprised to see how close the value obtained with this formula was to the exact value of $p(200)$ computed by MacMahon. It gave them the intuition that an exact formula for $p(n)$ can be found with a similar technique, and they proved the following formula:

$$p(n) = \frac{1}{2\sqrt{2}} \sum_{k=1}^{a\sqrt{n}} \sqrt{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}} \frac{\mathrm{d}}{\mathrm{d}n} \left(\exp\left(\frac{\pi\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}}{k} \right) \right),$$

where $\omega_{h,k}$ is a 24th root of unity, (h, k) denotes greatest common divisor of h and k, and a is an arbitrary constant, with the only condition that n be larger than some value $n_0(a)$ which depends on a. This formula is extremely precise. For example, it is sufficient to compute the first eight terms of the series to obtain $p(200) = 3972999029388$, which is the correct value.

However, there is a relation between n and a, so it is not an exact formula for $p(n)$ in the sense that we cannot directly substitute n in the formula and get the result. A few years later, in 1937, Rademacher [Rad37] managed to improve Hardy and Ramanujan's method to find an expression for $p(n)$ as a convergent series.

Theorem 2.9.

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{1/2}\right)}{\left(x - \frac{1}{24}\right)^{1/2}} \right]_{x=n},$$

where

$$A_k(n) = \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}}.$$

The technique they used to prove these formulas is called the Rogers-Ramanujan-Rademacher Circle Method. Its principle is the following. First we write $p(n)$ as the integral

$$p(n) = \frac{1}{2i\pi} \oint_\gamma \frac{P(x)}{x^{n+1}} \,\mathrm{d}x,$$

where P is the generating function for partitions and γ is any circle centred at the origin with radius smaller than 1. Then we use the fact that the singularities of the integrated function are the roots of unity, we divide the integration circle into small arcs according to which singularity is the closest, and we estimate the integral of the function on each of these arcs using the fact that P is (almost) a modular form.

In the 1920's, a few years after Ramanujan's death, Hardy and Littlewood published a series of papers entitled *Some problems of Partitio Numerorum* [HL20, HL23, HL25] in reference to the chapter about integer partitions in Euler's book [Eul48], in which they used the circle method to solve several important problems in additive number theory such as Waring's problem. The method proved very useful in additive combinatorics and has been used to prove important results such as Roth's theorem [Rot53], or the fact that every odd integer is the sum of at most five primes [Tao], to name only a few.

In 1933, Wright [Wri33] showed that if we are only interested in an asymptotic formula for the partition function and do not need an exact formula, then the Hardy-Ramanujan can be simplified by considering an arc around the dominant pole and showing that the asymptotic contribution of the integral on the remainder of the circle is negligible. Recently, this method has been used again to find asymptotic formulas for many functions related to partitions, in [BM13, BM14a, BM14b] to name a few. However, there was a mistake in the aforementioned papers, as the authors used the same asymptotic formula close to and far from the dominant pole, which was not valid far from it. Fortunately the proof can be fixed and the results are still correct. Apart from Wright's original papers, the method first appears in a correct version in [BDar].

The crank, introduced to explain the Ramanujan congruences, has also been the subject of asymptotic study. In 1989, Dyson conjectured the following asymptotic formula [Dys89]

Conjecture 2.10 (Dyson)**.** *As $n \to \infty$ we have*

$$M(m,n) \sim \frac{1}{4}\beta \operatorname{sech}^2\left(\frac{1}{2}\beta m\right) p(n)$$

with $\beta := \frac{\pi}{\sqrt{6n}}$.

Dyson knew that the formula was true with βm held fixed, but asked the question about the precise range of m (which may depend on n) in which this asymptotic holds and about the error term. For fixed m one can directly obtain asymptotic formulas since the generating function for $M(m,n)$ is the convolution of a modular form and a partial theta function [BM13]. However, Dyson's conjecture is a bivariate asymptotic, which is the source of the difficulty of this problem. While there exist several variants of the circle method which have been successfully applied to many problems, the existing techniques were not sufficient to solve this conjecture.

2.1.6. Overpartitions

Basic hypergeometric series (or q-series) are series constructed using the q-factorials

$$(a)_n = (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

for $n \in \mathbb{N} \cup \{\infty\}$. Identities such as the Rogers-Ramanujan identities or Euler's Pentagonal Number theorem are called q-series identities. Those identities play a significant role in combinatorics, number theory, representation theory, and mathematical physics, just to name a few. While the identities mentioned until now in this introduction have combinatorial interpretations in terms of partitions, other important identities, such as the q-binomial theorem [GR04]

$$\sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{2.1}$$

a q-analogue of the binomial theorem, do not. To be able to give a combinatorial interpretation of this identity and others, it is useful to consider a generalisation of partitions: overpartitions. An overpartition is a partition in which the last occurrence of a number may be overlined. It is equivalent to consider partitions in which the first occurrence of a number may be overlined (depending on the context, this convention may be more convenient). For example, there are 8 overpartitions of 3:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, 1+1+\overline{1}.$$

Though they were not called overpartitions at the time, they were already used in 1967 by Andrews [And67a] to give combinatorial interpretations of the q-binomial theorem, Heine's transformation and Lebesgue's identity. Then they were used in 1987 by Joichi and Stanton [JS87] in an algorithmic theory of bijective proofs of q-series identities. They have also been used in bijective proofs of Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation [Cor03, CL02]. It was Corteel and Lovejoy [CL04] who gave them their name in 2004 and revealed their generality by giving combinatorial interpretations for several q-series identities.

Beyond q-series identities, overpartitions went on to become a very interesting generalisation of partitions. Several partition identities have overpartition analogues or generalisations. For example, Lovejoy proved an overpartition analogue of Gordon's identity [Lov03], overpartition analogues for identities of Andrews-Santos and Gordon-Göllnitz [Lov04] and the following generalisation of Schur's theorem [Lov05b].

Theorem 2.11 (Lovejoy)**.** *Let $A(k,n)$ denote the number of overpartitions of n into parts congruent to 1 or 2 modulo 3 with k non-overlined parts. Let $B(k,n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_s$ of n, having k non-overlined parts and satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 3\chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_{i+1} \equiv 1,2 \mod 3, \\ 1 + 3\chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_{i+1} \equiv 0 \mod 3, \end{cases}$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all $k,n \geq 0$, $A(k,n) = B(k,n)$.

Lovejoy's work was followed by others and several new overpartition identities were found [CSS13, CM07, LM08].

Overpartitions also have interesting arithmetic properties [ACKOar, BL08, Mah04, Tre06] and are related to the fields of Lie algebras [KK04], mathematical physics [DLM03, FJM05a, FJM05b] and supersymmetric functions [DLM03].

2.2. Contributions of this thesis

Now that we have presented several aspects of the theory of partitions, let us present our contributions to some of them with the new results contained in this thesis.

2.2.1. Partition identities

Part II focuses on partition identities. In Chapter 4 we give three new proofs of Schur's theorem for overpartitions (Theorem 2.11). In Chapters 5 and 6 we generalise the two partition identities of Andrews to overpartitions. Finally in Chapter 7, we give a combinatorial proof and refinement of Siladić's theorem, a partition identity coming from the theory of vertex operator algebras. All these proofs use q-difference equations and recurrences, but very different techniques are used in each of them, showing the diversity of this method.

2.2.1.1. Schur's theorem

Let us recall Schur's theorem.

Theorem 2.12 (Schur)**.** *For any integer n, the number of partitions of n into parts congruent to 1 or -1 modulo 6 equals the number of partitions of n such that parts differ by at least 3 and no two consecutive multiples of 3 appear.*

Besides Schur's original proof [Sch26], Schur's theorem has been widely studied and a variety of proofs have been given. Bessenrodt [Bes91] and Bressoud [Bre80] proved it bijectively, Alladi and Gordon using the method of weighted words [AG93], and Andrews using recurrences [And67b, And68b, And71b].

As mentioned before, Schur's theorem has been generalised to overpartitions, Schur's theorem corresponding to the case $k = 0$ in Theorem 2.11. However, Lovejoy's theorem only had two different proofs. It was discovered using the method of weighted words [Lov05b] and subsequently proved bijectively [RP09]. It was not clear whether Andrews' proofs of Schur's theorem using recurrences could be adapted to prove Theorem 2.11.

In Chapter 4, we answer this question by giving three new proofs of Theorem 2.11 using recurrences, based on the three proofs of Andrews [And67b, And68b, And71b]. However the equations and techniques used to solve them are different and more intricate. These proofs are presented in the paper [Dou14b]. We already present a sketch of the proofs here.

First proof The first proof uses recurrences obtained by a combinatorial reasoning based on the smallest part of the overpartitions. The generating function for overpartitions counted by $A(k, n)$ is easy to compute and equals

$$\sum_{k,n\geq 0} A(k,n) d^k q^n = \prod_{n=0}^{\infty} \frac{(1+q^{3n+1})(1+q^{3n+2})}{(1-dq^{3n+1})(1-dq^{3n+2})}.$$

Our goal is to prove that the generating function for overpartitions counted by $B(k,n)$ is the same, but this is not as straightforward and we need to use recurrences and q-difference equations.

Letting $b_j(k,m,n)$ denote the number of overpartitions counted by $B(k,n)$ having m parts such that the smallest part exceeds j, we first obtain recurrence equations such as

$$b_0(k,m,n) - b_1(k,m,n) = b_0(k,m-1,n-3m+2) + b_0(k-1,m-1,n-1).$$

To do so, we observe that $b_0(k,m,n) - b_1(k,m,n)$ is the number of overpartitions counted by $b_0(k,m,n)$ such that the smallest part is equal to 1. We then remove the smallest part, and according to whether it was overlined or not, we deduce a condition on the second smallest part using the difference conditions.

Then, defining

$$f_i(x) = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} b_i(k,m,n)x^m d^k q^n,$$

we translate those recurrences into q-difference equations on the functions f_i, and after a few substitutions, we obtain a q-difference equation involving only f_0:

$$\begin{aligned}(1-dxq)(1-dxq^2)f_0(x) = (1 + xq + xq^2 + dxq^3 - dx^2q^3 - dx^2q^6)f_0(xq^3)\\ + xq^3(1-xq^3)f_0(xq^6).\end{aligned}$$

Our goal is then to find $f_0(1)$, which is exactly the generating function for overpartitions counted by $B(k,n)$. If $d=0$, which corresponds to the case of partitions, the equation can be solved quite easily. Here the presence of the variable d makes the equation much more complicated. To obtain $f_0(1)$, we define $F(x) := f_0(x)\prod_{k=0}^{\infty}\frac{(1-dxq^{3k+1})}{(1-xq^{3k})}$, deduce the q-difference equation satisfied by F, and translate it into a recurrence equation on (A_n), where $F(x) = \sum_{n=0}^{\infty} A_n x^n$. Namely,

$$\begin{aligned}(1-q^{3n})A_n = (1 + dq^2 + q^{3n-1})(1+q^{3n-2})A_{n-1}\\ - dq^2(1+q^{3n-2})(1+q^{3n-5})A_{n-2}.\end{aligned}$$

This leads us to define $a_n = A_n\prod_{k=0}^{n-1}\frac{1}{1+q^{3k+1}}$ to simplify the equation, and if we let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we get

$$(1-x)(1-dxq^2)f(x) = (1+xq^2)f(xq^3).$$

Iterating, we obtain

$$f(x) = \prod_{k=0}^{\infty} \frac{(1 + xq^{3k+2})}{(1 - xq^{3k})(1 - dxq^{3k+2})}.$$

Finally, we use Appell's Comparison Theorem [Die57, p. 101], stating that

$$\lim_{x \to 1^-} (1 - x) \sum_{n \geq 0} u_n x^n = \lim_{n \to \infty} u_n,$$

to trace our way back to $f_0(1)$. We have

$$\begin{aligned} \lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} \frac{A_n x^n}{\prod_{k=0}^{n-1} (1 + q^{3k+1})} &= \frac{\lim_{n \to \infty} A_n}{\prod_{k=0}^{\infty} (1 + q^{3k+1})} \\ &= \prod_{k=0}^{\infty} \frac{(1 + q^{3k+2})}{(1 - q^{3k+3})(1 - dq^{3k+2})}. \end{aligned}$$

Thus

$$\lim_{n \to \infty} A_n = \prod_{k=0}^{\infty} \frac{(1 + q^{3k+2})(1 + q^{3k+1})}{(1 - q^{3k+3})(1 - dq^{3k+2})},$$

and

$$f_0(x) = (1 - x) \prod_{k=0}^{\infty} \frac{(1 - xq^{3k+3})}{(1 - dxq^{3k+1})} \sum_{n=0}^{\infty} A_n x^n.$$

We apply Appell's Comparison Theorem again and we obtain

$$f_0(1) = \prod_{k=0}^{\infty} \frac{(1 + q^{3k+1})(1 + q^{3k+2})}{(1 - dq^{3k+1})(1 - dq^{3k+2})},$$

which completes the proof.

Second proof The second proof is similar to the first one, except that it relies on recurrences based on the largest part of the overpartitions.

We define $\psi_m(k, n)$ to be the number of overpartitions counted by $B(k, n)$ such that the largest part does not exceed m. By a combinatorial reasoning in which we remove the largest part of the overpartition, we obtain recurrences such as

$$\psi_{3m+1}(k, n) = \psi_{3m}(k, n) + \psi_{3m+1}(k - 1, n - 3m - 1) + \psi_{3m-2}(k, n - 3m - 1).$$

Then we define

$$a_m(q,d) = 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \psi_m(k,n) q^n d^k,$$

and going back and forth between q-difference equations and recurrences in a way similar to the first proof and finishing using Appell's Comparison Theorem too, we show that $\lim_{m\to\infty} a_m(q,d)$, which is the generating function for all overpartitions counted by $B(k,n)$, is indeed equal to

$$\prod_{n=0}^{\infty} \frac{(1+q^{3n+1})(1+q^{3n+2})}{(1-dq^{3n+1})(1-dq^{3n+2})}.$$

Third proof This proof also uses recurrences based on the largest part of overpartitions but we add one more variable counting the number of parts congruent to 1 or 2 modulo 3 plus two times the number of parts congruent to 0 modulo 3. This allows us to complete the proof without switching between recurrences and q-difference equations and using Appell's lemma, but makes equations slightly more complicated. However one can say that this is the most elementary proof of Schur's theorem for overpartitions.

Let $\psi_m(M,n,k)$ denote the number of overpartitions of n with k non-overlined parts, where M equals the number of parts congruent to 1 or 2 modulo 3 plus two times the number of parts congruent to 0 modulo 3, satisfying the difference conditions and such that the largest part does not exceed m. Using a reasoning similar to that of the second proof, we obtain equations such as

$$\begin{aligned}\psi_{3m+1}(M,k,n) = \psi_{3m}(M,k,n) &+ \psi_{3m+1}(M-1,k-1,n-3m-1)\\ &+ \psi_{3m-2}(M-1,k,n-3m-1).\end{aligned}$$

We define $a_m(q,d,t) = 1 + \sum_{M=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \psi_m(M,n,k) q^n d^k t^M$. We want to determine $\lim_{m\to\infty} a_m(q,d,t)$, the generating function for overpartitions counted by $B(k,n)$ having M parts where parts divisible by 3 are counted twice. We notice that $(a_{3m+3}(q,d,t))_{m\in\mathbb{N}}$ satisfies the same recurrence equation as $(a_{3m-1}(q,d,tq^3))_{m\in\mathbb{N}}$. Using the initial conditions we deduce that

$$a_{3m+3}(q,d,t) = \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} a_{3m-1}(q,d,tq^3).$$

Letting $m \to \infty$ and iterating, we get

$$\lim_{m\to\infty} a_m(q,d,t) = \prod_{n=0}^{\infty} \frac{(1+tq^{3n+1})(1+tq^{3n+2})}{(1-dtq^{3n+1})(1-dtq^{3n+2})}, \tag{2.2}$$

and the theorem is proved.

2.2.1.2. Andrews' theorems

In the first part of this introduction, we mentioned two partition identities of Andrews [And69a, And68a] which generalise Schur's theorem, but we only gave two particular cases (Theorems 2.6 and 2.7). We now introduce some notation in order to state them in their full generality.

Let $A = \{a(1), \dots, a(r)\}$ be a set of r distinct integers such that $\sum_{i=1}^{k-1} a(i) < a(k)$ for all $1 \leq k \leq r$ and the $2^r - 1$ possible sums of distinct elements of A are all distinct. We denote this set of sums by $A' = \{\alpha(1), \dots, \alpha(2^r - 1)\}$, where $\alpha(1) < \cdots < \alpha(2^r - 1)$. Let us notice that $\alpha(2^k) = a(k+1)$ for all $0 \leq k \leq r-1$ and that any α between $a(k)$ and $a(k+1)$ has largest summand $a(k)$. Let N be a positive integer with $N \geq \alpha(2^r - 1) = a(1) + \cdots + a(r)$. We further define $\alpha(2^r) = a(r+1) = N + a(1)$. Let A_N denote the set of positive integers congruent to some $a(i) \mod N$, $-A_N$ the set of positive integers congruent to some $-a(i) \mod N$, A'_N the set of positive integers congruent to some $\alpha(i) \mod N$ and $-A'_N$ the set of positive integers congruent to some $-\alpha(i) \mod N$. Let $\beta_N(m)$ be the least positive residue of $m \mod N$. If $\alpha \in A'$, let $w(\alpha)$ be the number of terms appearing in the defining sum of α and $v(\alpha)$ the smallest $a(i)$ appearing in this sum.

To understand this notation better, the reader might want to consider the example where $a(k) = 2^{k-1}$ for $1 \leq k \leq r$ and $\alpha(k) = k$ for $1 \leq k \leq 2^r - 1$.

We are now able to state Andrews' generalisations of Schur's theorem.

Theorem 2.13 (Andrews)**.** *Let $D(A_N; n)$ denote the number of partitions of n into distinct parts taken from A_N. Let $E(A'_N; n)$ denote the number of partitions of n into parts taken from A'_N of the form $n = \lambda_1 + \cdots + \lambda_s$, such that*

$$\lambda_i - \lambda_{i+1} \geq N w(\beta_N(\lambda_{i+1})) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then for all $n \geq 0$, $D(A_N; n) = E(A'_N; n)$.

Theorem 2.14 (Andrews)**.** *Let $F(-A_N; n)$ denote the number of partitions of n into distinct parts taken from $-A_N$. Let $G(-A'_N; n)$ denote the number of partitions of n into parts taken from $-A'_N$ of the form $n = \lambda_1 + \cdots + \lambda_s$, such that*

$$\lambda_i - \lambda_{i+1} \geq N w(\beta_N(-\lambda_i)) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

and $\lambda_s \geq N(w(\beta_N(-\lambda_s) - 1)$. Then for all $n \geq 0$, $F(-A_N; n) = G(-A'_N; n)$.

Andrews proved Theorem 2.13 [And69a] by extending his proof of Schur's theorem based on the smallest part of the partitions [And68b] and Theorem 2.14 [And68a] by extending his proof of Schur's theorem based on the largest part [And67b].

As Schur's theorem generalises to overpartitions, it was interesting to see whether it was possible to extend Andrews' theorems in their full generality to overpartitions too. We answer this question by proving the following generalisations.

Theorem 2.15. *Let $D(A_N; k, n)$ denote the number of overpartitions of n into parts taken from A_N, having k non-overlined parts. Let $E(A'_N; k, n)$ denote the number of overpartitions of n into parts taken from A'_N of the form $n = \lambda_1 + \cdots + \lambda_s$, having k non-overlined parts, such that*

$$\lambda_i - \lambda_{i+1} \geq N \left(w \left(\beta_N(\lambda_{i+1}) \right) - 1 + \chi(\overline{\lambda_{i+1}}) \right) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}),$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all $k, n \geq 0$, $D(A_N; k, n) = E(A'_N; k, n)$.

Theorem 2.16. *Let $F(-A_N; k, n)$ denote the number of overpartitions of n into parts taken from $-A_N$, having k non-overlined parts. Let $G(-A'_N; k, n)$ denote the number of overpartitions of n into parts taken from $-A'_N$ of the form $n = \lambda_1 + \cdots + \lambda_s$, having k non-overlined parts, such that*

$$\lambda_i - \lambda_i \geq N \left(w \left(\beta_N(\lambda_i) \right) - 1 + \chi(\overline{\lambda_{i+1}}) \right) + v(\beta_N(\lambda_i)) - \beta_N(\lambda_i),$$

$$\lambda_s \geq N(w(\beta_N(-\lambda_s)) - 1).$$

Then for all $k, n \geq 0$, $F(-A_N; k, n) = G(-A'_N; k, n)$.

Schur's theorem (resp. Schur's theorem for overpartitions) corresponds to $N = 3$, $r = 2$, $a(1) = 1$, $a(2) = 2$ in Theorems 2.13 and 2.14 (resp. Theorems 2.15 and 2.16). Again, the case $k = 0$ of Theorem 2.15 (resp. Theorem 2.16) gives Theorem 2.13 (resp. Theorem 2.14).

Let us illustrate Theorems 2.15 and 2.16 with examples in the case $N = 7$, $r = 3$, $a(1) = 1$, $a(2) = 2$, $a(3) = 4$. For Theorem 2.15, the overpartitions of 4 counted by $E(A'_7; k, 4)$ are 4, $\overline{4}$, $3 + 1$, $\overline{3} + 1$, $2 + 2$, $\overline{2} + 2$, $2 + 1 + 1$, $\overline{2} + 1 + 1$, $1 + 1 + 1 + 1$ and $\overline{1} + 1 + 1 + 1$. The overpartitions of 4 into parts congruent to 1,2 or 4 modulo 7 (counted by $D(A_7; k, 4)$) are 4, $\overline{4}$, $2 + 2$, $\overline{2} + 2$, $2 + 1 + 1$, $\overline{2} + 1 + 1$, $2 + \overline{1} + 1$, $\overline{2} + \overline{1} + 1$, $1 + 1 + 1 + 1$ and $\overline{1} + 1 + 1 + 1$. In both cases, we have one overpartition with 0 non-overlined parts, three overpartitions with 1 non-overlined part, three overpartitions with 2 non-overlined parts, two overpartitions with 3 non-overlined parts and one overpartition with 4 non-overlined parts. For Theorem 2.16, the overpartitions of 8 counted by $G(-A'_7; k, 8)$ are 8, $\overline{8}$, $5 + 3$ and $\overline{5} + 3$. The overpartitions of 8 into parts congruent to 3, 5 or 6 modulo 7 (counted by $F(-A_7; k, 8)$) are

$5+3$, $\overline{5}+3$, $5+\overline{3}$ and $\overline{5}+\overline{3}$. In both cases, we have one overpartition with 0 non-overlined parts, two overpartitions with 1 non-overlined part, and one overpartition with 2 non-overlined parts.

While the statements of Theorems 2.15 and 2.16 resemble those of Andrews' theorems, the proofs are considerably more intricate and involve a number of new ideas. Theorem 2.15 was proved in the paper [Douar] and Theorem 2.16 in [Dou15]. We present both Andrews' proofs and ours in Chapters 5 and 6, but let us give the general ideas of our proofs here.

To prove Theorem 2.15, we first give the q-differential equation satisfied by the generating function for overpartitions enumerated by $E(A'_N;k,n)$, using some combinatorial reasoning on the smallest part of the overpartition. Then we prove by induction on r that a function satisfying this q-difference equation is equal to

$$\prod_{k=1}^{r} \frac{(-q^{a(k)};q^N)_\infty}{(dq^{a(k)};q^N)_\infty},$$

which is the generating function for overpartitions counted by $D(A_N;k,n)$. To do so, we go back and forth from q-difference equations on generating functions to recurrence equations on their coefficients, to decrease the degree of the q-difference equation by one, somewhat like in the first proof of Schur's theorem for overpartitions.

The proof of Theorem 2.16 is rather similar. First, we give the recurrence equation satisfied by the generating function for overpartitions enumerated by $G(-A'_N;k,n)$ having their largest part $\leq m$, using some combinatorial reasoning on the largest part. Then we prove by induction on r that the limit when m tends to infinity of a function satisfying this recurrence equation is equal to

$$\prod_{j=1}^{r} \frac{(-q^{N-a(j)};q^N)_\infty}{(dq^{N-a(j)};q^N)_\infty},$$

which is the generating function for overpartitions counted by $F(-A_N;k,n)$.

2.2.1.3. Siladić's identity

In [Sil], Siladić proved the following theorem by studying representations of the twisted affine Lie algebra $A_2^{(2)}$.

Theorem 2.17 (Siladić). *The number of partitions $\lambda_1+\cdots+\lambda_s$ of an integer n into parts different from 2 such that the difference between two consecutive* *parts is at least* 5 *(ie.* $\lambda_i - \lambda_{i+1} \geq 5$*) and*

$$\begin{aligned}
\lambda_i - \lambda_{i+1} = 5 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \mod 16, \\
\lambda_i - \lambda_{i+1} = 6 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 2, \pm 6 \mod 16, \\
\lambda_i - \lambda_{i+1} = 7 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 3 \mod 16, \\
\lambda_i - \lambda_{i+1} = 8 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 4 \mod 16,
\end{aligned}$$

is equal to the number of partitions of n *into distinct odd parts.*

In Chapter 7, we give a combinatorial proof and refinement of Siladić's identity. This was presented in the paper [Dou14a]. Our refinement of Theorem 7.1 is the following:

Theorem 2.18. *For* $n \in \mathbb{N}$ *and* $k \in \mathbb{N}^*$*, let* $A(k,n)$ *denote the number of partitions* $\lambda_1 + \cdots + \lambda_s$ *of* n *such that* k *equals the number of odd parts plus twice the number of even parts, satisfying the following conditions:*

1. $\forall i \geq 1, \lambda_i \neq 2$,
2. $\forall i \geq 1, \lambda_i - \lambda_{i+1} \geq 5$,
3. $\forall i \geq 1$,

$$\begin{aligned}
\lambda_i - \lambda_{i+1} = 5 &\Rightarrow \lambda_i \equiv 1, 4 \mod 8, \\
\lambda_i - \lambda_{i+1} = 6 &\Rightarrow \lambda_i \equiv 1, 3, 5, 7 \mod 8, \\
\lambda_i - \lambda_{i+1} = 7 &\Rightarrow \lambda_i \equiv 0, 1, 3, 4, 6, 7 \mod 8, \\
\lambda_i - \lambda_{i+1} = 8 &\Rightarrow \lambda_i \equiv 0, 1, 3, 4, 5, 7 \mod 8.
\end{aligned}$$

For $n \in \mathbb{N}$ *and* $k \in \mathbb{N}^*$*, let* $B(k,n)$ *denote the number of partitions of* n *into* k *distinct odd parts. Then for all* $n \in \mathbb{N}$ *and* $k \in \mathbb{N}^*$*,* $A(k,n) = B(k,n)$*.*

As in Schur's theorem, the generating function for partitions counted by $B(k,n)$ is easy to compute and equals

$$\sum_{k,n \geq 0} B(k,n) t^k q^n = \prod_{k \geq 0} \left(1 + tq^{2k+1}\right),$$

but the generating function for partitions counted by $A(k,n)$ is harder to determine.

For $N \in \mathbb{N}$, $k, n \in \mathbb{N}^*$, let $a_N(k,n)$ denote the number of partitions counted by $A(k,n)$ with largest part at most N, and define

$$G_N(t,q) = 1 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_N(k,n) t^k q^n.$$

Thus $G_\infty(t,q) = \lim_{N\to\infty} G_N(t,q)$ is the generating function for the partitions counted by $A(k,n)$.

Our strategy is to show the following

Theorem 2.19. *For all $m \in \mathbb{N}^*$,*

$$G_{2m}(t,q) = (1+tq)G_{2m-3}(tq^2,q).$$

Indeed we can then let m go to infinity and deduce

$$\begin{aligned} G_\infty(t,q) =& (1+tq)G_\infty(tq^2,q) = (1+tq)(1+tq^3)G_\infty(tq^4,q) = \cdots \\ =& \prod_{k\geq 0}\left(1+tq^{2k+1}\right). \end{aligned}$$

To achieve this goal, we do the following. First, in Section 7.2.1 we give an equivalent formulation of Theorem 2.17 which is easier to manipulate in terms of partitions. Then in Section 7.2.2 we establish q-difference equations satisfied by the generating functions of partitions considered in Theorem 2.17. Finally in Section 7.2.3, we use those q-difference equations to prove Theorem 2.19 by induction.

2.2.2. Asymptotics and the two-variable circle method

In Part III, we focus on asymptotic aspects of the theory of partitions. First, in Chapter 8, we explain the principle of the Hardy-Ramanujan-Rademacher circle method [HR18b, Rad37] to compute the exact formula for $p(n)$ and in Chapter 9 we apply Wright's circle method [Wri33], which only gives an asymptotic formula but is much simpler, to compute an asymptotic formula for $p(n)$. Then, in Chapter 10, we apply Wright's circle method to give asymptotic formulas for two statistics on overpartitions with restricted odd differences. Finally, in Chapter 11, we present a new generalisation of Wright's circle method to Jacobi forms and mock Jacobi forms, which we call the two-variable circle method. In Chapter 12, we use this method to deduce an asymptotic formula for $M(m,n)$, therefore solving Dyson's conjecture, and in Chapter 13 we use it to find an asymptotic formula for $N(m,n)$.

2.2.2.1. Overpartitions with restricted odd differences

In Chapter 10, we study overpartitions where the difference between two successive parts may be odd only if the larger part is overlined, and use q-difference equations in order to compute a two-variable hypergeometric q-series representation of the corresponding generating function. This generating function specialises in one case to a modular form, and in another to a mixed mock modular form. We also consider the two-variable generating function for the same overpartitions with odd smallest part, and again find modular and mixed mock modular specializations. These modularity properties allow us to compute asymptotics using Wright's circle method. This was done in the paper [BDLM15].

Let $\overline{t}(n)$ denote the number of overpartitions where the difference between two successive parts may be odd only if the larger part is overlined, and if the smallest part is odd then it is overlined. Let $\overline{s}(n)$ denote the number of overpartitions counted by $\overline{t}(n)$ with odd smallest part. Thus $\overline{t}(4) = 8$ and $\overline{s}(4) = 4$, the 8 overpartitions counted by $\overline{t}(4)$ being

$$4,\, \overline{4},\, 3+\overline{1},\, \overline{3}+\overline{1},\, 2+2,\, 2+\overline{2},\, \overline{2}+1+\overline{1},\, 1+1+1+\overline{1}$$

and the 4 overpartitions counted by $\overline{s}(4)$ being

$$3+\overline{1},\, \overline{3}+\overline{1},\, \overline{2}+1+\overline{1},\, 1+1+1+\overline{1}.$$

First, we determine q-hypergeometric generating functions for $\overline{t}(m,n)$ and $\overline{s}(m,n)$, the number of overpartitions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) having m parts.

Theorem 2.20. *We have the following identities:*

$$\sum_{m,n\geq 0} \overline{t}(m,n)x^m q^n = \frac{(-xq)_\infty}{(xq)_\infty}\left(1+\sum_{n\geq 1}\frac{(-q^3;q^3)_{n-1}(-x)^n q^n}{(-q)_{n-1}(q^2;q^2)_n}\right), \tag{2.3}$$

$$\sum_{m,n\geq 1} \overline{s}(m,n)x^m q^n = \sum_{n\geq 1}\frac{(q^3;q^3)_{n-1}x^n q^n}{(q)_{n-1}(q^2;q^2)_n}. \tag{2.4}$$

The second identity follows from a straightforward combinatorial argument, but the first is more subtle and our proof depends on showing that both sides satisfy a certain q-difference equation.

When $x = 1$ in (2.3) or -1 in (2.4), then we have a modular form, and when $x = -1$ in (2.3) or 1 in (2.4), then we have the product of a modular form and a mock theta function, a so-called *mixed mock modular form* (see [LO13]). Define the mock theta functions [BL07] $\overline{\gamma}(q)$ and $\overline{\chi}(q)$ by

$$\overline{\gamma}(q) := \sum_{n\geq 0}\frac{(-1;q)_n(q;q)_n q^{\binom{n+1}{2}}}{(q^3;q^3)_n}$$

and

$$\overline{\chi}(q) := \sum_{n\geq 0} \frac{(-1;q)_n(-q;q)_n q^{\binom{n+1}{2}}}{(-q^3;q^3)_n}.$$

Let $\overline{t}_+(n)$ (resp. $\overline{s}_+(n)$) denote the number of overparititions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) with largest part even, and $\overline{t}_-(n)$ (resp. $\overline{s}_-(n)$) denote the number of overparititions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) with largest part odd.

Corollary 2.21. *We have*

$$\begin{aligned}\sum_{n\geq 0}\overline{t}(n)q^n &= \frac{\left(q^3;q^3\right)_\infty}{(q;q)_\infty\left(q^2;q^2\right)_\infty},\\ \sum_{n\geq 0}\left(\overline{t}_+(n)-\overline{t}_-(n)\right)q^n &= \frac{\left(-q^3;q^3\right)_\infty}{(-q;q)_\infty^3}\overline{\chi}(q),\\ 1+3\sum_{n\geq 1}\left(\overline{s}_+(n)-\overline{s}_-(n)\right)q^n &= \frac{\left(-q^3;q^3\right)_\infty}{(-q;q)_\infty^3},\\ 1+3\sum_{n\geq 1}\overline{s}(n)q^n &= \frac{\left(q^3;q^3\right)_\infty}{(q;q)_\infty\left(q^2;q^2\right)_\infty}\overline{\gamma}(q).\end{aligned}$$

Then we use these generating functions and Wright's circle method to deduce asymptotic formulas for $\overline{s}(n)$ and $\overline{t}_+(n)-\overline{t}_-(n)$, which are the cases from Corollary 2.21 in which the generating functions are mixed mock modular forms.

Theorem 2.22. *As $n\to\infty$, we have*

$$\overline{s}(n) \sim \frac{\sqrt{21}}{36n}\,e^{\frac{\pi\sqrt{7n}}{3}}, \tag{2.5}$$

$$\overline{t}_+(n)-\overline{t}_-(n) \sim (-1)^n\frac{\sqrt{3}}{18n^{-\frac{3}{4}}}\,e^{\frac{2\pi\sqrt{n}}{3}}. \tag{2.6}$$

Note that while in most examples in the literature the dominant pole in Wright's circle method is at $q=1$, for (2.6) it is at $q=-1$.

Chapter 10 is organized as follows. In Section 10.2 we prove Theorem 2.20 and Corollary 2.21 using analytic and combinatorial arguments. In Section 10.3 we prove Theorem 2.22 using Wright's Circle Method.

2.2.2.2. The two-variable circle method

In Chapter 11, we present a two-variable generalisation of Wright's circle method.

Bivariate asymptotics are generally much harder than univariate asymptotics, so it is no surprise that Dyson's conjecture (Conjecture 2.10) had been open since 1989. However, we give a new method which allows us to compute the bivariate asymptotics of the coefficients of Jacobi forms (and mock Jacobi forms) and solve Dyson's conjecture: the two-variable circle method.

Dyson's conjecture In Chapter 12, we prove that Dyson's conjecture is true. This was the object of the paper [BDar].

Theorem 2.23. *Dyson's conjecture is true. Precisely, if $|m| \leq \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$, we have as $n \to \infty$*

$$M(m,n) = \frac{\beta}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right)p(n)\left(1 + O\left(\beta^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right),$$

with $\beta := \frac{\pi}{\sqrt{6n}}$.

A straightforward calculation shows

Corollary 2.24. *Almost all partitions satisfy Dyson's conjecture. Precisely,*

$$\sharp\left\{\lambda \vdash n |\operatorname{crank}(\lambda)| \leq \frac{\sqrt{n}}{\pi\sqrt{6}}\log n\right\} \sim p(n).$$

Dyson's conjecture follows actually from a more general result concerning the coefficients $M_k(m,n)$ defined for $k \in \mathbb{N}$ by

$$\mathcal{C}_k(\zeta;q) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M_k(m,n)\zeta^m q^n := \frac{(q)_\infty^{2-k}}{(\zeta q)_\infty (\zeta^{-1}q)_\infty}.$$

Note that $M(m,n) = M_1(m,n)$. Denoting by $p_k(n)$ the number of partitions of n allowing k colors, we have

Theorem 2.25. *For k fixed and $|m| \leq \frac{1}{6\beta_k}\log n$, we have as $n \to \infty$*

$$M_k(m,n) = \frac{\beta_k}{4}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)p_k(n)\left(1 + O\left(\beta_k^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right),$$

with $\beta_k := \pi\sqrt{\frac{k}{6n}}$.

We note that for $k \geq 3$, the functions $C_k(\zeta;q)$ are known to be generating functions of Betti numbers of moduli spaces of Hilbert schemes on $(k-3)$−point blow-ups of the projective plane [Gö90]. Theorem 2.25 immediately gives the limiting profile of the Betti numbers for large second Chern class of the sheaves.

Chapter 12 is organised as follows. In Section 12.2, we recall basic facts on modular and Jacobi forms which are the base components of $\mathcal{C}_k$ and collect properties on Euler polynomials. In Section 12.3, we determine the asymptotic behaviour of $\mathcal{C}_k$. In Section 12.4, we use the two-variable circle method to finish the proof of Theorem 2.25. In Section 12.5, we illustrate Theorem 2.23 numerically.

Asymptotics for the rank In Chapter 13, we show that the two-variable circle method can also be used to prove that the same formula as in Dyson's conjecture holds for the rank. This was done in the paper [DMar]. The situation for the rank is more complicated than for the crank since the generating function is not a Jacobi form but a mock Jacobi form, which means roughly that there exists some non-holomorphic function such that its sum with the generating function has nice modular properties. Nonetheless the two-variable circle method works, even if some calculations become more complicated, and we prove

Theorem 2.26. *If $|m| \leq \frac{\sqrt{n}\log n}{\pi\sqrt{6}}$, we have as $n \to \infty$*

$$N(m,n) = \frac{\beta}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right) p(n) \left(1 + O\left(\beta^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right).$$

Chapter 13 is organised as follows. In Section 13.2, we prove some preliminary estimates for the rank generating function using the transformation properties of mock Jacobi forms. In Section 13.3, we use these results to prove the estimates close to and far from the dominant pole, and in Section 13.4, we apply the two-variable circle method to establish our main result Theorem 2.26.

2.2.3. An extension of q-binomial coefficients

In Chapter 14, we study an overpartition analogue of the q-binomial coefficients

$$\begin{bmatrix} M+N \\ N \end{bmatrix}_q = \frac{(q)_{M+N}}{(q)_M (q)_N}.$$

These polynomials have played many roles in combinatorics and number theory. They are generating functions for the number of inversions in permutations of a multi-set, the number of N dimensional subspaces of $M+N$ dimensional vector spaces over $\mathbb{F}_q$ and the number of partitions having restrictions on their largest part and their number of parts. They are also q-analogues of the classical binomial coefficients, which means that when we set $q = 1$ in q-binomial coefficients, we obtain the classical binomial coefficients.

As $\begin{bmatrix} M+N \\ N \end{bmatrix}_q$ is the generating function for the number of partitions fitting inside an $M \times N$ rectangle, i.e. with largest part $\leq M$ and number of parts $\leq N$ (for example, see [And84]), we define an overpartition analogue of q-binomial coefficients, which we will call *over q-binomial coefficients*, as the generating function for the number of overpartitions fitting inside an $M \times N$ rectangle.

Our first result is an exact formula for the over q-binomial coefficients $\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q$.

Theorem 2.27. *For positive integers M and N,*

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q = \sum_{k=0}^{\min\{M,N\}} q^{\frac{k(k+1)}{2}} \frac{(q)_{M+N-k}}{(q)_k (q)_{M-k} (q)_{N-k}}.$$

For example, from Theorem 2.27 we find that

$$\overline{\begin{bmatrix} 6 \\ 3 \end{bmatrix}} = 1 + 2q + 4q^2 + 8q^3 + 10q^4 + 12q^5 + 12q^6 + 8q^7 + 4q^8 + 2q^9,$$

and we can check that the 12 overpartitions of 5 fitting inside a 3×3 rectangle are the following.

$$3+2, \overline{3}+2, 3+\overline{2}, \overline{3}+\overline{2}, 3+1+1, \overline{3}+1+1, 3+1+\overline{1}$$
$$\overline{3}+1+\overline{1}, 2+2+1, 2+\overline{2}+1, 2+2+\overline{1}, 2+\overline{2}+\overline{1}.$$

As q-binomial coefficients satisfy simple recurrences which are q-analogues of Pascal's identity

$$\begin{bmatrix} M+N \\ N \end{bmatrix} = \begin{bmatrix} M+N-1 \\ N \end{bmatrix} + q^M \begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix},$$
$$\begin{bmatrix} M+N \\ N \end{bmatrix} = \begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix} + q^N \begin{bmatrix} M+N-1 \\ N \end{bmatrix},$$

over q-binomial coefficients also satisfy similar recurrences.

Theorem 2.28. *For positive integers M and N, we have*

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}} = \overline{\begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix}} + q^N \overline{\begin{bmatrix} M+N-1 \\ N \end{bmatrix}} + q^N \overline{\begin{bmatrix} M+N-2 \\ N-1 \end{bmatrix}},$$

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}} = \overline{\begin{bmatrix} M+N-1 \\ N \end{bmatrix}} + q^M \overline{\begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix}} + q^M \overline{\begin{bmatrix} M+N-2 \\ N-1 \end{bmatrix}}.$$

By employing over q-binomial coefficients, we can establish several identities. The Rogers-Ramanujan identities, introduced in Section 2.1.3, have been proved via various methods. Among them, one of the most elementary and beautiful is a proof by Andrews which uses recurrences satisfied by q-binomial coefficients [And88, And89]. Motivated by this proof, we find a Rogers-Ramanujan type identity for overpartitions.

Theorem 2.29. *Let $A(n)$ denote the number of overpartitions $\lambda_1 + \cdots + \lambda_\ell$ of n, where $\overline{1}$ cannot be a part, and satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 1, & \text{if } \lambda_i \text{ is not overlined,} \\ 2, & \text{if } \lambda_i \text{ is overlined.} \end{cases}$$

Let $B(n)$ denote the number of overpartitions of n with non-overlined parts $\equiv 2 \mod 4$ and $C(n)$ denote the number of partitions of n into parts $\not\equiv 0 \pmod 4$. Then for all non-negative integers n,

$$A(n) = B(n) = C(n).$$

Remark. *This is a special case of [Lov03, Theorem 1.2], which was generalized by Chen, Sang, and Shi [CSS13]. While the previous results were obtained by employing q-difference equations, we use the recurrence formulas for over q-binomial coefficients.*

The equality $B(n) = C(n)$ is easily proved by examining the generating functions, thus the important equality is $A(n) = B(n)$. Here we illustrate Theorem 2.29 for the case $n = 8$. There are 16 overpartitions satisfying the difference conditions:

$$8, \overline{8}, 7+1, \overline{7}+1, 6+2, \overline{6}+2, 6+\overline{2}, \overline{6}+\overline{2}, 5+3, \overline{5}+3,$$
$$5+\overline{3}, \overline{5}+\overline{3}, 5+2+1, \overline{5}+2+1, 4+3+1, 4+\overline{3}+1,$$

and there are also 16 overpartitions satisfying the congruence conditions:

$$\overline{8}, \overline{7}+\overline{1}, \overline{6}+\overline{2}, 6+2, \overline{6}+2, 6+\overline{2}, \overline{5}+\overline{3}, \overline{5}+\overline{2}+\overline{1}, \overline{5}+2+\overline{1}, \overline{4}+\overline{3}+\overline{1},$$
$$\overline{4}+2+2, \overline{4}+2+\overline{2}, \overline{3}+2+2+\overline{1}, \overline{3}+2+\overline{2}+\overline{1}, 2+2+2+2, 2+2+2+\overline{2}.$$

Chapter 14 is organised as follows. In Section 14.2, we give an exact expression for over q-binomial coefficients and recurrences analogous to the q-Pascal triangle (Proposition 2.28). In Section 14.3, we give various q-series identities which can be proved using over q-binomial coefficients. Finally in Section 14.4, we prove Theorem 2.29 using over q-binomial coefficients.

3. Preliminaries

3.1. Partitions and generating functions

Let us recall some basic facts about partitions, more formally than in the introduction.

Definition. A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \dots, \lambda_s$ such that $\lambda_1 + \cdots + \lambda_s = n$. The integers $\lambda_1, \dots, \lambda_s$ are called the *parts* of the partition.

Definition. Let n be a positive integer. Let $p(n)$ denote the number of partitions of n. The function p is called the *partition function*.

We notice that $p(n) = 0$ when n is negative, and we use the convention that $p(0) = 1$ (we consider that the empty sequence is the only partition of 0).

In this thesis, we use generating functions to prove partition identities and asymptotic formulas.

Definition. The *generating function* $f(q)$ for a sequence $(a_n)_{n\in\mathbb{N}}$ is the power series $f(q) = \sum_{n\geq 0} a_n q^n$, for $|q| < 1$.

We give a basic theorem about the generating function for partitions:

Theorem 3.1. *Let n be an integer, H a set of integers and let $p(n,H)$ denote the number of partitions of n whose parts lie in H. Then*

$$\sum_{n\geq 0} p(n,H) q^n = \prod_{n\in H} \frac{1}{(1-q^n)}.$$

Proof: Let us index the elements of H, so that $H = \{h_1, h_2, h_3, ...\}$. Then

$$\begin{aligned}
\prod_{n\in H} \frac{1}{(1-q^n)} &= \prod_{n\in H} \sum_{k\geq 0} q^{kn} \\
&= \sum_{k_1\geq 0} q^{k_1 h_1} \times \sum_{k_2\geq 0} q^{k_2 h_2} \times \sum_{k_3\geq 0} q^{k_3 h_3} \times \cdots \\
&= \sum_{k_1\geq 0} \sum_{k_2\geq 0} \sum_{k_3\geq 0} \cdots \; q^{k_1 h_1 + k_2 h_2 + k_3 h_3 + \cdots}.
\end{aligned}$$

Notice that the exponent of q is the partition

$$\underbrace{h_1 + \cdots + h_1}_{k_1 \text{ times}} + \underbrace{h_2 + \cdots + h_2}_{k_2 \text{ times}} + \underbrace{h_3 + \cdots + h_3}_{k_3 \text{ times}} + \cdots .$$

So q^n will appear in the summation exactly once for every partition of n into parts lying in H. Therefore

$$\prod_{n \in H} \frac{1}{(1-q^n)} = \sum_{n \geq 0} p(n,H) q^n.$$

□

So the generating function for partitions is:

$$\sum_{n \geq 0} p(n) q^n = \prod_{n \geq 1} \frac{1}{(1-q^n)}.$$

We now prove a similar theorem about partitions into distinct parts, following Euler's argument presented in the introduction.

Theorem 3.2. *Let n be an integer, H a set of integers and let $Q(n,H)$ denote the number of partitions of n into distinct parts whose parts lie in H. Then*

$$\sum_{n \geq 0} Q(n,H) q^n = \prod_{n \in H} (1+q^n).$$

Proof: This proof is similar to the proof of Theorem 3.1. Let us index the elements of H, so that $H = \{h_1, h_2, h_3, ...\}$. Then

$$\prod_{n \in H} (1+q^n) = \sum_{0 \leq k_1 \leq 1} \sum_{0 \leq k_2 \leq 1} \sum_{0 \leq k_3 \leq 1} \cdots \; q^{k_1 h_1 + k_2 h_2 + k_3 h_3 + \cdots}$$

The exponent of q is the partition $k_1 h_1 + k_2 h_2 + k_3 h_3 + \cdots$, where $k_i = 0$ or 1 for all $i \geq 1$. So q^n will appear in the summation exactly once for every partition of n into distinct parts lying in H. Therefore

$$\prod_{n \in H} (1+q^n) = \sum_{n \geq 0} Q(n,H) q^n.$$

□

So the generating function for partitions into distinct parts is:

$$\sum_{n\geq 0} Q(n)q^n = \prod_{n\geq 1}(1+q^n).$$

Finally, we give an expression for the generating function for overpartitions.

Definition. Let n be a positive integer. An *overpartition* of n is a partition of n in which the first occurrence of a number may be overlined.

Theorem 3.3. *Let n be an integer, H a set of integers and let $\overline{p}(n,H)$ denote the number of overpartitions of n whose parts lie in H. Then*

$$\sum_{n\geq 0} \overline{p}(n,H)q^n = \prod_{n\in H}\frac{1+q^n}{1-q^n}.$$

Proof: An overpartition can be seen as a pair formed by a partition (the non-overlined parts) and a partition into distinct parts (the overlined parts). Thus the generating function for overpartitions into parts lying in H is the product of the generating function for partitions into parts lying in H and the generating function for partitions into distinct parts lying in H. □

Until now we have only considered one-variable generating functions, but it can be useful to track another quantity in addition to the number partitioned, like the number of parts for example. This leads us to consider two-variable or three-variable generating functions.

Let us give the two-variable version of Theorem 3.1.

Theorem 3.4. *Let n and k be integers, H a set of integers and let $p(k,n;H)$ denote the number of partitions of n into k parts, whose parts lie in H. Then*

$$\sum_{n\geq 0}\sum_{k\geq 0} p(k,n;H)z^k q^n = \prod_{n\in H}\frac{1}{(1-zq^n)}.$$

Proof: Let us index the elements of H, so that $H=\{h_1,h_2,h_3,...\}$. Then, following the same principle as before, we have

$$\begin{aligned}\prod_{n\in H}\frac{1}{(1-zq^n)} &= \prod_{n\in H}\sum_{k\geq 0} z^k q^{kn}\\ &= \sum_{k_1\geq 0} z^{k_1}q^{k_1h_1}\times\sum_{k_2\geq 0} z^{k_2}q^{k_2h_2}\times\sum_{k_3\geq 0} z^{k_3}q^{k_3h_3}\times\cdots\\ &= \sum_{k_1\geq 0}\sum_{k_2\geq 0}\sum_{k_3\geq 0}\cdots\ z^{k_1+k_2+k_3}q^{k_1h_1+k_2h_2+k_3h_3+\cdots}.\end{aligned}$$

When the exponent of q is the partition

$$\underbrace{h_1 + \cdots + h_1}_{k_1 \text{ times}} + \underbrace{h_2 + \cdots + h_2}_{k_2 \text{ times}} + \underbrace{h_3 + \cdots + h_3}_{k_3 \text{ times}} + \cdots,$$

the exponent of z is $k_1 + k_2 + k_3 + \cdots$, then number of parts of this partition. The theorem is proved. □

Theorem 3.5. *Let n and k be integers, H a set of integers and let $Q(k, n; H)$ denote the number of partitions of n into k distinct parts whose parts lie in H. Then*

$$\sum_{n \geq 0} \sum_{k \geq 0} Q(k, n; H) z^k q^n = \prod_{n \in H} (1 + zq^n).$$

Combining Theorems 3.4 and 3.5, we give the following version of Theorem 3.3 tracking the number of overlined and non-overlined parts.

Theorem 3.6. *Let n, k, m be integers, H a set of integers and let $\overline{p}(k, m, n; H)$ denote the number of overpartitions of n, having k non-overlined parts and m overlined parts, whose parts lie in H. Then*

$$\sum_{n \geq 0} \sum_{m \geq 0} \sum_{k \geq 0} \overline{p}(k, m, n; H) d^k z^m q^n = \prod_{n \in H} \frac{1 + zq^n}{1 - dq^n}.$$

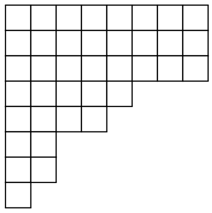
3.2. Gaussian polynomials (q-binomial coefficients)

In the previous section, we have given the generating function for partitions whose parts lie in some set. But what happens to the generating function if we give restrictions on the largest part and the number of parts? We will use Gaussian polynomials, a q-analogue of binomial coefficients to answer this question. But first, let us define a graphical representation of partitions.

Definition. Let λ be the partition $n = \lambda_1 + \cdots + \lambda_s$. The *Ferrers diagram* of λ is the arrangement of n square boxes in s rows, such that the boxes are left-justified and the i-th row contains λ_i boxes.

This definition is much clearer on an example. Figure 3.1 shows the Ferrers diagram of the partition $8 + 8 + 8 + 5 + 4 + 2 + 2 + 1$.

Figure 3.1.: The Ferrers diagram of the partition $8+8+8+5+4+2+2+1$.

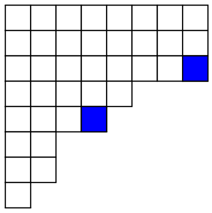


This graphical representation can be extended to overpartitions.

Definition. Let λ be the overpartition $n = \lambda_1 + \cdots + \lambda_s$. The *Ferrers diagram* of λ is the arrangement of n square boxes in s rows, such that the boxes are left-justified and the i-th row contains λ_i boxes, such that the last box of the i-th row is coloured if and only if λ_i is overlined.

Let us also illustrate this definition on an example. Figure 3.2 shows the Ferrers diagram of the overpartition $8+8+\overline{8}+5+\overline{4}+2+2+1$.

Figure 3.2.: The Ferrers diagram of the overpartition $8+8+\overline{8}+5+\overline{4}+2+2+1$.



Partitions having at most M parts all $\leq N$ are partitions whose Ferrers diagram fits inside an $M \times N$ box. Let us define their generating function.

Definition. Gaussian polynomials or *q-binomial coefficients* are defined as the generating functions

$$\begin{bmatrix} M+N \\ N \end{bmatrix}_q = \sum_{n\geq 0} p(n| \leq M \text{ parts, each} \leq N)q^n,$$

where $p(n| \leq M$ parts, each $\leq N)$ denotes the number of partitions of n fitting inside an $M \times N$ box. We will later omit the q in the notation and write $\begin{bmatrix} M+N \\ N \end{bmatrix}$.

Let us illustrate this on an example. Figure 3.3 shows all partitions fitting inside a 2×3 box. Thus we have

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}_q = 1q^0 + 1q^1 + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

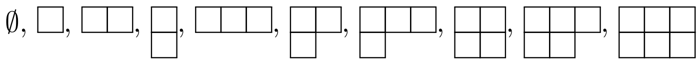


Figure 3.3.: Partitions fitting inside a 2×3 box.

We will now show that q-binomial coefficients are q-analogues of the classical binomial coefficients and that they satisfy similar properties. But let us first recall the definition of binomial coefficients.

Definition. Let j and n be non-negative integers. The *binomial coefficient* $\binom{n}{j}$ is defined as the number of ways to choose a subset of j elements from a set of n elements.

The binomial coefficients have the following simple explicit formula for $0 \leq j \leq n$

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}, \tag{3.1}$$

and satisfy a recurrence relation known as *Pascal's triangle*

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}. \tag{3.2}$$

The q-binomial coefficients also satisfy recurrences which are q-analogues of Pascal's identity

Proposition 3.7.

$$\begin{bmatrix} M+N \\ N \end{bmatrix} = \begin{bmatrix} M+N-1 \\ N \end{bmatrix} + q^M \begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix}, \tag{3.3}$$

$$\begin{bmatrix} M+N \\ N \end{bmatrix} = q^N \begin{bmatrix} M+N-1 \\ N \end{bmatrix} + \begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix}. \tag{3.4}$$

Proof: We prove (3.3). Equation (3.4) can be proved in a similar way. Let us consider a partition of n fitting inside an $M \times N$ box. Then either it has fewer than M parts or it has exactly M parts. In the first case it fits in fact inside an $(M-1) \times N$ box. In the second case, we remove the first column of the

Ferrers diagram (thus removing M squares) and we are left with a partition of $n-M$ fitting inside and $M \times (N-1)$ box. This gives

$$\begin{aligned} p(n| \leq M\text{parts, each} \leq N) &= p(n| \leq M-1\text{parts, each} \leq N) \\ &\quad + p(n-M| \leq M\text{parts, each} \leq N-1). \end{aligned}$$

Multiplying both sides by q^n and summing over n leads to (3.3). □

We can use these recurrences to prove an exact formula for q-binomial coefficients analogous to (3.1).

Proposition 3.8. *We have*

$$\begin{bmatrix} M+N \\ N \end{bmatrix}_q = \begin{cases} \frac{(q)_{M+N}}{(q)_M (q)_N} & \text{if } M, N \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We prove this formula by induction on $N+M$. Clearly we have $\begin{bmatrix} M+N \\ N \end{bmatrix}_q = 0$ if $M < 0$ or $N < 0$, because no partition can fit inside a box with a negative side, and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$ because the empty partition is the only one fitting inside a 0×0 rectangle. Let us now assume that the formula is true for $N+M = k \leq 0$ and prove it for $N+M = k+1$. By Proposition 3.7, we have

$$\begin{bmatrix} M+N \\ N \end{bmatrix} = \begin{bmatrix} M+N-1 \\ N \end{bmatrix} + q^M \begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix},$$

thus by the induction hypothesis

$$\begin{bmatrix} M+N \\ N \end{bmatrix} = \frac{(q)_{M+N-1}}{(q)_{M-1}(q)_N} + q^M \frac{(q)_{M+N-1}}{(q)_M (q)_{N-1}}.$$

Reducing to the same denominator gives

$$\begin{aligned} \begin{bmatrix} M+N \\ N \end{bmatrix} &= \frac{\left(1-q^M\right)(q)_{M+N-1}}{(q)_M (q)_N} + q^M \frac{\left(1-q^N\right)(q)_{M+N-1}}{(q)_M (q)_N} \\ &= \frac{\left(1-q^{M+N}\right)(q)_{M+N-1}}{(q)_M (q)_N} \\ &= \frac{(q)_{M+N}}{(q)_M (q)_N}. \end{aligned}$$

The theorem is proved. □

Letting $q \to 1$ in the formula of Proposition 3.8 gives $\frac{(M+N)!}{M!N!}$ so q-binomial coefficients are indeed q-analogues of binomial coefficients.

As binomial coefficients satisfy the binomial theorem

$$(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k,$$

q-binomial coefficients satisfy its following q-analogue [GR04].

Theorem 3.9 (q-binomial theorem)**.** *For all $n \geq 0$, we have*

$$\prod_{k=0}^{n-1} (1+tq^k) = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k.$$

Proof: For all $n \geq 0$, let us define

$$F_n := \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} t^k.$$

Using Proposition 3.7, we obtain

$$\begin{aligned} F_n :=& \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} t^k \left(\begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) \\ =& \sum_{k=0}^{n-1} q^{\frac{k(k-1)}{2}} t^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \sum_{k=1}^{n} q^{\frac{k(k-1)}{2}+n-k} t^k \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ =& F_{n-1} + \sum_{k=0}^{n-1} q^{\frac{k(k-1)}{2}+n-1} t^{k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\ =& \left(1+tq^{n-1}\right) F_{n-1}. \end{aligned}$$

Iterating and using the fact that $F_0 = 1$ gives the desired result. $\square$

3.3. Modular forms

Now that we have given the combinatorial basis of the theory of partitions, let us turn to its number theoretic basis: modular forms.

3.3.1. Basic facts

Let us start by stating a few general facts about modular forms. Further details and proofs can be found in [Kno70] or [Ono04].

Let us define $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$.

Let us now introduce a few important subgroups of $SL_2(\mathbb{Z})$.

Definition. Let N be a positive integer. We define the level N congruence subgroups as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N \right\},$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \mod N \text{ and } c \equiv 0 \mod N \right\},$$
$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \mod N \text{ and } c \equiv d \equiv 0 \mod N \right\}.$$

A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ acts on the upper half of the complex plane $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by the linear fractional transformation

$$Az = \frac{az + b}{cz + d}.$$

Definition. Let Γ be a subgroup of $SL_2(\mathbb{Z})$. Two complex numbers z_1 and z_2 are said to be *equivalent with respect to* Γ if there exists a matrix $A \in \Gamma$ such that $Az_1 = z_2$.

Definition. Let Γ be a subgroup of $SL_2(\mathbb{Z})$. A *fundamental domain* for Γ is an open subset $\mathcal{R}$ of $\mathcal{H}$ such that

1. one cannot find two distinct points of $\mathcal{R}$ which are equivalent with respect to Γ,
2. every point of $\mathcal{H}$ is equivalent to some point of $\overline{\mathcal{R}}$, the closure of $\mathcal{R}$.

The fundamental domains that we have just defined are far from being unique, there are even an infinitude of them for each subgroup Γ. Here is a classical example of fundamental domain.

Proposition 3.10. *The set*

$$\mathcal{R}(SL_2(\mathbb{Z})) = \{z \in \mathcal{H} : |z| > 1, |\mathrm{Re}(z)| < \frac{1}{2}\}$$

is a fundamental domain for $SL_2(\mathbb{Z})$.

Intuitively, a modular form of weight r on Γ is a function f well-defined and meromorphic on $\mathcal{H}$ which satisfies a growth condition at certain special points called cusps and the functional equations

$$f(Az) = \nu(A)(cz+d)^r f(z), \tag{3.5}$$

for all $z \in \mathcal{H}$ et $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Here $\nu(A)$ is a character for the group Γ, which we define now.

Definition. A function $\nu : \Gamma \to \mathbb{C}$ is a *character* for the group Γ and the weight r if

- for all $A \in \Gamma$, $|\nu(A)| = 1$,
- for all $A_1, A_2 \in \Gamma$, we have

$$\nu(A_1A_2)(c_3z+d_3)^r = \nu(A_1)\nu(A_2)(c_1A_2z+d_1)^r(c_2z+d_2)^r,$$

where $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ and $A_1A_2 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$.

Let us now study more precisely the growth conditions mentioned before. To do so, define the notion of cusp.

Definition. Let $\mathcal{R}$ be a fundamental domain of Γ. A *cusp* of Γ in $\mathcal{R}$ is a point $z \in \mathbb{Q}$ or $z = \infty$ such that $z \in \overline{\mathcal{R}}$, the closure of $\mathcal{R}$ in the Riemann sphere topology.

Remark. *When $\Gamma = SL_2(\mathbb{Z})$, there is only one cusp, which we traditionally choose to be at infinity.*

Starting from now, we consider the case $\Gamma = SL_2(\mathbb{Z})$ for simplicity.

Definition. Let f be a meromorphic function on $\mathcal{H}$ which satisfies the condition (3.5) for all $A \in SL_2(\mathbb{Z})$. Then it admits a Fourier expansion:

$$f(z) = \sum_{n=n_0}^{\infty} a_n q^n,$$

where $q = \mathrm{e}^{2i\pi z}$.

The function f is said to be *meromorphic (resp. holomorphic) at the cusp* if $n_0 > -\infty$ (resp. $n_0 \geq 0$).

We can finally give the rigorous definition of a modular form.

Definition. Let $r \in \frac{1}{2}\mathbb{Z}$. A *meromorphic (resp. holomorphic) modular form* of weight r with character ν on $SL_2(\mathbb{Z})$ is a function $f : \mathcal{H} \to \mathbb{C}$ which satisfies the following properties:

1. f is meromorphic (resp. holomorphic) on $\mathcal{H}$,
2. $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), f(Az) = \nu(A)(cz+d)^r f(z)$,
3. f is meromorphic (resp. holomorphic) at the cusp.

Remark. *To obtain the definition of a modular form on an other subgroup Γ of $SL_2(\mathbb{Z})$, one should require f to be meromorphic (resp. holomorphic) at all cusps, but the definition of Fourier expansion is slightly more complciated.*

3.3.2. Dedekind's η function

Now that we have defined modular forms, let us focus on one of them, Dedekind's η function, closely related to partitions.

Definition. For all $z \in \mathcal{H}$, define

$$\eta(z) := \mathrm{e}^{i\pi z/12} \prod_{k=1}^{\infty} \left(1 - \mathrm{e}^{2i\pi kz}\right).$$

Indeed, setting $q := \mathrm{e}^{2i\pi z}$, we see its link to the partition generating functions.

Theorem 3.11. *We have*

$$P(q) = \sum_{n \geq 0} p(n) q^n = q^{1/24} \frac{1}{\eta(z)}, \tag{3.6}$$

$$\sum_{n \geq 0} Q(n) q^n = q^{-1/24} \frac{\eta(2z)}{\eta(z)}, \tag{3.7}$$

$$\sum_{n \geq 0} \overline{p}(n) q^n = \frac{\eta(2z)}{\eta^2(z)}. \tag{3.8}$$

Proof: The proof of (3.6) is immediate by the definition of η and Theorem 3.1.

Let us prove (3.7). By Theorem 3.2, we have

$$\begin{aligned}\sum_{n\geq 0} Q(n)q^n &= \prod_{n\geq 1}(1+q^n)\\ &= \prod_{n\geq 1}\frac{(1+q^n)(1-q^n)}{(1-q^n)}\\ &= \prod_{n\geq 1}\frac{(1-q^{2n})}{(1-q^n)}\\ &= \frac{q^{-1/12}\eta(2z)}{q^{-1/24}\eta(z)}\\ &= q^{-1/24}\frac{\eta(2z)}{\eta(z)}.\end{aligned}$$

Finally, (3.8) is the product of (3.6) and (3.7). □

We can now state the theorem on which most of our asymptotic calculations from Chapter III rely.

Theorem 3.12. *Dedekind's η function is a holomorphic modular form of weight $\frac{1}{2}$ and character ν_η on $SL_2(\mathbb{Z})$, where for all $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$,*

$$\nu_\eta(A) = \begin{cases}\exp\left(\frac{bi\pi}{12}\right) & \text{if } c=0 \text{ and } d=1,\\ \exp\left(i\pi\left(\frac{a+d}{12c}+s(-d,c)-\frac{1}{4}\right)\right) & \text{if } c>0,\end{cases} \tag{3.9}$$

where $s(h,k)$ is Dedekind's sum:

$$s(h,k) = \sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\left\lfloor\frac{\mu}{k}\right\rfloor-\frac{1}{2}\right)\left(\frac{h\mu}{k}-\left\lfloor\frac{h\mu}{k}\right\rfloor-\frac{1}{2}\right).$$

Remark. *If we multiply both the numerator and the denominator of $\frac{az+b}{cz+d}$ by some constant, we can assume that $c>0$ or $c=0$ and $d=1$. Thus Theorem 3.12 treats all possible cases.*

Moreover the theory of eta-quotients allows one to build many more modular forms.

Definition. An *eta-quotient* is a function f of the form

$$f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta},$$

where the r_δ's are integers and $N\geq 1$.

Theorem 3.13. *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $k = \sum_{\delta|N} r_\delta \in \mathbb{Z}$, and also satisfies:*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \mod 24,$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \mod 24,$$

then f is a weight $\frac{k}{2}$ modular form on $\Gamma_0(N)$.

3.4. Mock theta functions and mock modular forms

In his 1920 letter to Hardy [Ram00], Ramanujan wrote that he had discovered 17 functions which he called "mock theta functions", among which were the following:

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2},$$
$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n},$$
$$\chi(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n^2}.$$

At the time, modular forms were called "theta functions". Ramanujan thought that these mock theta functions were very similar to modular forms, as they have an asymptotic expansion at the cusps, with maybe poles at these points, but they cannot be expressed in terms of classical modular forms. He also found interesting identities relating sums of mock theta functions to modular forms, such as

$$2\phi(-q) - f(q) = \frac{(q)_\infty}{(-q)_\infty^2}. \tag{3.10}$$

Since then, mock theta functions have been studied by several mathematicians ([And66, AH91, Hic88, Wat36] to name a few) but they were completely understood only in 2002 when Zwegers [Zwe02] showed that mock theta functions can be seen as the holomorphic part of a weak Maass form and built the whole theory of mock modular forms.

3.4.1. Definitions

Let us now define more precisely what mock modular forms are. Details can be found in [Zag09] or [Ono08].

Definition. Define the *Laplacian operator of order k* by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Definition. Let $k \in \frac{1}{2}\mathbb{Z}$ and Γ be a subgroup of $SL_2(\mathbb{Z})$. A *harmonic weak Maass form of weight k* is a continuous function f on the upper half plane $\mathcal{H}$ which satisfies the following properties:

1. $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, f(Az) = \nu(A)(cz+d)^{2-k} f(z)$,
2. f is an eigenvector of the Laplacian operator Δ_k and its eigenvalue is $\left(1 - \frac{k}{2}\right) \frac{k}{2}$,
3. f has at most linear exponential growth at each cusp of Γ.

If f is a harmonic weak Maass form of weight k, then

$$g := \sum_n b_n q^n = y^k \frac{\partial \overline{f}}{\partial \overline{z}}$$

is holomorphic and transforms like a modular form of weight k, but it may not be holomorphic at the cusps. We search for a function g^* with same image as g such that $f - g^*$ is holomorphic, which we obtain under the form

$$g^*(\tau) = \left(\frac{i}{2} \right)^{k-1} \int_{-\overline{\tau}}^{i\infty} (z+\tau)^{-k} \overline{g(-\overline{z})}\, dz = \sum_n n^{k-1} \overline{b_n} \beta_k(4ny) q^{-n+1},$$

where

$$\beta_k(t) = \int_t^{\infty} u^{-k} e^{-\pi u}\, du$$

is the incomplete Gamma function.

Definition. The function $h = f - g^*$ is called the *holomorphic part* of f.

Definition. A *mock modular form* is defined as the holomorphic part of some harmonic weak Maass form f.

The mock modular form h is holomorphic but not quite modular, whereas the harmonic weak Maass form $f = h + g^*$ transforms like a modular form but is not quite holomorphic. The function g is called the *shadow* of h.

Ramanujan's identities such as (3.10) can be explained by the fact that the non-holomorphic parts of the two mock modular forms in the sum cancel and leave us with a modular form.

We can also form interesting functions by multiplying modular forms and mock modular forms.

Definition. A *mixed mock modular form* is the product of a modular form and a mock modular form.

When one studies a mock modular form, one often tries to cancel one of its two parts to use either its modularity or its holomorphicity.

Part II.

Partition identities

4. Schur's theorem

Contents

This part focuses on partition identities. In this first chapter, we explain the three proofs of Schur's theorem (Theorem 2.4) due to Andrews [And67b, And68b, And71b] and we show how a combination of similar ideas and new ones can be used to prove Schur's theorem for overpartitions (Theorem 2.11) in three different ways. This was done in the paper [Dou14b].

4.1. Proofs using recurrences based on the largest part

4.1.1. Andrews' proof of Schur's theorem

Before Andrews published his first proof of Schur's theorem in 1967 [And67b], there were only two proofs in existence. The first one, by Schur himself, relied on recurrence relations for certain polynomials, and the second, due to Gleissberg[Gle28], was an arithmetic proof which was combinatorially quite intricate. Thus Andrews' proof can be considered as the first elementary proof of Schur's theorem. We explain it here in detail.

By Theorem 3.2, one easily sees that the generating function for partitions into distinct parts congruent to 1 or 2 modulo 3 is

$$\prod_{n=0}^{\infty}(1+q^{3n+1})(1+q^{3n+2}),$$

thus it suffices to prove that the generating function for partitions satisfying the difference conditions is the same. Let $\pi(n)$ denote the number of partitions of n of the form $n = \lambda_1 + \cdots + \lambda_s$ with $\lambda_i - \lambda_{i+1} \geq 3$ and $\lambda_i - \lambda_{i+1} > 3$ if $3 \mid \lambda_i$. So we want to prove that

$$d(q) := \sum_{n\geq 0} \pi(n) q^n = \prod_{n=0}^{\infty} (1+q^{3n+1})(1+q^{3n+2}).$$

Let $\pi_m(n)$ denote the number of partitions $n = \lambda_1 + \cdots + \lambda_s$ counted by $\pi(n)$ such that the largest part λ_1 does not exceed m. Then the following holds.

Lemma 4.1. *Let m and n be positive integers. We have*

$$\pi_{3m+1}(n) = \pi_{3m}(n) + \pi_{3m-2}(n-3m-1), \tag{4.1}$$

$$\pi_{3m+2}(n) = \pi_{3m+1}(n) + \pi_{3m-1}(n-3m-2), \tag{4.2}$$

$$\pi_{3m+3}(n) = \pi_{3m+2}(n) + \pi_{3m-1}(n-3m-3). \tag{4.3}$$

Proof: We give a proof of (4.1). The other equations can be proved in the same way. We break the set of partitions enumerated by $\pi_{3m+1}(n)$ into two sets, those with largest part less than $3m+1$ and those with largest part equal to $3m+1$. The first one is enumerated by $\pi_{3m}(n)$. The second is enumerated by $\pi_{3m-2}(n-3m-1)$. To see this, we remove the largest part, so the number partitioned becomes $n-3m-1$. The largest part $\lambda_1 = 3m+1$ is congruent to 1 modulo 3, so by the difference conditions, λ_2 has to be $\leq 3m-2$ and we obtain a partition counted by $\pi_{3m-2}(n-3m-1)$. □

Define, for $m \geq 1$ and $|q| < 1$,

$$d_m(q) = 1 + \sum_{n=1}^{\infty} \pi_m(n) q^n.$$

As $m \to \infty$, $d_m(q) \to d(q)$, the generating function for partitions with difference conditions.

By Equations (4.1), (4.2) and (4.3), we have

$$d_{3m+1}(q) = d_{3m}(q) + q^{3m+1} d_{3m-2}(q), \tag{4.4}$$

$$d_{3m+2}(q) = d_{3m+1}(q) + q^{3m+2} d_{3m-1}(q), \tag{4.5}$$

$$d_{3m+3}(q) = d_{3m+2}(q) + q^{3m+3} d_{3m-1}(q). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), we obtain:

$$d_{3m+2}(q) = \left(1 + q^{3m+1} + q^{3m+2}\right) d_{3m-1}(q) + q^{3m}(1 - q^{3m}) d_{3m-4}(q). \quad (4.7)$$

Let

$$\alpha_m(q) = d_{3m+2}(q).$$

Then $\alpha_m(q)$ is uniquely determined by the recurrence

$$\alpha_m(q) = \left(1 + q^{3m+1} + q^{3m+2}\right) \alpha_{m-1}(q) + q^{3m}(1 - q^{3m}) \alpha_{m-2}(q),$$

following from (4.7), and the initial conditions $\alpha_{-1}(q) = 1$, $\alpha_0(q) = 1 + q + q^2$.

Now, for $|x| < 1$ and $|q| < 1$, define $s_n(q)$ by

$$\prod_{n \geq 0} \frac{\left(1 + xq^{3n+1}\right)\left(1 + xq^{3n+2}\right)}{(1 - xq^{3n})} = \sum_{n \geq 0} s_n(q) x^n, \quad (4.8)$$

and let

$$S_n(q) = \prod_{j=1}^{n} \left(1 - q^{3j}\right) \times s_n(q).$$

Calling the left-hand side of (4.8) $f(x;q)$, we have

$$(1 - x) f(x;q) = (1 + xq)(1 + xq^2) f(xq^3; q).$$

Thus $s_0(q) = 1$, $s_1(q) = \frac{1}{(1-q)}$, and for $n \geq 1$,

$$\left(1 - q^{3n}\right) s_n(q) = \left(1 + q^{3n-2} + q^{3n-1}\right) s_{n-1}(q) + q^{3n-3} s_{n-2}(q).$$

Therefore $S_0(q) = 1$, $S_1(q) = \frac{1-q^3}{1-q} = 1 + q + q^2$, and

$$S_m(q) = \left(1 + q^{3m+1} + q^{3m+2}\right) S_{m-1}(q) + q^{3m}(1 - q^{3m-3}) S_{m-2}(q),$$

so $S_{n+1}(q) = \alpha_n(q)$ for all $n \geq 0$.

Thus for $|x| < 1$ and $|q| < 1$,

$$\prod_{n \geq 0} \frac{\left(1 + xq^{3n+1}\right)\left(1 + xq^{3n+2}\right)}{(1 - xq^{3n})} = \sum_{m \geq 0} \frac{\alpha_{m-1}(q) x^m}{\prod_{j=1}^{m}(1 - q^{3j})}.$$

To conclude, we need Appell's Comparison Theorem [Die57, p. 101].

Theorem 4.2 (Appell's Comparison Theorem). *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim\limits_{n\to\infty} a_n$ is finite. Then*

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n = \lim_{n\to\infty} a_n.$$

Applying this, we obtain

$$\begin{aligned}\prod_{n\geq 0}\frac{\left(1+q^{3n+1}\right)\left(1+q^{3n+2}\right)}{\left(1-q^{3n+3}\right)} &= \lim_{x\to 1^-}(1-x)\sum_{m\geq 0}\frac{\alpha_{m-1}(q)x^m}{\prod_{j=1}^{m}(1-q^{3j})}\\ &= \lim_{m\to\infty}\frac{\alpha_{m-1}(q)}{\prod_{j=1}^{m}(1-q^{3j})}\\ &= \frac{d(q)}{\prod_{j\geq 1}(1-q^{3j})}.\end{aligned}$$

Thus

$$d(q) = \prod_{n\geq 0}\left(1+q^{3n+1}\right)\left(1+q^{3n+2}\right),$$

and Schur's theorem is proved.

4.1.2. Proof of Schur's theorem for overpartitions

We now adapt Andrews' proof presented in the previous section to prove Schur's theorem for overpartitions. The first steps, up to the analogue of the recurrence equation (4.7) (Equation (4.22)), are somewhat similar to Andrews', though we need to take into consideration the fact that the parts of the overpartition may be overlined or not, with different difference conditions in each case, which leads to more complicated equations and an additional variable. However the method used to solve the recurrence equation (4.22) is quite different and more intricate, but uses Appell's Comparison Theorem as well.

By Theorem 3.3, one easily sees that the generating function for overpartitions into parts congruent to 1 or 2 modulo 3, where we keep track of the number of non-overlined parts, is

$$\prod_{n=0}^{\infty}\frac{(1+q^{3n+1})(1+q^{3n+2})}{(1-dq^{3n+1})(1-dq^{3n+2})}.$$

Thus it suffices to prove that the generating function for overpartitions satisfying the difference conditions is the same.

Let $\pi_m(k,n)$ denote the number of overpartitions counted by $B(k,n)$ such that the largest part does not exceed m and is overlined. Let $\phi_m(k,n)$ denote the number of overpartitions of n counted by $B(k,n)$ such that the largest part does not exceed m and is non-overlined.

Notice that for every $m,n \geq 1$, $k \geq 0$, we have

$$\pi_m(k-1,n) = \phi_m(k,n) \tag{4.9}$$

because we can either overline the largest part or not.

Let us start by giving equations similar to (4.1), (4.2) and (4.3).

Lemma 4.3. *Let m and n be positive integers, and k be a non-negative integer. Then we have*

$$\pi_{3m+1}(k,n) = \pi_{3m}(k,n) + \phi_{3m+1}(k,n-3m-1) + \pi_{3m-2}(k,n-3m-1), \tag{4.10}$$

$$\pi_{3m+2}(k,n) = \pi_{3m+1}(k,n) + \phi_{3m+2}(k,n-3m-2) + \pi_{3m-1}(k,n-3m-2), \tag{4.11}$$

$$\pi_{3m+3}(k,n) = \pi_{3m+2}(k,n) + \phi_{3m+2}(k,n-3m-3) + \pi_{3m-1}(k,n-3m-3), \tag{4.12}$$

$$\phi_{3m+1}(k,n) = \phi_{3m}(k,n) + \phi_{3m+1}(k-1,n-3m-1) + \pi_{3m-2}(k-1,n-3m-1), \tag{4.13}$$

$$\phi_{3m+2}(k,n) = \phi_{3m+1}(k,n) + \phi_{3m+2}(k-1,n-3m-2) + \pi_{3m-1}(k-1,n-3m-2), \tag{4.14}$$

$$\phi_{3m+3}(k,n) = \phi_{3m+2}(k,n) + \phi_{3m+2}(k-1,n-3m-3) + \pi_{3m-1}(k-1,n-3m-3). \tag{4.15}$$

Proof: We give a proof of (4.10). The other equations can be proved in the same way. We break the set of overpartitions enumerated by $\pi_{3m+1}(k,n)$ into two sets, those with largest part less than $3m+1$ and those with largest part equal to $3m+1$. The first one is enumerated by $\pi_{3m}(k,n)$. The second is enumerated by $\phi_{3m+1}(k,n-3m-1) + \pi_{3m-2}(k,n-3m-1)$. To see this, we remove the largest part, so the number partitioned becomes $n-3m-1$. The largest part was overlined so the number of remaining non-overlined parts is still k. If the second part is overlined, it has to be $\leq 3m-2$ and we obtain an overpartition counted by $\pi_{3m-2}(k,n-3m-1)$. If it is not overlined, it has to be $\leq 3m+1$ and we obtain an overpartition counted by $\phi_{3m+1}(k,n-3m-1)$. $\square$

For all m,n,k, let $\psi_m(k,n) = \pi_m(k,n) + \phi_m(k,n)$.

Adding (4.10) and (4.13), and using (4.9), we obtain

$$\psi_{3m+1}(k,n) = \psi_{3m}(k,n) + \psi_{3m+1}(k-1,n-3m-1) + \psi_{3m-2}(k,n-3m-1). \tag{4.16}$$

Adding (4.11) and (4.14), and using (4.9), we obtain

$$\psi_{3m+2}(k,n) = \psi_{3m+1}(k,n) + \psi_{3m+2}(k-1,n-3m-2) + \psi_{3m-1}(k,n-3m-2). \tag{4.17}$$

Adding (4.12) and (4.15), and using (4.9), we obtain

$$\psi_{3m+3}(k,n) = \psi_{3m+2}(k,n) + \psi_{3m+2}(k-1,n-3m-3) + \psi_{3m-1}(k,n-3m-3). \tag{4.18}$$

We define, for $m \geq 1$, $|q| < 1$, $|d| < 1$,

$$a_m(q,d) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \psi_m(k,n) q^n d^k,$$

and we set $a_0(q,d) = a_{-1}(q,d) = a_{-2}(q,d) = 1$ and $a_{-m}(q,d) = 0$ for $m \geq 3$.

As $m \to \infty$, $a_m(q,d) \to a(q,d)$ where $a(q,d)$ is the generating function for overpartitions counted by $B(k,n)$.

By (4.16), (4.17) and (4.18), we obtain:

$$(1-dq^{3m+1})a_{3m+1}(q,d) = a_{3m}(q,d) + q^{3m+1}a_{3m-2}(q,d), \tag{4.19}$$

$$(1-dq^{3m+2})a_{3m+2}(q,d) = a_{3m+1}(q,d) + q^{3m+2}a_{3m-1}(q,d), \tag{4.20}$$

$$a_{3m+3}(q,d) = (1+dq^{3m+3})a_{3m+2}(q,d) + q^{3m+3}a_{3m-1}(q,d). \tag{4.21}$$

Substituting (4.20) and (4.21) into (4.19), we obtain:

$$\begin{aligned} &(1-dq^{3m+1})(1-dq^{3m+2})a_{3m+2}(q,d) = \\ &(1+q^{3m+1}+q^{3m+2}+dq^{3m}-dq^{6m}-dq^{6m+3})a_{3m-1}(q,d) \\ &+q^{3m}(1-q^{3m})a_{3m-4}(q,d). \end{aligned} \tag{4.22}$$

Of course if we set $d = 0$, which corresponds to the case where all parts are overlined, in (4.22), we obtain (4.7).

Let

$$\alpha_m(q,d) = a_{3m+2}(q,d).$$

Then, $\alpha_{-1}(q,d) = 1$, $\alpha_{-2}(q,d) = 0$ and by (4.22) we have:

$$\begin{aligned} &(1-dq^{3m+1})(1-dq^{3m+2})\alpha_m(q,d) = \\ &(1+q^{3m+1}+q^{3m+2}+dq^{3m}-dq^{6m}-dq^{6m+3})\alpha_{m-1}(q,d) \\ &+q^{3m}(1-q^{3m})\alpha_{m-2}(q,d). \end{aligned} \tag{4.23}$$

The key in Andrews' proof was to divide $\alpha_n(q,d)$ by $\prod_{j=1}^{n}(1-q^{3j})$ to simplify the recurrence equation and get powers of q of at most $3m$ plus some constant, which can then be translated into a q-difference equation involving only $f(x;q)$ and $f(xq^3;q)$, which can be easily solved by iterating.

Here, because of the new variable d and the new factors it introduces in the left-hande side of (4.23) we cannot simply divide $\alpha_n(q,d)$ by $\prod_{j=1}^{n}(1-q^{3j})$, because then we would obtain terms in $q^{9m+constant}$ on the left-hand side and this would lead to a q-difference equation involving $f(x), f(xq^3), f(xq^6)$ and $f(xq^9)$, which would be quite difficult to solve. Unfortunately we cannot even multiply $\alpha_n(q,d)$ by anything to get powers of q of at most $3m$ plus some constant and find a first order q-difference equation, but we will be able to modify $\alpha_n(q,d)$ to get another second order recurrence equation, which leads to a second order q-difference equation which can be modified again in order to lead to a first order recurrence equation, which is easy to solve. We then conclude using Appell's Comparison Theorem twice.

Let

$$\beta_m(q,d) = \alpha_m(q,d)\prod_{k=1}^{m}\frac{(1-dq^{3k-1})}{(1-q^{3k})}.$$

Then $\beta_{-1}=1$, $\beta_{-2}=0$, and by (4.23), we obtain

$$\begin{aligned}&(1-dq^{3m-2})(1-q^{3m})\beta_m(q,d) = \\ &(1+q^{3m-1}+q^{3m-2}+dq^{3m-3}-dq^{6m-6}-dq^{6m-3})\beta_{m-1}(q,d) \\ &+q^{3m-3}(1-dq^{3m-4})\beta_{m-2}(q,d).\end{aligned} \tag{4.24}$$

Let us now translate this into a q-difference equation. For $|x|<1$, let

$$f(x) = \sum_{m=0}^{\infty}\beta_m(q,d)x^m.$$

Then $f(0)=1$ and by (4.24) we deduce

$$\begin{aligned}(1-x)f(x) &= (dq^{-2}+1+xq)(1+xq^2)f(xq^3) \\ &\quad - dq^{-2}(1+xq^2)(1+xq^5)f(xq^6).\end{aligned}$$

Let

$$f(x) = F(x)\prod_{k=1}^{\infty}(1+xq^{3k-1}).$$

Thus $F(0)=1$ and

$$(1-x)F(x) = (dq^{-2}+1+xq)F(xq^3) - dq^{-2}F(xq^6).$$

Now turn back to recurrence equations. Define s_n as

$$F(x) = \sum_{n=0}^{\infty} s_n x^n.$$

Then $s_0 = F(0) = 1$ and

$$(1 - dq^{3n-2})(1 - q^{3n})s_n = (1 + q^{3n-2})s_{n-1}.$$

So

$$s_n = \prod_{k=1}^{n} \frac{(1 + q^{3k-2})}{(1 - dq^{3k-2})(1 - q^{3k})}.$$

We have

$$\sum_{m=0}^{\infty} \beta_m(q,d)x^m = f(x) = \prod_{k=1}^{\infty}(1 + xq^{3k-1}) \sum_{n=0}^{\infty} s_n x^n.$$

Thus

$$\lim_{x \to 1^-} (1 - x) \sum_{m=0}^{\infty} \beta_m(q,d)x^m = \prod_{k=1}^{\infty}(1 + q^{3k-1}) \times \lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} s_n x^n.$$

Using Appell's Comparison Theorem we deduce that

$$\lim_{m \to \infty} \beta_m(q,d) = \prod_{k=1}^{\infty}(1 + q^{3k-1}) \times \lim_{n \to \infty} s_n = \prod_{k=1}^{\infty} \frac{(1 + q^{3k-1})(1 + q^{3k-2})}{(1 - dq^{3k-2})(1 - q^{3k})}.$$

Finally

$$\alpha_m(q,d) = \prod_{k=1}^{m} \frac{(1 - q^{3k})}{(1 - dq^{3k-1})} \beta_m(q,d).$$

So

$$\lim_{m \to \infty} \alpha_m(q,d) = \prod_{k=1}^{\infty} \frac{(1 + q^{3k-1})(1 + q^{3k-2})}{(1 - dq^{3k-1})(1 - dq^{3k-2})},$$

which completes the proof of Schur's theorem for overpartitions.

4.2. Proofs using recurrences based on the smallest part

4.2.1. Andrews' proof of Schur's theorem

While the proofs of Section 4.1 relied on recurrences based on the largest part of the partitions (or overpartitions), here we focus on proofs relying on recurrences based on the smallest part of the partitions. We start by presenting the second of Andrews' proofs of Schur's theorem [And68b], which he published not long after the proof of the previous section [And67b].

Let $b_j(m,n)$ denote the number of partitions of n into m parts such that $n = \lambda_1 + \cdots + \lambda_m$, $\lambda_i - \lambda_{i+1} \geq 3$, and if $3 \mid \lambda_i$, then $\lambda_i - \lambda_{i+1} > 3$, such that the smallest part λ_m is larger than j. We have the following recurrences.

Lemma 4.4. *For $m, n \geq 1$, we have*

$$b_0(m,n) - b_1(m,n) = b_0(m-1, n-3m+2), \tag{4.25}$$

$$b_1(m,n) - b_2(m,n) = b_1(m-1, n-3m+1), \tag{4.26}$$

$$b_2(m,n) - b_3(m,n) = b_3(m-1, n-3m), \tag{4.27}$$

$$b_3(m,n) = b_0(m, n-3m). \tag{4.28}$$

Proof: We prove (4.27). The other identities can be proved in the same way. Now $b_2(m,n) - b_3(m,n)$ denotes the number of partitions with difference conditions such that the smallest part is equal to 3. By the difference conditions, the second smallest part λ_{m-1} is at least 7. We subtract 3 from every part. The new partition will satisfy the same difference conditions, as they only depend on the value of parts modulo 3. The number of parts is reduced to $m-1$, the number partitioned is reduced to $n-3m$, and the smallest part is now ≥ 4. So we obtain the partitions enumerated by $b_3(m-1, n-3m)$. $\square$

For $\mid q \mid < 1$, let us define

$$f_i(x) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_i(m,n) x^m q^n.$$

We want to find $f_0(1)$, which is the generating function for all partitions with the difference conditions of Schur's theorem.

Translating (4.25), (4.26), (4.27) and (4.28) into q-difference equations on the generating functions leads to

$$f_0(x) - f_1(x) = xq f_0(xq^3), \tag{4.29}$$

$$f_1(x) - f_2(x) = xq^2 f_1(xq^3), \tag{4.30}$$
$$f_2(x) - f_3(x) = xq^3 f_3(xq^3), \tag{4.31}$$
$$f_3(x) = f_0(xq^3). \tag{4.32}$$

Thus by (4.29),

$$f_1(x) = f_0(x) - xq f_0(xq^3). \tag{4.33}$$

By (4.31) and (4.32),

$$f_2(x) = f_0(xq^3) + xq^3 f_0(xq^6). \tag{4.34}$$

Substituting (4.33) and (4.34) into (4.30), we obtain:

$$f_0(x) = (1 + xq + xq^2) f_0(xq^3) + xq^3(1 - xq^3) f_0(xq^6). \tag{4.35}$$

Let

$$F(x) = f_0(x) \prod_{n=0}^{\infty} \frac{1}{(1 - xq^{3n})}.$$

Then by (4.35),

$$(1 - x)F(x) = (1 + xq + xq^2)F(xq^3) + xq^3 F(xq^6).$$

Now that the highest power of x in this equation is 1, we can translate it into a first order recurrence equation.

Let $F(x) = \sum_{n=0}^{\infty} A_n(q)x^n$. Then $A_0(q) = 1$ and

$$A_n(q) - A_{n-1}(q) = q^{3n}A_n(q) + q^{3n-2}A_{n-1}(q) + q^{3n-1}A_{n-1}(q) + q^{6n-3}A_{n-1}(q),$$

which can be rewritten as

$$A_n(q) = \frac{(1 + q^{3n-1})(1 + q^{3n-2})}{(1 - q^{3n})} A_{n-1}(q). \tag{4.36}$$

Then, iterating (4.36) and using the fact that $A_0(q) = 1$ gives

$$A_n(q) = \prod_{j=1}^{n} \frac{(1 + q^{3j-1})(1 + q^{3j-2})}{(1 - q^{3j})}.$$

Hence

$$f_0(x) = \prod_{n=0}^{\infty}(1 - xq^{3n}) \sum_{m=0}^{\infty} x^m \prod_{j=1}^{m} \frac{(1 + q^{3j-1})(1 + q^{3j-2})}{(1 - q^{3j})}.$$

In his original proof, Andrews used Watson's q-analogue of Whipple's theorem [GR04]. However, here we conclude using Appell's Comparison Theorem as well, as it seems simpler.

By Appell's Comparison Theorem, we obtain

$$\begin{aligned}
f_0(1) &= \lim_{x\to 1}(1-x)\prod_{n=1}^{\infty}(1-xq^{3n})\sum_{n=0}^{\infty}x^n\prod_{j=1}^{n}\frac{(1+q^{3j-1})(1+q^{3j-2})}{(1-q^{3j})} \\
&= \prod_{n=1}^{\infty}(1-q^{3n})\prod_{j=1}^{\infty}\frac{(1+q^{3j-1})(1+q^{3j-2})}{(1-q^{3j})} \\
&= \prod_{j=1}^{\infty}(1+q^{3j-1})(1+q^{3j-2}).
\end{aligned}$$

Thus $f_0(1)$ is the generating function for partitions into distinct parts congruent to 1 or 2 modulo 3, which concludes the proof.

4.2.2. Proof of Schur's theorem for overpartitions

We now present a proof of Schur's theorem for overpartitions based on Andrews' proof. As in our proof of Secton 4.1, the beginning of the proof, up to the analogue of the q-difference equation (4.35) (Equation (4.47)) is similar to Andrews', except that we keep track of the number of non-overlined parts in the partition. Again, the method to solve this q-difference equation is different from Andrews' and requires new techniques, which are similar to those used in our proof of Schur's theorem for overpartitions of Section 4.1.

Let $b_j(k,m,n)$ denote the number of overpartitions $n=\lambda_1+\cdots+\lambda_m$ counted by $B(k,n)$ having m parts such that the smallest part λ_m is larger than j. The following recurrence equations hold.

Lemma 4.5. *For $m,n \geq 1$, we have*

$$b_0(k,m,n)-b_1(k,m,n) = b_0(k,m-1,n-3m+2)+b_0(k-1,m-1,n-1), \quad (4.37)$$

$$b_1(k,m,n)-b_2(k,m,n) = b_1(k,m-1,n-3m+1)+b_1(k-1,m-1,n-2), \quad (4.38)$$

$$b_2(k,m,n)-b_3(k,m,n) = b_3(k,m-1,n-3m)+b_0(k-1,m-1,n-3m), \quad (4.39)$$

$$b_3(k,m,n) = b_0(k,m,n-3m). \quad (4.40)$$

Proof: We observe that $b_{i-1}(k,m,n)-b_i(k,m,n)$ is the number of overpartitions counted by $b_{i-1}(k,m,n)$ such that the smallest part is equal to i. We begin by treating (4.37) : If $\lambda_m = \overline{1}$, then $\lambda_{m-1} \geq 4$. In that case we remove the $\overline{1}$ and subtract 3 from each remaining part. The number of parts is reduced to $m-1$, the number of non-overlined parts is still k, and the number partitioned is now $n-3m+2$. So we have an overpartition counted by $b_0(k, m-1, n-3m+2)$. If $\lambda_m = 1$, then $\lambda_{m-1} \geq 1$. In that case we remove λ_m. The number of parts is reduced to $m-1$, the number of non-overlined parts is reduced to $k-1$, and the number partitioned is now $n-1$. We have an overpartition counted by $b_0(k-1, m-1, n-1)$. Equations (4.38), (4.39) and (4.40) can be proved in the same way. $\square$

For $|x| < 1$, $|d| < 1$, $|q| < 1$, define

$$f_i(x, d, q) = f_i(x) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} b_i(k, m, n) x^m d^k q^n.$$

We want to find $f_0(1)$, which is the generating function for overpartitions counted by $B(k, n)$.

Translating (4.37), (4.38), (4.39) and (4.40) into q-difference equations on the generating functions leads to

$$f_0(x) - f_1(x) = xqf_0(xq^3) + dxqf_0(x), \tag{4.41}$$

$$f_1(x) - f_2(x) = xq^2 f_1(xq^3) + dxq^2 f_1(x), \tag{4.42}$$

$$f_2(x) - f_3(x) = xq^3 f_3(xq^3) + dxq^3 f_0(xq^3), \tag{4.43}$$

$$f_3(x) = f_0(xq^3). \tag{4.44}$$

Thus by (4.41),

$$f_1(x) = (1 - dxq) f_0(x) - xqf_0(xq^3). \tag{4.45}$$

By (4.43) and (4.44),

$$f_2(x) = (1 + dxq^3) f_0(xq^3) + xq^3 f_0(xq^6). \tag{4.46}$$

Substituting (4.45) and (4.46) into (4.42), we obtain

$$\begin{aligned}(1 - dxq)(1 - dxq^2) f_0(x) = (1 + xq + xq^2 + dxq^3 - dx^2q^3 - dx^2q^6) f_0(xq^3) \\ + xq^3(1 - xq^3) f_0(xq^6).\end{aligned} \tag{4.47}$$

Again, if we set $d = 0$ in (4.47), we find (4.35).

The key in Andrews' proof was to divide $f_0(x)$ by $\prod_{n \geq 0}(1 - xq^{3n})$ to simplify the q-difference equation and get powers of x of at most 1, so it can be translated into a first order recurrence equation, which can be solved easily.

Here, as in Section 4.1, because of the new variable d and the new factors in x^2 it introduces in (4.47), we cannot simplify the equation so easily. However we are able to modify $f_0(x)$ to get another second order q-difference equation, still with some terms in x^2, but which leads to a second order recurrence equation which can be modified again in order to lead to a first order q-difference equation, which is easy to solve. As in Section 4.1, we conclude using Appell's Comparison Theorem twice.

Let

$$F(x) = f_0(x) \prod_{k=0}^{\infty} \frac{(1 - dxq^{3k+1})}{(1 - xq^{3k})}.$$

Then by (4.47),

$$\begin{aligned}(1-x)(1-dxq^2)F(x) = {} & (1 + xq + xq^2 + dxq^3 - dx^2q^3 - dx^2q^6)F(xq^3) \\ & + xq^3(1 - dxq^4)F(xq^6).\end{aligned}$$

Let us define A_n as

$$F(x) = \sum_{n=0}^{\infty} A_n x^n.$$

Then $A_0 = F(0) = f_0(0) = 1$ and

$$\begin{aligned}& A_n - A_{n-1} - dq^2A_{n-1} + dq^2A_{n-2} = \\ & q^{3n}A_n + (q^{3n-2} + q^{3n-1} + dq^{3n})A_{n-1} \\ & - (dq^{3n-3} + dq^{3n})A_{n-2} + q^{6n-3}A_{n-1} \\ & - dq^{6n-5}A_{n-2}.\end{aligned}$$

Factorising, we obtain

$$\begin{aligned}(1 - q^{3n})A_n = {} & (1 + dq^2 + q^{3n-1})(1 + q^{3n-2})A_{n-1} \\ & - dq^2(1 + q^{3n-2})(1 + q^{3n-5})A_{n-2}.\end{aligned}$$

Now let us define a_n as

$$A_n = a_n \prod_{k=0}^{n-1}(1 + q^{3k+1}).$$

Then $a_0 = A_0 = 1$ and

$$(1 - q^{3n})a_n = (1 + dq^2 + q^{3n-1})a_{n-1} - dq^2a_{n-2}.$$

Finally define

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Thus $f(0) = a_0 = 1$ and

$$(1-x)(1-dxq^2)f(x) = (1+xq^2)f(xq^3). \tag{4.48}$$

Iterating (4.48) and using the fact that $f(0) = 1$ leads to

$$f(x) = \prod_{k=0}^{\infty} \frac{(1+xq^{3k+2})}{(1-xq^{3k})(1-dxq^{3k+2})}.$$

Next,

$$\sum_{n=0}^{\infty} \frac{A_n x^n}{\prod_{k=0}^{n-1}(1+q^{3k+1})} = \sum_{n=0}^{\infty} a_n x^n = f(x) = \prod_{k=0}^{\infty} \frac{(1+xq^{3k+2})}{(1-xq^{3k})(1-dxq^{3k+2})}.$$

By Appell's Comparison Theorem,

$$\begin{aligned}
\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} \frac{A_n x^n}{\prod_{k=0}^{n-1}(1+q^{3k+1})} &= \frac{\lim\limits_{n\to\infty} A_n}{\prod_{k=0}^{\infty}(1+q^{3k+1})} \\
&= \prod_{k=0}^{\infty} \frac{(1+q^{3k+2})}{(1-q^{3k+3})(1-dq^{3k+2})}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} A_n = \prod_{k=0}^{\infty} \frac{(1+q^{3k+2})(1+q^{3k+1})}{(1-q^{3k+3})(1-dq^{3k+2})}.$$

Next,

$$f_0(x) = \prod_{k=0}^{\infty} \frac{(1-xq^{3k})}{(1-dxq^{3k+1})} \sum_{n=0}^{\infty} A_n x^n = (1-x)\prod_{k=0}^{\infty} \frac{(1-xq^{3k+3})}{(1-dxq^{3k+1})} \sum_{n=0}^{\infty} A_n x^n.$$

We apply Appell's Comparison Theorem again and obtain

$$\begin{aligned}
f_0(1) &= \prod_{k=0}^{\infty} \frac{(1-xq^{3k+3})}{(1-dxq^{3k+1})} \lim_{n\to\infty} A_n \\
&= \prod_{k=0}^{\infty} \frac{(1+q^{3k+1})(1+q^{3k+2})}{(1-dq^{3k+1})(1-dq^{3k+2})}.
\end{aligned}$$

This completes the proof.

4.3. Proofs based on the largest part and parts counted twice

4.3.1. Andrews' proof of Schur's theorem

Let us now turn to the third and simplest of Andrews' proofs of Schur's theorem, published in 1971 [And71b]. The beginning of this proof follows the same principle as the proof of Section 4.1, except that parts congruent to 0 modulo 3 are counted twice. This actually proves the following refinement of Schur's theorem, due to Gleissberg [Gle28].

Theorem 4.6 (Gleissberg)**.** *Let $C(m,n)$ denote the number of partitions of n into m distinct parts congruent to 1 or 2 modulo 3. Let $D(m,n)$ denote the number of partitions of n into m parts (where parts divisible by 3 are counted twice), where parts differ by at least 3 and no two consecutive multiples of three appear. Then for all $m,n \geq 0$, $C(m,n) = D(m,n)$.*

Let $\pi_m(M,n)$ denote the number of partitions counted by $D(M,n)$ such that the largest part does not exceed m. We have the following recurrence equations.

Lemma 4.7. *Let M, m, n be positive integers. Then*

$$\pi_{3m+1}(M,n) = \pi_{3m}(M,n) + \pi_{3m-2}(M-1, n-3m-1), \tag{4.49}$$

$$\pi_{3m+2}(M,n) = \pi_{3m+1}(M,n) + \pi_{3m-1}(M-1, n-3m-2), \tag{4.50}$$

$$\pi_{3m+3}(M,n) = \pi_{3m+2}(M,n) + \pi_{3m-1}(M-2, n-3m-3). \tag{4.51}$$

Proof: We give a proof of (4.49). The other equations can be proved in the same way. We break the set of partitions enumerated by $\pi_{3m+1}(M,n)$ into two sets, those with largest part less than $3m+1$ and those with largest part equal to $3m+1$. The first one is enumerated by $\pi_{3m}(M,n)$. The second is enumerated by $\pi_{3m-2}(M-1, n-3m-1)$. To see this, we remove the largest part, so the number partitioned becomes $n-3m-1$. Because of the difference conditions, the second largest part is $\leq 3m-2$ and we obtain an overpartition counted by $\pi_{3m-2}(M-1, n-3m-1)$. □

Define, for $|q| < 1$, $|t| < 1$,

$$a_m(t,q) = 1 + \sum_{M=1}^{\infty} \sum_{n=1}^{\infty} \pi_m(M,n) t^M q^n.$$

As $m \to \infty$, $a_m(t,q) \to a(t,q)$ where $a(t,q)$ is the generating function for partitions counted by $D(M,n)$.

By (4.49), (4.50) and (4.51), we obtain

$$a_{3m+1}(t,q) = a_{3m}(t,q) + tq^{3m+1}a_{3m-2}(t,q), \tag{4.52}$$

$$a_{3m+2}(t,q) = a_{3m+1}(t,q) + tq^{3m+2}a_{3m-1}(t,q), \tag{4.53}$$

$$a_{3m+3}(t,q) = a_{3m+2}(t,q) + t^2q^{3m+3}a_{3m-1}(t,q). \tag{4.54}$$

By (4.52), (4.53) and (4.54) we obtain:

$$\begin{aligned} a_{3m+2}(t,q) =&(1+tq^{3m+1}+tq^{3m+2})a_{3m-1}(t,q) \\ &+t^2q^{3m}(1-q^{3m})a_{3m-4}(t,q). \end{aligned} \tag{4.55}$$

Replacing m by $m-1$ in (4.55) we obtain

$$\begin{aligned} a_{3m-1}(t,q) =&(1+tq^{3m-1}+tq^{3m-2})a_{3m-4}(t,q) \\ &+t^2q^{3m-3}(1-q^{3m-3})a_{3m-7}(t,q). \end{aligned} \tag{4.56}$$

Now multiply (4.56) by t^2q^{3m+3} and add the result with (4.55), hence

$$\begin{aligned} &a_{3m+2}(t,q) + t^2q^{3m+3}a_{3m-1}(t,q) = \\ &\quad (1+tq^{3m+1}+tq^{3m+2})(a_{3m-1}(t,q)+t^2q^{3m+3}a_{3m-4}(t,q)) \\ &\quad +t^2q^{3m+3}(1-q^{3m-3})(a_{3m-4}(t,q)+t^2q^{3m-3}a_{3m-7}(t,q)). \end{aligned} \tag{4.57}$$

Thus by (4.54) and (4.57),

$$\begin{aligned} a_{3m+3}(t,q) =&(1+tq^{3m+1}+tq^{3m+2})a_{3m}(t,q) \\ &+t^2q^{3m+3}(1-q^{3m-3})a_{3m-3}(t,q). \end{aligned} \tag{4.58}$$

Now let us replace t by tq^3 in (4.56). This gives

$$\begin{aligned} a_{3m-1}(tq^3,q) =&(1+tq^{3m+1}+tq^{3m+2})a_{3m-4}(tq^3,q) \\ &+t^2q^{3m+3}(1-q^{3m-3})a_{3m-7}(tq^3,q). \end{aligned} \tag{4.59}$$

So $a_{3m+3}(t,q)$ satisfies the same recurrence equation as $a_{3m-1}(tq^3,q)$.
Furthermore, by (4.52), (4.53) and (4.54), we obtain:

$$a_3(t,q) = (1+tq)(1+tq^2) = (1+tq)(1+tq^2)a_{-1}(tq^3,q), \tag{4.60}$$

and

$$\begin{aligned} a_6(t,q) &= (1+tq)(1+tq^2)(1+tq^4+tq^5-dt^2q^9) \\ &= (1+tq)(1+tq^2)a_2(tq^3,q). \end{aligned} \tag{4.61}$$

Using the recurrence equation satisfied by $a_{3m+3}(t,q)$ and $a_{3m-1}(tq^3,q)$ and (4.60) and (4.61), by mathematical induction, we have for all $m \geq 0$:

$$a_{3m+3}(t,q) = (1+tq)(1+tq^2)a_{3m-1}(tq^3,q).$$

So, letting $m \to \infty$, we obtain

$$\lim_{m\to\infty} a_m(t,q) = (1+tq)(1+tq^2)\lim_{m\to\infty} a_m(tq^3,q). \tag{4.62}$$

Iteration of (4.62) shows that:

$$\lim_{m\to\infty} a_m(t,q) = \prod_{n=0}^{\infty}(1+tq^{3n+1})(1+tq^{3n+2}). \tag{4.63}$$

This completes the proof.

4.3.2. Proof of Schur's theorem for overpartitions

Now let us adapt Andrews' proof to prove the following refinement of Schur's theorem for overpartitions, which can already be deduced from Lovejoy's proof [Lov05b].

Theorem 4.8 (Lovejoy)**.** *Let $A(k,m,n)$ denote the number of overpartitions of n into m parts congruent to 1 or 2 modulo 3 with k non-overlined parts. Let $B(k,m,n)$ denote the number of overpartitions of n having k non-overlined parts, where m equals the number of parts congruent to 1 or 2 modulo 3 plus twice the number of parts congruent to 0 modulo 3, and satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 3\chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_{i+1} \equiv 1,2 \mod 3, \\ 1 + 3\chi(\overline{\lambda_{i+1}}) \text{ if } \lambda_{i+1} \equiv 0 \mod 3, \end{cases}$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all $k,m,n \geq 0$, $A(k,m,n) = B(k,m,n)$.

Here the proof is quite similar to Andrews'.

Let $\pi_m(k,M,n)$ denote the number of overpartitions counted by $B(k,M,n)$ such that the largest part does not exceed m and is overlined. Let $\phi_m(k,M,n)$ denote the number of overpartitions counted by $B(k,M,n)$ such that the largest part does not exceed m and is non-overlined.

Notice that for every $M,m,n \geq 1$, $k \geq 0$, we have

$$\pi_m(M,k-1,n) = \phi_m(M,k,n), \tag{4.64}$$

because we can either overline the largest part or not. We have the following recurrence equations.

Lemma 4.9. *Let M, m, n be positive integers and k be a non-negative integer. Then*

$$\begin{aligned}\pi_{3m+1}(k,M,n) = \pi_{3m}(k,M,n) + \phi_{3m+1}(k,M-1,n-3m-1)\\ + \pi_{3m-2}(k,M-1,n-3m-1),\end{aligned} \tag{4.65}$$

$$\begin{aligned}\pi_{3m+2}(k,M,n) = \pi_{3m+1}(k,M,n) + \phi_{3m+2}(k,M-1,n-3m-2)\\ + \pi_{3m-1}(k,M-1,n-3m-2),\end{aligned} \tag{4.66}$$

$$\begin{aligned}\pi_{3m+3}(k,M,n) = \pi_{3m+2}(k,M,n) + \phi_{3m+2}(k,M-2,n-3m-3)\\ + \pi_{3m-1}(k,M-2,n-3m-3),\end{aligned} \tag{4.67}$$

$$\begin{aligned}\phi_{3m+1}(M,k,n) = \phi_{3m}(k,M,n) + \phi_{3m+1}(k-1,M-1,n-3m-1)\\ + \pi_{3m-2}(k-1,M-1,n-3m-1),\end{aligned} \tag{4.68}$$

$$\begin{aligned}\phi_{3m+2}(k,M,n) = \phi_{3m+1}(k,M,n) + \phi_{3m+2}(k-1,M-1,n-3m-2)\\ + \pi_{3m-1}(k-1,M-1,n-3m-2),\end{aligned} \tag{4.69}$$

$$\begin{aligned}\phi_{3m+3}(k,M,n) = \phi_{3m+2}(k,M,n) + \phi_{3m+2}(k-1,M-2,n-3m-3)\\ + \pi_{3m-1}(k-1,M-2,n-3m-3).\end{aligned} \tag{4.70}$$

Proof: We give a proof of (4.65). The other equations can be proved in the same way. We break the set of overpartitions enumerated by $\pi_{3m+1}(k,M,n)$ into two sets, those with largest part less than $3m+1$ and those with largest part equal to $3m+1$. The first one is enumerated by $\pi_{3m}(k,M,n)$. The second is enumerated by $\phi_{3m+1}(k,M-1,n-3m-1) + \pi_{3m-2}(k,M-1,n-3m-1)$. To see this, we remove the largest part, so the number partitioned becomes $n-3m-1$. The largest part was overlined so the number of remaining non-overlined parts is still k and the number of parts is now $M-1$. If the second part is overlined, it has to be $\leq 3m-2$ and we obtain an overpartition counted by $\pi_{3m-2}(k,M-1,n-3m-1)$. If it is not overlined, it has to be $\leq 3m+1$ and we obtain an overpartition counted by $\phi_{3m+1}(k,M-1,n-3m-1)$. □

For all k, M, m, n, let $\psi_m(k,M,n) = \pi_m(k,M,n) + \phi_m(k,M,n)$.
Adding (4.65) and (4.68), and using (4.64), we obtain

$$\begin{aligned}\psi_{3m+1}(k,M,n) = \psi_{3m}(k,M,n) + \psi_{3m+1}(k-1,M-1,n-3m-1)\\ + \psi_{3m-2}(k,M-1,n-3m-1).\end{aligned} \tag{4.71}$$

Adding (4.66) and (4.69), and using (4.64), we obtain

$$\begin{aligned}\psi_{3m+2}(k,M,n) = \psi_{3m+1}(k,M,n) + \psi_{3m+2}(k-1,M-1,n-3m-2)\\ + \psi_{3m-1}(k,M-1,n-3m-2).\end{aligned} \tag{4.72}$$

Adding (4.67) and (4.70), and using (4.64), we obtain

$$\begin{aligned}\psi_{3m+3}(k,M,n) = \psi_{3m+2}(k,M,n) + \psi_{3m+2}(k-1,M-2,n-3m-3)\\ + \psi_{3m-1}(k,M-2,n-3m-3).\end{aligned} \tag{4.73}$$

We define, for $|q|<1$, $|d|<1$, $|t|<1$,

$$a_m(d,t,q) = 1 + \sum_{k=0}^{\infty}\sum_{M=1}^{\infty}\sum_{n=1}^{\infty} \psi_m(k,M,n) d^k t^M q^n.$$

As $m \to \infty$, $a_m(d,t,q) \to a(d,t,q)$ where $a(d,t,q)$ is the generating function for overpartitions counted by $B(k,M,n)$.

By (4.71), (4.72) and (4.73), we obtain

$$(1-dtq^{3m+1})a_{3m+1}(d,t,q) = a_{3m}(d,t,q) + tq^{3m+1}a_{3m-2}(d,t,q), \tag{4.74}$$

$$(1-dtq^{3m+2})a_{3m+2}(d,t,q) = a_{3m+1}(d,t,q) + tq^{3m+2}a_{3m-1}(d,t,q), \tag{4.75}$$

$$a_{3m+3}(d,t,q) = (1+dt^2q^{3m+3})a_{3m+2}(d,t,q) + t^2q^{3m+3}a_{3m-1}(d,t,q). \tag{4.76}$$

By (4.74), (4.75) and (4.76) we obtain:

$$\begin{aligned}&(1-dtq^{3m+1})(1-dtq^{3m+2})a_{3m+2}(d,t,q) =\\ &(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m}-dt^2q^{6m}-dt^2q^{6m+3})a_{3m-1}(d,t,q)\\ &+t^2q^{3m}(1-q^{3m})a_{3m-4}(d,t,q).\end{aligned} \tag{4.77}$$

Replacing m by $m-1$ and t by tq^3 in (4.77), we obtain:

$$\begin{aligned}&(1-dtq^{3m+1})(1-dtq^{3m+2})a_{3m-1}(d,tq^3,q) =\\ &(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m+3}-dt^2q^{6m}-dt^2q^{6m+3})a_{3m-4}(d,tq^3,q)\\ &+t^2q^{3m+3}(1-q^{3m-3})a_{3m-7}(d,tq^3,q).\end{aligned} \tag{4.78}$$

As in Andrews' proof, we want to prove that $a_{3m+3}(d,t,q)$ satisfies the same recurrence equation (4.78) as $a_{3m-1}(d, tq^3, q)$. Using (4.76) we have

$$
\begin{aligned}
&(1-dtq^{3m+1})(1-dtq^{3m+2})a_{3m+3}(d,t,q)\\
&-(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m+3}-dt^2q^{6m}-dt^2q^{6m+3})a_{3m}(d,t,q)\\
&-t^2q^{3m+3}(1-q^{3m-3})a_{3m-3}(d,t,q)\\
=\,&(1-dtq^{3m+1})(1-dtq^{3m+2})\times\\
&\big((1+dt^2q^{3m+3})a_{3m+2}(q,d,t)+t^2q^{3m+3}a_{3m-1}(d,t,q)\big)\\
&-(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m+3}-dt^2q^{6m}-dt^2q^{6m+3})\times\\
&\big((1+dt^2q^{3m})a_{3m-1}(q,d,t)+t^2q^{3m}a_{3m-4}(d,t,q)\big)\\
&-t^2q^{3m+3}(1-q^{3m-3})\times\\
&\big((1+dt^2q^{3m-3})a_{3m-4}(d,t,q)+t^2q^{3m-3}a_{3m-7}(d,t,q)\big)\,.
\end{aligned}
\tag{4.79}
$$

Substituting (4.77) into (4.79), and after simplification we obtain

$$
\begin{aligned}
&(1-dtq^{3m+1})(1-dtq^{3m+2})a_{3m+3}(d,t,q)\\
&-(1+tq^{3m+1}+tq^{3m+2}+dt^2q^{3m+3}-dt^2q^{6m}-dt^2q^{6m+3})a_{3m}(d,t,q)\\
&-t^2q^{3m+3}(1-q^{3m-3})a_{3m-3}(d,t,q)\\
=\,&t^2q^{3m+3}[(1-dtq^{3m-1})(1-dtq^{3m-2})a_{3m-1}(d,t,q)\\
&-(1+tq^{3m-1}+tq^{3m-2}+dt^2q^{3m-3}-dt^2q^{6m-3}-dt^2q^{6m-6})a_{3m-4}(d,t,q)\\
&-t^2q^{3m-3}(1-q^{3m-3})a_{3m-7}(d,t,q)]\\
=\,&0,
\end{aligned}
$$

by (4.77) in which we have replaced m by $m-1$.

So $a_{3m+3}(d, t, q)$ satisfies the same recurrence equation as $a_{3m-1}(d, tq^3, q)$.

Furthermore, by (4.74), (4.75) and (4.76), we obtain:

$$
a_3(d,t,q)=\frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)}=\frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)}a_{-1}(d,tq^3,q), \tag{4.80}
$$

and

$$
\begin{aligned}
a_6(d,t,q)&=\frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)}\frac{(1+tq^4+tq^5-dt^2q^9)}{(1-dtq^4)(1-dtq^5)}\\
&=\frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)}a_2(d,tq^3,q).
\end{aligned}
\tag{4.81}
$$

Using the recurrence equation satisfied by $a_{3m+3}(d, t, q)$ and $a_{3m-1}(d, tq^3, q)$ and (4.80) and (4.81), by mathematical induction, we have for all $m \geq 0$:

$$
a_{3m+3}(d,t,q)=\frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)}a_{3m-1}(d,tq^3,q).
$$

So, letting $m \to \infty$, we obtain

$$\lim_{m\to\infty} a_m(d,t,q) = \frac{(1+tq)(1+tq^2)}{(1-dtq)(1-dtq^2)} \lim_{m\to\infty} a_m(d,tq^3,q). \tag{4.82}$$

Iteration of (4.82) shows that:

$$\lim_{m\to\infty} a_m(d,t,q) = \prod_{n=0}^{\infty} \frac{(1+tq^{3n+1})(1+tq^{3n+2})}{(1-dtq^{3n+1})(1-dtq^{3n+2})}. \tag{4.83}$$

This completes our final proof of Schur's theorem for overpartitions.

5. Generalisation of a theorem of Andrews

Contents

5.1. Andrews' first generalisation of Schur's theorem

5.1.1. Statement of the theorem

Now that we have studied Andrews' three proofs of Schur's theorem, let us focus on a more general theorem of Andrews, proved in [And69a], which he found by generalising his proof of Schur's theorem based on the smallest part of the partition [And68b] explained in Section 4.2.

First recall some notation due to Andrews already defined in the introduction. Let $A = \{a(1), \dots, a(r)\}$ be a set of r distinct integers such that $\sum_{i=1}^{k-1} a(i) < a(k)$ for all $1 \leq k \leq r$ and the $2^r - 1$ possible sums of distinct elements of A are all distinct. We denote this set of sums by $A' = \{\alpha(1), \dots, \alpha(2^r-1)\}$, where $\alpha(1) < \cdots < \alpha(2^r-1)$. Notice that $\alpha(2^k) = a(k+1)$ for all $0 \leq k \leq r-1$ and that any α between $a(k)$ and $a(k+1)$ has largest summand $a(k)$. Let N be a positive integer with $N \geq \alpha(2^r-1) = a(1) + \cdots + a(r)$. Let A_N denote the set of positive integers congruent to some $a(i) \mod N$ and A'_N the set of positive integers congruent to some $\alpha(i) \mod N$. Let $\beta_N(m)$ be the least positive residue of $m \mod N$. If $\alpha \in A'$, let $w(\alpha)$ be the number of terms appearing in the defining sum of α and $v(\alpha)$ the smallest $a(i)$ appearing in this sum. Let us also define $\alpha(2^r) := a(r+1) = N + a(1)$.

To have a better intuitive idea of these notations, it might be useful to consider the example where $a(k) = 2^{k-1}$ for $1 \leq k \leq r$ and $\alpha(k) = k$ for $1 \leq k \leq 2^r - 1$.

Let us also recall Andrews' theorem (Theorem 2.13 from the introduction). Schur's theorem corresponds to the case $N = 3$, $r = 2$, $a(1) = 1, a(2) = 2$.

Theorem 5.1 (Andrews)**.** *Let $D(A_N;n)$ denote the number of partitions of n into distinct parts taken from A_N. Let $E(A'_N;n)$ denote the number of partitions of n into parts taken from A'_N of the form $n = \lambda_1 + \cdots + \lambda_s$, such that*

$$\lambda_i - \lambda_{i+1} \geq Nw(\beta_N(\lambda_{i+1})) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then for all $n \geq 0$, $D(A_N;n) = E(A'_N;n)$.

This theorem led to a number of important developments in combinatorics [All97, CL06, Yee08] and also plays a natural role in group representation theory [AO91] and quantum algebra [Oh15].

5.1.2. Proof

We now present Andrews' proof of Theorem 5.1.

As in the proofs of Schur's theorem, by Theorem 3.2, the generating function for partitions enumerated by $D(A_N;n)$ is

$$\sum_{n\geq 0} D(A_N;n)q^n = \prod_{k=1}^{r} (-q^{a(k)};q^N)_\infty,$$

where we recall that $(a;q)_\infty = \prod_{k\geq 0}(1-aq^k)$. So we only need to show that the generating function for partitions enumerated by $E(A'_N;n)$ is the same. First, we give the q-difference equation satisfied by the generating function for partitions enumerated by $E(A'_N;n)$, where we also keep track on the number of parts. Then we conclude by switching to a recurrence equation and using Appell's Comparison Theorem as in the proof of Schur's theorem from Section 4.2.

Let $p_{\alpha(i)}(m,n)$ denote the number of partitions counted by $E(A'_N;n)$ having m parts such that the smallest part is at least $\alpha(i)$.

The following lemma holds.

Lemma 5.2. *If* $1 \leq i \leq 2^r - 1$, *then*

$$p_{\alpha(i)}(m,n) - p_{\alpha(i+1)}(m,n) = p_{v(\alpha(i))}\big(m-1, n-(m-1)Nw(\alpha(i)) - \alpha(i)\big), \tag{5.1}$$

$$p_{\alpha(2^r)}(m,n) = p_{a(1)}(m, n-mN). \tag{5.2}$$

Proof: Let us start by proving (5.1).

We observe that $p_{\alpha(i)}(m,n) - p_{\alpha(i+1)}(m,n)$ is the number of partitions of the form $n = \lambda_1 + \cdots + \lambda_m$ enumerated by $E(A'_N; n)$ such that the smallest part is equal to $\alpha(i)$. By the definition of $E(A'_N; n)$, we have

$$\lambda_{m-1} \geq \alpha(i) + Nw(\alpha(i)) + v(\alpha(i)) - \alpha(i) = Nw(\alpha(i)) + v(\alpha(i)).$$

We remove $\lambda_m = \alpha(i)$ and subtract $Nw(\alpha(i))$ from each remaining part. The number of parts is reduced to $m-1$, the number partitioned is now

$$n - (m-1)Nw(\alpha(i)) - \alpha(i),$$

and the smallest part is now $\geq v(\alpha(i))$. Therefore we have a partition counted by $p_{v(\alpha(i))}(k, m-1, n-(m-1)Nw(\alpha(i)) - \alpha(i))$.

To prove (5.2), we consider a partition enumerated by $p_{\alpha(2^r)}(m,n)$ and subtract N from each part. As $p_{\alpha(2^r)}(m,n) = p_{N+a(1)}(m,n)$, we obtain a partition enumerated by $p_{a(1)}(m, n-mN)$. $\square$

For $|x| < 1$ and $|q| < 1$, define

$$f_{\alpha(i)}(x,q) = f_{\alpha(i)}(x) := 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} p_{\alpha(i)}(m,n) x^m q^n.$$

We want to find $f_{a(1)}(1)$, which is the generating function for all partitions counted by $E(A'_N; n)$. To do so, we establish a q-difference equation for $f_{a(1)}(x)$. Let us first give some relations between generating functions.

Lemma 5.2 directly implies

Lemma 5.3. *If* $1 \leq i \leq 2^r - 1$, *then*

$$f_{\alpha(i)}(x) - f_{\alpha(i+1)}(x) = xq^{\alpha(i)} f_{v(\alpha(i))}\left(xq^{Nw(\alpha(i))}\right), \tag{5.3}$$

$$f_{\alpha(2^r)}(x) = f_{a(1)}\left(xq^N\right). \tag{5.4}$$

Adding Equations (5.3) together for $1 \leq i \leq 2^{k-1}-1$ and using the fact that $\alpha\left(2^{k-1}\right) = a(k)$, we obtain

$$f_{a(1)}(x) - f_{a(k)}(x) = \sum_{\alpha < a(k)} xq^{\alpha} f_{v(\alpha)}\left(xq^{Nw(\alpha)}\right). \tag{5.5}$$

Let us now add Equations (5.3) together for $2^{k-2} \leq i \leq 2^{k-1}-1$. This gives

$$f_{a(k-1)}(x) - f_{a(k)}(x) = \sum_{a(k-1) \leq \alpha < a(k)} xq^{\alpha} f_{v(\alpha)}\left(xq^{Nw(\alpha)}\right). \tag{5.6}$$

Every $a(k-1) < \alpha < a(k)$ is of the form $\alpha = a(k-1)+\alpha'$, with $\alpha' < a(k-1)$. Hence we can rewrite (5.6) as

$$\begin{aligned}
&f_{a(k-1)}(x) - f_{a(k)}(x) \\
&= xq^{a(k-1)} f_{a(k-1)}\left(xq^N\right) + q^{a(k-1)-N} \sum_{\alpha' < a(k-1)} xq^{\alpha'+N} f_{v(\alpha')}\left(xq^{N(w(\alpha')+1)}\right) \\
&= xq^{a(k-1)} f_{a(k-1)}\left(xq^N\right) + q^{a(k-1)-N}\left(f_{a(1)}\left(xq^N\right) - f_{a(k-1)}\left(xq^N\right)\right) \\
&= q^{a(k-1)-N} f_{a(1)}\left(xq^N\right) - q^{a(k-1)-N}\left(1-xq^N\right) f_{a(k-1)}\left(xq^N\right),
\end{aligned} \tag{5.7}$$

where the second equality follows from (5.5).

We now state the key lemma which will lead to the desired q-difference equation.

Lemma 5.4. *For $1 \leq k \leq r+1$, we have*

$$f_{a(1)}(x) = f_{a(k)}(x) + \sum_{j=1}^{k-1} \left(\sum_{\substack{\alpha < a(k) \\ w(\alpha)=j}} xq^{\alpha} \right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right). \tag{5.8}$$

Proof: We prove this lemma by induction on k.

For $k=1$, this reduces to $f_{a(1)}(x) = f_{a(1)}(x)$, which is clearly true.

Let us assume that (5.8) is true for some $1 \leq k \leq r$ and show it also holds for $k+1$. We have

$$\begin{aligned}
f_{a(1)}(x) - f_{a(k+1)}(x) &= \left(f_{a(1)}(x) - f_{a(k)}(x)\right) + \left(f_{a(k)}(x) - f_{a(k+1)}(x)\right) \\
&= \sum_{j=1}^{k-1} \left(\sum_{\substack{\alpha < a(k) \\ w(\alpha)=j}} xq^{\alpha} \right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right)
\end{aligned}$$

$$+ q^{a(k)-N} f_{a(1)}\left(xq^N\right) - q^{a(k)-N}\left(1-xq^N\right) f_{a(k)}(xq^N),$$

by the induction hypothesis and Equation (5.7). Thus

$$\begin{aligned}
&f_{a(1)}(x) - f_{a(k+1)}(x) \\
&= \sum_{j=1}^{k-1}\left(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}} xq^{\alpha}\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right) + q^{a(k)-N} f_{a(1)}\left(xq^N\right) \\
&- q^{a(k)-N}\left(1-xq^N\right)\times \\
&\left(f_{a(1)}(xq^N) - \sum_{j=1}^{k-1}\left(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}} xq^{N+\alpha}\right)\prod_{h=1}^{j-1}\left(1-xq^{(h+1)N}\right) f_{a(1)}\left(xq^{(j+1)N}\right)\right) \\
&= \sum_{j=1}^{k-1}\left(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}} xq^{\alpha}\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right) + xq^{a(k)} f_{a(1)}\left(xq^N\right) \\
&+ \sum_{j=1}^{k-1}\left(\sum_{\substack{a(k)<\alpha'<a(k+1)\\ w(\alpha')=j+1}} xq^{\alpha'}\right)\prod_{h=1}^{j}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{(j+1)N}\right) \\
&= \sum_{j=1}^{k-1}\left(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j}} xq^{\alpha}\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right) \\
&+ \sum_{j=1}^{k}\left(\sum_{\substack{a(k)\le\alpha'<a(k+1)\\ w(\alpha')=j}} xq^{\alpha'}\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right) \\
&= \sum_{j=1}^{k}\left(\sum_{\substack{\alpha<a(k+1)\\ w(\alpha)=j}} xq^{\alpha}\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right).
\end{aligned}$$

We obtain (5.8) for $k+1$ and the lemma is proved. $\square$

We are now ready to prove Theorem 5.1 by finding $f_{a(1)}(1)$ in a similar way to the proof of Section 4.2.

By Lemma 5.4 for $k = r+1$ and (5.2), we have the following q-difference equation

$$f_{a(1)}(x) = f_{a(1)}(xq^N) + \sum_{j=1}^{r} \left(\sum_{\substack{\alpha < a(r+1) \\ w(\alpha)=j}} xq^{\alpha} \right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right). \tag{5.9}$$

Let

$$F(x) = f_{a(1)}(x) \prod_{k=0}^{\infty} \frac{1}{(1 - xq^{Nk})}.$$

Then by (5.9),

$$(1-x)F(x) = F(xq^N) + \sum_{j=1}^{r} \left(\sum_{\substack{\alpha < a(r+1) \\ w(\alpha)=j}} xq^{\alpha} \right) F\left(xq^{jN}\right).$$

Now the highest power of x in this equation is 1, so we can translate it into a first order recurrence equation.

Let us define $B_n(q)$ as

$$F(x) = \sum_{n=0}^{\infty} B_n(q) x^n.$$

Then $B_0(q) = 1$ and

$$B_n(q) - B_{n-1}(q) = q^{Nn} B_n(q) + \left(\sum_{\alpha \in A'} q^{(n-1)w(\alpha)N + \alpha} \right) B_{n-1}(q).$$

Defining $\alpha_0 = 0$, $w(\alpha_0) = 0$ and $A^* = A' \cup \{\alpha_0\}$, this can be rewritten as

$$\begin{aligned}
(1 - q^{Nn}) B_n(q) &= \left(\sum_{\alpha \in A^*} q^{(n-1)w(\alpha)N + \alpha} \right) B_{n-1}(q) \\
&= \left(\sum q^{(n-1)jN + a(i_1) + \cdots + a(i_j)} \right) B_{n-1}(q) \\
&= \left(1 + q^{(n-1)N + a(1)}\right) \cdots \left(1 + q^{(n-1)N + a(r)}\right) B_{n-1}(q).
\end{aligned} \tag{5.10}$$

Then, iterating (5.10) and using the fact that $B_0(q) = 1$ gives

$$B_n(q) = \prod_{j=0}^{n-1} \frac{\left(1 + q^{Nj + a(1)}\right) \cdots \left(1 + q^{Nj + a(r)}\right)}{\left(1 - q^{N(j+1)}\right)}.$$

By Appell's Comparison Theorem, we obtain

$$\begin{aligned}
f_{a(1)}(1) &= \lim_{x\to 1}(1-x)\prod_{k=1}^{\infty}(1-xq^{Nk})\sum_{n=0}^{\infty}x^n\prod_{j=0}^{n-1}\frac{\left(1+q^{Nj+a(1)}\right)\cdots\left(1+q^{Nj+a(r)}\right)}{\left(1-q^{N(j+1)}\right)}\\
&= \prod_{k=1}^{\infty}(1-q^{Nk})\prod_{j=0}^{\infty}\frac{\left(1+q^{Nj+a(1)}\right)\cdots\left(1+q^{Nj+a(r)}\right)}{\left(1-q^{N(j+1)}\right)}\\
&= \prod_{j=0}^{\infty}\left(1+q^{Nj+a(1)}\right)\cdots\left(1+q^{Nj+a(r)}\right).
\end{aligned}$$

Thus $f_{a(1)}(1)$ is the generating function for partitions into distinct parts taken from A_N, which concludes the proof.

5.2. A generalisation of Andrews' theorem to overpartitions

5.2.1. Statement of the theorem

Now that we have presented Andrews' proof of his generalisation of Schur's theorem, let us generalise it to overpartitions by proving the following theorem (already stated as Theorem 2.15 in the introduction). This led to the publication [Douar].

Theorem 5.5. *Let $D(A_N;k,n)$ denote the number of overpartitions of n into parts taken from A_N, having k non-overlined parts. Let $E(A'_N;k,n)$ denote the number of overpartitions of n into parts taken from A'_N of the form $n = \lambda_1 + \cdots + \lambda_s$, having k non-overlined parts, such that*

$$\lambda_i - \lambda_{i+1} \geq N\left(w\left(\beta_N(\lambda_{i+1})\right) - 1 + \chi(\overline{\lambda_{i+1}})\right) + v(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}),$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all $k,n \geq 0$, $D(A_N;k,n) = E(A'_N;k,n)$.

When $k=0$, which means that all parts are overlined, this becomes Andrews' theorem, and when $N=3$, $r=2$, $a(1)=1$, $a(2)=2$, we obtain Schur's theorem for overpartitions.

We now prove Theorem 5.5. As in the proofs of Schur's theorem for overpartitions, by Theorem 3.3, the generating function for overpartitions enumerated by $D(A_N;k,n)$ is easy to compute and equals

$$\sum_{k\geq 0}\sum_{n\geq 0}D(A_N;k,n)q^n = \prod_{k=1}^{r}\frac{(-q^{a(k)};q^N)_\infty}{(dq^{a(k)};q^N)_\infty}.$$

So we only need to show that the generating function for partitions enumerated by $E(A'_N;k,n)$ is the same. However the presence of a denominator makes the proof of Theorem 5.5 much more intricate than that of Theorem 5.1 as we need to use induction and a new technique to eliminate one of the products $\frac{(-q^{a(k)};q^N)_\infty}{(dq^{a(k)};q^N)_\infty}$ by switching back and forth from q-difference equations to recurrences. We proceed as follows.

First, we give the q-difference equation satisfied by the generating function for overpartitions enumerated by $E(A'_N;k,n)$ with an added variable keeping track of the number of parts. Then we prove by induction on r that a function satisfying this q-difference equation is equal to $\prod_{k=1}^r \frac{(-q^{a(k)};q^N)_\infty}{(dq^{a(k)};q^N)_\infty}$, which is the generating function for overpartitions counted by $D(A_N;k,n)$.

5.2.2. The q-difference equation satisfied by the generating function

Let $p_{\alpha(i)}(k,m,n)$ denote the number of overpartitions counted by $E(A'_N;k,n)$ having m parts such that the smallest part is $\geq \alpha(i)$. Then we have the following recurrences.

Lemma 5.6. *If $1 \leq i \leq 2^r-1$, then*

$$\begin{aligned} p_{\alpha(i)}(k,m,n) &- p_{\alpha(i+1)}(k,m,n) \\ &= p_{v(\alpha(i))}\big(k,m-1,n-(m-1)Nw(\alpha(i))-\alpha(i)\big) \\ &+p_{v(\alpha(i))}\big(k-1,m-1,n-(m-1)N(w(\alpha(i))-1)-\alpha(i)\big), \end{aligned} \tag{5.11}$$

$$p_{\alpha(2^r)}(k,m,n) = p_{a(1)}(k,m,n-mN). \tag{5.12}$$

Proof: We first prove (5.11). We observe that $p_{\alpha(i)}(k,m,n)-p_{\alpha(i+1)}(k,m,n)$ is the number of overpartitions of the form $n=\lambda_1+\cdots+\lambda_m$ enumerated by $E(A'_n;k,n)$ such that the smallest part is equal to $\alpha(i)$.

If $\lambda_m = \overline{\alpha(i)}$ is overlined, then by the definition of $E(A'_N;k,n)$,

$$\lambda_{m-1} \geq \alpha(i) + Nw(\alpha(i)) + v(\alpha(i)) - \alpha(i) = Nw(\alpha(i)) + v(\alpha(i)).$$

In that case we remove $\lambda_m = \overline{\alpha(i)}$ and subtract $Nw(\alpha(i))$ from each remaining part. The number of parts is reduced to $m-1$, the number of non-overlined parts is still k, and the number partitioned is now $n-(m-1)Nw(\alpha(i))-\alpha(i)$. Moreover the smallest part is now $\geq v(\alpha(i))$. Therefore we have an overpartition counted by $p_{v(\alpha(i))}(k,m-1,n-(m-1)Nw(\alpha(i))-\alpha(i))$.

If $\lambda_m = \alpha(i)$ is not overlined, then by the definition of $E(A'_N;k,n)$,

$$\lambda_{m-1} \geq N\left(w(\alpha(i))-1\right) + v(\alpha(i)).$$

In that case we remove $\lambda_m = \alpha(i)$ and subtract $N(w(\alpha(i)) - 1)$ from each remaining part. The number of parts is reduced to $m-1$, the number of non-overlined parts is reduced $k-1$, and the number partitioned is now $n - (m-1)N(w(\alpha(i)) - 1) - \alpha(i)$. Moreover the smallest part is now $\geq v(\alpha(i))$. Therefore we have an overpartition counted by

$$p_{v(\alpha(i))}(k-1, m-1, n-(m-1)N(w(\alpha(i))-1) - \alpha(i)).$$

To prove (5.12), we consider a partition enumerated by $p_{\alpha(2^r)}(k,m,n)$ and subtract N from each part. As $p_{\alpha(2^r)}(k,m,n) = p_{N+a(1)}(k,m,n)$, we obtain a partition enumerated by $p_{a(1)}(k,m,n-N)$. □

For $|d| < 1$, $|x| < 1$, $|q| < 1$, define

$$f_{\alpha(i)}(d,x,q) = f_{\alpha(i)}(x) := 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} p_{\alpha(i)}(k,m,n) d^k x^m q^n.$$

We want to find $f_{a(1)}(1)$, which is the generating function for all overpartitions counted by $E(A'_N; k, n)$. To do so, we establish a q-difference equation relating $f_{a(1)}\left(xq^{jN}\right)$, for $j \geq 0$. Let us start by giving some relations between generating functions.

Lemma 5.6 directly implies

Lemma 5.7. *If $1 \leq i \leq 2^r - 1$, then*

$$\begin{aligned} f_{\alpha(i)}(x) - f_{\alpha(i+1)}(x) = {} & xq^{\alpha(i)} f_{v(\alpha(i))}\left(xq^{Nw(\alpha(i))}\right) \\ & + dxq^{\alpha(i)} f_{v(\alpha(i))}\left(xq^{N\left(w(\alpha(i))-1\right)}\right), \end{aligned} \tag{5.13}$$

$$f_{\alpha(2^r)}(x) = f_{a(1)}\left(xq^N\right). \tag{5.14}$$

Adding Equations (5.13) together for $1 \leq i \leq 2^{k-1} - 1$ and using the fact that $\alpha\left(2^{k-1}\right) = a(k)$, we obtain

$$f_{a(1)}(x) - f_{a(k)}(x) = \sum_{\alpha < a(k)} \left(xq^{\alpha} f_{v(\alpha)}\left(xq^{Nw(\alpha)}\right) + dxq^{\alpha} f_{v(\alpha)}\left(xq^{N(w(\alpha)-1)}\right)\right). \tag{5.15}$$

Let us now add Equations (5.13) together for $2^{k-2} \leq i \leq 2^{k-1} - 1$. This gives

$$f_{a(k-1)}(x) - f_{a(k)}(x) = \sum_{a(k-1)\le\alpha<a(k)} \left(xq^{\alpha} f_{v(\alpha)}\left(xq^{Nw(\alpha)}\right) + dxq^{\alpha} f_{v(\alpha)}\left(xq^{N(w(\alpha)-1)}\right)\right). \tag{5.16}$$

Every $a(k-1) < \alpha < a(k)$ is of the form $\alpha = a(k-1) + \alpha'$, with $\alpha' < a(k-1)$. Hence we can rewrite (5.16) as

$$\begin{aligned} f_{a(k-1)}(x) - f_{a(k)}(x) &\\ &= xq^{a(k-1)} f_{a(k-1)}\left(xq^N\right) + dxq^{a(k-1)} f_{a(k-1)}(x) \\ &+ q^{a(k-1)-N} \times \\ &\sum_{\alpha'<a(k-1)} \left(xq^{\alpha'+N} f_{v(\alpha')}\left(xq^{N(w(\alpha')+1)}\right) + dxq^{\alpha'+N} f_{v(\alpha')}\left(xq^{Nw(\alpha')}\right)\right) \\ &= xq^{a(k-1)} f_{a(k-1)}\left(xq^N\right) + dxq^{a(k-1)} f_{a(k-1)}(x) \\ &+ q^{a(k-1)-N}\left(f_{a(1)}\left(xq^N\right) - f_{a(k-1)}\left(xq^N\right)\right), \end{aligned}$$

where the last equality follows from (5.15).

Thus

$$\begin{aligned} f_{a(k)}(x) = \left(1 - dxq^{a(k-1)}\right) f_{a(k-1)}(x) - q^{a(k-1)-N} f_{a(1)}\left(xq^N\right) \\ + q^{a(k-1)-N}\left(1 - xq^N\right) f_{a(k-1)}\left(xq^N\right). \end{aligned} \tag{5.17}$$

We are now ready to state the key lemma which will lead to the desired q-difference equation. Note that q-binomial coefficients (among other things) appear in the q-difference equation, which wasn't the case in Andrews' proof.

Lemma 5.8. *For $1 \le k \le r+1$, we have*

$$\begin{aligned} &\prod_{j=1}^{k-1}\left(1 - dxq^{a(j)}\right) f_{a(1)}(x) = f_{a(k)}(x) + \\ &\sum_{j=1}^{k-1}\left(\sum_{m=0}^{k-j-1} d^m \sum_{\substack{\alpha<a(k)\\ w(\alpha)=j+m}} xq^{\alpha}\left((-x)^{m-1}\begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N} + (-x)^m \begin{bmatrix} j+m \\ m \end{bmatrix}_{q^N}\right)\right) \\ &\times \prod_{h=1}^{j-1}\left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right). \end{aligned} \tag{5.18}$$

Proof: We prove this lemma by induction on k. For $k=1$, this reduces to $f_{a(1)}(x)=f_{a(1)}(x)$. Let us assume that (5.18) is true for some $1\leq k\leq r$ and show it also holds for $k+1$. In the following let

$$s_k(x) := \sum_{j=1}^{k-1}\left(\sum_{m=0}^{k-j-1} d^m \sum_{\substack{\alpha<a(k)\\ w(\alpha)=j+m}} xq^{\alpha}\left((-x)^{m-1}\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^N}+(-x)^m\begin{bmatrix} j+m\\ m\end{bmatrix}_{q^N}\right)\right) \times\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right).$$

Therefore we want to prove that

$$\prod_{j=1}^{k}\left(1-dxq^{a(j)}\right) f_{a(1)}(x) = f_{a(k+1)}(x)+s_{k+1}(x).$$

We have

$$\begin{aligned}
&\prod_{j=1}^{k}\left(1-dxq^{a(j)}\right) f_{a(1)}(x)-f_{a(k+1)}(x)\\
&=\left(1-dxq^{a(k)}\right)\left(\prod_{j=1}^{k-1}\left(1-dxq^{a(j)}\right) f_{a(1)}(x)-f_{a(k)}(x)\right)\\
&\quad+\left(1-dxq^{a(k)}\right) f_{a(k)}(x)-f_{a(k+1)}(x)\\
&=\left(1-dxq^{a(k)}\right) s_k(x)\\
&\quad+q^{a(k)-N} f_{a(1)}\left(xq^{N}\right)-q^{a(k)-N}\left(1-xq^{N}\right) f_{a(k)}(xq^{N}),
\end{aligned}$$

where the last equality follows from the induction hypothesis and (5.17). Thus

$$\begin{aligned}
&\prod_{j=1}^{k}\left(1-dxq^{a(j)}\right) f_{a(1)}(x)-f_{a(k+1)}(x)\\
&=\left(1-dxq^{a(k)}\right) s_k(x)+q^{a(k)-N} f_{a(1)}\left(xq^{N}\right)\\
&\quad-q^{a(k)-N}\left(1-xq^{N}\right)\left(\prod_{j=1}^{k-1}\left(1-dxq^{N+a(j)}\right) f_{a(1)}\left(xq^{N}\right)-s_k\left(xq^{N}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - dxq^{a(k)}\right) s_k(x) + q^{a(k)-N} \left(1 - xq^N\right) s_k\left(xq^N\right) \\
&\quad + q^{a(k)-N} \left(1 - \left(1 - xq^N\right) \prod_{j=1}^{k-1} \left(1 - dxq^{N+a(j)}\right)\right) \\
&\quad \times f_{a(1)}\left(xq^N\right) \\
&= \left(1 - dxq^{a(k)}\right) s_k(x) + q^{a(k)-N} \left(1 - xq^N\right) s_k\left(xq^N\right) \\
&\quad + q^{a(k)-N} \left(1 - \left(1 - xq^N\right) \left(1 + \sum_{m=1}^{k-1} \sum_{\substack{\alpha < a(k) \\ w(\alpha)=m}} (-dxq^N)^m q^\alpha\right)\right) \\
&\quad \times f_{a(1)}\left(xq^N\right) \\
&= \left(1 - dxq^{a(k)}\right) s_k(x) + q^{a(k)-N} \left(1 - xq^N\right) s_k\left(xq^N\right) \\
&\quad + q^{a(k)-N} \left(xq^N + \sum_{m=1}^{k-1} d^m \sum_{\substack{\alpha < a(k) \\ w(\alpha)=m}} xq^{\alpha+N} \left((-xq^N)^{m-1} + (-xq^N)^m\right)\right) \\
&\quad \times f_{a(1)}\left(xq^N\right) \\
&= \left(1 - dxq^{a(k)}\right) s_k(x) + q^{a(k)-N} \left(1 - xq^N\right) s_k\left(xq^N\right) \\
&\quad + \left(xq^{a(k)} + \sum_{m=1}^{k-1} d^m \sum_{\substack{a(k) < \alpha' < a(k+1) \\ w(\alpha')=m+1}} xq^{\alpha'} \left((-xq^N)^{m-1} + (-xq^N)^m\right)\right) \\
&\quad \times f_{a(1)}\left(xq^N\right) \\
&= \sum_{j=1}^{k-1} \left(\sum_{m=0}^{k-j-1} d^m \sum_{\substack{\alpha < a(k) \\ w(\alpha)=j+m}} xq^\alpha \left((-x)^{m-1} \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N}\right.\right. \\
&\quad \left.\left. + \ (-x)^m \begin{bmatrix} j+m \\ m \end{bmatrix}_{q^N}\right)\right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{k-1}\left(\sum_{m=0}^{k-j-1} d^{m+1} \sum_{\substack{\alpha<a(k)\\ w(\alpha)=j+m}} xq^{a(k)+\alpha}\left((-x)^m \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N}\right.\right.\\
&\qquad \left.\left.+(-x)^{m+1}\begin{bmatrix} j+m \\ m \end{bmatrix}_{q^N}\right)\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right)\\
&+q^{a(k)-N}\left(1-xq^N\right)\\
&\quad\times\sum_{j=1}^{k-1}\left(\sum_{m=0}^{k-j-1} d^{m} \sum_{\substack{\alpha<a(k)\\ w(\alpha)=j+m}} xq^{\alpha+N}\left((-xq^N)^{m-1} \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N}\right.\right.\\
&\qquad \left.\left.+(-xq^N)^{m}\begin{bmatrix} j+m \\ m \end{bmatrix}_{q^N}\right)\right)\prod_{h=1}^{j-1}\left(1-xq^{(h+1)N}\right) f_{a(1)}\left(xq^{(j+1)N}\right)\\
&+\left(xq^{a(k)}+\sum_{m=1}^{k-1} d^m \sum_{\substack{a(k)<\alpha'<a(k+1)\\ w(\alpha')=m+1}} xq^{\alpha'}\left((-xq^N)^{m-1}+(-xq^N)^m\right)\right)\\
&\quad\times f_{a(1)}\left(xq^N\right)\\
&=\sum_{j=1}^{k-1}\left(\sum_{m=0}^{k-j-1} d^{m} \sum_{\substack{\alpha<a(k)\\ w(\alpha)=j+m}} xq^{\alpha}\left((-x)^{m-1} \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N}\right.\right.\\
&\qquad \left.\left.+(-x)^{m}\begin{bmatrix} j+m \\ m \end{bmatrix}_{q^N}\right)\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right)\\
&+\sum_{j=1}^{k-1}\left(\sum_{m=1}^{k-j} d^{m} \sum_{\substack{a(k)<\alpha<a(k+1)\\ w(\alpha)=j+m}} xq^{\alpha}\left((-x)^{m-1} \begin{bmatrix} j+m-2 \\ m-2 \end{bmatrix}_{q^N}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad +(-x)^m \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N} \Bigg) \Bigg) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right) \\
&+ \sum_{j=2}^{k} \left(\sum_{m=0}^{k-j} d^m \sum_{\substack{a(k)<\alpha<a(k+1) \\ w(\alpha)=j+m}} xq^{\alpha} \left((-xq^N)^{m-1} \begin{bmatrix} j+m-2 \\ m-1 \end{bmatrix}_{q^N} \right.\right. \\
&\qquad \left.\left. +(-xq^N)^m \begin{bmatrix} j+m-1 \\ m \end{bmatrix}_{q^N} \right) \right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right) \\
&+ \left(xq^{a(k)} + \sum_{m=1}^{k-1} d^m \sum_{\substack{a(k)<\alpha'<a(k+1) \\ w(\alpha')=m+1}} xq^{\alpha'} \left((-xq^N)^{m-1} + (-xq^N)^m \right) \right) \\
&\quad \times f_{a(1)}\left(xq^N\right) \\
&= \sum_{j=1}^{k-1} \left(\sum_{m=0}^{k-j-1} d^m \sum_{\substack{\alpha<a(k) \\ w(\alpha)=j+m}} xq^{\alpha} \left((-x)^{m-1} \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N} \right.\right. \\
&\qquad \left.\left. +(-x)^m \begin{bmatrix} j+m \\ m \end{bmatrix}_{q^N} \right) \right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right) \\
&+ \sum_{j=1}^{k-1} \left(\sum_{m=1}^{k-j} d^m \sum_{\substack{a(k)<\alpha<a(k+1) \\ w(\alpha)=j+m}} xq^{\alpha} \left((-x)^{m-1} \begin{bmatrix} j+m-2 \\ m-2 \end{bmatrix}_{q^N} \right.\right. \\
&\qquad \left.\left. +(-x)^m \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N} \right) \right) \prod_{h=1}^{j-1} \left(1 - xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right)
\end{aligned}
$$

$$+\sum_{j=1}^{k}\left(\sum_{m=0}^{k-j} d^m \sum_{\substack{a(k)\leq\alpha<a(k+1)\\ w(\alpha)=j+m}} xq^{\alpha}\left((-xq^N)^{m-1}\begin{bmatrix} j+m-2\\ m-1\end{bmatrix}_{q^N}\right.\right.$$

$$\left.\left.+(-xq^N)^m\begin{bmatrix} j+m-1\\ m\end{bmatrix}_{q^N}\right)\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right)$$

$$=\sum_{j=1}^{k-1}\left[\sum_{\substack{\alpha<a(k+1)\\ w(\alpha)=j}} xq^{\alpha}\right.$$

$$+\sum_{m=1}^{k-j} d^m\left(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=j+m}} xq^{\alpha}\right.$$

$$\times\left((-x)^{m-1}\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^N}+(-x)^m\begin{bmatrix} j+m\\ m\end{bmatrix}_{q^N}\right)$$

$$+\sum_{\substack{a(k)<\alpha<a(k+1)\\ w(\alpha)=j+m}} xq^{\alpha}$$

$$\times\left((-x)^{m-1}\begin{bmatrix} j+m-2\\ m-2\end{bmatrix}_{q^N}+(-x)^{m-1}q^{N(m-1)}\begin{bmatrix} j+m-2\\ m-1\end{bmatrix}_{q^N}\right.$$

$$\left.\left.\left.+\ (-x)^m\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^N}+(-x)^m q^{Nm}\begin{bmatrix} j+m-1\\ m\end{bmatrix}_{q^N}\right)\right)\right]$$

$$\times\prod_{h=1}^{j-1}\left(1-xq^{hN}\right)f_{a(1)}\left(xq^{jN}\right)$$

$$+\,xq^{a(1)+\cdots+a(k)}\prod_{h=1}^{k-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{kN}\right).$$

Thus by (3.4) of Proposition 3.7, we obtain

$$\prod_{j=1}^{k}\left(1-dxq^{a(j)}\right) f_{a(1)}(x)-f_{a(k+1)}(x)$$

$$
\begin{aligned}
=&\sum_{j=1}^{k-1}\left(\sum_{m=0}^{k-j} d^m \sum_{\substack{\alpha<a(k+1)\\ w(\alpha)=j+m}} xq^\alpha \left((-x)^{m-1}\begin{bmatrix} j+m-1 \\ m-1\end{bmatrix}_{q^N}\right.\right.\\
&\left.\left. + (-x)^m \begin{bmatrix} j+m \\ m\end{bmatrix}_{q^N}\right)\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right)\\
&+ xq^{a(1)+\cdots+a(k)}\prod_{h=1}^{k-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{kN}\right)\\
=&\sum_{j=1}^{k}\left(\sum_{m=0}^{k-j} d^m \sum_{\substack{\alpha<a(k+1)\\ w(\alpha)=j+m}} xq^\alpha \left((-x)^{m-1}\begin{bmatrix} j+m-1 \\ m-1\end{bmatrix}_{q^N}\right.\right.\\
&\left.\left. +(-x)^m \begin{bmatrix} j+m \\ m\end{bmatrix}_{q^N}\right)\right)\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right)\\
=&s_{k+1}(x).
\end{aligned}
$$

This completes the proof. $\square$

Now, by setting $k = r+1$ in Lemma 5.8, and using (5.14), we obtain the desired q-difference equation.

$$
\begin{aligned}
&\prod_{j=1}^{r}\left(1-dxq^{a(j)}\right) f_{a(1)}(x) = f_{a(1)}(xq^N)\\
&+\sum_{j=1}^{r}\left(\sum_{m=0}^{r-j} d^m \sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m}} xq^\alpha \left((-x)^{m-1}\begin{bmatrix} j+m-1 \\ m-1\end{bmatrix}_{q^N} + (-x)^m \begin{bmatrix} j+m \\ m\end{bmatrix}_{q^N}\right)\right)\\
&\qquad\times\prod_{h=1}^{j-1}\left(1-xq^{hN}\right) f_{a(1)}\left(xq^{jN}\right).
\end{aligned}
\tag{eq$_{N,r}$}
$$

We now need to evaluate $f_{a(1)}(1)$, which we recall is the generating function for the overpartitions with difference conditions counted by $E(A'_N; k, n)$.

5.2.3. Evaluating $f_{a(1)}(1)$ by induction

In this section, we evaluate $f_{a(1)}(1)$. In our case, as our q-difference equation is much more complicated than Andrews', it is not sufficient to proceed as in his proof. But we solve the problem by proving the following theorem by induction on r.

Theorem 5.9. *Let r be a positive integer. Then for every $N \geq \alpha(2^r - 1)$, for every function f satisfying $(\mathrm{eq}_{N,r})$ and the initial condition $f(0) = 1$, we have*

$$f(1) = \prod_{k=1}^{r} \frac{(-q^{a(k)}; q^N)_\infty}{(dq^{a(k)}; q^N)_\infty}.$$

The idea of the proof is to start from a function satisfying $(\mathrm{eq}_{N,r})$ and to do some transformations to relate it to a function satisfying $(\mathrm{eq}_{N,r-1})$ in order to use the induction hypothesis. To simplify the proof, we split it into several lemmas.

Lemma 5.10. *Let f and F be two functions such that*

$$F(x) := f(x) \prod_{n=0}^{\infty} \frac{1 - dxq^{Nn+a(r)}}{1 - xq^{Nn}}.$$

Then $f(0) = 1$ and f satisfies $(\mathrm{eq}_{N,r})$ if and only if $F(0) = 1$ and F satisfies the following q-difference equation

$$\begin{aligned}
&\left(1 + \sum_{j=1}^{r} \left(d^{j-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha) = j-1}} q^\alpha + d^j \sum_{\substack{\alpha < a(r) \\ w(\alpha) = j}} q^\alpha \right) (-x)^j \right) F(x) \\
&= F\left(xq^N\right) + \sum_{j=1}^{r} \sum_{l=1}^{r} \sum_{k=0}^{\min(j-1,l-1)} c_{k,j} b_{l-k,j} (-1)^{l-1} x^l F\left(xq^{jN}\right),
\end{aligned} \tag{$\mathrm{eq}'_{N,r}$}$$

where

$$c_{k,j} := q^{N\frac{k(k+1)}{2} + ka(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} d^k,$$

and

$$b_{m,j} := \left(d^{m-1} \sum_{\substack{\alpha < a(r+1) \\ w(\alpha) = j+m-1}} q^\alpha + d^m \sum_{\substack{\alpha < a(r+1) \\ w(\alpha) = j+m}} q^\alpha \right) \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N}.$$

Proof: Directly plugging the definition of f into $(\mathrm{eq}_{N,r})$, we get

$$(1-x)\prod_{j=1}^{r-1}\left(1-dxq^{a(j)}\right)F(x)=F(xq^N)$$
$$+\sum_{j=1}^{r}\left(\sum_{m=0}^{r-j}d^m\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m}}xq^\alpha\left((-x)^{m-1}\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^N}+(-x)^m\begin{bmatrix}j+m\\ m\end{bmatrix}_{q^N}\right)\right)$$
$$\times\prod_{h=1}^{j-1}\left(1-dxq^{hN+a(r)}\right)F\left(xq^{jN}\right).$$

With the conventions that

$$\sum_{\substack{\alpha<a(r)\\ w(\alpha)=n}}q^\alpha=0 \text{ for } n\geq r,$$

and

$$\sum_{\substack{\alpha<a(r)\\ w(\alpha)=0}}q^\alpha=1,$$

this can be reformulated as

$$\left(1+\sum_{j=1}^{r}\left(d^{j-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j-1}}q^\alpha+d^j\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j}}q^\alpha\right)(-x)^j\right)F(x)=F(xq^N)$$
$$+\sum_{j=1}^{r}\left(\sum_{m=1}^{r-j+1}\left(d^{m-1}\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m-1}}q^\alpha+d^m\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m}}q^\alpha\right)\right.$$
$$\left.\times\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^N}(-1)^{m-1}x^m\right)$$
$$\times\left(\sum_{k=0}^{j-1}q^{N\frac{k(k-1)}{2}+ka(r)}\begin{bmatrix}j-1\\ k\end{bmatrix}_{q^N}d^k(-x)^k\right)F\left(xq^{jN}\right),$$

because of the q-binomial theorem (Theorem 3.9),

$$\prod_{k=0}^{n-1}(1+q^k t)=\sum_{k=0}^{n} q^{\frac{k(k-1)}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q t^k,$$

in which we replace q by q^N, n by $j-1$ and t by $-dxq^{N+a(r)}$. Finally, noting that $b_{l-k,j}=0$ if $j+l-k-1\geq r$, we can rewrite this as ($\mathrm{eq}'_{N,r}$). Moreover, $F(0)=f(0)=1$ and the lemma is proved. $\square$

We can directly transform ($\mathrm{eq}'_{N,r}$) into a recurrence equation on the coefficients of the generating function F.

Lemma 5.11. *Let F be a function and $(A_n)_{n\in\mathbb{N}}$ a sequence such that*

$$F(x)=:\sum_{n=0}^{\infty} A_n x^n.$$

Then F satisfies ($\mathrm{eq}'_{N,r}$) and the initial condition $F(0)=1$ if and only if $A_0=1$ and $(A_n)_{n\in\mathbb{N}}$ satisfies the following recurrence equation

$$\left(1-q^{nN}\right)A_n=\sum_{m=1}^{r}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{\alpha}+d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{\alpha}\right.$$
$$\left.+\sum_{j=1}^{r}\sum_{k=0}^{\min(j-1,m-1)} c_{k,j}b_{m-k,j}q^{jN(n-m)}\right)(-1)^{m+1}A_{n-m}. \tag{$\mathrm{rec}_{N,r}$}$$

Proof: By the definition of $(A_n)_{n\in\mathbb{N}}$ and ($\mathrm{eq}'_{N,r}$), we have

$$\left(1-q^{nN}\right)A_n=\sum_{j=1}^{r}\left(d^{j-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j-1}} q^{\alpha}+d^{j}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j}} q^{\alpha}\right)(-1)^{j+1}A_{n-j}$$
$$+\sum_{j=1}^{r}\sum_{l=1}^{r}\sum_{k=0}^{\min(j-1,l-1)} c_{k,j}b_{l-k,j}q^{jN(n-l)}(-1)^{l+1}A_{n-l}.$$

Relabelling the summation indices and factorising leads to ($\mathrm{rec}_{N,r}$). Moreover, $A_n=F(0)=1$. This completes the proof. $\square$

Let us now do some transformations starting from ($\mathrm{eq}_{N,r-1}$).

Lemma 5.12. *Let g and G be two functions such that*

$$G(x) := g(x) \prod_{n=0}^{\infty} \frac{1}{1-xq^{Nn}}.$$

Then g satisfies ($\mathrm{eq}_{N,r-1}$) and the initial condition $g(0) = 1$ if and only if $G(0) = 1$ and G satisfies the following q-difference equation

$$\begin{aligned}
&\left(1+\sum_{j=1}^{r}\left(d^{j-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j-1}} q^{\alpha} + d^{j}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j}} q^{\alpha}\right)(-x)^{j}\right) G(x) = G\left(xq^{N}\right)\\
&+\sum_{j=1}^{r}\sum_{m=1}^{r-j}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{\alpha} + d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{\alpha}\right)\\
&\quad\times \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N} (-1)^{m+1} x^{m} G\left(xq^{jN}\right).
\end{aligned} \qquad (\mathrm{eq}''_{N,r-1})$$

Proof: Using the definition of G and ($\mathrm{eq}_{N,r-1}$), we get

$$\begin{aligned}
&(1-x)\prod_{j=1}^{r-1}\left(1-dxq^{a(j)}\right) G(x) = G(xq^{N})\\
&+\sum_{j=1}^{r-1}\left(\sum_{m=0}^{r-j-1} d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} xq^{\alpha}\left((-x)^{m-1}\begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^N}\right.\right.\\
&\qquad\left.\left. +(-x)^{m}\begin{bmatrix} j+m \\ m \end{bmatrix}_{q^N}\right)\right) G\left(xq^{jN}\right).
\end{aligned}$$

Then, as in the proof of Lemma 5.10, this can be reformulated as ($\mathrm{eq}''_{N,r-1}$), and $G(0) = g(0) = 1$. $\square$

Again, let us translate this into a recurrence equation on the coefficients of the generating function G.

Lemma 5.13. *Let G be a function and $(a_n)_{n\in\mathbb{N}}$ be a sequence such that*

$$G(x) =: \sum_{n=0}^{\infty} a_n x^n.$$

Then G satisfies ($\mathrm{eq}''_{N,r-1}$) *and the initial condition $G(0) = 1$ if and only if $a_0 = 1$ and $(a_n)_{n\in\mathbb{N}}$ satisfies the following recurrence equation*

$$\begin{aligned}\left(1-q^{nN}\right) a_n = &\\ \sum_{m=1}^{r}\sum_{j=0}^{r-1} &\left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{\alpha}\right)\\ &\times \begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^N} q^{jN(n-m)}(-1)^{m+1} a_{n-m}.\end{aligned} \qquad (\mathrm{rec}''_{N,r-1})$$

Proof: By the definition of $(a_n)_{n\in\mathbb{N}}$ and ($\mathrm{eq}''_{N,r-1}$), we have

$$\begin{aligned}\left(1-q^{nN}\right) a_n = &\sum_{m=1}^{r}\left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{\alpha}\right)(-1)^{m+1} a_{n-m}\\ +\sum_{m=1}^{r-1}\sum_{j=1}^{r-1} &\left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{\alpha}\right)\\ &\times \begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^N} q^{jN(n-m)}(-1)^{m+1} a_{n-m}.\end{aligned}$$

As the summand of the second term equals 0 when $m = r$, we can equivalently write that the second sum is taken over m going from 1 to r. Then we observe that the first term corresponds to $j = 0$ in the second term, and factorising gives exactly ($\mathrm{rec}_{N,r}$). Moreover, $a_n = G(0) = 1$. This completes the proof. □

Let us do a final transformation and obtain yet another recurrence equation.

Lemma 5.14. *Let $(a_n)_{n\in\mathbb{N}}$ and $(A'_n)_{n\in\mathbb{N}}$ be two sequences such that*

$$A'_n := a_n \prod_{k=0}^{n-1}\left(1+q^{Nk+a(r)}\right).$$

Then $(a_n)_{n\in\mathbb{N}}$ *satisfies* $(\mathrm{rec}''_{N,r-1})$ *and the initial condition* $a_0 = 1$ *if and only if* $A'_0 = 1$ *and* $(A'_n)_{n\in\mathbb{N}}$ *satisfies the following recurrence equation*

$$\begin{aligned}\left(1-q^{nN}\right)A'_n = \sum_{m=1}^{r}\Bigg(&\sum_{\nu=0}^{r-1}\sum_{\mu=0}^{\min(m-1,\nu)} f_{m,\mu}e_{m,\nu-\mu}q^{\nu N(n-m)}\\ &+ q^{a(r)}\sum_{\nu=1}^{r}\sum_{\mu=0}^{\min(m-1,\nu-1)} f_{m,\mu}e_{m,\nu-\mu-1}q^{\nu N(n-m)}\Bigg)(-1)^{m+1}A'_{n-m},\end{aligned} \tag{$\mathrm{rec}'_{N,r-1}$}$$

where

$$e_{m,j} := \left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{\alpha} + d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{\alpha}\right)\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^N},$$

and

$$f_{m,k} := q^{N\frac{k(k+1)}{2}+ka(r)}\begin{bmatrix} m-1\\ k\end{bmatrix}_{q^N}.$$

Proof: By the definition of $(A'_n)_{n\in\mathbb{N}}$, we have

$$\begin{aligned}\left(1-q^{nN}\right)A'_n =& \sum_{m=1}^{r}\sum_{j=0}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{\alpha} + d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{\alpha}\right)\\ &\times \begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^N} q^{jN(n-m)}(-1)^{m+1}\prod_{k=1}^{m}\left(1+q^{N(n-k)+a(r)}\right)A'_{n-m}.\end{aligned}$$

Furthermore

$$\begin{aligned}\prod_{k=1}^{m}\left(1+q^{N(n-k)+a(r)}\right) &= \prod_{k=0}^{m-1}\left(1+q^{Nk+N(n-m)+a(r)}\right)\\ &= \left(1+q^{N(n-m)+a(r)}\right)\prod_{k=1}^{m-1}\left(1+q^{Nk+N(n-m)+a(r)}\right)\\ &= \left(1+q^{N(n-m)+a(r)}\right)\sum_{k=0}^{m-1}q^{N\frac{k(k+1)}{2}+kN(n-m)+ka(r)}\begin{bmatrix} m-1\\ k\end{bmatrix}_{q^N},\end{aligned}$$

where the last equality follows from Theorem 3.9. Therefore

$$\begin{aligned}
&\left(1-q^{nN}\right)A'_n\\
&=\sum_{m=1}^{r}\left(\sum_{j=0}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}}q^{\alpha}+d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}}q^{\alpha}\right)\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^N}q^{jN(n-m)}\right.\\
&\left.\times\left(1+q^{N(n-m)+a(r)}\right)\sum_{k=0}^{m-1}q^{N\frac{k(k+1)}{2}+kN(n-m)+ka(r)}\begin{bmatrix}m-1\\ k\end{bmatrix}_{q^N}\right)(-1)^{m+1}A'_{n-m}\\
&=\sum_{m=1}^{r}\left[\sum_{j=0}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}}q^{\alpha}+d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}}q^{\alpha}\right)\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^N}q^{jN(n-m)}\right.\\
&\times\sum_{k=0}^{m-1}q^{N\frac{k(k+1)}{2}+kN(n-m)+ka(r)}\begin{bmatrix}m-1\\ k\end{bmatrix}_{q^N}\\
&+\sum_{j=0}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}}q^{\alpha}+d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}}q^{\alpha}\right)\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^N}q^{(j+1)N(n-m)}\\
&\left.\times q^{a(r)}\times\sum_{k=0}^{m-1}q^{N\frac{k(k+1)}{2}+kN(n-m)+ka(r)}\begin{bmatrix}m-1\\ k\end{bmatrix}_{q^N}\right](-1)^{m+1}A'_{n-m}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(1-q^{nN}\right)A'_n=\sum_{m=1}^{r}&\left(\sum_{j=0}^{r-1}e_{m,j}q^{jN(n-m)}\sum_{k=0}^{m-1}f_{m,k}q^{kN(n-m)}\right.\\
&\left.+\,q^{a(r)}\sum_{j=1}^{r}e_{m,j-1}q^{jN(n-m)}\sum_{k=0}^{m-1}f_{m,k}q^{kN(n-m)}\right)(-1)^{m+1}A'_{n-m}.
\end{aligned}$$

Rearranging leads to ($\mathrm{rec}'_{N,r-1}$). As always, $A'_0=a_0=1$. The lemma is proved. $\square$

We now want to show that $(A_n)_{n\in\mathbb{N}}$ and $(A'_n)_{n\in\mathbb{N}}$ are in fact equal.

Lemma 5.15. *Let $(A_n)_{n\in\mathbb{N}}$ and $(A'_n)_{n\in\mathbb{N}}$ be defined as in Lemmas 5.11 and 5.14. Then for every $n \in \mathbb{N}$, $A_n = A'_n$.*

Proof: To prove the equality, it is sufficient to show that for every $1 \le m \le r$, the coefficient of $(-1)^{m+1}A_{n-m}$ in $(\mathrm{rec}_{N,r})$ is the same as the coefficient of $(-1)^{m+1}A'_{n-m}$ in $(\mathrm{rec'}_{N,r-1})$. Let $m \in \{1,\dots,r\}$,

$$\begin{aligned}
S_m &:= \left[(-1)^{m+1}A_{n-m}\right](\mathrm{rec}_{N,r})\\
&= d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{\alpha} + d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{\alpha} + \sum_{j=1}^{r}\sum_{k=0}^{\min(j-1,m-1)} c_{k,j}b_{m-k,j}q^{jN(n-m)},
\end{aligned}$$

and

$$\begin{aligned}
S'_m &:= \left[(-1)^{m+1}A'_{n-m}\right](\mathrm{rec'}_{N,r-1})\\
&= \sum_{\nu=0}^{r-1}\sum_{\mu=0}^{\min(m-1,\nu)} f_{m,\mu}e_{m,\nu-\mu}q^{\nu N(n-m)}\\
&\quad + q^{a(r)}\sum_{\nu=1}^{r}\sum_{\mu=0}^{\min(m-1,\nu-1)} f_{m,\mu}e_{m,\nu-\mu-1}q^{\nu N(n-m)}\\
&= f_{m,0}e_{m,0} + \sum_{\nu=1}^{r}\left(\sum_{\mu=0}^{\min(m-1,\nu)} f_{m,\mu}e_{m,\nu-\mu}\right.\\
&\qquad\qquad \left. + q^{a(r)}\sum_{\mu=0}^{\min(m-1,\nu-1)} f_{m,\mu}e_{m,\nu-\mu-1}\right) q^{\nu N(n-m)},
\end{aligned}$$

where the last equality follows from the fact that $e_{m,r-\mu} = 0$ for all μ, as $\mu \le m-1$ so the sums are over α such that $\alpha < a(r)$ and $w(\alpha) \ge r$, which is impossible.

Let us first notice that

$$f_{m,0}e_{m,0} = d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{\alpha} + d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{\alpha}.$$

Now define

$$T_{m,j} := \sum_{k=0}^{\min(j-1,m-1)} c_{k,j}b_{m-k,j},$$

and

$$T'_{m,j} := \sum_{k=0}^{\min(m-1,j)} f_{m,k} e_{m,j-k} + q^{a(r)} \sum_{k=0}^{\min(m-1,j-1)} f_{m,k} e_{m,j-k-1}.$$

The only thing left to do is to show that for every $1 \leq j \leq r$,

$$T_{m,j} = T'_{m,j}.$$

We have

$$\begin{aligned}
c_{k,j} b_{m-k,j} &= q^{N\frac{k(k+1)}{2}+ka(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} \\
&\quad \times \left(d^{m-1} \sum_{\substack{\alpha < a(r+1) \\ w(\alpha)=j+m-k-1}} q^\alpha + d^m \sum_{\substack{\alpha < a(r+1) \\ w(\alpha)=j+m-k}} q^\alpha \right) \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} \\
&= \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-1}} q^\alpha + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k}} q^\alpha \right) q^{N\frac{k(k+1)}{2}+ka(r)} \\
&\quad \times \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} \\
&\quad + q^{a(r)} \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-2}} q^\alpha + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-1}} q^\alpha \right) \\
&\quad \times q^{N\frac{k(k+1)}{2}+ka(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N},
\end{aligned} \tag{5.19}$$

in which the last equality follows from separating the sums over α according to whether α contains $a(r)$ as a summand or not.

We also have

$$\begin{aligned}
f_{m,k} e_{m,j-k} &= q^{N\frac{k(k+1)}{2}+ka(r)} \begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^N} \\
&\quad \times \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-1}} q^\alpha + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k}} q^\alpha \right) \begin{bmatrix} j+m-k-1 \\ m-1 \end{bmatrix}_{q^N},
\end{aligned} \tag{5.20}$$

and

$$q^{a(r)} f_{m,k} e_{m,j-k-1} = q^{N\frac{k(k+1)}{2}+(k+1)a(r)} \begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^N} \times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-2}} q^\alpha + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^\alpha \right) \begin{bmatrix} j+m-k-2 \\ m-1 \end{bmatrix}_{q^N}. \tag{5.21}$$

By a simple calculation using the exact formula for q-binomial coefficients (Proposition 3.8), we get the following result. For all $j, k, m \in \mathbb{N}$,

$$\begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-1 \end{bmatrix}_{q^N} = \begin{bmatrix} j \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N}. \tag{5.22}$$

Using (5.22), we obtain

$$\begin{aligned} T'_{m,j} =& \chi(j \leq m-1)\ q^{N\frac{j(j+1)}{2}+ja(r)} \\ & \times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=m-1}} q^\alpha + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=m}} q^\alpha \right) \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^N} \\ & + \sum_{k=0}^{\min(m-1,j-1)} q^{N\frac{k(k+1)}{2}+ka(r)} \begin{bmatrix} j \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} \\ & \times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^\alpha + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k}} q^\alpha \right) \\ & + \sum_{k=0}^{\min(m-1,j-1)} q^{N\frac{k(k+1)}{2}+(k+1)a(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-2 \\ m-k-1 \end{bmatrix}_{q^N} \\ & \times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-2}} q^\alpha + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^\alpha \right). \end{aligned}$$

By (3.3) of Proposition 3.7, we have

$$\begin{bmatrix} j \\ k \end{bmatrix}_{q^N} = \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} + q^{N(j-k)} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^N},$$

$$\begin{bmatrix} j+m-k-2 \\ m-k-1 \end{bmatrix}_{q^N} = \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} - q^{Nj} \begin{bmatrix} j+m-k-2 \\ m-k-2 \end{bmatrix}_{q^N}.$$

This allows us to rewrite $T'_{m,j}$ as

$$\begin{aligned}
T'_{m,j} =& \chi(j \le m-1)\ q^{N\frac{j(j+1)}{2}+ja(r)} \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^\alpha + d^m \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^\alpha \right) \\
&\times \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^N} \\
&+ \sum_{k=0}^{\min(m-1,j-1)} q^{N\frac{k(k+1)}{2}+ka(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} \\
&\times \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-1}} q^\alpha + d^m \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k}} q^\alpha \right) \\
&+ \sum_{k=0}^{\min(m-1,j-1)} q^{N\frac{k(k+1)}{2}+ka(r)+N(j-k)} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} \\
&\times \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-1}} q^\alpha + d^m \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k}} q^\alpha \right) \\
&+ \sum_{k=0}^{\min(m-1,j-1)} q^{N\frac{k(k+1)}{2}+(k+1)a(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} \\
&\times \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-2}} q^\alpha + d^m \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-1}} q^\alpha \right) \\
&- \sum_{k=0}^{\min(m-2,j-1)} q^{N\frac{k(k+1)}{2}+(k+1)a(r)+Nj} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-2 \\ m-k-2 \end{bmatrix}_{q^N} \\
&\times \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-2}} q^\alpha + d^m \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-1}} q^\alpha \right).
\end{aligned}$$

By (5.19), the sum of the second and fourth term in the sum above is exactly equal to Tm, j. Let X denote the sum of the third and fifth term. We now want to show that

$$X+\chi(j \leq m-1)\ q^{N\frac{j(j+1)}{2}+ja(r)} \times \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{\alpha} \right) \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^N} = 0.$$

By the change of variable $k' = k+1$ in the fourth sum, we get

$$\begin{aligned}
X = & \sum_{k=0}^{\min(m-1,j-1)} q^{N\frac{k(k-1)}{2}+ka(r)+Nj} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} \\
& \times \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-1}} q^{\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k}} q^{\alpha} \right) \\
& - \sum_{k=1}^{\min(m-1,j)} q^{N\frac{k(k-1)}{2}+ka(r)+Nj} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^N} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^N} \\
& \times \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-1}} q^{\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k}} q^{\alpha} \right) \\
= & \begin{cases} 0, \text{ if } j \geq m, \\ -q^{N\frac{j(j+1)}{2}+ja(r)} \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{\alpha} \right) \left[\begin{smallmatrix} m-1 \\ m-j-1 \end{smallmatrix}\right]_{q^N}, \\ \text{otherwise} \end{cases} \\
= & -\chi(j \leq m-1)\ q^{N\frac{j(j+1)}{2}+ja(r)} \\
& \times \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{\alpha} \right) \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^N}.
\end{aligned}$$

This completes the proof. □

We can finally turn to the proof of Theorem 5.9.

Proof of Theorem 5.9: Let us start by the initial case $r = 1$. Let $N \geq a(1)$ and f such that

$$\left(1 - dxq^{a(1)}\right) f(x) = f\left(xq^N\right) + xq^{a(1)} f\left(xq^N\right). \qquad (\mathrm{eq}_{N,1})$$

Then

$$f(x) = \frac{1 + xq^{a(1)}}{1 - dxq^{a(1)}} f\left(xq^N\right). \qquad (5.23)$$

Iterating (5.23), we get

$$f(x) = \prod_{n=0}^{\infty} \frac{1 + xq^{Nn+a(1)}}{1 - dxq^{Nn+a(1)}} f(0).$$

Thus

$$f(1) = \frac{(-q^{a(1)}; q^N)_\infty}{(dq^{a(1)}; q^N)_\infty}.$$

Now assume that Theorem 5.9 is true for some $r - 1 \geq 1$. We want to show that it is true for r too. Let $N \geq \alpha\left(2^r - 1\right)$ and f be a function with $f(0) = 1$ satisfying $(\mathrm{eq}_{N,r})$. Let

$$F(x) := f(x) \prod_{n=0}^{\infty} \frac{1 - dxq^{Nn+a(r)}}{1 - xq^{Nn}}.$$

By Lemma 5.10, $F(0) = 1$ and F satisfies $(\mathrm{eq}'_{N,r})$. Now let

$$F(x) =: \sum_{n=0}^{\infty} A_n x^n.$$

Then by Lemma 5.11 $A_0 = 1$ and $(A_n)_{n\in\mathbb{N}}$ satisfies $(\mathrm{rec}_{N,r})$.
But by Lemma 5.15, $(A_n)_{n\in\mathbb{N}}$ also satisfies $({\mathrm{rec}'}_{N,r-1})$. Now let

$$A_n =: a_n \prod_{k=0}^{n-1} \left(1 + q^{Nk+a(r)}\right).$$

By Lemma 5.14, $a_0 = 1$ and $(a_n)_{n\in\mathbb{N}}$ satisfies $({\mathrm{rec}''}_{N,r-1})$. Let

$$G(x) := \sum_{n=0}^{\infty} a_n x^n.$$

By Lemma 5.13, $G(0)=1$ and G satisfies $(\mathrm{eq}''_{N,r-1})$. Finally let

$$g(x) := G(x)\prod_{n=0}^{\infty}\left(1-xq^{Nn}\right).$$

By Lemma 5.12, $g(0)=1$ and g satisfies $(\mathrm{eq}_{N,r-1})$. Now N is still larger than $\alpha\left(2^{r-1}-1\right)$ and we can use the induction hypothesis which gives

$$g(1)=\prod_{k=1}^{r-1}\frac{(-q^{a(k)};q^N)_\infty}{(dq^{a(k)};q^N)_\infty}. \tag{5.24}$$

By Appell's Comparison Theorem,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}a_nx^n\\ &= \lim_{x\to 1^-}(1-x)G(x)\\ &= \lim_{x\to 1^-}(1-x)\frac{g(x)}{\prod_{n=0}^{\infty}(1-xq^{Nn})}\\ &= \frac{g(1)}{\prod_{n=1}^{\infty}(1-q^{nN})}.\end{aligned}$$

Thus

$$\lim_{n\to\infty} A_n = \prod_{k=0}^{\infty}\left(1+q^{Nk+a(r)}\right)\frac{g(1)}{\prod_{n=1}^{\infty}(1-q^{nN})}.$$

Therefore, by Appell's Comparison Theorem again,

$$\begin{aligned}\lim_{x\to 1^-}(1-x)F(x) &= \lim_{n\to\infty}A_n\\ &= \prod_{k=0}^{\infty}\left(1+q^{Nk+a(r)}\right)\frac{g(1)}{\prod_{n=1}^{\infty}(1-q^{nN})}.\end{aligned} \tag{5.25}$$

Finally,

$$\begin{aligned}f(1) &= \lim_{x\to 1^-} f(x)\\ &= \lim_{x\to 1^-}\prod_{n=0}^{\infty}\frac{1-xq^{Nn}}{1-dxq^{Nn+a(r)}}F(x)\\ &= \frac{\prod_{n=1}^{\infty}\left(1-q^{Nn}\right)}{\prod_{n=0}^{\infty}\left(1-dq^{Nn+a(r)}\right)}\prod_{k=0}^{\infty}\left(1+q^{Nk+a(r)}\right)\frac{g(1)}{\prod_{n=1}^{\infty}(1-q^{nN})} \quad \text{by (5.25)}\end{aligned}$$

$$= \frac{\left(-q^{a(r)}; q^N\right)_\infty}{\left(dq^{a(r)}; q^N\right)_\infty} g(1).$$

Then by (5.24),

$$f(1) = \prod_{k=1}^{r} \frac{(-q^{a(k)}; q^N)_\infty}{(dq^{a(k)}; q^N)_\infty}.$$

This completes the proof. $\square$

Now Theorem 5.5 is a simple corollary of Theorem 5.9.

Proof of Theorem 5.5: By Lemma 5.8, $f_a(1)$ satisfies $(\mathrm{eq}_{N,r})$. Therefore the generating function for overpartitions counted by $E(A'_N; k, n)$ is equal to

$$f_{a(1)}(1) = \prod_{k=1}^{r} \frac{(-q^{a(k)}; q^N)_\infty}{(dq^{a(k)}; q^N)_\infty},$$

which is also the generating function for overpartitions counted by $D(A_N; k, n)$. Thus for all $k, n \geq 0$, $D(A_N; k, n) = E(A'_N; k, n)$ and the theorem is proved. $\square$

6. Generalisation of a second theorem of Andrews

Contents

6.1. Andrews' second generalisation of Schur's theorem

6.1.1. Statement of the theorem

As the ideas in Andrews' proof of Schur's theorem based on the smallest part of the partition generalised to prove Theorem 5.1, it was natural to wonder if it could also be proved by generalising the ideas of the proof based on the largest part. Surprisingly enough, the ideas could indeed be generalised but Andrews found that it led to a completely different generalisation [And68a].

We use the same notation as in Chapter 5. Let $-A_N$ denote the set of positive integers congruent to some $-a(i) \mod N$ and $-A'_N$ the set of positive integers congruent to some $-\alpha(i) \mod N$. Let us recall the second of Andrews' generalisations (stated as Theorem 2.14 in the introduction). Again, Schur's theorem corresponds to the case $N = 3$, $r = 2$, $a(1) = 1, a(2) = 2$.

Theorem 6.1 (Andrews)**.** *Let $F(-A_N; n)$ denote the number of partitions of n into distinct parts taken from $-A_N$. Let $G(-A'_N; n)$ denote the number of partitions of n into parts taken from $-A'_N$ of the form $n = \lambda_1 + \cdots + \lambda_s$, such that*

$$\lambda_i - \lambda_{i+1} \geq Nw(\beta_N(-\lambda_i)) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

and $\lambda_s \geq N(w(\beta_N(-\lambda_s)) - 1)$.

Then for all $n \geq 0$, $F(-A_N; n) = G(-A'_N; n)$.

6.1.2. Proof

We now present Andrews' proof of Theorem 6.1.

As in the proofs of Schur's theorem, by Theorem 3.2, the generating function for partitions enumerated by $F(-A_N; n)$ is

$$\sum_{n\geq 0} F(-A_N; n) q^n = \prod_{k=1}^{r} (-q^{N-a(k)}; q^N)_\infty.$$

So we only need to show that the generating function for partitions enumerated by $G(-A'_N; n)$ is the same. First, we give the recurrence equation satisfied by the generating function for partitions enumerated by $G(-A'_N; n)$, with an added condition on the largest part. Then we conclude by switching to a recurrence equation and using Appell's Comparison Theorem as in the proof of Schur's theorem from Section 4.2.

As in the proof of Schur's theorem from Section 4.1, we start by giving recurrences based on the largest part of the partition. Let $\pi_m(n)$ denote the number of partitions counted by $G(-A'_N; n)$ such that no part exceeds m.

The following lemma holds.

Lemma 6.2. *If* $j \geq 1$, *then*

$$\pi_{jN-\alpha(m)}(n) = \pi_{jN-\alpha(m+1)}(n) + \pi_{jN-w(\alpha(m))N-v(\alpha(m))}(n - jN + \alpha(m)). \tag{6.1}$$

Proof: We break the partitions counted by $\pi_{jN-\alpha(m)}(n)$ into two sets : those with largest part $< jN - \alpha(m)$ and those with largest part equal to $jN - \alpha(m)$. The first set is counted by $\pi_{jN-\alpha(m+1)}(n)$, and the second by $\pi_{jN-w(\alpha(m))N-v(\alpha(m))}(n - jN + \alpha(m))$. To see this, let us consider a partition $n = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ counted by $\pi_{jN-\alpha(m)}(n)$ with largest part equal to $jN - \alpha(m)$. Now remove its largest part $\lambda_1 = jN - \alpha(m)$. The number partitioned becomes $n - jN + \alpha(m)$. By the difference conditions, we have

$$\begin{aligned}\lambda_2 &\leq \lambda_1 - w(\alpha(m))N - v(\alpha(m)) + \alpha(m)\\ &\leq jN - w(\alpha(m))N - v(\alpha(m)),\end{aligned}$$

and we obtain a partition counted by $\pi_{jN-w(\alpha(m))N-v(\alpha(m))}(n-jN+a(m))$. $\square$

We define, for $m \geq 0$, $|q| < 1$,

$$d_m = d_m(q) := 1 + \sum_{n=1}^{\infty} \pi_m(n) q^n,$$

and to have valid equations even when m is negative, we set

$$d_m = \begin{cases} 1 \text{ for } -N \leq m < 0, \\ 0 \text{ for } m < -N. \end{cases}$$

This definition is consistent with (6.1) and the condition that

$$\lambda_s \geq N(w(\beta_N(-\lambda_s)) - 1).$$

We want to find $\lim_{m \to \infty} d_m$, which is the generating function for all partitions counted by $G(-A'_N; n)$. To do so, we establish a recurrence equation relating $g_{(j-i)N-a(1)}$, for $0 \leq i \leq r$. Let us start by giving some relations between generating functions.

Lemma 6.2 directly implies

Lemma 6.3. *We have*

$$d_{jN-\alpha(m)} = d_{jN-\alpha(m+1)} + q^{jN-\alpha(m)} d_{jN-w(\alpha(m))N-v(\alpha(m))}. \tag{6.2}$$

Let $1 \leq k \leq r+1$. Adding Equations (5.13) together for $1 \leq m \leq 2^{k-1} - 1$, using the fact that $\alpha\left(2^{k-1}\right) = a(k)$, we obtain

$$d_{jN-a(1)} = d_{jN-a(k)} + \sum_{\alpha < a(k)} q^{jN-\alpha} d_{(j-w(\alpha))N-v(\alpha)}. \tag{6.3}$$

Let us now add Equations (6.2) together for $2^{k-2} \leq m \leq 2^{k-1} - 1$. This gives

$$d_{jN-a(k)} = d_{jN-a(k+1)} + \sum_{a(k) \leq \alpha < a(k+1)} q^{jN-\alpha} d_{(j-w(\alpha))N-v(\alpha)}. \tag{6.4}$$

Every $a(k) < \alpha < a(k+1)$ is of the form $\alpha = a(k) + \alpha'$, with $\alpha' < a(k)$. Hence we can rewrite (6.4) as

$$\begin{aligned} d_{jN-a(k)} - d_{jN-a(k+1)} = {} & q^{jN-a(k)} d_{(j-1)N-a(k)} \\ & + \sum_{\alpha' < a(k)} q^{jN-a(k)-\alpha'} d_{(j-w(\alpha')-1)N-v(\alpha')} \end{aligned}$$

$$\begin{aligned}&= q^{jN-a(k)} d_{(j-1)N-a(k)} \\ &\quad + q^{N-a(k)} \left(d_{(j-1)N-a(1)} - d_{(j-1)N-a(k)} \right),\end{aligned}$$

where the last equality follows from (6.3). Thus

$$\begin{aligned} d_{jN-a(k)} &= d_{jN-a(k+1)} + q^{N-a(k)} d_{(j-1)N-a(1)} \\ &\quad - q^{N-a(k)} \left(1 - q^{(j-1)N}\right) d_{(j-1)N-a(k)}. \end{aligned} \tag{6.5}$$

Now we can give the recurrence equation satisfied by $(g_{jN-a(1)})_{j\in\mathbb{N}}$.

Lemma 6.4. *For $1 \leq k \leq r+1$, we have*

$$d_{jN-a(1)} = d_{jN-a(k)} + \sum_{i=1}^{k-1} \left(\sum_{\substack{\alpha < a(k) \\ w(\alpha)=i}} q^{jN-\alpha} \right) \prod_{h=1}^{i-1} \left(1 - q^{(j-h)N}\right) d_{(j-i)N-a(1)}. \tag{6.6}$$

Proof: Let us prove this lemma by induction on k. For $k=1$, (6.6) reduces to $d_{jN-a(1)} = d_{jN-a(1)}$. We assume that the lemma is true for some $1 \leq k \leq r$ and we show that it is also true for $k+1$. We have

$$\begin{aligned} & d_{jN-a(1)} - d_{jN-a(k+1)} \\ &\quad = \left(d_{jN-a(1)} - d_{jN-a(k)}\right) + \left(d_{jN-a(k)} - d_{jN-a(k+1)}\right) \\ &\quad = d_{jN-a(1)} - d_{jN-a(k)} + q^{N-a(k)} d_{(j-1)N-a(1)} \\ &\qquad - q^{N-a(k)} \left(1 - q^{(j-1)N}\right) d_{(j-1)N-a(k)} \\ &\quad = d_{jN-a(1)} - d_{jN-a(k)} + q^{N-a(k)} d_{(j-1)N-a(1)} \\ &\qquad - q^{N-a(k)} \left(1 - q^{(j-1)N}\right) \left(d_{(j-1)N-a(1)} \vphantom{\sum_{\substack{\alpha \\ w}}} \right. \\ &\qquad \left. - \sum_{i=1}^{k-1} \left(\sum_{\substack{\alpha < a(k) \\ w(\alpha)=i}} q^{(j-1)N-\alpha} \right) \prod_{h=1}^{i-1} \left(1 - q^{(j-h-1)N}\right) d_{(j-i-1)N-a(1)} \right) \\ &\quad = d_{jN-a(1)} - d_{jN-a(k)} + q^{jN-a(k)} d_{(j-1)N-a(1)} \end{aligned}$$

$$+\sum_{i=1}^{k-1}\left(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=i}} q^{jN-\alpha-a(k)}\right)\prod_{h=1}^{i}\left(1-q^{(j-h)N}\right)d_{(j-i-1)N-a(1)}.$$

If α runs over the elements of A'_N less that $a(k)$, then $\alpha+a(k)$ runs over the elements strictly between $a(k)$ and $a(k+1)$. Thus

$$\begin{aligned}
&d_{jN-a(1)}-d_{jN-a(k+1)}\\
&\quad=d_{jN-a(1)}-d_{jN-a(k)}\\
&\qquad+\sum_{i=0}^{k-1}\left(\sum_{\substack{a(k)\le\alpha'<a(k+1)\\ w(\alpha')=i+1}} q^{jN-\alpha'}\right)\prod_{h=1}^{i}\left(1-q^{(j-h)N}\right)d_{(j-i-1)N-a(1)}\\
&\quad=\sum_{i=1}^{k-1}\left(\sum_{\substack{\alpha<a(k)\\ w(\alpha)=i}} q^{jN-\alpha-a(k)}\right)\prod_{h=1}^{i-1}\left(1-q^{(j-h)N}\right)d_{(j-i)N-a(1)}\\
&\qquad+\sum_{i=1}^{k}\left(\sum_{\substack{a(k)\le\alpha'<a(k+1)\\ w(\alpha')=i}} q^{jN-\alpha'}\right)\prod_{h=1}^{i-1}\left(1-q^{(j-h)N}\right)d_{(j-i)N-a(1)}\\
&\quad=\sum_{i=1}^{k}\left(\sum_{\substack{\alpha'<a(k+1)\\ w(\alpha')=i}} q^{jN-\alpha'}\right)\prod_{h=1}^{i-1}\left(1-q^{(j-h)N}\right)d_{(j-i)N-a(1)}.
\end{aligned}$$

We obtain (6.6) for $k+1$, the lemma is proved. □

Writing $t_j := d_{jN-a(1)}$ and setting $k=r+1$ in Lemma 6.4, we obtain the desired recurrence equation

$$t_j=t_{j-1}+\sum_{i=1}^{r}\left(\sum_{\substack{\alpha\in A'\\ w(\alpha)=i}} q^{jN-\alpha}\right)\prod_{h=1}^{i-1}\left(1-q^{(j-h)N}\right)t_{j-i}, \tag{6.7}$$

with the initial conditions $t_0=1$ and $t_{-n}=0$ for $n>0$.

Now we proceed as in the proof of Schur's theorem and define

$$s_j = \frac{t_j}{\prod_{k=1}^{j}(1-q^{Nk})}.$$

Then $s_0 = 1$, $s_{-n} = 0$ for $n > 0$, and

$$(1-q^{Nj})s_j = s_{j-1} + \sum_{i=1}^{r} \left(\sum_{\substack{\alpha \in A' \\ w(\alpha)=i}} q^{jN-\alpha} \right) s_{j-i}.$$

Let

$$f(x) = \sum_{j \leq 0} s_j x^j.$$

Then $f(0) = 1$ and

$$(1-x)f(x) = \left(1 + xq^{N-a(1)}\right) \cdots \left(1 + xq^{N-a(r)}\right) f\left(xq^N\right).$$

Iterating leads to

$$f(x) = \frac{\prod_{k=1}^{r}(-xq^{N-a(k)};q^N)_\infty}{(x;q^N)_\infty}.$$

Now we conclude using Appell's Comparison Theorem. We have

$$\begin{aligned}
\lim_{j\to\infty} t_j &= \prod_{k=1}^{\infty}\left(1-q^{Nk}\right) \lim_{j\to\infty} s_j \\
&= \prod_{k=1}^{\infty}\left(1-q^{Nk}\right) \lim_{x\to 1^-} \frac{\prod_{k=1}^{r}(-xq^{N-a(k)};q^N)_\infty}{(xq^N;q^N)_\infty} \\
&= \prod_{k=1}^{r}(-q^{N-a(k)};q^N)_\infty.
\end{aligned}$$

This is the generating function for partitions enumerated by $F(-A_n;n)$ and the theorem is proved.

6.2. A generalisation of Andrews' second theorem to overpartitions

6.2.1. Statement of the theorem

Now that we have given a proof of Andrews' second generalisation of Schur's theorem, we show that it can also be generalised to overpartitions by proving the following theorem (stated as Theorem 2.16 in the introduction). This was done in the publication [Dou15].

Theorem 6.5. *Let $F(-A_N;k,n)$ denote the number of overpartitions of n into parts taken from $-A_N$, having k non-overlined parts. Let $G(-A'_N;k,n)$ denote the number of overpartitions of n into parts taken from $-A'_N$ of the form $n = \lambda_1 + \cdots + \lambda_s$, having k non-overlined parts, such that*

$$\lambda_i - \lambda_{i+1} \geq \big(Nw\left(\beta_N(-\lambda_i)\right) - 1 + \chi(\overline{\lambda_{i+1}})\big) + v(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i),$$

$$\lambda_s \geq N(w(\beta_N(-\lambda_s)) - 1),$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all $k,n \geq 0$, $F(-A_N;k,n) = G(-A'_N;k,n)$.

When $k=0$, which means that all parts are overlined, this becomes Andrews' theorem, and when $N=3$, $r=2$, $a(1)=1$, $a(2)=2$, we obtain Schur's theorem for overpartitions.

The proof of Theorem 6.5 uses ideas similar to the proof of Theorem 5.5, in the sense that it relies on going back and forth from q-difference equations to recurrence equations. First, we give the recurrence equation satisfied by the generating function for overpartitions enumerated by $G(-A'_N;k,n)$ having their largest part $\leq m$, using some combinatorial reasoning on the largest part. Then we prove by induction on r that the limit when m tends to infinity of a function satisfying this recurrence equation is equal to $\prod_{j=1}^{r} \frac{(-q^{N-a(j)};q^N)_\infty}{(dq^{N-a(j)};q^N)_\infty}$, which is the generating function for overpartitions counted by $F(-A_N;k,n)$.

6.2.2. The recurrence equation

In this section, we establish the recurrence equation satisfied by the generating function for overpartitions enumerated by $G(-A'_N;k,n)$ having their largest part $\leq m$.

Let $n,m \in \mathbb{N}^*$, $k \in \mathbb{N}$. Let $\pi_m(k,n)$ denote the number of overpartitions counted by $G(-A'_N;k,n)$ such that the largest part is $\leq m$ and overlined. Let $\phi_m(k,n)$ denote the number of overpartitions counted by $G(-A'_N;k,n)$ such that the largest part is $\leq m$ and non-overlined. Then $\psi_m(k,n) := \pi_m(k,n) + \phi_m(k,n)$ is the number of overpartitions counted by $G(-A'_N;k,n)$ with largest part $\leq m$.

Then the following holds.

Lemma 6.6. *We have*

$$\begin{aligned}\psi_{jN-\alpha(m)}(k,n) &- \psi_{jN-\alpha(m+1)}(k,n)\\ &= \psi_{jN-w(\alpha(m))N-v(\alpha(m))}(k,n-jN+\alpha(m)) \\ &+\psi_{jN-(w(\alpha(m))-1)N-v(\alpha(m))}(k-1,n-jN+\alpha(m)).\end{aligned} \tag{6.8}$$

Proof: Let us first prove the following equation:

$$\begin{aligned}\pi_{jN-\alpha(m)}(k,n) = {}& \pi_{jN-\alpha(m+1)}(k,n) \\ &+ \pi_{jN-w(\alpha(m))N-v(\alpha(m))}(k,n-jN+\alpha(m)) \\ &+ \phi_{jN-(w(\alpha(m))-1)N-v(\alpha(m))}(k,n-jN+\alpha(m)).\end{aligned} \tag{6.9}$$

We break the overpartitions counted by $\pi_{jN-\alpha(m)}(k,n)$ into two sets : those with largest part $< jN-\alpha(m)$ and those with largest part equal to $jN-\alpha(m)$. The first set is counted by $\pi_{jN-\alpha(m+1)}(k,n)$, and the second by

$$\begin{aligned}&\pi_{jN-w(\alpha(m))N-v(\alpha(m))}(k,n-jN+a(m)) \\ &+ \phi_{jN-(w(\alpha(m))-1)N-v(\alpha(m))}(k,n-jN+a(m)).\end{aligned}$$

To see this, let us consider an overpartition $n = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ counted by $\pi_{jN-\alpha(m)}(k,n)$ with largest part equal to $jN-\alpha(m)$. Now remove its largest part $\lambda_1 = \overline{jN-\alpha(m)}$. The number partitioned becomes $n-jN+\alpha(m)$. The largest part was overlined so the number of non-overlined parts is still k. If λ_2 was overlined, then we have

$$\begin{aligned}\lambda_2 &\le \lambda_1 - w(\alpha(m))N - v(\alpha(m)) + \alpha(m) \\ &\le jN - w(\alpha(m))N - v(\alpha(m)),\end{aligned}$$

and we obtain an overpartition counted by $\pi_{jN-w(\alpha(m))N-v(\alpha(m))}(k,n-jN+a(m))$. If λ_2 was not overlined, then we have

$$\begin{aligned}\lambda_2 &\le \lambda_1 - (w(\alpha(m))-1)N - v(\alpha(m)) + \alpha(m) \\ &\le jN - (w(\alpha(m))-1)N - v(\alpha(m)),\end{aligned}$$

and we obtain an overpartition counted by $\phi_{jN-(w(\alpha(m))-1)N-v(\alpha(m))}(k,n-jN+a(m))$.

In the same way we can prove the following

$$\begin{aligned}\phi_{jN-\alpha(m)}(k,n) = {}& \phi_{jN-\alpha(m+1)}(k,n) \\ &+ \pi_{jN-w(\alpha(m))N-v(\alpha(m))}(k-1,n-jN+\alpha(m)) \\ &+ \phi_{jN-(w(\alpha(m))-1)N-v(\alpha(m))}(k-1,n-jN+\alpha(m)).\end{aligned} \tag{6.10}$$

Adding Equations (6.9) and (6.10) and noting that for all m, n, k, $\pi_m(k-1, n) = \phi_m(k, n)$ (we can either overline the largest part or not), we obtain Equation (6.8). $\square$

We define, for $m \geq 1$, $|q| < 1$, $|d| < 1$,

$$g_m = g_m(q,d) := 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \psi_m(k,n) q^n d^k,$$

and for all $0 \leq k \leq r-1$, we set $g_{-m}(q,d) = (-d)^k$ for all $kN \leq m \leq (k+1)N$. This definition is consistent with (6.8) and the condition that $\lambda_s \geq N(w(\beta_N(-\lambda_s)) - 1)$.

We want to find $\lim_{m\to\infty} g_m$, which is the generating function for all overpartitions counted by $G(-A'_N; k, n)$. To do so, we establish a recurrence equation relating $g_{(m-j)N-a(1)}$, for $0 \leq j \leq r$. Let us start by giving some relations between generating functions.

Lemma 6.6 directly implies

Lemma 6.7. *We have*

$$\begin{aligned} g_{jN-\alpha(m)} = g_{jN-\alpha(m+1)} &+ q^{jN-\alpha(m)} g_{jN-w(\alpha(m))N-v(\alpha(m))} \\ &+ dq^{jN-\alpha(m)} g_{jN-(w(\alpha(m))-1)N-v(\alpha(m))}. \end{aligned} \tag{6.11}$$

Let $1 \leq k \leq r+1$. Adding Equations (6.11) together for $1 \leq m \leq 2^{k-1}-1$, using the fact that $\alpha\left(2^{k-1}\right) = a(k)$, we obtain

$$\begin{aligned} g_{jN-a(1)} &= g_{jN-a(k)} \\ &+ \sum_{\alpha < a(k)} \left(q^{jN-\alpha} g_{(j-w(\alpha))N-v(\alpha)} + dq^{jN-\alpha} g_{(j-w(\alpha)+1)N-v(\alpha)} \right). \end{aligned} \tag{6.12}$$

Let us now add Equations (6.11) together for $2^{k-2} \leq m \leq 2^{k-1}-1$. This gives

$$\begin{aligned} g_{jN-a(k)} &= g_{jN-a(k+1)} \\ &+ \sum_{a(k) \leq \alpha < a(k+1)} \left(q^{jN-\alpha} g_{(j-w(\alpha))N-v(\alpha)} + dq^{jN-\alpha} g_{(j-w(\alpha)+1)N-v(\alpha)} \right). \end{aligned} \tag{6.13}$$

Every $a(k) < \alpha < a(k+1)$ is of the form $\alpha = a(k) + \alpha'$, with $\alpha' < a(k)$. Hence we can rewrite (6.13) as

$$\begin{aligned}
&g_{jN-a(k)} - g_{jN-a(k+1)}\\
&= q^{jN-a(k)} g_{(j-1)N-a(k)} + dq^{jN-a(k)} g_{jN-a(k)}\\
&+ \sum_{\alpha' < a(k)} \left(q^{jN-a(k)-\alpha'} g_{(j-w(\alpha')-1)N-v(\alpha')} + dq^{jN-a(k)-\alpha'} g_{(j-w(\alpha'))N-v(\alpha')} \right)\\
&= q^{jN-a(k)} g_{(j-1)N-a(k)} + dq^{jN-a(k)} g_{jN-a(k)}\\
&+ q^{N-a(k)} \left(g_{(j-1)N-a(1)} - g_{(j-1)N-a(k)} \right),
\end{aligned}$$

where the last equality follows from (6.12).

Thus

$$\begin{aligned}
\left(1 - dq^{jN-a(k)}\right) g_{jN-a(k)} = g_{jN-a(k+1)} &+ q^{N-a(k)} g_{(j-1)N-a(1)}\\
&- q^{N-a(k)} \left(1 - q^{(j-1)N}\right) g_{(j-1)N-a(k)}.
\end{aligned} \tag{6.14}$$

We are now ready to state the key lemma which will lead to the recurrence equation satisfied by $(g_{\ell N-a(1)})_{\ell \in \mathbb{N}}$.

Lemma 6.8. *For $1 \leq k \leq r+1$, we have*

$$\begin{aligned}
&\prod_{j=1}^{k-1} \left(1 - dq^{\ell N-a(j)}\right) g_{\ell N-a(1)} = g_{\ell N-a(k)}\\
&+\sum_{j=1}^{k-1} \left(\sum_{m=0}^{k-j-1} d^m \sum_{\substack{\alpha < a(k)\\ w(\alpha)=j+m}} q^{\ell N-\alpha} \left((-1)^{m-1} q^{\ell(m-1)N} \begin{bmatrix} j+m-1\\ m-1 \end{bmatrix}_{q^{-N}} \right.\right.\\
&\qquad \left.\left. +(-1)^m q^{\ell m N} \begin{bmatrix} j+m\\ m \end{bmatrix}_{q^{-N}} \right) \right) \prod_{h=1}^{j-1} \left(1 - q^{(\ell-h)N}\right) g_{(\ell-j)N-a(1)}.
\end{aligned} \tag{6.15}$$

Proof: To prove this lemma, it is sufficient to replace q by q^{-1}, then x by $q^{\ell}N$ and finally $f_{a(i)}\left(q^{mN}\right)$ by $g_{mN-a(i)}$ in the proof of Lemma 5.8 of the proof of the first generalisation in Section 5. □

Writing $u_\ell := g_{\ell N-a(1)}$ and setting $k = r+1$ in Lemma 6.8, we obtain the desired recurrence equation

$$\prod_{j=1}^{r}\left(1-dq^{\ell N-a(j)}\right)u_\ell = u_{\ell-1}$$
$$+\sum_{j=1}^{r}\left(\sum_{m=0}^{r-j} d^m \sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m}} q^{\ell N-\alpha}\left((-1)^{m-1}q^{\ell(m-1)N}\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^{-N}}\right.\right. \qquad (\mathrm{rec}_{N,r})$$
$$\left.\left.+(-1)^m q^{\ell mN}\begin{bmatrix} j+m\\ m\end{bmatrix}_{q^{-N}}\right)\right)\prod_{h=1}^{j-1}\left(1-q^{(\ell-h)N}\right)u_{\ell-j},$$

with the initial conditions $u_{-k}=(-d)^k$ for all $0\leq k\leq r-1$.

6.2.3. Evaluating $\lim\limits_{\ell\to\infty} u_\ell$ by induction

In this section, we evaluate $\lim\limits_{\ell\to\infty} u_\ell$, which is the generating function for partitions counted by $G(-A'_N;k,n)$. To do so, we prove the following theorem by induction on r.

Theorem 6.9. *Let r be a positive integer. Then for every $N\geq\alpha(2^r-1)$, for every sequence $(u_m)_{m\in\mathbb{N}}$ satisfying $(\mathrm{rec}_{N,r})$ and the initial condition $u_0=1$, we have*

$$\lim_{\ell\to\infty} u_\ell = \prod_{k=1}^{r}\frac{(-q^{N-a(k)};q^N)_\infty}{(dq^{N-a(k)};q^N)_\infty}.$$

The idea of the proof is to start from a function satisfying $(\mathrm{rec}_{N,r})$ and to do some transformations to obtain a function satisfying $(\mathrm{rec}_{N,r-1})$ in order to use the induction hypothesis. To simplify the proof, we split it into several lemmas.

Lemma 6.10. *Let (u_m) and (β_m) be two sequences such that for all $m\in\mathbb{N}$,*

$$\beta_m := u_m\prod_{j=1}^{m}\frac{1-dq^{jN-a(r)}}{1-q^{jN}}$$

Then $u_0=1$ and (u_m) satisfies $(\mathrm{rec}_{N,r})$ if and only if $\beta_0=1$ and (β_m) satisfies *the following recurrence equation*

$$\left(1+\sum_{j=1}^{r}\left(d^{j-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j-1}}q^{-\alpha}+d^{j}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j}}q^{-\alpha}\right)(-1)^{j}q^{j\ell N}\right)\beta_\ell$$
$$=\beta_{\ell-1}+\sum_{j=1}^{r}\sum_{h=1}^{r}\sum_{k=0}^{\min(j-1,h-1)}c_{k,j}b_{h-k,j}(-1)^{h+1}q^{h\ell N}\beta_{\ell-j}, \qquad (\mathrm{rec}'_{N,r})$$

where

$$c_{k,j}:=q^{-N\frac{k(k+1)}{2}-ka(r)}\begin{bmatrix}j-1\\ k\end{bmatrix}_{q^{-N}}d^{k},$$

and

$$b_{m,j}:=\left(d^{m-1}\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m-1}}q^{-\alpha}+d^{m}\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m}}q^{-\alpha}\right)\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^{-N}}.$$

Proof: Directly plugging the definition of (β_m) into $(\mathrm{rec}_{N,r})$, we get

$$(1-q^{\ell N})\prod_{j=1}^{r-1}\left(1-dq^{\ell N-a(j)}\right)\beta_\ell=\beta_{\ell-1}$$
$$+\sum_{j=1}^{r}\left(\sum_{m=0}^{r-j}d^{m}\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m}}q^{\ell N-\alpha}\left((-1)^{m-1}q^{\ell(m-1)N}\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^{-N}}\right.\right.$$
$$\left.\left.+(-1)^{m}q^{\ell mN}\begin{bmatrix}j+m\\ m\end{bmatrix}_{q^{-N}}\right)\right)\prod_{h=1}^{j-1}\left(1-dq^{(\ell-h)N-a(r)}\right)\beta_{\ell-j}.$$

With the conventions that

$$\sum_{\substack{\alpha<a(r)\\ w(\alpha)=n}}q^{-\alpha}=0 \text{ for } n\geq r,$$

and

$$\sum_{\substack{\alpha<a(r)\\ w(\alpha)=0}}q^{-\alpha}=1,$$

this can be reformulated as

$$\left(1+\sum_{j=1}^{r}\left(d^{j-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j-1}}q^{-\alpha}+d^{j}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j}}q^{-\alpha}\right)(-1)^{j}q^{j\ell N}\right)\beta_{\ell}=\beta_{\ell-1}$$

$$+\sum_{j=1}^{r}\left(\sum_{m=1}^{r-j+1}\left(d^{m-1}\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m-1}}q^{-\alpha}+d^{m}\sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m}}q^{-\alpha}\right)\begin{bmatrix}j+m-1\\ m-1\end{bmatrix}_{q^{-N}}\right.$$

$$\left.\times(-1)^{m-1}q^{m\ell N}\right)\left(\sum_{k=0}^{j-1}q^{-N\frac{k(k-1)}{2}-ka(r)}\begin{bmatrix}j-1\\ k\end{bmatrix}_{q^{-N}}d^{k}(-1)^{k}q^{k\ell N}\right)\beta_{\ell-j},$$

because of the q-binomial theorem (Theorem 3.9),

$$\prod_{k=0}^{n-1}(1+q^{k}t)=\sum_{k=0}^{n}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\ k\end{bmatrix}_{q}t^{k},$$

in which we replace q by q^{-N}, n by $j-1$ and t by $-dq^{(\ell-1)N-a(r)}$. Finally, noting that $b_{l-k,j}=0$ if $j+l-k-1\geq r$, we can rewrite this as $(\mathrm{rec}'_{N,r})$. Moreover, $\beta_0=u_0=1$ and the lemma is proved. □

We can directly transform $(\mathrm{rec}'_{N,r})$ into a q-difference equation on the generating function for (β_m).

Lemma 6.11. *Let (β_m) be a sequence and f a function such that*

$$f(x):=\sum_{n=0}^{\infty}\beta_n x^n.$$

Then (β_m) satisfies $(\mathrm{rec}'_{N,r})$ and the initial condition $\beta_0=1$ if and only if

$f(0)=1$ and f satisfies the following recurrence equation

$$(1-x)f(x) = \sum_{m=1}^{r} \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{-\alpha} + d^{m} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{\alpha} \right.$$
$$\left. + \sum_{j=1}^{r} \sum_{k=0}^{\min(j-1,m-1)} c_{k,j} b_{m-k,j} x^j q^{mjN} \right) (-1)^{m+1} f(xq^{mN}). \tag{eq$_{N,r}$}$$

Proof: By the definition of f and (rec$'_{N,r}$), we have

$$(1-x)f(x) = \sum_{j=1}^{r} \left(d^{j-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j-1}} q^{-\alpha} + d^{j} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j}} q^{-\alpha} \right) (-1)^{j+1} f\left(xq^{jN}\right)$$
$$+ \sum_{j=1}^{r} \sum_{h=1}^{r} \sum_{k=0}^{\min(j-1,h-1)} c_{k,j} b_{h-k,j} (-1)^{h+1} x^j q^{hjN} f\left(xq^{hN}\right).$$

Relabelling the summation indices and factorising leads to (eq$_{N,r}$). Moreover, $f(0)=\beta_0=1$. This completes the proof. □

Let us now do some transformations starting from (rec$_{N,r-1}$).

Lemma 6.12. *Let (μ_n) and (s_n) be two sequences such that for all n,*

$$s_n := \mu_n \prod_{k=1}^{n} \frac{1}{1-q^{Nk}}.$$

Then (μ_n) satisfies (rec$_{N,r-1}$) and the initial condition $\mu_0=1$ if and only if

$s_0 = 1$ and (s_n) satisfies the following recurrence equation

$$\left(1+\sum_{j=1}^{r}\left(d^{j-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j-1}} q^{-\alpha}+d^{j}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j}} q^{-\alpha}\right)(-1)^{j}q^{j\ell N}\right)s_\ell = s_{\ell-1}$$
$$+\sum_{j=1}^{r}\sum_{m=1}^{r-j}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{-\alpha}+d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{-\alpha}\right)$$
$$\times\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^{-N}}(-1)^{m+1}q^{m\ell N}s_{\ell-j}. \tag{$\mathrm{rec}''_{N,r-1}$}$$

Proof: Using the definition of (s_n) and $(\mathrm{rec}_{N,r-1})$, we get

$$(1-q^{\ell N})\prod_{j=1}^{r-1}\left(1-dq^{\ell N-a(j)}\right)s_\ell = s_{\ell-1}$$
$$+\sum_{j=1}^{r-1}\left(\sum_{m=0}^{r-j-1}d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{\ell N-\alpha}\left((-1)^{m-1}q^{\ell(m-1)N}\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^{-N}}\right.\right.$$
$$\left.\left.+(-1)^{m}q^{\ell mN}\begin{bmatrix} j+m\\ m\end{bmatrix}_{q^{-N}}\right)\right)s_{\ell-j}.$$

Then, as in the proof of Lemma 6.10, this can be reformulated as $(\mathrm{rec}''_{N,r-1})$, and $s_0=\mu_0=1$. □

Again, let us translate this into a recurrence equation on the generating function for (s_n).

Lemma 6.13. *Let (s_n) be a sequence and G be a function such that*

$$G(x) := \sum_{n=0}^{\infty} s_n x^n.$$

Then (s_n) *satisfies* $(\mathrm{rec}''_{N,r-1})$ *and the initial condition* $s_0 = 1$ *if and only if* $G(0) = 1$ *and* G *satisfies the following* q*-difference equation*

$$(1-x)\,G(x) = \sum_{m=1}^{r}\sum_{j=0}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{-\alpha} + d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{-\alpha}\right) \times \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^{-N}} (-1)^{m+1} x^{j} q^{jmN} G\left(xq^{mN}\right). \tag{$\mathrm{eq}''_{N,r-1}$}$$

Proof: By the definition of G and $(\mathrm{rec}''_{N,r-1})$, we have

$$\begin{aligned}(1-x)\,G(x) = {} & \sum_{m=1}^{r}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{-\alpha} + d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{-\alpha}\right)(-1)^{m+1} G\left(xq^{mN}\right) \\ & + \sum_{m=1}^{r-1}\sum_{j=1}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{-\alpha} + d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{-\alpha}\right) \\ & \times \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^{-N}} (-1)^{m+1} x^{j} q^{jmN} G\left(xq^{mN}\right).\end{aligned}$$

As the summand of the second term equals 0 when $m = r$, we can equivalently write that the second sum is taken over m going from 1 to r. Then we observe that the first term corresponds to $j = 0$ in the second term, and factorising gives exactly $(\mathrm{eq}''_{N,r-1})$. Moreover, $G(0) = s_0 = 1$. This completes the proof. □

Let us do a final transformation and obtain yet another q-difference equation.

Lemma 6.14. *Let* G *and* g *be two sequences such that*

$$g(x) := G(x)\prod_{k=1}^{\infty}\left(1 + xq^{kN-a(r)}\right).$$

Then G *satisfies* $(\mathrm{eq}''_{N,r-1})$ *and the initial condition* $G(0) = 1$ *if and only if*

$g(0) = 1$ *and* g *satisfies the following q-difference equation*

$$(1-x)\,g(x) = \sum_{m=1}^{r} \left(\sum_{\nu=0}^{r-1} \sum_{\mu=0}^{\min(m-1,\nu)} f_{m,\mu} e_{m,\nu-\mu} x^{\nu} q^{\nu m N} \right.$$

$$\left. + \, q^{-a(r)} \sum_{\nu=1}^{r} \sum_{\mu=0}^{\min(m-1,\nu-1)} f_{m,\mu} e_{m,\nu-\mu-1} x^{\nu} q^{\nu m N} \right) (-1)^{m+1} g\left(xq^{mN}\right), \tag{eq$'_{N,r-1}$}$$

where

$$e_{m,j} := \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha) = j+m-1}} q^{-\alpha} + d^{m} \sum_{\substack{\alpha < a(r) \\ w(\alpha) = j+m}} q^{-\alpha} \right) \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^{-N}},$$

and

$$f_{m,k} := q^{-N\frac{k(k+1)}{2} - ka(r)} \begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^{-N}}.$$

Proof: By the definition of g, we have

$$(1-x)g(x) = \sum_{m=1}^{r} \sum_{j=0}^{r-1} \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha) = j+m-1}} q^{-\alpha} + d^{m} \sum_{\substack{\alpha < a(r) \\ w(\alpha) = j+m}} q^{-\alpha} \right)$$

$$\times \begin{bmatrix} j+m-1 \\ m-1 \end{bmatrix}_{q^{-N}} (-1)^{m+1} \prod_{k=1}^{m} \left(1 + xq^{kN-a(r)}\right) x^{j} q^{jmN} g\left(xq^{mN}\right).$$

Furthermore

$$\begin{aligned}
\prod_{k=1}^{m} \left(1 + xq^{kN-a(r)}\right) &= \prod_{k=0}^{m-1} \left(1 + xq^{(m-k)N-a(r)}\right) \\
&= \left(1 + xq^{mN-a(r)}\right) \prod_{k=1}^{m-1} \left(1 + xq^{(m-k)N-a(r)}\right) \\
&= \left(1 + xq^{mN-a(r)}\right) \sum_{k=0}^{m-1} x^{k} q^{kmN - N\frac{k(k+1)}{2} - ka(r)} \begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^{-N}},
\end{aligned}$$

where the last equality follows from Theorem 3.9. Therefore

$$
\begin{aligned}
&(1-x)\,g(x)\\
&=\sum_{m=1}^{r}\left[\sum_{j=0}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{-\alpha}+d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{-\alpha}\right)\right.\\
&\quad\times\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^{-N}} x^{j}q^{jmN}\times\left(1+xq^{mN-a(r)}\right)\\
&\quad\left.\times\sum_{k=0}^{m-1}x^{k}q^{kmN-N\frac{k(k+1)}{2}-ka(r)}\begin{bmatrix} m-1\\ k\end{bmatrix}_{q^{-N}}\right](-1)^{m+1}g\left(xq^{mN}\right)\\
&=\sum_{m=1}^{r}\left[\sum_{j=0}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{-\alpha}+d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{-\alpha}\right)\right.\\
&\quad\times\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^{-N}} x^{j}q^{jmN}\\
&\quad\times\sum_{k=0}^{m-1}x^{k}q^{kmN-N\frac{k(k+1)}{2}-ka(r)}\begin{bmatrix} m-1\\ k\end{bmatrix}_{q^{-N}}\\
&\quad+\sum_{j=0}^{r-1}\left(d^{m-1}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-1}} q^{-\alpha}+d^{m}\sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m}} q^{-\alpha}\right)\\
&\quad\times q^{-a(r)}\begin{bmatrix} j+m-1\\ m-1\end{bmatrix}_{q^{-N}} x^{j+1}q^{(j+1)mN}\\
&\quad\left.\times\sum_{k=0}^{m-1}x^{k}q^{kmN-N\frac{k(k+1)}{2}-ka(r)}\begin{bmatrix} m-1\\ k\end{bmatrix}_{q^{-N}}\right](-1)^{m+1}g\left(xq^{mN}\right).
\end{aligned}
$$

Thus

$$
(1-x)\,g(x)=\sum_{m=1}^{r}\left(\sum_{j=0}^{r-1}e_{m,j}x^{j}q^{jmN}\sum_{k=0}^{m-1}f_{m,k}x^{k}q^{kmN}\right.
$$

$$+ q^{-a(r)} \sum_{j=1}^{r} e_{m,j-1} x^j q^{jmN} \sum_{k=0}^{m-1} f_{m,k} x^k q^{kmN} \Bigg) (-1)^{m+1} g\left(xq^{mN}\right).$$

Rearranging leads to ($\mathrm{eq'}_{N,r-1}$). As always, $g(0) = G(0) = 1$. The lemma is proved. □

We now want to show that f and g are in fact equal.

Lemma 6.15. *Let f and g be defined as in Lemmas 6.11 and 6.14. Then $f = g$.*

Proof: To prove the equality, it is sufficient to show that for every $1 \leq m \leq r$, the coefficient of $(-1)^{m+1} f\left(xq^{mN}\right)$ in ($\mathrm{eq}_{N,r}$) is the same as the coefficient of $(-1)^{m+1} g\left(xq^{mN}\right)$ in ($\mathrm{eq'}_{N,r-1}$). Let $m \in \{1, \ldots, r\}$,

$$\begin{aligned} S_m &:= \left[(-1)^{m+1} f\left(xq^{mN}\right)\right] (\mathrm{eq}_{N,r}) \\ &= d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha) = m-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha) = m}} q^{-\alpha} + \sum_{j=1}^{r} \sum_{k=0}^{\min(j-1,m-1)} c_{k,j} b_{m-k,j} x^j q^{jmN}, \end{aligned}$$

and

$$\begin{aligned} S'_m &:= \left[(-1)^{m+1} g\left(xq^{mN}\right)\right] (\mathrm{eq'}_{N,r-1}) \\ &= \sum_{\nu=0}^{r-1} \sum_{\mu=0}^{\min(m-1,\nu)} f_{m,\mu} e_{m,\nu-\mu} x^\nu q^{\nu mN} \\ &\quad + q^{-a(r)} \sum_{\nu=1}^{r} \sum_{\mu=0}^{\min(m-1,\nu-1)} f_{m,\mu} e_{m,\nu-\mu-1} x^\nu q^{\nu mN} \\ &= f_{m,0} e_{m,0} + \sum_{\nu=1}^{r} \Bigg(\sum_{\mu=0}^{\min(m-1,\nu)} f_{m,\mu} e_{m,\nu-\mu} \\ &\qquad + q^{-a(r)} \sum_{\mu=0}^{\min(m-1,\nu-1)} f_{m,\mu} e_{m,\nu-\mu-1} \Bigg) x^\nu q^{\nu mN}, \end{aligned}$$

where the last equality follows from the fact that $e_{m,r-\mu} = 0$ for all μ, as $\mu \leq m-1$ so the sums are over α such that $\alpha < a(r)$ and $w(\alpha) \geq r$, which is impossible.

Let us first notice that

$$f_{m,0}e_{m,0} = d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r)\\ w(\alpha)=m}} q^{-\alpha}.$$

Now define

$$T_{m,j} := \sum_{k=0}^{\min(j-1,m-1)} c_{k,j} b_{m-k,j},$$

and

$$T'_{m,j} := \sum_{k=0}^{\min(m-1,j)} f_{m,k}e_{m,j-k} + q^{-a(r)} \sum_{k=0}^{\min(m-1,j-1)} f_{m,k}e_{m,j-k-1}.$$

The only thing left to do is to show that for every $1 \leq j \leq r$,

$$T_{m,j} = T'_{m,j}.$$

We have

$$\begin{aligned}
&c_{k,j}b_{m-k,j}\\
&= q^{-N\frac{k(k+1)}{2}-ka(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}}\\
&\quad \times \left(d^{m-1} \sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m-k-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r+1)\\ w(\alpha)=j+m-k}} q^{-\alpha} \right)\\
&= \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k}} q^{-\alpha} \right) q^{-N\frac{k(k+1)}{2}-ka(r)}\\
&\quad \times \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}}\\
&+ q^{-a(r)} \left(d^{m-1} \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-2}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r)\\ w(\alpha)=j+m-k-1}} q^{-\alpha} \right) q^{-N\frac{k(k+1)}{2}-ka(r)}\\
&\quad \times \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}},
\end{aligned} \tag{6.16}$$

in which the last equality follows from separating the sums over α according to whether α contains $a(r)$ as a summand or not.

We also have

$$f_{m,k}e_{m,j-k} = q^{-N\frac{k(k+1)}{2}-ka(r)} \begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^{-N}} \times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k}} q^{-\alpha} \right) \begin{bmatrix} j+m-k-1 \\ m-1 \end{bmatrix}_{q^{-N}}, \tag{6.17}$$

and

$$q^{-a(r)} f_{m,k}e_{m,j-k-1} = q^{-N\frac{k(k+1)}{2}-(k+1)a(r)} \begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^{-N}} \times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-2}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} \right) \begin{bmatrix} j+m-k-2 \\ m-1 \end{bmatrix}_{q^{-N}}. \tag{6.18}$$

By a simple calculation using the definition of q-binomial coefficients, we get the following result. For all $j,k,m \in \mathbb{N}$,

$$\begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-1 \end{bmatrix}_{q^{-N}} = \begin{bmatrix} j \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}}. \tag{6.19}$$

Using (6.19), we obtain

$$\begin{aligned} T'_{m,j} = {} & \chi(j \le m-1)\; q^{-N\frac{j(j+1)}{2}-ja(r)} d^{m-1} \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^{-N}} \\ & \times \left(\sum_{\substack{\alpha<a(r) \\ w(\alpha)=m-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=m}} q^{-\alpha} \right) \\ & + \sum_{k=0}^{\min(m-1,j-1)} q^{-N\frac{k(k+1)}{2}-ka(r)} \begin{bmatrix} j \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}} \\ & \times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k}} q^{-\alpha} \right) \end{aligned}$$

$$+\sum_{k=0}^{\min(m-1,j-1)} q^{-N\frac{k(k+1)}{2}-(k+1)a(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-2 \\ m-k-1 \end{bmatrix}_{q^{-N}}$$
$$\times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-2}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} \right).$$

By (3.3) of Proposition 3.7, we have

$$\begin{bmatrix} j \\ k \end{bmatrix}_{q^{-N}} = \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^{-N}} + q^{N(k-j)} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^{-N}},$$
$$\begin{bmatrix} j+m-k-2 \\ m-k-1 \end{bmatrix}_{q^{-N}} = \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}} - q^{-Nj} \begin{bmatrix} j+m-k-2 \\ m-k-2 \end{bmatrix}_{q^{-N}}.$$

This allows us to rewrite $T'_{m,j}$ as

$$T'_{m,j} = \chi(j \le m-1)\; q^{-N\frac{j(j+1)}{2}-ja(r)} \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^{-N}}$$
$$\times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=m-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=m}} q^{-\alpha} \right)$$
$$+\sum_{k=0}^{\min(m-1,j-1)} q^{-N\frac{k(k+1)}{2}-ka(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}}$$
$$\times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k}} q^{-\alpha} \right)$$
$$+\sum_{k=0}^{\min(m-1,j-1)} q^{-N\frac{k(k+1)}{2}-ka(r)+N(k-j)} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}}$$
$$\times \left(d^{m-1} \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha<a(r) \\ w(\alpha)=j+m-k}} q^{-\alpha} \right)$$
$$+\sum_{k=0}^{\min(m-1,j-1)} q^{-N\frac{k(k+1)}{2}-(k+1)a(r)} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}}$$

$$
\times \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-2}} q^{-\alpha} + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} \right)
$$

$$
- \sum_{k=0}^{\min(m-2,j-1)} q^{-N\frac{k(k+1)}{2}-(k+1)a(r)-Nj} \begin{bmatrix} j-1 \\ k \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-2 \\ m-k-2 \end{bmatrix}_{q^{-N}}
$$

$$
\times \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-2}} q^{-\alpha} + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} \right).
$$

By (6.16), the sum of the second and fourth term in the sum above is exactly equal to Tm, j. Let X denote the sum of the third and fifth term. We now want to show that

$$
X + \chi(j \leq m-1)\ q^{-N\frac{j(j+1)}{2}-ja(r)} \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^{-N}}
$$

$$
\times \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha)=m-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha)=m}} q^{-\alpha} \right) = 0.
$$

By the change of variable $k' = k+1$ in the fourth sum, we get

$$
X = \sum_{k=0}^{\min(m-1,j-1)} q^{-N\frac{k(k-1)}{2}-ka(r)-Nj} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}}
$$

$$
\times \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k}} q^{-\alpha} \right)
$$

$$
- \sum_{k=1}^{\min(m-1,j)} q^{-N\frac{k(k-1)}{2}-ka(r)-Nj} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^{-N}} \begin{bmatrix} j+m-k-1 \\ m-k-1 \end{bmatrix}_{q^{-N}}
$$

$$
\times \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha)=j+m-k}} q^{-\alpha} \right)
$$

$$= \begin{cases} 0, & \text{if } j \geq m, \\ -\, q^{-N\frac{j(j+1)}{2} - ja(r)} \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^{-N}} & \\ \quad \times \left(d^{m-1} \sum\limits_{\substack{\alpha < a(r) \\ w(\alpha) = m-1}} q^{-\alpha} + d^m \sum\limits_{\substack{\alpha < a(r) \\ w(\alpha) = m}} q^{-\alpha} \right), & \text{otherwise} \end{cases}$$

$$= -\chi(j \leq m-1)\; q^{-N\frac{j(j+1)}{2} - ja(r)} \begin{bmatrix} m-1 \\ m-j-1 \end{bmatrix}_{q^{-N}}$$
$$\times \left(d^{m-1} \sum_{\substack{\alpha < a(r) \\ w(\alpha) = m-1}} q^{-\alpha} + d^m \sum_{\substack{\alpha < a(r) \\ w(\alpha) = m}} q^{-\alpha} \right).$$

This completes the proof. □

We can finally turn to the proof of Theorem 6.9.

Proof of Theorem 6.9: Let us start by the initial case $r = 1$. Let $N \geq a(1)$ and (u_m) such that $u_0 = 1$ and

$$\left(1 - dq^{\ell N - a(1)}\right) u_\ell = \left(1 + q^{\ell N - a(1)}\right) u_{\ell - 1}. \qquad (\mathrm{rec}_{N,1})$$

Then

$$u_\ell = \frac{(-q^{N-a(1)}; q^N)_\ell}{(dq^{N-a(1)}; q^N)_\ell}.$$

Taking the limit as ℓ tends to infinity gives the desired result.

Now assume that Theorem 6.9 is true for some $r - 1 \geq 1$. We want to show that it is true for r too. Let $N \geq \alpha(2^r - 1)$, and $(u_m)_{m \in \mathbb{N}}$ satisfy $(\mathrm{rec}_{N,r})$ and the initial condition $u_0 = 1$. For all m, let

$$\beta_m := u_m \prod_{j=1}^{m} \frac{1 - dq^{jN - a(r)}}{1 - q^{jN}}.$$

Then $\beta_0 = 1$ and by Lemma 6.10, (β_m) satisfies $(\mathrm{rec}'_{N,r})$. Now let

$$f(x) := \sum_{n=0}^{\infty} \beta_n x^n.$$

Then by Lemma 6.11, $f(0) = 1$ and f satisfies ($\mathrm{eq}_{N,r}$). But by Lemma 6.15, f also satisfies (${\mathrm{eq}'}_{N,r-1}$). Now let

$$G(x) := \frac{f(x)}{\prod_{k=1}^{\infty}\left(1 + xq^{kN-a(r)}\right)}.$$

By Lemma 6.14, $G(0) = 1$ and G satisfies (${\mathrm{eq}''}_{N,r-1}$). Let

$$G(x) =: \sum_{n=0}^{\infty} s_n x^n.$$

By Lemma 6.13, $s_0 = 1$ and (s_n) satisfies ($\mathrm{rec}''_{N,r-1}$). Finally let

$$\mu_n := s_n \prod_{k=1}^{n}\left(1 - q^{Nk}\right).$$

By Lemma 6.12, $\mu_0 = 1$ and (μ_n) satisfies ($\mathrm{rec}_{N,r-1}$). Now N is still larger than $\alpha\left(2^{r-1} - 1\right)$ and we can use the induction hypothesis which gives

$$\lim_{\ell\to\infty} \mu_\ell = \prod_{k=1}^{r-1} \frac{(-q^{N-a(k)}; q^N)_\infty}{(dq^{N-a(k)}; q^N)_\infty}.. \tag{6.20}$$

Therefore by the definition of (s_ℓ),

$$\lim_{\ell\to\infty} s_\ell = \frac{1}{(q^N; q^N)_\infty} \prod_{k=1}^{r-1} \frac{(-q^{N-a(k)}; q^N)_\infty}{(dq^{N-a(k)}; q^N)_\infty}.$$

We have

$$\begin{aligned} \sum_{m=0}^{\infty} \beta_m x^m &= f(x) \\ &= \prod_{k=1}^{\infty}\left(1 + xq^{kN-a(r)}\right) G(x) \qquad (6.21)\\ &= \prod_{k=1}^{\infty}\left(1 + xq^{kN-a(r)}\right) \sum_{m=0}^{\infty} s_m x^m. \end{aligned}$$

We multiply both sides of (6.21) by $(1 - x)$ and we apply Appell's Comparison Theorem. We obtain

$$\lim_{\ell\to\infty} \beta_\ell = \prod_{k=1}^{\infty}\left(1 + q^{kN-a(r)}\right) \lim_{\ell\to\infty} s_\ell$$

$$= \frac{(-q^{N-a(r)}; q^N)_\infty}{(q^N; q^N)_\infty} \prod_{k=1}^{r-1} \frac{(-q^{N-a(k)}; q^N)_\infty}{(dq^{N-a(k)}; q^N)_\infty}.$$

Thus by the definition of (β_ℓ), we have

$$\lim_{\ell\to\infty} u_\ell = \prod_{j=1}^{\infty} \frac{1-q^{jN}}{1-dq^{jN-a(r)}} \lim_{\ell\to\infty} \beta_\ell$$
$$= \frac{(-q^{N-a(r)}; q^N)_\infty}{(dq^{N-a(r)}; q^N)_\infty} \prod_{k=1}^{r-1} \frac{(-q^{N-a(k)}; q^N)_\infty}{(dq^{N-a(k)}; q^N)_\infty}.$$

Theorem 6.9 is proved. □

Now Theorem 6.5 is a simple corollary of Theorem 6.9.

Proof of Theorem 6.5: We have that $\lim_{\ell\to\infty} u_\ell$, which is the generating function for partitions counted by $G(-A'_N; k, n)$, is equal to $\prod_{k=1}^{r} \frac{(-q^{N-a(k)};q^N)_\infty}{(dq^{N-a(k)};q^N)_\infty}$, which is the generating function for partitions counted by $F(-A_N; k, n)$. Thus for all $k, n \geq 0$,

$$F(-A_N; n, k) = G(-A'_N; n, k),$$

and Theorem 6.5 is proved. □

7. Siladić's identity

Contents

7.1. Statement of the theorem

Now that we have proved several overpartition identities related to Schur's theorem, let us turn to a partition identity coming from the theory of Lie algebras, Siladić's identity (already stated as Theorem 2.17 in the introduction).

Theorem 7.1 (Siladić)**.** *The number of partitions $\lambda_1 + \cdots + \lambda_s$ of an integer n into parts different from 2 such that the difference between two consecutive parts is at least 5 (ie. $\lambda_i - \lambda_{i+1} \geq 5$) and*

$$\begin{aligned}
\lambda_i - \lambda_{i+1} = 5 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \mod 16, \\
\lambda_i - \lambda_{i+1} = 6 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 2, \pm 6 \mod 16, \\
\lambda_i - \lambda_{i+1} = 7 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 3 \mod 16, \\
\lambda_i - \lambda_{i+1} = 8 &\Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 4 \mod 16,
\end{aligned}$$

is equal to the number of partitions of n into distinct odd parts.

This theorem was originally proved by studying representations of the twisted affine Lie algebra $A_2^{(2)}$. Here, we give a combinatorial proof and refinement of Siladić's identity by proving the following theorem [Dou14a], already stated as Theorem 2.18 in the introduction.

Theorem 7.2. *For $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, let $A(k,n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n such that k equals the number of odd part plus twice the number of even parts, satisfying the following conditions:*

1. $\forall i \geq 1, \lambda_i \neq 2$,

2. $\forall i \geq 1, \lambda_i - \lambda_{i+1} \geq 5$,

3. $\forall i \geq 1$,

$$\begin{aligned}
\lambda_i - \lambda_{i+1} = 5 &\Rightarrow \lambda_i \equiv 1, 4 \mod 8,\\
\lambda_i - \lambda_{i+1} = 6 &\Rightarrow \lambda_i \equiv 1, 3, 5, 7 \mod 8,\\
\lambda_i - \lambda_{i+1} = 7 &\Rightarrow \lambda_i \equiv 0, 1, 3, 4, 6, 7 \mod 8,\\
\lambda_i - \lambda_{i+1} = 8 &\Rightarrow \lambda_i \equiv 0, 1, 3, 4, 5, 7 \mod 8.
\end{aligned}$$

For $k, n \in \mathbb{N}$, let $B(k,n)$ denote the number of partitions of n into k distinct odd parts. Then for all $k, n \in \mathbb{N}$, $A(k,n) = B(k,n)$.

7.2. Proof

We now prove Theorem 7.2.

As in Schur's theorem, the generating function for partitions counted by $B(k,n)$ is easy to compute and equals

$$\sum_{k,n \geq 0} B(k,n) t^k q^n = \prod_{k \geq 0} \left(1 + tq^{2k+1}\right),$$

so we only need to show that the generating function for partitions counted by $A(k,n)$ is the same. We proceed as follows. In Section 7.2.1 we give an equivalent formulation of Theorem 7.1 which is easier to manipulate in terms of partitions. In Section 7.2.2 we establish q-difference equations satisfied by the generating functions for partitions counted by $A(k,n)$. Finally, we use those q-difference equations to prove Theorem 7.2 by induction.

7.2.1. Reformulating the problem

Our idea is to find q-difference equations which we will use to prove Theorem 7.1, but its original formulation is not very convenient to manipulate combinatorially because it gives conditions on the sum of two consecutive parts of the partition. Therefore we will transform those conditions into conditions that only involve one part at a time.

Lemma 7.3. *Conditions*

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \mod 16, \tag{7.1}$$

$$\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 2, \pm 6 \mod 16, \tag{7.2}$$

$$\lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 3 \mod 16, \tag{7.3}$$

$$\lambda_i - \lambda_{i+1} = 8 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 4 \mod 16, \tag{7.4}$$

are respectively equivalent to conditions

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i \equiv 1, 4 \mod 8, \tag{7.5}$$

$$\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i \equiv 1, 3, 5, 7 \mod 8, \tag{7.6}$$

$$\lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 6, 7 \mod 8, \tag{7.7}$$

$$\lambda_i - \lambda_{i+1} = 8 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 5, 7 \mod 8. \tag{7.8}$$

Proof: Let us prove the first equivalence. The others are proved in exactly the same way. We have

$$\begin{aligned}
&\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \mod 16\\
\Leftrightarrow\ &\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv 1, 15, 5, 11, 7, 9 \mod 16\\
\Leftrightarrow\ &\lambda_i - \lambda_{i+1} = 5 \Rightarrow 2\lambda_i = \lambda_i + \lambda_{i+1} + \lambda_i - \lambda_{i+1} \not\equiv 6, 4, 10, 0, 12, 14 \mod 16\\
\Leftrightarrow\ &\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i \not\equiv 3, 2, 5, 0, 6, 7 \mod 8\\
\Leftrightarrow\ &\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i \equiv 1, 4 \mod 8.
\end{aligned}$$

Therefore condition (7.1) is equivalent to condition (7.5). □

By Lemma 7.3, Theorem 7.1 is equivalent to the following theorem.

Theorem 7.4. *The number of partitions $\lambda_1 + \cdots + \lambda_s$ of an integer n into parts different from 2 such that difference between two consecutive parts is at least 5 (i.e.. $\lambda_i - \lambda_{i+1} \geq 5$) and*

$$\begin{aligned}
&\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i \equiv 1, 4 \mod 8,\\
&\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i \equiv 1, 3, 5, 7 \mod 8,\\
&\lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 6, 7 \mod 8,\\
&\lambda_i - \lambda_{i+1} = 8 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 5, 7 \mod 8,
\end{aligned}$$

is equal to the number of partitions of n into distinct odd parts.

Moreover for every n, the sets of partitions are exactly the same as those in Theorem 7.1, so this is just a reformulation of the same theorem.

7.2.2. Obtaining q-difference equations

Now that we have stated Theorem 7.1 in a more convenient manner, we can establish our q-difference equations and prove Theorem 7.2.

For $n \in N$, $k \in \mathbb{N}^*$, let $a_N(k,n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ counted by $A(k,n)$ such that the largest part λ_1 is at most N. Let also $e_N(k,n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ counted by $A(k,n)$ such that the largest part λ_1 is equal to N. We define, for $|t| < 1$, $|q| < 1$, $N \in \mathbb{N}^*$,

$$G_N(t,q) = 1 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_N(k,n) t^k q^n.$$

Thus $G_\infty(t,q) = \lim_{N \to \infty} G_N(t,q)$ is the generating function for the partitions counted by $A(k,n)$.

Our goal is to show that

$$\forall N \in \mathbb{N}^*, G_{2N}(t,q) = (1+tq) G_{2N-3}(tq^2, q).$$

Indeed we can then let N go to infinity and deduce

$$G_\infty(t,q) = (1+tq) G_\infty(tq^2,q) = (1+tq)(1+tq^3) G_\infty(tq^4,q) = \cdots,$$

which means that

$$G_\infty(t,q) = \prod_{k=0}^{\infty} \left(1 + tq^{2k+1}\right),$$

which is the generating function for partitions counted by $B(k,n)$.

Let us now state some q-difference equations that we will use throughout our proof in Section 7.2.3. We have the following identities:

Lemma 7.5. *For all $k, n, N \in \mathbb{N}^*$,*

$$a_{8N}(k,n) = a_{8N-1}(k,n) + a_{8N-7}(k-2, n-8N), \tag{7.9}$$

$$a_{8N+1}(k,n) = a_{8N}(t,q) + a_{8N-4}(k-1, n-(8N+1)), \tag{7.10}$$

$$a_{8N+2}(k,n) = a_{8N+1}(k,n) + a_{8N-7}(k-2, n-(8N+2)), \tag{7.11}$$

$$a_{8N+3}(k,n) = a_{8N+2}(k,n) + a_{8N-3}(k-1, n-(8N+3)), \tag{7.12}$$

$$\begin{aligned} a_{8N+4}(k,n) &= a_{8N+3}(k,n) + a_{8N-3}(k-2, n-(8N+4)) \\ &\quad + a_{8N-7}(k-3, n-(16N+3)), \end{aligned} \tag{7.13}$$

$$\begin{aligned} a_{8N+5}(k,n) &= a_{8N+4}(k,n) + a_{8N-3}(k-1, n-(8N+5)) \\ &\quad + a_{8N-7}(k-2, n-(16N+4)), \end{aligned} \tag{7.14}$$

$$a_{8N+6}(k,n) = a_{8N+5}(k,n) + a_{8N-3}(k-2, n-(8N+6)) \\ + a_{8N-7}(k-3, n-(16N+5)), \tag{7.15}$$

$$a_{8N+7}(k,n) = a_{8N+6}(k,n) + a_{8N+1}(k-1, n-(8N+7)). \tag{7.16}$$

Proof: We prove Equations (7.9) and (7.13). Equations (7.10), (7.11), (7.12) and (7.16) are proved in the same way as Equation (7.9), and Equations (7.14) and (7.15) in the same way as Equation (7.13).

Let us prove (7.9).We divide the set of partitions enumerated by $a_{8N}(k,n)$ into two sets, those with largest part less than $8N$ and those with largest part equal to $8N$. Thus

$$a_{8N}(k,n) = a_{8N-1}(k,n) + e_{8N}(k,n).$$

Let us now consider a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ counted by $e_{8N}(k,n)$. By Conditions (7.5)-(7.8), $\lambda_1 - \lambda_2 \geq 7$, therefore $\lambda_2 \leq 8N - 7$. Let us remove the largest part $\lambda_1 = 8N$. The largest part is now $\lambda_2 \leq 8N - 7$, the number partitioned is $n - 8N$, and we removed an even part so k becomes $k - 2$. We obtain a partition counted by $a_{8N-7}(k-2, n-8N)$. This process is reversible, because we can add a part equal to $8N$ to any partition counted by $a_{8N-7}(k-2, n-8N)$ and obtain a partition counted by $e_{8N}(k,n)$ so we have a bijection between partitions counted by $e_{8N}(k,n)$ and those counted by $a_{8N-7}(k-2, n-8N)$. Therefore

$$e_{8N}(k,n) = a_{8N-7}(k-2, n-8N)$$

for all $k, n, N \in \mathbb{N}^*$ and (7.9) is proved.

Let us now prove (7.13). Again let us divide the set of partitions enumerated by $a_{8N+4}(k,n)$ into two sets, those with largest part less than $8N+4$ and those with largest part equal to $8N+4$. Thus

$$a_{8N+4}(k,n) = a_{8N+3}(k,n) + e_{8N+4}(k,n).$$

Let us now consider a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ counted by $e_{8N+4}(k,n)$. By Conditions (7.5)-(7.8), $\lambda_1 - \lambda_2 = 5$ or $\lambda_1 - \lambda_2 \geq 7$, therefore $\lambda_2 = 8N - 1$ or $\lambda_2 \leq 8N - 3$. Let us remove the largest part $\lambda_1 = 8N + 4$. If $\lambda_2 = 8N - 1$, we obtain a partition counted by $e_{8N-1}(k-1, n-(8N+5))$. If $\lambda_2 \leq 8N - 3$, we obtain a partition counted by $a_{8N-3}(k-1, n-(8N+4))$. This process is also reversible and the following holds:

$$e_{8N+4}(k,n) = e_{8N-1}(k-1, n-(8N+4)) + a_{8N-3}(k-1, n-(8N+4))$$

Moreover, again by removing the largest part, we can prove that

$$e_{8N-1}(k-1, n-(8N+4)) = a_{8N-7}(k-2, n-(16N+3)).$$

This concludes the proof of (7.13). $\square$

The equations of Lemma 7.5 lead to the following q-difference equations:

Lemma 7.6. *For all* $N \in \mathbb{N}^*$,

$$G_{8N}(t,q) = G_{8N-1}(t,q) + t^2 q^{8N} G_{8N-7}(t,q), \tag{7.17}$$

$$G_{8N+1}(t,q) = G_{8N}(t,q) + tq^{8N+1} G_{8N-4}(t,q), \tag{7.18}$$

$$G_{8N+2}(t,q) = G_{8N+1}(t,q) + t^2 q^{8N+2} G_{8N-7}(t,q), \tag{7.19}$$

$$G_{8N+3}(t,q) = G_{8N+2}(t,q) + tq^{8N+3} G_{8N-3}(t,q), \tag{7.20}$$

$$G_{8N+4}(t,q) = G_{8N+3}(t,q) + t^2 q^{8N+4} G_{8N-3}(t,q) + t^3 q^{16N+3} G_{8N-7}(t,q), \tag{7.21}$$

$$G_{8N+5}(t,q) = G_{8N+4}(t,q) + tq^{8N+5} G_{8N-3}(t,q) + t^2 q^{16N+4} G_{8N-7}(t,q), \tag{7.22}$$

$$G_{8N+6}(t,q) = G_{8N+5}(t,q) + t^2 q^{8N+6} G_{8N-3}(t,q) + t^3 q^{16N+5} G_{8N-7}(t,q), \tag{7.23}$$

$$G_{8N+7}(t,q) = G_{8N+6}(t,q) + tq^{8N+7} G_{8N+1}(t,q). \tag{7.24}$$

Some more q-difference equations will be stated in the proof of Section 7.2.3 as their interest arises from the proof itself.

Even if we use the idea of counting certain parts twice as in Andrews' proof of Schur's and our proof of Schur's theorem for overpartitions from Section 4.3, the consequent number of equations (here we have 8 equations while there were only 3 equations in the proofs above mentioned) makes it difficult to find directly a recurrence equation satisfied by $G_{8N}(t,q)$ and use the same method. Therefore we proceed differently as shown in next section.

7.2.3. The induction

In this section we prove the following theorem by induction:

Theorem 7.7. *For all* $m \in \mathbb{N}^*$,

$$G_{2m}(t,q) = (1+tq) G_{2m-3}(tq^2, q). \tag{7.25}$$

7.2.3.1. Initialisation

First we need to check some initial cases.

With the initial conditions

$$G_0(t,q) = 1,$$

$$G_1(t,q) = 1 + tq,$$

$$G_2(t,q) = 1 + tq,$$

$$G_3(t,q) = G_2(t,q) + tq^3,$$
$$G_4(t,q) = G_3(t,q) + t^2q^4,$$
$$G_5(t,q) = G_4(t,q) + tq^5,$$
$$G_6(t,q) = G_5(t,q) + t^2q^6,$$
$$G_7(t,q) = G_6(t,q) + tq^7 + t^2q^8,$$

and Equations (7.17)-(7.24), we use MAPLE to check that Theorem 7.7 is verified for $m = 1, \dots, 8$.

Let us now assume that Theorem 7.7 is true for all $k \leq m-1$ and show that Equation (7.25) is also satisfied for m. To do so, we will consider 4 different cases: $m \equiv 0 \mod 4$, $m \equiv 1 \mod 4$, $m \equiv 2 \mod 4$ and $m \equiv 3 \mod 4$.

7.2.3.2. First case: $m \equiv 0 \mod 4$

We start by studying the case where $m = 4N$ with $N \geq 2$. We want to prove that

$$G_{8N}(t,q) = (1+tq)G_{8N-3}(tq^2,q).$$

Replacing N by $N-1$ in (7.24) and substituting into (7.17), we obtain

$$G_{8N}(t,q) = G_{8N-2}(t,q) + \left(tq^{8N-1} + t^2q^{8N}\right) G_{8N-7}(t,q). \tag{7.26}$$

We now replace N by $N-1$ in (7.18) and substitute into (7.26). This gives

$$\begin{aligned} G_{8N}(t,q) = G_{8N-2}(t,q) &+ (1+tq)tq^{8N-1}G_{8N-8}(t,q) \\ &+ (1+tq)t^2q^{16N-8}G_{8N-12}(t,q). \end{aligned}$$

Then by the induction hypothesis,

$$\begin{aligned} G_{8N}(t,q) = (1+tq) \big(&G_{8N-5}(tq^2,q) + (1+tq)tq^{8N-1}G_{8N-11}(tq^2,q) \\ &+(1+tq)t^2q^{16N-8}G_{8N-15}(tq^2,q)\big) . \end{aligned} \tag{7.27}$$

Replacing N by $N-1$ and t by tq^2 in (7.21), we obtain

$$\begin{aligned} G_{8N-4}(tq^2,q) = G_{8N-5}(tq^2,q) &+ t^2q^{8N}G_{8N-11}(tq^2,q) \\ &+ t^3q^{16N-7}G_{8N-15}(tq^2,q). \end{aligned} \tag{7.28}$$

Replacing N by $N-1$ and t by tq^2 in (7.21) gives

$$\begin{aligned} G_{8N-3}(tq^2,q) = G_{8N-4}(tq^2,q) &+ tq^{8N-1}G_{8N-11}(tq^2,q) \\ &+ t^2q^{16N-8}G_{8N-15}(tq^2,q). \end{aligned} \tag{7.29}$$

Adding (7.28) and (7.29), we get

$$\begin{aligned}G_{8N-3}(tq^2,q) &= G_{8N-5}(tq^2,q) + (1+tq)tq^{8N-1}G_{8N-11}(tq^2,q)\\&\quad + (1+tq)t^2q^{16N-8}G_{8N-15}(tq^2,q).\end{aligned}$$

Thus by (7.27),we deduce that

$$G_{8N}(t,q) = (1+tq)G_{8N-3}(tq^2,q).$$

It remains now to treat the cases $m \equiv 1,2,3 \mod 4$.

7.2.3.3. Second case: $m \equiv 1 \mod 4$

We now assume that $m = 4N+1$ with $N \geq 2$ and prove that

$$G_{8N+2}(t,q) = (1+tq)G_{8N-1}(tq^2,q).$$

Replacing N by $N-1$ in (7.24), we obtain

$$G_{8N-1}(t,q) = G_{8N-2}(t,q) + tq^{8N-1}G_{8N-7}(t,q). \tag{7.30}$$

Replacing N by $N-1$ in (7.23) and substituting in (7.30), we get

$$\begin{aligned}G_{8N-1}(t,q) &= G_{8N-3}(t,q) + tq^{8N-1}G_{8N-7}(t,q)\\&\quad + t^2q^{8N-2}G_{8N-11}(t,q) + t^3q^{16N-11}G_{8N-15}(t,q).\end{aligned} \tag{7.31}$$

Then replacing t by tq^2 in (7.31), we obtain the following equation:

$$\begin{aligned}G_{8N-1}(tq^2,q) &= G_{8N-3}(tq^2,q) + tq^{8N+1}G_{8N-7}(tq^2,q)\\&\quad + t^2q^{8N+2}G_{8N-11}(tq^2,q) + t^3q^{16N-5}G_{8N-15}(tq^2,q).\end{aligned} \tag{7.32}$$

Thus we want to prove that

$$\begin{aligned}G_{8N+2}(t,q) &= G_{8N}(t,q) + tq^{8N+1}G_{8N-4}(t,q)\\&\quad + t^2q^{8N+2}G_{8N-8}(t,q) + t^3q^{16N-5}G_{8N-12}(tq^2,q).\end{aligned}$$

in order to be able to use the induction hypothesis. We will need a few new equations to do so.

By definition, for all $n,k,N \in \mathbb{N}^*$,

$$a_{8N+2}(k,n) = a_{8N}(k,n) + e_{8N+1}(k,n) + e_{8N+2}(k,n). \tag{7.33}$$

We need formulas for $e_{8N+1}(k,n)$ and $e_{8N+2}(k,n)$.

Lemma 7.8. *For all $n, k, N \in \mathbb{N}^*$,*

$$e_{8N+1}(k,n) = a_{8N-4}(k-1, n-(8N+1)), \tag{7.34}$$

$$e_{8N+2}(k,n) = a_{8N-8}(k-2, n-(8N+2)) + a_{8N-12}(k-3, n-(16N-5)), \tag{7.35}$$

Proof:

- Proof of (7.34):
  Let us consider a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ counted by $e_{8N+1}(k,n)$. By conditions (7.5)-(7.8), $\lambda_1 - \lambda_2 \geq 5$, therefore $\lambda_2 \leq 8N-4$. Therefore if we remove the largest part, we obtain a partition counted by $a_{8N-4}(k-1, n-(8N+1))$.
- Proof of (7.35):
  Let us consider a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ counted by $e_{8N+2}(k,n)$. By conditions (7.5)-(7.8), $\lambda_1 - \lambda_2 \geq 9$, therefore $\lambda_2 \leq 8N-7$. Therefore if we remove the largest part, we obtain a partition counted by $a_{8N-7}(k-2, n-(8N+2))$. So $e_{8N+2}(k,n) = a_{8N-7}(k-2, n-(8N+2))$, and by definition

  $$e_{8N+2}(k,n) = a_{8N-8}(k-2, n-(8N+2)) + e_{8N-7}(k-2, n-(8N+2)).$$

  Let us now consider a partition $\mu_1 + \mu_2 + \cdots + \mu_r$ counted by $e_{8N-7}(k-2, n-(8N+2))$. By conditions (7.5)-(7.8), $\mu_1 - \mu_2 \geq 5$, therefore $\mu_2 \leq 8N-12$. If we remove the largest part $\mu_1 = 8N-7$, we obtain a partition counted by $a_{8N-12}(k-3, n-(8N+2)-(8N-7))$. Thus

  $$e_{8N+2}(k,n) = a_{8N-8}(k-2, n-(8N+2)) + a_{8N-12}(k-3, n-(16N-5)).$$

$\square$

Now by Lemma 7.8 and (7.33), for all $k, n, N \in \mathbb{N}*$,

$$\begin{aligned} a_{8N+2}(k,n) = {} & a_{8N}(k,n) + a_{8N-4}(k-1, n-(8N+1)) \\ & + a_{8N-8}(k-2, n-(8N+2)) + a_{8N-12}(k-3, n-(16N-5)). \end{aligned}$$

This leads to the desired q-difference equation:

$$\begin{aligned} G_{8N+2}(t,q) = {} & G_{8N}(t,q) + tq^{8N+1} G_{8N-4}(t,q) \\ & + t^2 q^{8N+2} G_{8N-8}(t,q) + t^3 q^{16N-5} G_{8N-12}(tq^2, q). \end{aligned}$$

By the induction hypothesis, the result from the last subsection and (7.32), we show

$$G_{8N+2}(t,q) = (1+tq) G_{8N-1}(t,q).$$

Let us now turn to the case $m \equiv 2 \mod 4$.

7.2.3.4. Third case: $m \equiv 2 \mod 4$

We suppose that $m = 4N + 2$ with $N \geq 2$ and prove that

$$G_{8N+4}(t,q) = (1+tq)G_{8N+1}(tq^2,q).$$

Substituting (7.17) into (7.18), we have

$$G_{8N+1}(t,q) = G_{8N-1}(t,q) + tq^{8N+1}G_{8N-4}(t,q) + t^2q^{8N}G_{8N-7}(t,q). \tag{7.36}$$

Replacing N by $N-1$ in (7.21) and substituting in (7.36), we have

$$\begin{aligned} G_{8N+1}(t,q) = G_{8N-1}(t,q) + tq^{8N+1}G_{8N-5}(t,q) + t^2q^{8N}G_{8N-7}(t,q) \\ + t^3q^{16N-3}G_{8N-11}(t,q) + t^4q^{24N-12}G_{8N-15}(t,q). \end{aligned} \tag{7.37}$$

Then replacing t by tq^2 in (7.37), we obtain the following equation:

$$\begin{aligned} G_{8N+1}(tq^2,q) = G_{8N-1}(tq^2,q) + tq^{8N+3}G_{8N-5}(tq^2,q) + t^2q^{8N+4}G_{8N-7}(tq^2,q) \\ + t^3q^{16N+3}G_{8N-11}(tq^2,q) + t^4q^{24N-4}G_{8N-15}(tq^2,q). \end{aligned} \tag{7.38}$$

Thus we want to prove that

$$\begin{aligned} G_{8N+4}(t,q) = G_{8N+2}(t,q) + tq^{8N+3}G_{8N-2}(t,q) + t^2q^{8N+4}G_{8N-4}(t,q) \\ + t^3q^{16N+3}G_{8N-8}(t,q) + t^4q^{24N-4}G_{8N-12}(t,q). \end{aligned}$$

Again we need new equations to do so.

By definition, for all $n, k, N \in \mathbb{N}^*$,

$$a_{8N+4}(k,n) = a_{8N+2}(k,n) + e_{8N+3}(k,n) + e_{8N+4}(k,n). \tag{7.39}$$

We need formulas for $e_{8N+3}(k,n)$ and $e_{8N+4}(k,n)$.

Lemma 7.9. *For all $n, k, N \in \mathbb{N}^*$,*

$$e_{8N+3}(k,n) = a_{8N-2}(k-1, n-(8N+3)) - e_{8N-3}(k-2, n-(8N+4)), \tag{7.40}$$

$$\begin{aligned} e_{8N+4}(k,n) = a_{8N-4}(k-2, n-(8N+4)) + e_{8N-3}(k-2, n-(8N+4)) \\ + a_{8N-8}(k-3, n-(16N+3)) + a_{8N-12}(k-4, n-(24N-4)). \end{aligned} \tag{7.41}$$

Proof:

- Proof of (7.40):
  In the same way as before, by conditions (7.5)-(7.8),
  $$e_{N+3}(k,n) = a_{8N-3}(k-1, n-(8N+3)).$$
  Thus by definition
  $$e_{8N+3}(k,n) = a_{8N-2}(k-1, n-(8N+3)) - e_{8N-2}(k-1, n-(8N+3)).$$
  Now let us consider a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ counted by $e_{8N-2}(k-1, n-(8N+3))$. By Conditions (7.5)-(7.8), $\lambda_1 - \lambda_2 = 7$ or $\lambda_1 - \lambda_2 \geq 9$, therefore $\lambda_2 = 8N-9$ or $\lambda_2 \leq 8N-11$. Let us remove the largest part $\lambda_1 = 8N-2$. If $\lambda_2 = 8N-9$, we obtain a partition counted by $e_{8N-9}(k-3, n-(16N+1))$. If $\lambda_2 \leq 8N-11$, we obtain a partition counted by $a_{8N-11}(k-3, n-(16N+1))$. Thus the following holds:
  $$\begin{aligned} e_{8N-2}(k-1, n-(8N+3)) &= e_{8N-9}(k-3, n-(16N+1)) \\ &\quad + a_{8N-11}(k-3, n-(16N+1)). \end{aligned}$$
  In the exact same way we can show that
  $$\begin{aligned} e_{8N-3}(k-2, n-(8N+4)) &= e_{8N-9}(k-3, n-(16N+1)) \\ &\quad + a_{8N-11}(k-3, n-(16N+1)). \end{aligned}$$
  Therefore
  $$e_{8N-2}(k-1, n-(8N+3)) = e_{8N-3}(k-2, n-(8N+4)),$$
  and (7.40) is proved.
- Proof of (7.41):
  Now let us consider a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ counted by $e_{8N+4}(k,n)$. By conditions (7.5)-(7.8), $\lambda_1 - \lambda_2 = 5$ or $\lambda_1 - \lambda_2 \geq 7$. Therefore by removing the largest part, we obtain
  $$\begin{aligned} e_{8N+4}(k,n) &= a_{8N-4}(k-2, n-(8N+4)) \\ &\quad + e_{8N-3}(k-2, n-(8N+4)) + e_{8N-1}(k-2, n-(8N+4)). \end{aligned}$$
  By similar reasoning,
  $$\begin{aligned} &e_{8N-1}(k-2, n-(8N+4)) \\ &= a_{8N-8}(k-3, n-(16+3)) + e_{8N-7}(k-3, n-(16N+3)) \\ &= a_{8N-8}(k-3, n-(16+3)) + a_{8N-12}(k-4, n-(24N-4)). \end{aligned}$$
  Equation (7.41) is proved.

□

Now by Lemma 7.9 and (7.39), for all $k, n, N \in \mathbb{N}*$,

$$\begin{aligned} a_{8N+4}(k,n) &= a_{8N}(k,n) + a_{8N-2}(k-1, n-(8N+3)) \\ &\quad + a_{8N-4}(k-2, n-(8N+4)) + a_{8N-8}(k-3, n-(16N+3)) \\ &\quad + a_{8N-12}(k-4, n-(24N-4)). \end{aligned}$$

This leads to the desired q-difference equation:

$$\begin{aligned} G_{8N+4}(t,q) &= G_{8N+2}(t,q) + tq^{8N+3}G_{8N-2}(t,q) + t^2q^{8N+4}G_{8N-4}(t,q) \\ &\quad + t^3q^{16N+3}G_{8N-8}(t,q) + t^4q^{24N-4}G_{8N-12}(t,q). \end{aligned}$$

By the induction hypothesis and (7.38), we show

$$G_{8N+4}(t,q) = (1+tq)G_{8N+1}(t,q).$$

We can now treat the last case.

7.2.3.5. Fourth case: $m \equiv 3 \mod 4$

Finally, we suppose that $m = 4N+3$ with $N \geq 2$ and prove that

$$G_{8N+6}(t,q) = (1+tq)\, G_{8N+3}(tq^2, q).$$

Replacing t by tq^2 in (7.19) and (7.20) leads to

$$G_{8N+2}(tq^2,q) = G_{8N+1}(tq^2,q) + t^2q^{8N+6}G_{8N-7}(tq^2,q), \tag{7.42}$$

$$G_{8N+3}(tq^2,q) = G_{8N+2}(tq^2,q) + tq^{8N+5}G_{8N-3}(tq^2,q). \tag{7.43}$$

Adding (7.42) and (7.43) we obtain:

$$G_{8N+3}(tq^2,q) = G_{8N+1}(tq^2,q) + tq^{8N+5}G_{8N-3}(tq^2,q) + t^2q^{8N+6}G_{8N-7}(tq^2,q). \tag{7.44}$$

We now want to show that

$$G_{8N+6}(t,q) = G_{8N+4}(t,q) + tq^{8N+5}G_{8N}(t,q) + t^2q^{8N+6}G_{8N-4}(t,q).$$

By definition we have

$$a_{8N+6}(k,n) = a_{8N+4}(k,n) + e_{8N+5}(k,n) + e_{8N+6}(k,n). \tag{7.45}$$

In a similar manner as above, by conditions (7.5)-(7.8) and removing the largest part, we show that

$$\begin{aligned} e_{8N+5}(k,n) &= a_{8N}(k-1,n-(8N+5)) \\ &\quad - e_{8N}(k-1,n-(8N+5)) - e_{8N-2}(k-1,n-(8N+5)), \end{aligned} \tag{7.46}$$

and

$$\begin{aligned} e_{8N+6}(k,n) &= a_{8N-4}(k-2,n-(8N+6)) \\ &\quad + e_{8N-1}(k-2,n-(8N+6)) + e_{8N-3}(k-2,n-(8N+6)). \end{aligned} \tag{7.47}$$

Yet again by the same method we show that

$$e_{8N-1}(k-2,n-(8N+6)) = a_{8N-7}(k-3,n-(16N+5)),$$

and

$$e_{8N}(k-1,n-(8N+5)) = a_{8N-7}(k-3,n-(16N+5)).$$

Therefore

$$e_{8N}(k-1,n-(8N+5)) = e_{8N-1}(k-2,n-(8N+6)).$$

And in the same way

$$\begin{aligned} e_{8N-3}(k-2,n-(8N+6)) &= e_{8N-9}(k-3,n-(16N+3)) \\ &\quad + a_{8N-11}(k-3,n-(16N+3)), \end{aligned}$$

and

$$\begin{aligned} e_{8N-2}(k-1,n-(8N+5)) &= e_{8N-9}(k-3,n-(16N+3)) \\ &\quad + a_{8N-11}(k-3,n-(16N+3)). \end{aligned}$$

Therefore

$$e_{8N-2}(k-1,n-(8N+5)) = e_{8N-3}(k-2,n-(8N+6)).$$

So by summing (7.46) and (7.47) and replacing in (7.45), we get

$$a_{8N+6}(k,n) = a_{8N+4}(k,n)+a_{8N}(k-1,n-(8N+5))+a_{8N-4}(k-2,n-(8N+6)),$$

which gives in terms of generating functions

$$G_{8N+6}(t,q) = G_{8N+4}(t,q) + tq^{8N+5}G_{8N}(t,q) + t^2q^{8N+6}G_{8N-4}(t,q).$$

By (7.44), the results from the last two subsections and the induction hypothesis,

$$G_{8N+6}(t,q) = (1+tq)\,G_{8N+3}(tq^2,q).$$

This concludes the proof of Theorem 7.7.

7.2.4. Final argument

By Theorem 7.7, we have for all $N \in \mathbb{N}^*$,

$$G_{2N}(t,q) = (1+tq)G_{2N-3}(tq^2,q).$$

So, if we let $N \to \infty$, we obtain:

$$G_{\infty}(t,q) = (1+tq)\,G_{\infty}(tq^2,q). \tag{7.48}$$

Iteration of (7.48) shows that:

$$G_{\infty}(t,q) = \prod_{k=0}^{\infty}\left(1+tq^{2k+1}\right).$$

This completes the proof of Theorem 7.2.

Part III.

Asymptotics

8. The Hardy-Ramanujan-Rademacher circle method

Contents

This part focuses on asymptotic formulas for functions related to integer partitions and on several variants of the Hardy-Ramanujan circle method. In this first chapter, we compute an exact formula for $p(n)$ using the Rogers-Ramanujan-Rademacher circle method. More details can be found in [And84].

8.1. Introduction

Hardy and Ramanujan [HR18b] were the first to study $p(n)$ analytically and they proved the following formula:

$$p(n) = \frac{1}{2\sqrt{2}} \sum_{k=1}^{a\sqrt{n}} \sqrt{k} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}} \frac{d}{dn} \left(\exp\left(\frac{\pi \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}}{k} \right) \right),$$

where $\omega_{h,k}$ is a 24th root of unity, (h,k) denotes greatest common divisor of h and k, and a is an arbitrary constant, with the only condition that n be larger than some value $n_0(a)$ which depends on a. This formula is very precise, but there is a relation between n and a, so it is not an exact formula for $p(n)$ in the sense that we cannot directly substitute n in the formula and get the result.

Rademacher [Rad37] managed to improve Hardy and Ramanujan's method to find an expression for $p(n)$ as a convergent series.

Theorem 8.1.

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{1/2}\right)}{\left(x - \frac{1}{24}\right)^{1/2}} \right]_{x=n}, \tag{8.1}$$

where

$$A_k(n) = \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}}.$$

The k-th term in the sum of (8.1) is $O(\exp\left(\pi\sqrt{\frac{2}{3}}\frac{\sqrt{n}}{k}\right))$. Thus, considering only the first term of the sum, we can deduce the asymptotic formula for $p(n)$ first established by Hardy and Ramanujan [HR18b].

Theorem 8.2. *As n tends to infinity,*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Let us give Rademacher's proof of Theorem 8.1, explained in Andrews' book [And84]. The method relies on the fact that the generating function for partitions is (up to some some power of q), the inverse of the Dedekind eta function, which is a modular form as seen in Section 3.3.2. First we give a transformation formula for $P(q)$, the generating function for partitions. Then, using Cauchy's theorem, we write $p(n)$ as an integral on a circle, and finally we divide this circle in small arcs and give an estimate on the integral on each of these arcs to find the formula for $p(n)$.

8.2. A transformation formula for $P(q)$

Theorem 3.11 shows that the generating function for partitions $P(q)$ is closely related to the Dedekind η function. Writing $q = e^{2\pi i \tau}$, we have

$$P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)}.$$

Thus we use the modular transformation properties of η to obtain a transformation formula for $P(q)$.

Theorem 8.3. *Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, and h, h', k be integers such that*

$$hh' \equiv -1 \mod k.$$

Then

$$P\left(\exp\left(\frac{2i\pi(h+iz)}{k}\right)\right) = \omega_{h,k} z^{\frac{1}{2}} \exp\left(\frac{\pi(z^{-1}-z)}{12k}\right) \times P\left(\exp\left(\frac{2i\pi(h'+iz^{-1})}{k}\right)\right),$$

where if $z = |z|\mathrm{e}^{i\theta}$ with $-\pi < \theta \leq \pi$, then $z^{\frac{1}{2}} = |z|^{\frac{1}{2}}\mathrm{e}^{\frac{i\theta}{2}}$ and

$$\omega_{h,k} = \exp\left(i\pi s(h,k)\right),$$

where $s(h,k)$ is Dedekind's sum defined in the statement of Theorem 3.12.

Proof: As $\mathrm{Re}(z) > 0$, we have $\frac{h+iz}{k} \in \mathcal{H}$. Thus by Theorem 3.12,

$$\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \eta\left(A\frac{h+iz}{k}\right) = \nu_\eta(A)\left(c\frac{h+iz}{k}+d\right)^{1/2}\eta\left(\frac{h+iz}{k}\right).$$

In particular $\frac{-1-hh'}{k} \in \mathbb{Z}$ because $hh' \equiv -1 \mod k$. So the theorem is valid with

$$A = \begin{pmatrix} h' & \frac{-1-hh'}{k} \\ k & -h \end{pmatrix},$$

and we obtain

$$\eta\left(\frac{h'\frac{h+iz}{k} + \frac{-1-hh'}{k}}{k\frac{h+iz}{k} - h}\right) = \nu_\eta\left(\begin{pmatrix} h' & \frac{-1-hh'}{k} \\ k & -h \end{pmatrix}\right)\left(k\frac{h+iz}{k} - h\right)^{1/2}\eta\left(\frac{h+iz}{k}\right),$$

so

$$\eta\left(\frac{h'+iz^{-1}}{k}\right) = \exp\left(i\pi\left(\frac{h'-h}{12k} + s(h,k)\right)\right) z^{1/2}\eta\left(\frac{h+iz}{k}\right). \tag{8.2}$$

By (3.6) and (8.2),

$$
\begin{aligned}
&P\left(\exp\left(2i\pi\frac{h+iz}{k}\right)\right)\\
&=\exp\left(i\pi\left(\frac{h'-h}{12k}+s(h,k)\right)\right)z^{1/2}\\
&\times\exp\left(\frac{i\pi}{12}\left(\frac{h+iz}{k}-\frac{h'+iz^{-1}}{k}\right)\right)P\left(\exp\left(2i\pi\frac{h'+iz^{-1}}{k}\right)\right)\\
&=\exp\left(i\pi\left(\frac{h'-h}{12k}+s(h,k)\right)\right)\exp\left(\frac{i\pi(h-h')}{12k}\right)\\
&\times z^{1/2}\exp\left(\frac{\pi(z^{-1}-z)}{12k}\right)P\left(\exp\left(2i\pi\frac{h'+iz^{-1}}{k}\right)\right)\\
&=\omega_{h,k}z^{1/2}\exp\left(\frac{\pi(z^{-1}-z)}{12k}\right)P\left(\exp\left(2i\pi\frac{h'+iz^{-1}}{k}\right)\right),
\end{aligned}
$$

where

$$\omega_{h,k}=\exp\left(i\pi s(h,k)\right).$$

The theorem is proved. □

8.3. An expression of $p(n)$ as an integral on a circle

Now that we have a transformation formula for $P(q)$, let us use it in the circle method. It relies on the expression of $p(n)$ as the integral of a complex function on a circle of radius less than 1 (the singularities of the function being located on the unit circle) which is then divided in small arcs on which the value of the integral can be easily estimated.

Let us recall Cauchy's theorem.

Theorem 8.4. *Let U be a simply connected open set of the complex plane $\mathbb{C}$, $f : U \to \mathbb{C}$ be an infinitely differentiable function, $D = \{z :\mid z - z_0 \mid\leq r\}$ be a disc included in U, and γ be the boundary of D.*

Then for all $a \in \mathring{D}$, for all $n \in \mathbb{N}$,

$$f^{(n)}(a)=\frac{n!}{2i\pi}\oint_{\gamma}\frac{f(x)}{(x-a)^{n+1}}\,dx.$$

Applying Cauchy's theorem to P the generating function for partitions, defined on the complex numbers of modulus less than 1, with γ a circle centred at the origin with radius $\rho < 1$, and $a = 0$, we obtain the following formula.

$$P^{(n)}(0) = \frac{n!}{2i\pi} \oint_\gamma \frac{P(x)}{x^{n+1}}\, dx.$$

But by the definition of P, we have $p(n) = \frac{P^{(n)}(0)}{n!}$. Thus for all $n \in \mathbb{N}$,

$$p(n) = \frac{1}{2i\pi} \oint_\gamma \frac{P(x)}{x^{n+1}}\, dx. \tag{8.3}$$

Our goal is now to evaluate the integral from (8.3). The function $\prod_{k=1}^{N} \frac{1}{1-z^k}$ has a pole of order N at $z = 1$, a pole of order $\lfloor \frac{N}{2} \rfloor$ at $z = -1$, poles of order $\lfloor \frac{N}{3} \rfloor$ at $z = \exp(\frac{2i\pi}{3})$ and $z = \exp(\frac{4i\pi}{3})$, and more generally poles of order $\lfloor \frac{N}{k} \rfloor$ at all the points $z = \exp(\frac{2i\pi h}{k})$ such that h is coprime to k.

Furthermore Theorem 8.3 allows one to know precisely the behaviour of $P(x)$ close to $\exp(2i\pi h/k)$. As $z \to 0$ with $\mathrm{Re}(z) > 0$,

$$P\left(\exp\left(\frac{2i\pi(h+iz)}{k}\right)\right) \sim \omega_{h,k} z^{1/2} \exp\left(\frac{\pi(z^{-1}-z)}{12k}\right). \tag{8.4}$$

So it seems intuitive to divide the integration circle γ in segments according from which point $\exp(2i\pi h/k)$ we are the closest. To do so we need a few facts about Farey sequences [HW08].

Definition. The *Farey sequence* F_N of order N is the increasing sequence of irreducible fractions between 0 and 1 whose denominator does not exceed N.

For example the Farey sequence of order 5 is

$$F_5 = \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

The next two theorems are equivalent and state the characteristic properties of Farey sequences.

Theorem 8.5. *If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two consecutive terms of F_N, then*

$$kh' - hk' = 1.$$

Theorem 8.6. *If $\frac{h}{k}$, $\frac{h'}{k'}$ and $\frac{h''}{k''}$ are three consecutive terms of F_N, then*

$$\frac{h'}{k'} = \frac{h+h''}{k+k''}.$$

Thus if $\frac{h}{k}$ and $\frac{h'}{k'}$ are two consecutive terms of F_N, the rational number with the smallest denominator located between $\frac{h}{k}$ and $\frac{h'}{k'}$ is $\frac{h+h'}{k+k'}$, called the *mediant* of $\frac{h}{k}$ and $\frac{h'}{k'}$. We will use mediants as the extremities of the arcs to divide the integration circle γ.

If $\frac{h_0}{k_0}$, $\frac{h}{k}$ and $\frac{h_1}{k_1}$ are three consecutive terms of F_N, define

$$\theta'_{h,k} = \frac{h}{k} - \frac{h_0+h}{k_0+k},$$

$$\theta''_{h,k} = \frac{h_1+h}{k_1+k} - \frac{h}{k}.$$

Let us also define

$$\theta''_{0,1} = \theta'_{1,1} = \frac{1}{N+1}.$$

Then by 8.5, we have

$$\theta'_{h,k} = \frac{1}{k(k_0+k)}, \tag{8.5}$$

$$\theta''_{h,k} = \frac{1}{k(k_1+k)}. \tag{8.6}$$

By (8.3) with the change of variable $x = \rho\exp(2i\pi\phi)$, we get

$$\begin{aligned} p(n) &= \rho^{-n}\int_0^1 P\left(\rho\exp(2i\pi\phi)\right)\exp(-2i\pi n\phi)\,d\phi \\ &= \rho^{-n}\sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} P\left(\rho\exp\left(\frac{2i\pi h}{k}+2i\pi\phi\right)\right) \\ &\qquad\qquad \times\exp\left(-\frac{2i\pi nh}{k}-2i\pi n\phi\right)\,d\phi. \end{aligned} \tag{8.7}$$

Define $\rho := \exp\left(-\frac{2\pi}{N^2}\right)$, $z := k(N^{-2}-i\phi)$, and apply Theorem 8.3. We obtain

$$\begin{aligned} p(n) &= \exp\left(\frac{2\pi n}{N^2}\right)\sum_{k=1}^{N}\sum_{\substack{0\le h<k\\(h,k)=1}}\exp\left(-\frac{2i\pi hn}{k}\right)\omega_{h,k} \\ &\times\int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{1/2}\exp\left(\frac{\pi(z^{-1}-z)}{12k}\right)P\left(\exp\left(\frac{2i\pi(h'+iz^{-1})}{k}\right)\right)\exp\left(-2i\pi n\phi\right)\,d\phi. \end{aligned} \tag{8.8}$$

As z tends to 0 with $\mathrm{Re}(z) > 0$, $\exp\left(\frac{2i\pi(h'+iz^{-1})}{k}\right)$ quickly converges to 0. Thus $P\left(\exp\left(\frac{2i\pi(h'+iz^{-1})}{k}\right)\right)$ converges to $P(0) = 1$. To evaluate (8.8), let us $P(x)$ by $1 + (P(x) - 1)$ in the integral, which gives

$$p(n) = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \exp\left(\frac{2\pi n}{N^2}\right) \sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \exp\left(-\frac{2i\pi hn}{k}\right) \omega_{h,k} \times \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{1/2} \exp\left(\frac{\pi(z^{-1} - z)}{12k} - 2i\pi n\phi\right) d\phi,$$

and

$$\Sigma_2 = \exp\left(\frac{2\pi n}{N^2}\right) \sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \exp\left(-\frac{2i\pi hn}{k}\right) \omega_{h,k} \times \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{1/2} \exp\left(\frac{\pi(z^{-1} - z)}{12k}\right) \left(P\left(\exp\left(\frac{2i\pi(h' + iz^{-1})}{k}\right)\right) - 1\right) \times \exp\left(-2i\pi n\phi\right) d\phi.$$

We now want to show that Σ_2 is negligible compared to Σ_1.

8.4. Σ_2 is negligible compared to Σ_1

In this section, we verify that Σ_2 is indeed negligible compared to Σ_1. We have

$$\begin{aligned} &\left| z^{1/2} \exp\left(\frac{\pi(z^{-1} - z)}{12k}\right) \left(P\left(\exp\left(\frac{2i\pi(h' + iz^{-1})}{k}\right)\right) - 1\right) \right| \\ &\quad \le |z^{1/2}| \left| \exp\left(\frac{\pi(z^{-1} - z)}{12k}\right) \sum_{m=1}^{\infty} p(m) \exp\left(\frac{2i\pi m(h' + iz^{-1})}{k}\right) \right| \\ &\quad \le |z^{1/2}| \exp\left(\frac{-\pi \mathrm{Re}(z)}{12k}\right) \sum_{m=1}^{\infty} p(m) \exp\left(\frac{-2\pi m \mathrm{Re}(z^{-1})}{k} + \frac{\pi \mathrm{Re}(z^{-1}}{12k}\right) \\ &\quad \le |z^{1/2}| \exp\left(\frac{-\pi}{12N^2}\right) \sum_{m=1}^{\infty} p(m) \exp\left(-2\pi \mathrm{Re}(z^{-1}) \frac{m - 1/24}{k}\right). \end{aligned} \tag{8.9}$$

Furthermore

$$\frac{1}{z} = \frac{1}{kN^{-2} - ik\phi} = \frac{N^{-2} + i\phi}{k(N^{-4} + \phi^2)}.$$

But, by the properties of the mediants of Farey sequences (8.5) and (8.6), all the $\theta'_{h,k}$ and $\theta''_{h,k}$ satisfy $\frac{1}{2kN} \leq \theta_{h,k} < \frac{1}{kN}$. Thus

$$\frac{1}{k}\mathrm{Re}(z^{-1}) = \frac{N^{-2}}{k^2(N^{-4} + \phi^2)} > \frac{N^{-2}}{k^2N^{-4} + N^{-2}} = \frac{1}{1 + k^2N^{-2}} \geq \frac{1}{2}, \tag{8.10}$$

and

$$|z^{1/2}| = (k^2N^{-4} + k^2\phi^2)^{1/2} < 2^{1/4}N^{-1/2}. \tag{8.11}$$

By (8.9), (8.10) and (8.11), we get

$$\begin{aligned}
|\Sigma_2| \leq& \exp\left(\frac{2\pi n}{N^2}\right) \sum_{k=1}^{N} \sum_{\substack{0\leq h<k \\ (h,k)=1}} 2^{1/4}N^{-1/2} \exp\left(-\frac{\pi}{12N^2}\right) \\
&\times \sum_{m=1}^{\infty} p(m) \exp\left(-\pi\left(m - \frac{1}{24}\right)\right) \int_{-\theta'_{h,k}}^{\theta''_{h,k}} d\phi. \\
\leq& \exp\left(\frac{2\pi n}{N^2} - \frac{\pi}{12N^2}\right) 2^{1/4}N^{-1/2} \sum_{m=1}^{\infty} p(m) \exp\left(-\pi\left(m - \frac{1}{24}\right)\right) \\
\leq& CN^{-1/2} \exp\left(\frac{2\pi n}{N^2}\right) \underset{N\to 0}{\longrightarrow} 0.
\end{aligned} \tag{8.12}$$

So Σ_2 is indeed negligible compared to Σ_1.

8.5. Estimating Σ_1

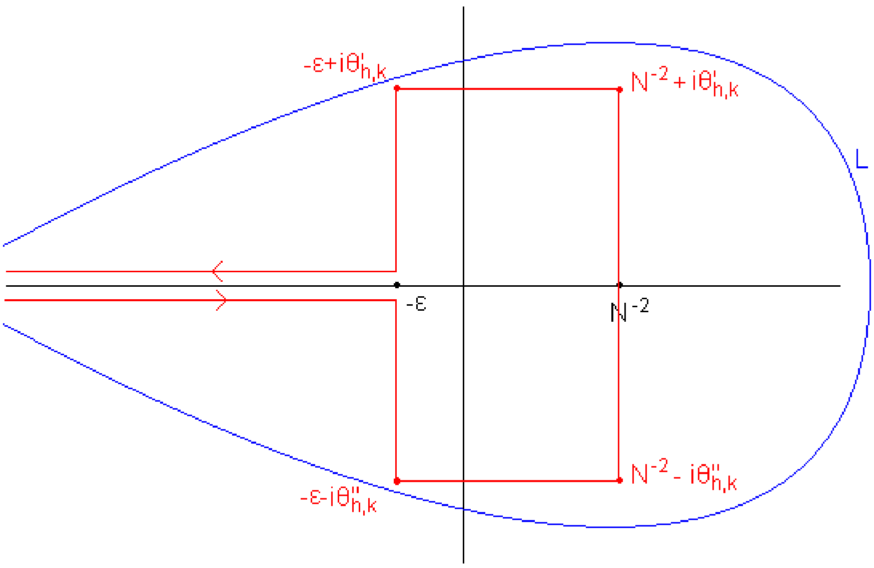
Now that we have shown that the asymptotic contribution only comes from Σ_1, let us estimate it. Define

$$I_{h,k} = \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{1/2} \exp\left(\frac{\pi(z^{-1} - z)}{12k} - 2i\pi n\phi\right) d\phi.$$

Thus

$$\Sigma_1 = \exp\left(\frac{2\pi n}{N^2}\right) \sum_{k=1}^{N} \sum_{\substack{0\leq h<k \\ (h,k)=1}} \exp\left(-\frac{2i\pi hn}{k}\right) \omega_{h,k} I_{h,k}.$$

Figure 8.1.: Two integration paths



Define $\omega := N^{-2} - i\phi$, so $z = k\omega$. We have

$$\begin{aligned} I_{h,k} &= \int_{N^{-2}+i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}} (k\omega)^{1/2} \exp\left(\frac{\pi}{12k}\left(\frac{1}{k\omega} - k\omega\right) + 2\pi n(\omega - N^{-2})\right) i\, d\omega \\ &= \exp(-2\pi n N^{-2}) k^{1/2} (-i) \int_{N^{-2}-i\theta''_{h,k}}^{N^{-2}+i\theta'_{h,k}} g(\omega)\, d\omega, \end{aligned} \tag{8.13}$$

with

$$g(\omega) = \omega^{1/2} \exp\left(\frac{\pi}{12k^2\omega} + 2\pi\omega\left(n - \frac{1}{24}\right)\right).$$

The function g is analytic in the complex plane cut along the negative real axis. Thus, by Cauchy's integral theorem, the integral of g on the blue or the red path from Figure 8.1 are equal. Here ϵ is some real number strictly between 0 and N^{-2}.

Thus

$$\underbrace{\int_{-\infty}^{(0+)} g}_{L_k} = \underbrace{\int_{-\infty}^{-\epsilon} g}_{I_1} + \underbrace{\int_{-\epsilon}^{-\epsilon - i\theta''_{h,k}} g}_{I_2} + \underbrace{\int_{-\epsilon - i\theta''_{h,k}}^{N^{-2} - i\theta''_{h,k}} g}_{I_3} + \int_{N^{-2} + i\theta''_{h,k}}^{N^{-2} - i\theta'_{h,k}} g$$
$$+ \underbrace{\int_{N^{-2} + i\theta'_{h,k}}^{-\epsilon + i\theta'_{h,k}} g}_{I_4} + \underbrace{\int_{-\epsilon + i\theta'_{h,k}}^{-\epsilon} g}_{I_5} + \underbrace{\int_{-\epsilon}^{-\infty} g}_{I_6},$$

so

$$\exp(2\pi n N^{-2}) I_{h,k} = -ik^{1/2} \left(L_k - I_1 - I_2 - I_3 - I_4 - I_5 - I_6 \right),$$

where L_k is the integral of g on the blue path (L) of Figure 8.1.

We now want to determine the limits of I_2, I_3, I_4 and I_5 as ϵ tends to 0, in order to show that they are all negligible compared to L_k, I_1 et I_6. We only treat the case of I_2 in detail, the other integrals being very similar.

By the change of variable $\omega = i\nu - \epsilon$,

$$I_2 = i \int_0^{-\theta''_{h,k}} g(i\nu - \epsilon)\, d\nu.$$

Thus

$$\begin{aligned} |I_2| &= \left| \int_0^{-\theta''_{h,k}} (i\nu - \epsilon)^{1/2} \exp\left[\frac{\pi}{12k^2(i\nu - \epsilon)} + 2\pi \left(n - \frac{1}{24} \right) (i\nu - \epsilon) \right] d\nu \right| \\ &\leq \int_0^{-\theta''_{h,k}} (\nu^2 + \epsilon^2)^{1/4} \exp\left[\frac{\pi}{12k^2} \mathrm{Re}\left(\frac{1}{i\nu - \epsilon} \right) \right] \exp\left[-2\pi \left(n - \frac{1}{24} \right) \epsilon \right] d|\nu| \end{aligned}$$

Furthermore $\frac{1}{i\nu - \epsilon} = \frac{-i\nu - \epsilon}{\nu^2 + \epsilon^2}$, so $\mathrm{Re}\left(\frac{1}{i\nu - \epsilon} \right) < 0$ and

$$\exp\left[\frac{\pi}{12k^2} \mathrm{Re}\left(\frac{1}{i\nu - \epsilon} \right) \right] < 1.$$

Therefore

$$\begin{aligned} |I_2| &< \exp\left[-2\pi \left(n - \frac{1}{24} \right) \epsilon \right] \int_0^{-\theta''_{h,k}} (\nu^2 + \epsilon^2)^{1/4} \, d|\nu| \\ &< \exp\left[-2\pi \left(n - \frac{1}{24} \right) \epsilon \right] \left({\theta''_{h,k}}^2 + \epsilon^2 \right)^{1/4} \theta''_{h,k} \\ &< \exp\left[-2\pi \left(n - \frac{1}{24} \right) \epsilon \right] \left(\frac{1}{k^2 N^2} + \epsilon^2 \right)^{1/4} \frac{1}{kN} \underset{\epsilon \to 0}{\longrightarrow} k^{-3/2} N^{-3/2}. \end{aligned}$$

The integral $|I_5|$ can be bounded in the same way, so it is also dominated by $k^{-3/2}N^{-3/2}$ as $\epsilon \to 0$.

Integrals $|I_3|$ and $|I_4|$ are $O\left(k^{-1/2}N^{-5/2}\exp\left(\frac{2\pi n}{N^2}\right)\right)$ as $\epsilon \to 0$.

Moreover

$$\begin{aligned}
I_1 + I_6 =& \int_{-\infty}^{-\epsilon} |\omega|^{1/2} \exp\left(-\frac{\pi i}{2}\right) \exp\left(\frac{\pi}{12k^2\omega} + 2\pi\omega\left(n - \frac{1}{24}\right)\right) d\omega \\
&+ \int_{-\epsilon}^{-\infty} |\omega|^{1/2} \exp\left(\frac{\pi i}{2}\right) \exp\left(\frac{\pi}{12k^2\omega} + 2\pi\omega\left(n - \frac{1}{24}\right)\right) d\omega \\
=& - 2i \int_{\epsilon}^{\infty} t^{1/2} \exp\left(-\frac{\pi}{12k^2 t} - 2\pi t\left(n - \frac{1}{24}\right)\right) dt,
\end{aligned}$$

by the change of variable $u = -t$.

Defining $H_k = \int_0^\infty t^{1/2}\exp\left(-\frac{\pi}{12k^2t} - 2\pi t\left(n - \frac{1}{24}\right)\right) dt$, we get as $\epsilon \to 0$:

$$\begin{aligned}
\exp(2\pi n N^{-2}) I_{h,k} =& - ik^{1/2} L_k + 2k^{1/2} H_k \\
&+ O(k^{-1}N^{-3/2}) + O\left(\exp\left(\frac{2\pi n}{N^2}\right) N^{-5/2}\right).
\end{aligned} \tag{8.14}$$

Let $\psi_k(n) := -ik^{1/2}L_k + 2k^{1/2}H_k$. Now

$$\begin{aligned}
p(n) =& \Sigma_1 + \Sigma_2 \\
=& \sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \exp\left(-\frac{2i\pi h n}{k}\right) \omega_{h,k} \psi_k(n) + O\left(\sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ (h,k)=1}} k^{-1}N^{-3/2}\right) \\
&+ O\left(\sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \exp\left(\frac{2\pi n}{N^2}\right) N^{-5/2}\right) + O\left(\exp\left(\frac{2\pi n}{N^2}\right) N^{-1/2}\right) \\
=& \sum_{k=1}^{N} A_k(n)\psi_k(n) + O\left(\exp\left(\frac{2\pi n}{N^2}\right) N^{-1/2}\right).
\end{aligned}$$

Consequently, as N tends to infinity,

$$p(n) = \sum_{k=1}^{\infty} A_k(n)\psi_k(n). \tag{8.15}$$

8.6. An expression for $\psi_k(n)$

The last thing to do in order to establish (8.1) is to show that

$$\psi_k(n) = -ik^{1/2}L_k + 2k^{1/2}H_k = \frac{k^{1/2}}{\pi\sqrt{2}}\left[\frac{d}{dx}\frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{1/2}\right)}{\left(x-\frac{1}{24}\right)^{1/2}}\right]_{x=n}.$$

8.6.1. An expression for L_k

We first search for an expression for L_k.

$$\begin{aligned}
-iL_k &= -i\int_{-\infty}^{(0+)} \omega^{1/2}\exp\left(\frac{\pi}{12k^2\omega}+2\pi\omega\left(n-\frac{1}{24}\right)\right)d\omega\\
&= -i\int_{-\infty}^{(0+)} \omega^{1/2}\exp\left(2\pi\omega\left(n-\frac{1}{24}\right)\right)\sum_{m=0}^{\infty}\frac{\left(\frac{\pi}{12k^2\omega}\right)^m}{m!}\,d\omega\\
&= -i\sum_{m=0}^{\infty}\frac{\left(\frac{\pi}{12k^2}\right)^m}{m!}\int_{-\infty}^{(0+)} \omega^{\frac{1}{2}-m}\exp\left(2\pi\omega\left(n-\frac{1}{24}\right)\right)d\omega\\
&= -i\sum_{m=0}^{\infty}\frac{\left(\frac{\pi}{12k^2}\right)^m}{m!}\left(2\pi\left(n-\frac{1}{24}\right)\right)^{m-\frac{3}{2}}\int_{-\infty}^{(0+)} z^{\frac{1}{2}-m}\exp(z)\,dz,
\end{aligned}$$

where the last equality follows from the change of variable $z = 2\pi\omega\left(n-\frac{1}{24}\right)$.

Hankel proved the following formula relating the Gamma function and the integral above.

Theorem 8.7. *For every integer m,*

$$\frac{1}{\Gamma\left(m-\frac{1}{2}\right)} = \frac{1}{2i\pi}\int_{-\infty}^{(0+)} z^{\frac{1}{2}-m}\exp(z)\,dz,$$

where Γ is defined for all $z\in\mathbb{C}$ such that $\mathrm{Re}(z)>0$ by

$$\Gamma : z \mapsto \int_0^{\infty} t^{z-1}\exp(-t)\,dt.$$

Therefore

$$-iL_k = (2\pi)^{-1/2}\left(n-\frac{1}{24}\right)^{-3/2}\sum_{m=0}^{\infty}\frac{\left(\pi^2\left(n-\frac{1}{24}\right)/(6k^2)\right)^m}{m!\Gamma(m-\frac{1}{2})}. \tag{8.16}$$

An important property of the Gamma function is the following. For all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, we have $\Gamma(z+1) = z\Gamma(z)$. Moreover $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, so for all $m \in \mathbb{N}^*$, we have

$$\begin{aligned}\Gamma\left(m - \frac{1}{2}\right) = &\left(m - \frac{3}{2}\right)\left(m - \frac{5}{2}\right)\dots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\\ =&\frac{\sqrt{\pi}}{2^{m-1}}(2m-3)(2m-5)\dots 1\\ =&\frac{(2m-2)!\sqrt{\pi}}{(m-1)!2^{2m-2}}.\end{aligned}$$

Thus

$$\frac{\left(\frac{1}{4}Y^2\right)^m}{m!\Gamma\left(m-\frac{1}{2}\right)} = \frac{Y^{2m}2^{2m-2}(m-1)!}{2^{2m}m!(2m-2)!\sqrt{\pi}} = \frac{Y^{2m}}{4m(2m-2)!\sqrt{\pi}} = \frac{Y^{2m}(2m-1)}{2(2m)!\sqrt{\pi}}.$$

Therefore

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{4}Y^2\right)^m}{m!\Gamma\left(m-\frac{1}{2}\right)} =&\frac{1}{2\sqrt{\pi}}\left(-1 + Y^2\sum_{m=1}^{\infty}\frac{Y^{2m-2}(2m-1)}{(2m)!}\right)\\ =&\frac{1}{2\sqrt{\pi}}\left(-1 + Y^2\sum_{m=1}^{\infty}\frac{d}{dY}\left(\frac{Y^{2m-1}}{(2m)!}\right)\right)\\ =&\frac{1}{2\sqrt{\pi}}\left(-1 + Y^2\frac{d}{dY}\left(\sum_{m=0}^{\infty}\frac{Y^{2m+1}}{(2m+2)!}\right)\right)\\ =&\frac{1}{2\sqrt{\pi}}\left(-1 + Y^2\frac{d}{dY}\left(\frac{\cosh Y - 1}{Y}\right)\right)\end{aligned}$$

Let us go back to (8.16) with $Y = \frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{1/2}$. We have

$$\begin{aligned}
-iL_k &= (2\pi)^{-1/2}\left(n-\frac{1}{24}\right)^{-3/2}\frac{1}{2\sqrt{\pi}}\left[-1+Y^2\frac{d}{dY}\left(\frac{\cosh Y-1}{Y}\right)\right]_{x=n}\\
&= 2^{-3/2}\pi^{-1}\left(n-\frac{1}{24}\right)^{-3/2}\left[Y^2\frac{d}{dY}\left(\frac{\cosh Y}{Y}\right)\right]_{x=n}\\
&= 2^{-3/2}\pi^{-1}\left(n-\frac{1}{24}\right)^{-3/2}\left[Y^2\frac{dx}{dY}\times\frac{d}{dx}\left(\frac{\cosh Y}{Y}\right)\right]_{x=n}\\
&= 2^{-3/2}\pi^{-1}\left(n-\frac{1}{24}\right)^{-3/2}\left[\frac{3k^2Y^3}{\pi^2}\frac{d}{dx}\left(\frac{\cosh Y}{Y}\right)\right]_{x=n}\\
&= 3^{-1/2}k^{-1}\left[\frac{d}{dx}\left(\frac{\cosh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{1/2}\right)}{\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{1/2}}\right)\right]_{x=n}\\
&= \frac{1}{\pi\sqrt{2}}\left[\frac{d}{dx}\left(\frac{\cosh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{1/2}\right)}{\left(x-\frac{1}{24}\right)^{1/2}}\right)\right]_{x=n}.
\end{aligned} \tag{8.17}$$

We obtained the desired expression for L_k.

8.6.2. An expression for H_k

We now want to establish a simple expression for

$$H_k=\int_0^\infty t^{1/2}\exp\left(-\frac{\pi}{12k^2t}-2\pi t\left(n-\frac{1}{24}\right)\right)dt,$$

in order to prove the result on $\psi_k(n)$. Let us start by some integral calculation.

For all $a,c>0$, we have, by the change of variable $t=u^2$,

$$\begin{aligned}
\int_0^\infty \exp\left(-c^2t-a^2t^{-1}\right)t^{-1/2}\,dt &= 2\int_0^\infty \exp\left(-c^2u^2-a^2u^{-2}\right)du\\
&= 2\exp(-2ac)I,
\end{aligned} \tag{8.18}$$

where

$$I=\int_0^\infty \exp\left(-\left(cu-\frac{a}{u}\right)^2\right)du.$$

By the change of variable $u=\frac{a}{cz}$,

$$I=\frac{a}{c}\int_0^\infty \exp\left(-\left(cz-\frac{a}{z}\right)^2\right)\frac{dz}{z^2}.$$

Adding both expressions of I and multiplying by c, we get:

$$2cI = \int_0^\infty \exp\left(-\left(cz - \frac{a}{z}\right)^2\right)\left(c + \frac{a}{z^2}\right)\,dz.$$

Finally, by the change of variable $x = cz - \frac{a}{z}$,

$$2cI = \int_{-\infty}^\infty \exp(-x^2)\,dx = \sqrt{\pi}.$$

Thus by (8.18),

$$\int_0^\infty \exp\left(-c^2t - a^2t^{-1}\right) t^{-1/2}\,dt = \frac{\sqrt{\pi}}{c}\exp(-2ac).$$

Differentiating this expression with respect to c, we obtain

$$\int_0^\infty t^{1/2}\exp\left(-c^2t - a^2t^{-1}\right)\,dt = \frac{-\sqrt{\pi}}{2c}\frac{d}{dc}\left(\frac{\exp(-2ac)}{c}\right). \tag{8.19}$$

Let us apply (8.19) with $c = \left(2\pi\left(n - \frac{1}{24}\right)\right)^{1/2}$ et $a = \left(\frac{\pi}{3}\right)^{1/2}\frac{1}{2k}$. We have

$$\begin{aligned} H_k &= \frac{-1}{4\left(\pi\left(n-\frac{1}{24}\right)\right)^{1/2}}\left[\frac{d}{dy}\left(\frac{\exp(-\frac{\pi}{k}\left(\frac{2}{3}\right)^{1/2}y)}{(2\pi)^{1/2}y}\right)\right]_{y=(n-1/24)^{1/2}} \\ &= \frac{-1}{2^{3/2}\pi}\left[\frac{d}{dx}\left(\frac{\exp(-\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{1/2}}{\left(x-\frac{1}{24}\right)^{1/2}}\right)\right]_{x=n}. \end{aligned} \tag{8.20}$$

8.6.3. Final expression

Now that we have found simple expressions for L_k and H_k, we can turn back to $\psi_k(n)$. By (8.17) and (8.20), we have

$$\begin{aligned} \psi_k(n) &= -ik^{1/2}L_k + 2k^{1/2}H_k \\ &= \frac{k^{1/2}}{\pi\sqrt{2}}\left[\frac{d}{dx}\left(\frac{\cosh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{1/2}\right)}{\left(x-\frac{1}{24}\right)^{1/2}}\right)\right]_{x=n} \\ &\quad - \frac{k^{1/2}}{\pi\sqrt{2}}\left[\frac{d}{dx}\left(\frac{\exp(-\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{1/2}}{\left(x-\frac{1}{24}\right)^{1/2}}\right)\right]_{y=n} \\ &= \frac{k^{1/2}}{\pi\sqrt{2}}\left[\frac{d}{dx}\left(\frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{1/2}\right)}{\left(x-\frac{1}{24}\right)^{1/2}}\right)\right]_{x=n}. \end{aligned}$$

Thus by (8.15), we find

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{1/2}\right)}{\left(x - \frac{1}{24}\right)^{1/2}} \right]_{x=n},$$

which is exactly (8.1). Theorem 8.1 is proved.

9. Wright's circle method

Contents

9.1. Introduction

In the last chapter, we explained the classical circle method. Now, we describe Wright's variant of the circle method, which only allows one to obtain an asymptotic formula and not an exact one, but is much simpler. We also start by expressing our function as an integral using Cauchy's theorem, but then we only divide the circle in two arcs: a small arc around the dominant pole which gives the main asymptotic contribution, and the rest of the circle which gives a negligible asymptotic contribution.

Wright originally applied his method to weighted partitions [Wri33], but here we apply it to the partition function $p(n)$ in order to make the comparison with the classical circle method easier. It has since been used many times to prove asymptotic formulas for functions related to partitions [BM13, BM14a, BM14b].

As in the previous chapter, let us start by writing $p(n)$ as an integral using Cauchy's theorem.

$$p(n) = \frac{1}{2i\pi} \oint_{\mathcal{C}} \frac{P(q)}{q^{n+1}}\, dq,$$

where $\mathcal{C}$ can be any circle centred at the origin, with radius less than 1. Here we choose $\mathcal{C}$ to have radius $e^{\frac{-\pi}{\sqrt{6n}}}$, as it is the correct value for the upcoming calculations. Wright's version of the circle method [Wri33] consists of showing that the major asymptotic contribution to this integral comes from the region around the dominant cusp $q = 1$. Therefore we split $\mathcal{C}$ into two arcs, $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_1$ is the counter-clockwise arc from phase $\frac{-\pi}{\sqrt{6n}}$ to $\frac{\pi}{\sqrt{6n}}$ and $\mathcal{C}_2$ is its complement in $\mathcal{C}$. Thus we have

$$M(m,n) = M + E,$$

with

$$M := \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{P(q)}{q^{n+1}}dq,$$

$$E := \frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{P(q)}{q^{n+1}}dq.$$

We will show that the integral E is negligible compared to M, which will give the asymptotic formula of Theorem 8.2. To do this, we are interested in the asymptotic behaviour of $P(q)$ when q is close to 1 and when it is far from it. In the following, we write $q = e^{2\pi i\tau}$.

9.2. Asymptotic behaviour of $P(q)$

9.2.1. Close to the dominant pole

We first study the asymptotic behaviour of $P(q)$ close to the dominant pole $q = 1$. To do so, let us write $\tau = \frac{iz}{2\pi}$, with $z = \frac{\pi}{\sqrt{6n}}(1 + ix)$, so $q = e^{\frac{-\pi}{\sqrt{6n}}(1+ix)}$. Thus $\mathcal{C}_1$ corresponds to $|x| \leq 1$ and $\mathcal{C}_2$ to $1 \leq |x| \leq \sqrt{6n}$.

We prove the following asymptotic formula.

Theorem 9.1. *Assume that $|x| \leq 1$. As n tends to infinity,*

$$P(q) = \sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}} + O\left(n^{\frac{-3}{4}}e^{\pi\sqrt{\frac{n}{6}}}\right).$$

Proof: As a particular case of Theorem 3.12, we have the transformation formula

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau). \tag{9.1}$$

Thus

$$
\begin{aligned}
P(q) &= \frac{q^{\frac{1}{24}}}{\eta(\tau)} \\
&= \sqrt{-i\tau}\,\frac{q^{\frac{1}{24}}}{\eta\left(\frac{-1}{\tau}\right)} \\
&= \sqrt{-i\tau}\,\frac{e^{\frac{2\pi i\tau}{24}}}{e^{\frac{-2i\pi}{24\tau}}\prod_{k\geq 1}\left(1-e^{\frac{-2k\pi i}{\tau}}\right)} \\
&= \sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}}\left(1+O(|z|)\right)\left(1+O\left(\left|e^{\frac{-2\pi i}{\tau}}\right|\right)\right) \\
&= \sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}}+O\left(|z|^{\frac{3}{2}}e^{\frac{\pi^2}{6}\mathrm{Re}\left(\frac{1}{z}\right)}\right).
\end{aligned}
$$

But $\mathrm{Re}\left(\frac{1}{z}\right)=\frac{\sqrt{6n}}{\pi}\frac{1}{1+x^2}\leq\frac{\sqrt{6n}}{\pi}$. Therefore

$$
P(q)=\sqrt{\frac{z}{2\pi}}e^{\frac{\pi^2}{6z}}+O\left(n^{\frac{-3}{4}}e^{\pi\sqrt{\frac{n}{6}}}\right).
$$

The theorem is proved. □

9.2.2. Far from the dominant pole

We also need an asymptotic estimation of $P(q)$ far from the dominant pole (when $1\leq|x|\leq\sqrt{6n}$). To do so, we prove a general lemma which we will also need in Chapters 12 and 13.

Lemma 9.2. *Let $P(q)=\frac{q^{\frac{1}{24}}}{\eta(\tau)}$ be the generating function for partitions. Assume that $\tau=u+iv\in\mathbb{H}$. For $Mv\leq|u|\leq\frac{1}{2}$ and $v\to 0$, we have that*

$$
|P(q)|\ll\sqrt{v}\exp\left[\frac{1}{v}\left(\frac{\pi}{12}-\frac{1}{2\pi}\left(1-\frac{1}{\sqrt{1+M^2}}\right)\right)\right].
$$

Proof: Let us write the following Taylor expansion

$$
\log(P(q))=-\sum_{n=1}^{\infty}\log(1-q^n)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{nm}}{m}=\sum_{m=1}^{\infty}\frac{q^m}{m(1-q^m)}.
$$

Therefore we have the estimate

$$\begin{aligned} |\log(P(q))| &\leq \sum_{m=1}^{\infty} \frac{|q|^m}{m|1-q^m|} \\ &\leq \frac{|q|}{|1-q|} - \frac{|q|}{1-|q|} + \sum_{m=1}^{\infty} \frac{|q|^m}{m(1-|q|^m)} \\ &= \log(P(|q|)) - |q| \left(\frac{1}{1-|q|} - \frac{1}{|1-q|} \right). \end{aligned}$$

For $Mv \leq |u| \leq \frac{1}{4}$, we have $\cos(2\pi u) \leq \cos(2\pi Mv)$. Therefore

$$|1-q|^2 = 1 - 2e^{-2\pi v}\cos(2\pi u) + e^{-4\pi v} \geq 1 - 2e^{-2\pi v}\cos(2\pi Mv) + e^{-4\pi v}.$$

By a Taylor expansion around $v = 0$ we find that

$$|1-q| \geq 2\pi v\sqrt{1+M^2} + O(v^2). \tag{9.2}$$

When $\frac{1}{4} \leq |u| \leq \frac{1}{2}$, we have $\cos(2\pi u) \leq 0$. Therefore

$$|1-q| \geq 1.$$

When $v \to 0$, this is asymptotically larger than (9.2). Hence, for all $Mv \leq |u| \leq \frac{1}{2}$,

$$|1-q| \geq 2\pi v\sqrt{1+M^2} + O(v^2). \tag{9.3}$$

Furthermore we have

$$1 - |q| = 1 - e^{-2\pi v} = 2\pi v + O(v^2). \tag{9.4}$$

By the transformation formula for η, we have:

$$P(|q|) = \frac{e^{\frac{-2\pi v}{24}}}{\eta(iv)} = \sqrt{v}e^{\frac{\pi}{12v}}\left(1 + O(v)\right).$$

Thus

$$\log(P(|q|)) = \frac{\pi}{12v} + \frac{1}{2}\log(v) + O(v). \tag{9.5}$$

Combining (9.3), (9.4) and (9.5), we finally obtain

$$\begin{aligned} |\log(P(q))| &\leq \log(P(|q|)) - \frac{1}{2\pi v}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)(1 + O(v)) \\ &= \frac{\pi}{12v} + \frac{1}{2}\log(v) + O(v) - \frac{1}{2\pi v}\left(1 - \frac{1}{\sqrt{1+M^2}}\right) + O(1) \\ &= \frac{1}{v}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right) + \frac{1}{2}\log(v) + O(1). \end{aligned}$$

Exponentiating yields the desired result. $\square$

Applying Lemma 9.2 with $M = 1$, $u = \frac{-x}{2\sqrt{6n}}$ and $v = \frac{1}{2\sqrt{6n}}$, we obtain the following bound for $|P(q)|$ far from the dominant pole.

Theorem 9.3. *Assume that $1 \leq |x| \leq \sqrt{6n}$. As n tends to infinity,*

$$|P(q)| \ll n^{\frac{-1}{4}} e^{\pi\sqrt{\frac{n}{6}} - \frac{1}{\pi}\sqrt{\frac{3n}{2}}}.$$

9.3. The circle method

Now that we have good estimates for P close to the dominant cusp and far from it, we can estimate the integrals M and E.

9.3.1. The main arc

Let us start by investigating the asymptotic behaviour of

$$M = \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{P(q)}{q^{n+1}} dq.$$

We prove the following.

Theorem 9.4. *As $n \to \infty$,*

$$M = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} + O\left(n^{\frac{-5}{4}} e^{\pi\sqrt{\frac{2n}{3}}}\right).$$

Proof: By the change of variable $q = e^{\frac{\pi}{\sqrt{6n}}(1+ix)}$ and Theorem 9.1, we get

$$\begin{aligned}
M &= \frac{1}{2\sqrt{6n}}\int_{-1}^{1} P(q) e^{\pi\sqrt{\frac{n}{6}}(1+ix)} dx \\
&= \frac{1}{2\sqrt{6n}}\int_{-1}^{1}\left(\sqrt{\frac{1+ix}{2\sqrt{6n}}} e^{\pi\sqrt{\frac{n}{6}}\frac{1}{1+ix}} + O\left(n^{\frac{-3}{4}} e^{\pi\sqrt{\frac{n}{6}}}\right)\right) e^{\pi\sqrt{\frac{n}{6}}(1+ix)} dx \\
&= \frac{1}{2^{\frac{3}{2}}(6n)^{\frac{3}{4}}}\int_{-1}^{1}\sqrt{1+ix}\, e^{\pi\sqrt{\frac{n}{6}}\left(\frac{1}{1+ix}+1+ix\right)} dx + O\left(n^{\frac{-5}{4}} e^{\pi\sqrt{\frac{2n}{3}}}\right).
\end{aligned}$$

We want to rewrite M in terms of Bessel functions, defined by

$$I_{-s-1}(2u) := \frac{1}{2\pi i}\int_{\Gamma} t^s e^{\pi u\left(t+\frac{1}{t}\right)} dt,$$

where Γ is Hankel's standard contour that begins in the lower-half plane at $-\infty$, goes counter-clockwise around the origin and then goes back to $-\infty$ in the upper-half plane.

To do this, let us introduce an auxiliary function

$$P_s := \frac{1}{2\pi i} \int_{1-i}^{1+i} v^s e^{\pi\sqrt{\frac{n}{6}}\left(v+\frac{1}{v}\right)} dv.$$

The following lemma relates P_s and $I_{-s-1}\left(\pi\sqrt{\frac{2n}{3}}\right)$.

Lemma 9.5. *As $n \to \infty$, we have*

$$P_s - I_{-s-1}\left(\pi\sqrt{\frac{2n}{3}}\right) \ll e^{\frac{\pi}{2}\sqrt{\frac{3n}{2}}}.$$

Proof: We set Γ to be the piecewise linear path consisting of the segments

$$\gamma_4 : \left(-\infty - \frac{i}{2}, -1 - \frac{i}{2}\right), \gamma_3 : \left(-1 - \frac{i}{2}, -1 - i\right),$$
$$\gamma_2 : (-1-i, 1-i), \gamma_1 : (1-i, 1+i),$$

then followed by the corresponding mirror images γ_2', γ_3' and γ_4'. Since $P_s = \int_{\gamma_1}$, we must show that the integrals on the other segments are bounded as claimed.

We have

$$\begin{aligned}\int_{\gamma_4} &= \frac{1}{2\pi i} \int_{-\infty}^{-1} e^{\pi\sqrt{\frac{n}{6}}\left(t-\frac{i}{2}+\frac{1}{t-\frac{i}{2}}\right)} \left(t - \frac{i}{2}\right)^s dt \\ &\ll \int_1^{\infty} e^{-\pi\sqrt{\frac{n}{6}}t} \left|-t - \frac{i}{2}\right|^s dt \\ &\ll n^{\frac{-1}{2}} e^{-\pi\sqrt{\frac{n}{6}}},\end{aligned}$$

where the final inequality follows from a simple bound for the incomplete Gamma function.

We also have

$$\begin{aligned}\int_{\gamma_3} &= \frac{1}{2\pi i} \int_{\frac{1}{2}}^{1} e^{-\pi\sqrt{\frac{n}{6}}\left(1+it+\frac{1}{1+it}\right)} (-1-it)^s dt \\ &\ll \int_{\frac{1}{2}}^{1} e^{-\pi\sqrt{\frac{n}{6}}\left(1+\frac{1}{1+t^2}\right)} |1+it|^s dt \\ &\ll e^{-\pi\sqrt{\frac{n}{6}}}.\end{aligned}$$

And finally

$$\begin{aligned}\int_{\gamma_2} &= \frac{1}{2\pi i}\int_{-1}^{1} e^{\pi\sqrt{\frac{n}{6}}\left(t-i+\frac{1}{t-i}\right)}(t-i)^s dt\\ &\ll \int_{-1}^{1} e^{\pi\sqrt{\frac{n}{6}}\left(t+\frac{t}{t^2+1}\right)}|t-i|^s dt\\ &\ll e^{\frac{\pi}{2}\sqrt{\frac{3n}{2}}}.\end{aligned}$$

□

Let us turn back to M. By the change of variable $v = 1 + ix$, we get

$$\begin{aligned}M &= \frac{1}{i2^{\frac{3}{2}}(6n)^{\frac{3}{4}}}\int_{1-i}^{1+i} \sqrt{v}e^{\pi\sqrt{\frac{n}{6}}\left(\frac{1}{v}+v\right)}dv + O\left(n^{\frac{-5}{4}}e^{\pi\sqrt{\frac{2n}{3}}}\right)\\ &= \frac{\pi}{\sqrt{2}(6n)^{\frac{3}{4}}}\left(I_{\frac{-3}{2}}\left(\pi\sqrt{\frac{2n}{3}}\right) + O\left(e^{\frac{\pi}{2}\sqrt{\frac{3n}{2}}}\right)\right) + O\left(n^{\frac{-5}{4}}e^{\pi\sqrt{\frac{2n}{3}}}\right).\end{aligned}$$

There is a simple formula for the asymptotic of the Bessel function (see (4.12.7) in [AAR01]). For every l, as $x \to \infty$,

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right). \tag{9.6}$$

Therefore

$$\begin{aligned}M &= \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}} + O\left(n^{\frac{-3}{2}}e^{\pi\sqrt{\frac{2n}{3}}}\right) + O\left(n^{\frac{-5}{4}}e^{\pi\sqrt{\frac{2n}{3}}}\right)\\ &= \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}} + O\left(n^{\frac{-5}{4}}e^{\pi\sqrt{\frac{2n}{3}}}\right).\end{aligned}$$

This completes the proof. □

9.3.2. The error arc

It remains now to verify that the error integral E is indeed negligible compared to M.

Theorem 9.6. *As $n \to \infty$,*

$$E \ll n^{\frac{1}{4}}e^{\pi\sqrt{\frac{2n}{3}}-\frac{1}{\pi}\frac{\sqrt{3n}}{2}}.$$

Proof: By Theorem 9.3, we have

$$\begin{aligned}|E| &= \left|\frac{1}{2\pi}\int_{1<|x|<\sqrt{6n}} P(q)e^{\pi\sqrt{\frac{n}{6}}(1+ix)}dx\right| \\ &\ll \int_{1<|x|<\sqrt{6n}} n^{\frac{-1}{4}}e^{\pi\sqrt{\frac{n}{6}}-\frac{1}{\pi}\sqrt{\frac{3n}{2}}}e^{\pi\sqrt{\frac{n}{6}}}dx \\ &\ll n^{\frac{1}{4}}e^{\pi\sqrt{\frac{2n}{3}}-\frac{1}{\pi}\frac{\sqrt{3n}}{2}}.\end{aligned}$$

$\square$

This is exponentially small compared to

$$M = \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}} + O\left(n^{\frac{-5}{4}}e^{\pi\sqrt{\frac{2n}{3}}}\right).$$

Thus

$$p(n) = M + E \underset{n\to\infty}{\sim} \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}.$$

10. Overpartitions with restricted odd differences

Contents

10.1. Introduction

In this chapter, we study overpartitions where the difference between two successive parts may be odd only if the larger part is overlined. We use q-difference equations in order to compute a two-variable hypergeometric q-series representation of the corresponding generating function, and we use Wright's circle method to compute asymptotic formulas.

Let us recall some definitions already given in the introduction.

Let $\overline{t}(n)$ denote the number of overpartitions where (i) the difference between two successive parts may be odd only if the larger part is overlined, and (ii) if the smallest part is odd then it is overlined. Let $\overline{s}(n)$ denote the number of overpartitions counted by $\overline{t}(n)$ but with odd smallest part. Let $\overline{t}_+(n)$ (resp. $\overline{s}_+(n)$) denote the number of overparititions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) with largest part even, and $\overline{t}_-(n)$ (resp. $\overline{s}_-(n)$) denote the number of overparititions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) with largest part odd.

With the above notation, we have the following asymptotic formulas.

Theorem 10.1. *As $n \to \infty$, we have*

$$\overline{t}(n) \sim \frac{\sqrt{21}}{36n}\, e^{\frac{\pi\sqrt{7n}}{3}}, \tag{10.1}$$

$$\overline{s}(n) \sim \frac{\sqrt{21}}{36n}\, e^{\frac{\pi\sqrt{7n}}{3}}, \tag{10.2}$$

$$\overline{t}_+(n) - \overline{t}_-(n) \sim (-1)^n \frac{\sqrt{3}}{18n^{\frac{3}{4}}}\, e^{\frac{2\pi\sqrt{n}}{3}}, \tag{10.3}$$

$$\overline{s}_+(n) - \overline{s}_-(n) \sim (-1)^n \frac{\sqrt{3}}{18n^{\frac{3}{4}}}\, e^{\frac{2\pi\sqrt{n}}{3}}. \tag{10.4}$$

Since the generating functions for $\overline{t}(n)$ (10.12) and $\overline{s}_+(n)-\overline{s}_-(n)$ (10.14) are (up to some power of q) weakly holomorphic modular forms of non-positive weight, Rademacher and Zuckerman's refinement of the Hardy-Ramanujan Circle Method applies [RZ38] and we can directly find their asymptotic. However the generating functions for $\overline{s}(n)$ and $\overline{t}_+(n)-\overline{t}_-(n)$ are mixed mock modular forms, which makes asymptotic computations more difficult. Therefore we will only prove (10.2) and (10.3) using Wright's circle method. In the circle method to compute (10.2), the dominant cusp is $q = 1$ as in the case of partitions, but for (10.3) the major arc is instead centred at $q = -1$.

10.2. Generating functions

We first determine generating functions for $\overline{t}(m, n)$ and $\overline{s}(m, n)$, the number of overpartitions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) having m parts using the analytic and combinatorial theory of q-difference equations. We prove the following

Theorem 10.2. *We have the following identities:*

$$\sum_{m,n\geq 0} \overline{t}(m,n)x^m q^n = \frac{(-xq)_\infty}{(xq)_\infty}\left(1 + \sum_{n\geq 1} \frac{(-q^3;q^3)_{n-1}(-x)^n q^n}{(-q)_{n-1}(q^2;q^2)_n}\right), \tag{10.5}$$

$$\sum_{m,n\geq 1} \overline{s}(m,n)x^m q^n = \sum_{n\geq 1} \frac{(q^3;q^3)_{n-1}x^n q^n}{(q)_{n-1}(q^2;q^2)_n}. \tag{10.6}$$

Proof: We begin by considering (10.5). Define $\overline{H}(x;q)$ by

$$\overline{H}(x;q) := \sum_{m,n\geq 0} \overline{t}(m,n)x^m q^n.$$

Now, if λ is an overpartition counted by $\bar{t}(m,n)$, then its smallest part is either $\overline{1}$ or at least 2. In the first case, we may remove the $\overline{1}$ along with any other 1s in λ and then subtract 1 from each remaining part to obtain an overpartition counted by $\bar{t}(m-a,n-m)$, where $a \geq 1$ is the number of 1s occurring in λ. In the second case, we may remove any 2s in λ and then subtract 2 from each remaining part to obtain an overpartition counted by $\bar{t}(m-b,n-2m)$, where $b \geq 0$ is the number of 2s occurring in λ. Note that if the smallest part in this new partition is odd, then it must have correspondingly occurred as an overlined part in λ. In other words, we have

$$\overline{H}(x;q) = \frac{xq}{1-xq}\overline{H}(xq;q) + \frac{1+xq^2}{1-xq^2}\overline{H}\left(xq^2;q\right). \tag{10.7}$$

Together with the fact that $\overline{H}(0;q)=1$, this uniquely defines $\overline{H}(x;q)$.

We claim that the right-hand side of (10.5) also satisfies (10.7) with the same initial condition. To see this, first define

$$\overline{K}(x;q) = \overline{K}(x) := 1 + \sum_{n\geq 1} \frac{\prod_{j=1}^{n-1}\left(1-q^j+q^{2j}\right)}{(q^2;q^2)_n} x^n q^n.$$

We have that

$$\begin{aligned}
\overline{K}\left(xq^2\right) &= 1 - \sum_{n\geq 1} \frac{x^n q^n \prod_{j=1}^{n-1}\left(1-q^j+q^{2j}\right)}{(q^2;q^2)_n}\left(1-q^{2n}-1\right) \\
&= \overline{K}(x) - \sum_{n\geq 1} \frac{x^n q^n \prod_{j=1}^{n-1}\left(1-q^j+q^{2j}\right)}{(q^2;q^2)_{n-1}} \\
&= \overline{K}(x) - \sum_{n\geq 0} \frac{x^{n+1} q^{n+1} \prod_{j=1}^{n-1}\left(1-q^j+q^{2j}\right)\left(1-q^n+q^{2n}\right)}{(q^2;q^2)_n} \\
&= \overline{K}(x) - xq\overline{K}(x) + xq\overline{K}(xq) - xq\overline{K}\left(xq^2\right)
\end{aligned}$$

so that

$$\overline{K}(x) = -\frac{xq}{1-xq}\overline{K}(xq) + \frac{1+xq}{1-xq}\overline{K}\left(xq^2\right). \tag{10.8}$$

Now, the right-hand side of (10.5) is $\overline{G}(x;q)$, where

$$\overline{G}(x;q) = \overline{G}(x) := \frac{(-xq)_\infty}{(xq)_\infty}\overline{K}(-x),$$

and applying (10.8) to $\overline{G}(x;q)$ gives

$$\overline{G}(x;q) = \frac{xq}{1-xq}\overline{G}(xq;q) + \frac{1+xq^2}{1-xq^2}\overline{G}\left(xq^2;q\right). \tag{10.9}$$

Comparing (10.7) and (10.9) and noting the initial condition $\overline{G}(0;q) = 1$ yields (10.5).

Next we consider (10.6). Here we require the notion of the conjugate of an overpartition, which is obtained by reading the columns of the Ferrers diagram (an overlined part is designated by a mark at the end of a row). The right-hand side of (10.6) is

$$\sum_{n\geq 1} \frac{\left(q^3;q^3\right)_{n-1} x^n q^n}{(q)_{n-1}\left(q^2;q^2\right)_n} = \sum_{m\geq 1} \frac{x^m q^m}{1-q^{2m}} \prod_{j=1}^{m-1} \left(\frac{q^j}{1-q^j} + \frac{1}{1-q^{2j}}\right).$$

Here the mth summand is the generating function for overpartitions where the largest part m occurs overlined and an odd number of times, while each part less than m occurs an even number of times if it does not occur overlined. Conjugating we obtain an overpartition counted by $\overline{s}(m,n)$. This is illustrated in Figure 10.1.

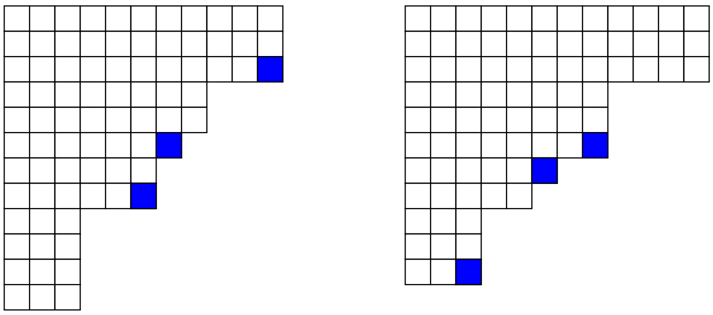


Figure 10.1.: The overpartition $(11, 11, \overline{11}, 8, 8, \overline{7}, 6, \overline{6}, 3, 3, 3, 3)$ and its conjugate $(12, 12, 12, 8, 8, \overline{8}, \overline{6}, 5, 3, 3, \overline{3})$.

This establishes (10.6) and completes the proof of Theorem 10.2. $\square$

Now we use identities from the theory of hypergeometric q-series in order to evaluate the generating functions for overpartitions with odd differences.

When $x = 1$ in (10.5) or -1 in (10.6), then we have a modular form, and when $x = -1$ in (10.5) or 1 in (10.6), then we have a mixed mock modular form (see [LO13]). Define the mock theta functions $\overline{\gamma}(q)$ and $\overline{\chi}(q)$ by

$$\overline{\gamma}(q) := \sum_{n\geq 0} \frac{(-1;q)_n (q;q)_n q^{\binom{n+1}{2}}}{(q^3;q^3)_n} \tag{10.10}$$

and

$$\overline{\chi}(q) := \sum_{n\geq 0} \frac{(-1;q)_n (-q;q)_n q^{\binom{n+1}{2}}}{(-q^3;q^3)_n}. \tag{10.11}$$

We have the following formulas for our generating functions as modular forms or mixed mock modular forms.

Corollary 10.3. *We have*

$$\sum_{n\geq 0} \overline{t}(n) q^n = \frac{\left(q^3;q^3\right)_\infty}{(q;q)_\infty \left(q^2;q^2\right)_\infty}, \tag{10.12}$$

$$\sum_{n\geq 0} \left(\overline{t}_+(n) - \overline{t}_-(n)\right) q^n = \frac{\left(-q^3;q^3\right)_\infty}{(-q;q)_\infty^3} \overline{\chi}(q), \tag{10.13}$$

$$1 + 3\sum_{n\geq 1} \left(\overline{s}_+(n) - \overline{s}_-(n)\right) q^n = \frac{\left(-q^3;q^3\right)_\infty}{(-q;q)_\infty^3}, \tag{10.14}$$

$$1 + 3\sum_{n\geq 1} \overline{s}(n) q^n = \frac{\left(q^3;q^3\right)_\infty}{(q;q)_\infty \left(q^2;q^2\right)_\infty} \overline{\gamma}(q). \tag{10.15}$$

Proof: To prove this corollary, we require the q-Gauss summation formula [GR04, Equation (1.5.1)],

$$\sum_{n\geq 0} \frac{(a)_n (b)_n (c/ab)^n}{(c)_n (q)_n} = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}, \tag{10.16}$$

and a certain ${}_3\phi_2$ transformation [GR04, Appendix III, Equation (III.10)],

$$\begin{aligned} \sum_{n\geq 0} \frac{(aq/bc)_n (d)_n (e)_n}{(q)_n (aq/b)_n (aq/c)_n} \left(\frac{aq}{de}\right)^n &= \frac{(aq/d)_\infty (aq/e)_\infty (aq/bc)_\infty}{(aq/b)_\infty (aq/c)_\infty (aq/de)_\infty} \\ &\quad \times \sum_{n\geq 0} \frac{(aq/de)_n (b)_n (c)_n}{(q)_n (aq/d)_n (aq/e)_n} \left(\frac{aq}{bc}\right)^n. \end{aligned} \tag{10.17}$$

Setting $x = 1$ in (10.5) and applying (10.16) with $a = 1/b = -\zeta_3$ (where $\zeta_3 := e^{\frac{2\pi i}{3}}$) and $c = -q$, we obtain

$$\begin{aligned}
\sum_{n\geq 0} \bar{t}(n)q^n &= \frac{(-q)_\infty}{(q)_\infty} \sum_{n\geq 0} \frac{(-\zeta_3)_n\,(-\zeta_3^{-1})_n(-q)^n}{(q)_n(-q)_n} \\
&= \frac{(-q)_\infty}{(q)_\infty} \times \frac{(\zeta_3 q)_\infty\left(\zeta_3^{-1}q\right)_\infty}{(-q)_\infty^2} \\
&= \frac{\left(q^3;q^3\right)_\infty}{(q)_\infty\left(q^2;q^2\right)_\infty},
\end{aligned}$$

which proves (10.12).

For the next case we set $x = -1$ in (10.5) and invoke (10.17) with $a = -b = 1$, $d = 1/e = -\zeta_3$, and $c \to \infty$, which yields

$$\begin{aligned}
\sum_{n\geq 0} \left(\bar{t}_+(n) - \bar{t}_-(n)\right) q^n &= \frac{(q)_\infty}{(-q)_\infty} \sum_{n\geq 0} \frac{(-\zeta_3)_n\left(-\zeta_3^{-1}\right)_n q^n}{(q)_n(-q)_n} \\
&= \frac{(q)_\infty}{(-q)_\infty} \times \frac{(-\zeta_3 q)_\infty\left(-\zeta_3^{-1}q\right)_\infty}{(q)_\infty(-q)_\infty} \sum_{n\geq 0} \frac{(-1)_n q^{\binom{n+1}{2}}}{(-\zeta_3 q)_n\,(-\zeta_3^{-1}q)_n} \\
&= \frac{\left(-q^3;q^3\right)_\infty}{(-q)_\infty^3}\overline{\chi}(q).
\end{aligned}$$

The final two equations in the statement of the corollary are proven similarly. For (10.14) we set $x = -1$ in (10.6) and apply (10.16) with $a = 1/b = \zeta_3$ and $c = -q$, whereas for (10.15) we set $x = 1$ in (10.6) and appeal to (10.17) in the case $a = -b = 1$, $d = 1/e = \zeta_3$, and $c \to \infty$. $\square$

We remark that (10.12) can also be established by a simple combinatorial argument, as in the proof of (10.6). Indeed, conjugating an overpartition λ counted by $\bar{t}(n)$ gives an overpartition with the property that if $\overline{m}$ does not occur, then m occurs an even number of times. Hence

$$\sum_{n\geq 0} \bar{t}(n)q^n = \prod_{m\geq 1}\left(\frac{q^m}{(1-q^m)} + \frac{1}{(1-q^{2m})}\right) = \frac{\left(q^3;q^3\right)_\infty}{\left(q^2;q^2\right)_\infty(q)_\infty}.$$

This argument easily generalizes. Let $\bar{t}^{(k)}(n)$ denote the number of overpartitions of n where (i) consecutive parts differ by a multiple of $(k+1)$ unless the larger of the two is overlined, and (ii) the smallest part is overlined unless it is divisible by $k+1$. Then we have

$$\sum_{n\geq 0} \bar{t}^{(k)}(n)q^n = \prod_{m\geq 1}\left(\frac{q^m}{(1-q^m)} + \frac{1}{(1-q^{km})}\right) = \frac{\left(q^{k+1};q^{k+1}\right)_\infty}{\left(q^k;q^k\right)_\infty (q)_\infty}.$$

10.3. Wright's Circle Method and the proof of Theorem 10.1

In this section we apply Wright's circle method to find the asymptotic formulas (10.2) and (10.3) of Theorem 10.1.

For notational convenience we recall (10.13) and (10.15) and write

$$f_1(q) := 1 + 3\sum_{n\geq 1} \overline{s}(n)q^n = \frac{\left(q^3;q^3\right)_\infty}{(q;q)_\infty \left(q^2;q^2\right)_\infty}\overline{\gamma}(q), \tag{10.18}$$

$$f_2(q) := \sum_{n\geq 0}\left(\bar{t}_+(n) - \bar{t}_-(n)\right)q^n = \frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3}\overline{\chi}(q).$$

10.3.1. Asymptotic behaviour of $f_1(q)$

We begin with $\overline{s}(n)$, as the asymptotic analysis is analogous to case of $p(n)$ presented in Chapter 9, with the dominant pole at $q=1$. Throughout the section we use the standard parametrisation $q = e^{2\pi i\tau}$, where $\tau = x+iy$,$y>0$ and $\frac{-1}{2} \leq x \leq \frac{1}{2}$. Recalling (10.10), we use an alternative expression for the mock theta function, which follows from equation (1.3) in [BL07], namely

$$\overline{\gamma}(q) = \frac{(-q)_\infty}{(q)_\infty}\left(1 + 6\sum_{k\geq 1}\frac{(-1)^k q^{k^2+k}}{(1-\zeta_3 q^k)(1-\zeta_3^{-1}q^k)}\right), \tag{10.19}$$

with $\zeta_3 := e^{\frac{2\pi i}{3}}$.

10.3.1.1. Close to the dominant pole $q=1$

We show that the overall asymptotic behaviour of $f_1(q)$ is largely controlled by the singularities of the infinite product from (10.18), so the dominant pole is at $q=1$. The following result describes the behaviour of $f_1(q)$ in a neighbourhood of $q=1$.

Theorem 10.4. *Assume that* $y = \frac{\sqrt{7}}{12\sqrt{n}}$, *and let* $M > 0$ *be fixed. If* $|x| \leq My$, *then, as* $n \to \infty$,

$$f_1(q) = \sqrt{-6i\tau}e^{\frac{7\pi i}{72\tau}} + O\left(n^{-\frac{3}{4}}e^{\frac{\pi\sqrt{7n}}{6}}\right). \tag{10.20}$$

Proof: In order to determine the behaviour of $f_1(q)$ near $q = 1$, we begin by studying $\overline{\gamma}(q)$ near this point. By Taylor's Theorem, we have

$$\overline{\gamma}(q) = \overline{\gamma}(1) + O(|\tau|),$$

and by (10.10), we directly calculate

$$\overline{\gamma}(1) = \sum_{k\geq 0}\left(\frac{2}{3}\right)^k = 3.$$

Since $|x| \leq My$, we therefore have the following expansion as $n \to \infty$:

$$\overline{\gamma}(q) = 3 + O\left(n^{-\frac{1}{2}}\right). \tag{10.21}$$

We next determine the asymptotic behaviour of the infinite product, using its modularity. The modular inversion formula for Dedekind's η function (9.1) gives, as n tends to ∞,

$$\begin{aligned}
\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} &= \frac{\eta(3\tau)}{\eta(\tau)\eta(2\tau)} \\
&= \sqrt{\frac{-2i\tau}{3}}\frac{\eta\left(\frac{-1}{3\tau}\right)}{\eta\left(\frac{-1}{\tau}\right)\eta\left(\frac{-1}{2\tau}\right)} \\
&= \sqrt{\frac{-2i\tau}{3}}\frac{e^{\frac{-i\pi}{12\times 3\tau}}}{e^{\frac{-i\pi}{12\tau}}e^{\frac{-i\pi}{12\times 2\tau}}}\prod_{k\geq 1}\frac{\left(1-e^{\frac{-2ik\pi}{3\tau}}\right)}{\left(1-e^{\frac{-2ik\pi}{\tau}}\right)\left(1-e^{\frac{-2ik\pi}{2\tau}}\right)} \\
&= \sqrt{\frac{-2i\tau}{3}}e^{\frac{7i\pi}{72\tau}}\left(1+O\left(\left|e^{\frac{-2i\pi}{3\tau}}\right|\right)\right) \\
&= \sqrt{\frac{-2i\tau}{3}}e^{\frac{7\pi i}{72\tau}}\left(1+O\left(e^{-\frac{8\pi}{(M^2+1)}\sqrt{\frac{n}{7}}}\right)\right).
\end{aligned}$$

Combining this with (10.21) gives

$$f_1(q) = \sqrt{-6i\tau}e^{\frac{7\pi i}{72\tau}} + O\left(n^{-\frac{3}{4}}e^{\frac{\pi\sqrt{7n}}{6}}\right),$$

which completes the proof. $\square$

10.3.1.2. Far from the dominant pole

We now give an uniform bound for $f_1(q)$ far from $q = 1$.

Theorem 10.5. *Assume that $y = \frac{\sqrt{7}}{12\sqrt{n}}$, and let $M > 0$ be fixed. If $My < |x| \leq \frac{1}{2}$, then as $n \to \infty$,*

$$|f_1(q)| \ll n^{\frac{3}{4}} \exp\left(\frac{8\pi\sqrt{n}}{3\sqrt{7}}\left(1 - \frac{27}{4\pi^2}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right)\right).$$

Proof: We begin by considering the sum in (10.19). Bounding each term in the sum absolutely, we get

$$\left|\sum_{k\geq 1} \frac{(-1)^k q^{k^2+k}}{\left(1-\zeta_3 q^k\right)\left(1-\zeta_3^{-1} q^k\right)}\right| \ll \frac{1}{(1-|q|)^2} \sum_{k\geq 1} |q|^{k^2} \ll n^{\frac{5}{4}}, \tag{10.22}$$

where the final bound follows by an integral comparison.

Simplifying all of the infinite products from (10.18) and (10.19), we have

$$f_1(q) = \frac{\left(q^3; q^3\right)_\infty}{(q;q)_\infty^3} \cdot O\left(n^{\frac{5}{4}}\right).$$

For convenience denote the above infinite product by $g_1(q)$. We calculate

$$\begin{aligned}\log\left(g_1(q)\right) &= \sum_{m\geq 1}\left(\frac{3q^m}{m\left(1-q^m\right)} - \frac{q^{3m}}{m\left(1-q^{3m}\right)}\right)\\ &= \sum_{m\geq 1}\left(\frac{3q^{3m-2}}{(3m-2)\left(1-q^{3m-2}\right)} + \frac{3q^{3m-1}}{(3m-1)\left(1-q^{3m-1}\right)}\right).\end{aligned}$$

This implies that

$$\begin{aligned}\left|\log\left(g_1(q)\right)\right| &\leq \sum_{m\geq 1}\left(\frac{3|q|^{3m+1}}{(3m+1)\left(1-|q|^{3m+1}\right)} + \frac{3|q|^{3m-1}}{(3m-1)\left(1-|q|^{3m-1}\right)}\right)\\ &\qquad + \frac{3|q|}{|1-q|}\\ &= \log\left(g_1\left(|q|\right)\right) - 3|q|\left(\frac{1}{1-|q|} - \frac{1}{|1-q|}\right).\end{aligned} \tag{10.23}$$

Using the transformation (9.1), we find that

$$g_1(|q|) = \frac{\eta(3iy)}{\eta(iy)^3} = \frac{y}{\sqrt{3}} e^{\frac{2\pi}{9y}}\left(1 + O\left(e^{\frac{-2\pi}{3y}}\right)\right). \tag{10.24}$$

To bound the other terms, we use the fact that $|x| > My$, which implies that $\cos(2\pi x) < \cos(2\pi My) \leq 1$. Thus

$$|1-q|^2 = 1 - 2e^{-2\pi y}\cos(2\pi x) + e^{-4\pi y} > 1 - 2e^{-2\pi y}\cos(2\pi My) + e^{-4\pi y};$$

calculating the Taylor expansion of the final expression (around $y = 0$) then gives

$$|1-q| = c_0 + c_1 y + O\left(y^2\right), \tag{10.25}$$

with $c_0 \geq 0$ and $c_1 > 2\pi\sqrt{1+M^2}$. Furthermore, we have

$$1 - |q| = 1 - e^{-2\pi y} = 2\pi y + O\left(y^2\right). \tag{10.26}$$

Converting (10.25) and (10.26) to Laurent series and combining with (10.23) and (10.24), we obtain

$$|g_1(q)| \ll y \exp\left[\frac{1}{y}\left(\frac{2\pi}{9} - \frac{3}{2\pi}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right)\right].$$

This completes the proof of the theorem.

□

A short calculation shows that the bound in Theorem 10.5 is indeed an error term as long as $M > \left(\left(\frac{12}{12-\pi^2}\right)^2 - 1\right)^{1/2} = 5.543\ldots$. So we will use the following corollary in the circle method.

Corollary 10.6. *If $M \geq 6$ and $My < |x| \leq \frac{1}{2}$, then for some $\varepsilon > 0$ we have*

$$f_1(q) = O\left(e^{\frac{1}{y}\left(\frac{7\pi}{72} - \varepsilon\right)}\right).$$

10.3.2. Wright's Circle Method and the asymptotic formula for $\overline{s}(n)$

We now apply Wright's circle method and prove (10.2) in Theorem 10.1. By Cauchy's Theorem, we have for all $n \geq 1$,

$$3\overline{s}(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f_1(q)}{q^{n+1}} dq = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_1\left(e^{-\frac{\pi\sqrt{7}}{6\sqrt{n}} + 2\pi i x}\right) e^{\frac{\pi\sqrt{7n}}{6} - 2\pi i n x} dx, \tag{10.27}$$

where the contour is the counterclockwise traversal of the circle $\mathcal{C} := \{|q| = e^{-2\pi y}\}$, and $y = \frac{\sqrt{7}}{12\sqrt{n}}$ as in the previous section. We separate (10.27) in two integrals, writing $3\overline{s}(n) = I' + I''$, where

$$I' := \int\limits_{|x| \leq My} f_1\left(e^{-\frac{\pi\sqrt{7}}{6\sqrt{n}} + 2\pi i x}\right) e^{\frac{\pi\sqrt{7n}}{6} - 2\pi i n x} dx$$

and

$$I'' := \int\limits_{My<|x|\leq\frac{1}{2}} f_1\left(e^{-\frac{\pi\sqrt{7}}{6\sqrt{n}}+2\pi ix}\right)e^{\frac{\pi\sqrt{7n}}{6}-2\pi inx}dx.$$

It turns out that I' provides the main asymptotic contribution, while I'' is an error term.

As in Chapter 9, we approximate I' by Bessel functions. Assuming that $M > 0$ is fixed and that $u > 0$ is a real variable, we introduce the auxiliary function

$$P_s(u) := \frac{1}{2\pi i}\int_{1-Mi}^{1+Mi} v^s e^{u\left(v+\frac{1}{v}\right)}dv.$$

Finally, we note a result relating P_s to a Bessel function, which is proven in essentially the same manner as Lemma 9.5 in Chapter 9.

Lemma 10.7. *As $n \to \infty$,*

$$P_s = I_{-s-1}(2u) + O\left(e^u\right).$$

Proof of (10.2) **of Theorem 10.1:** Making the change of variables

$$v = -\frac{i\tau}{y} = -\frac{12\sqrt{n}i}{\sqrt{7}},$$

and using Theorem 10.4, we write I' as

$$\begin{aligned} I' &= -iy\int_{1-Mi}^{1+Mi} f_1\left(e^{-\frac{\pi\sqrt{7}v}{6\sqrt{n}}}\right)e^{\frac{\pi\sqrt{7n}v}{6}}dv \qquad (10.28)\\ &= -i\sqrt{6}y^{\frac{3}{2}}\int_{1-Mi}^{1+Mi}\left(\sqrt{v}e^{\frac{\pi\sqrt{7}}{6\sqrt{n}v}} + O\left(n^{-\frac{3}{4}}e^{\frac{\pi\sqrt{7n}}{6}}\right)\right)e^{\frac{\pi\sqrt{7n}v}{6}}dv. \end{aligned}$$

Combining (10.20), (9.6), and Lemma 10.7 (with $u = \frac{\pi\sqrt{7}}{6\sqrt{n}}$ and $s = \frac{1}{2}$), we find that, as $n \to \infty$,

$$I' = \frac{\sqrt{21}}{12}\frac{e^{\frac{\pi\sqrt{7n}}{3}}}{n} + O\left(n^{-\frac{5}{4}}e^{\frac{\pi\sqrt{7n}}{3}}\right).$$

We now turn to I'' and show that it is negligible compared to I'. Applying Corollary 10.6 and taking the absolute value of the rest of the integrand, we directly obtain that, for some $\varepsilon > 0$,

$$I'' \ll e^{\frac{\sqrt{7n}\pi}{3}(1-\varepsilon)}.$$

Thus I'' is exponentially smaller than I', and this completes the proof. $\square$

10.3.3. Asymptotic behaviour of $f_2(q)$

In this section we determine the asymptotic behaviour of $f_2(q)$ close to the dominant cusp and far from it. The difference from the previous applications of Wright's circle method is that the dominant cusp is now $q = -1$, which can be seen by comparing the singularities of the product $\frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3}$. To deal with this change, let us modify the notation and write $q = -e^{\pi i z}$, where $z = x + iy$ and $y > 0$.

Recalling (10.11) (and [BL07]), we again provide an alternative series,

$$\overline{\chi}(q) = \frac{(-q)_\infty}{(q)_\infty}\left(1 + 2\sum_{k\geq 1}\frac{(-1)^k q^{k^2+k}}{(1+\zeta_3 q^k)(1+\zeta_3^{-1}q^k)}\right). \tag{10.29}$$

10.3.3.1. Close to the dominant pole $q = -1$

Let us start by studying $f_2(q)$ close to its dominant pole. We have the following result.

Theorem 10.8. *Assume that $y = \frac{1}{3\sqrt{n}}$, and let $M > 0$ be fixed. If $|x| \leq My$, then as $n \to \infty$,*

$$f_2(q) = \frac{1}{3}e^{\frac{i\pi}{9z}} + O\left(n^{\frac{-1}{2}}e^{\frac{\pi\sqrt{n}}{3}}\right).$$

Proof: In order to determine the behaviour of $f_1(q)$ near $q = -1$, we begin by studying $\overline{\chi}(q)$ near this point. By Taylor's theorem, we have

$$\overline{\chi}(-e^{i\pi z}) = \overline{\chi}(-1) + O(|z|),$$

and by (10.11), we obtain

$$\overline{\chi}(-1) = 1 - \frac{2}{3} + \sum_{n\geq 2} 0 = \frac{1}{3}.$$

Thus

$$\overline{\chi}(q) = \frac{1}{3} + O\left(n^{\frac{-1}{2}}\right). \tag{10.30}$$

Now let us turn to the asymptotic behaviour of the product $\frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3}$ close to $q = -1$. Let us recall the following modular transformation formulas for Dedekind's eta function, obtained via Theorem 3.12.

$$
\begin{aligned}
\eta\left(\frac{1+z}{2}\right) &= \frac{-1}{\sqrt{-iz}}\eta\left(\frac{1-z^{-1}}{2}\right),\\
\eta(1+z) &= \frac{e^{\frac{i\pi}{12}}}{\sqrt{-iz}}\eta(-z^{-1}),\\
\eta(3+3z) &= \frac{e^{\frac{i\pi}{4}}}{\sqrt{-3iz}}\eta(-(3z)^{-1}),\\
\eta\left(\frac{3+3z}{2}\right) &= \frac{-e^{\frac{i\pi}{12}}}{\sqrt{-3iz}}\eta\left(\frac{1-(3z)^{-1}}{2}\right).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3} &= \frac{\eta(3+3z)\eta^3\left(\frac{1+z}{2}\right)}{\eta\left(\frac{3+3z}{2}\right)\eta^3(1+z)}\\
&= \frac{\frac{e^{\frac{i\pi}{4}}}{\sqrt{-3iz}}\eta(-(3z)^{-1})\frac{-1}{\sqrt{-iz}^3}\eta^3\left(\frac{1-z^{-1}}{2}\right)}{\frac{-e^{\frac{i\pi}{12}}}{\sqrt{-3iz}}\eta\left(\frac{1-(3z)^{-1}}{2}\right)\frac{e^{\frac{i\pi}{4}}}{\sqrt{-iz}^3}\eta^3(-z^{-1})}\\
&= e^{\frac{-i\pi}{12}}\frac{\eta(-(3z)^{-1})\eta^3\left(\frac{1-z^{-1}}{2}\right)}{\eta\left(\frac{1-(3z)^{-1}}{2}\right)\eta^3(-z^{-1})}\\
&= e^{\frac{-i\pi}{12}}\frac{e^{\frac{i\pi}{12}\left(\frac{-1}{3z}\right)}e^{\frac{3i\pi}{12}\left(\frac{1-z^{-1}}{2}\right)}}{e^{\frac{i\pi}{12}\left(\frac{1-(3z)^{-1}}{2}\right)}e^{\frac{-3i\pi}{12z}}}\left(1+O\left(\left|e^{\frac{-i\pi}{3z}}\right|\right)\right)\\
&= e^{\frac{i\pi}{9z}}\left(1+O\left(e^{-\frac{\pi\sqrt{n}}{M^2+1}}\right)\right).
\end{aligned}
$$

Multiplying with (10.30) yields the desired result. □

10.3.3.2. Far from the dominant pole

We now give an uniform bound for $f_2(q)$ far from $q=-1$.

Theorem 10.9. *Assume that $y=\frac{1}{3\sqrt{n}}$, and let $M>0$ be fixed. If $My<|x|\leq 1$, then as $n\to\infty$,*

$$
|f_2(q)| \ll n\exp\left(\frac{7\pi\sqrt{n}}{12}\left(1-\frac{36}{7\pi^2}\left(1-\frac{1}{\sqrt{1+M^2}}\right)\right)\right).
$$

Proof: Using a similar argument as in (10.22), we conclude that

$$\left|\sum_{k\geq 1}\frac{(-1)^k q^{k^2+k}}{(1+\zeta_3 q^k)\left(1+\zeta_3^{-1}q^k\right)}\right| \ll n^{\frac{5}{4}}.$$

Combining the products from (10.13) and (10.29), we therefore write

$$g_2(q) := \frac{\left(-q^3;q^3\right)_\infty}{(q)_\infty\left(-q;q\right)_\infty^2} = \frac{(q)_\infty\left(q^6;q^6\right)_\infty}{(q^2;q^2)_\infty^2\left(q^3;q^3\right)_\infty}.$$

After some simplification, we obtain

$$\log\left(g_2(q)\right) = \sum_{m\geq 1}\left(-\frac{q^m}{m\left(1+q^m\right)} + \frac{q^{3m}}{m\left(1-q^{6m}\right)}\right).$$

We therefore have the bound

$$\begin{aligned}\left|\log\left(g_2(q)\right)\right| &\leq \sum_{m\geq 1}\left(\frac{|q|^m}{m\left(1-|q|^m\right)} + \frac{|q|^{3m}}{m\left(1-|q|^{6m}\right)}\right) - |q|\left(\frac{1}{1-|q|} - \frac{1}{|1+q|}\right)\\ &= \log\left(\frac{\left(|q|^6;|q|^6\right)_\infty}{\left(|q|;|q|\right)_\infty\left(|q|^3;|q|^3\right)_\infty}\right) - |q|\left(\frac{1}{1-|q|} - \frac{1}{|1+q|}\right).\end{aligned}$$

Once again Theorem 3.12 implies that

$$\frac{\left(|q|^6;|q|^6\right)_\infty}{\left(|q|;|q|\right)_\infty\left(|q|^3;|q|^3\right)_\infty} = \sqrt{\frac{y}{2}}e^{\frac{7\pi}{36y}}\left(1+O(y)\right).$$

Now we use the fact that $|x| > My$, which implies that

$$\cos(\pi x) = \cos\left(\pi|x|\right) = -\cos\left(\pi - x\right) > -\cos(\pi My),$$

from which we conclude

$$|1+q|^2 = 1 + 2e^{-\pi y}\cos(\pi x) + e^{-2\pi y} > 1 - 2e^{-\pi y}\cos(\pi My) + e^{-2\pi y}.$$

The remainder of the proof proceeds analogously to the arguments following (10.25). □

Again, the choice $M = 6$ is sufficient to give an error term in the circle method.

Corollary 10.10. *If $My < |x| \leq 1$ and $M \geq 6$, then for some $\varepsilon > 0$ we have*

$$f_2(q) = O\left(e^{\frac{1}{y}\left(\frac{\pi}{9}-\varepsilon\right)}\right).$$

10.3.4. Wright's Circle Method and the asymptotic formula for $\overline{t}_+(n) - \overline{t}_-(n)$

Proof of (10.3)**:** We proceed as in Section 10.3.2 and in Chapter 9, with the difference that the parametrisation is now around $q = -1$. By Cauchy's Theorem, we again have that for all $n \geq 1$,

$$\overline{t}_+(n) - \overline{t}_-(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f_2(q)}{q^{n+1}} dq = \frac{(-1)^{n+1}}{2} \int_{-1}^{1} f_2\left(-e^{-\pi y + \pi i x}\right) e^{\pi n y - \pi i n x} dx,$$

where the contour is the counter-clockwise traversal of the circle $\mathcal{C} := \{|q| = e^{-\pi y}\}$, and $y = \frac{1}{3\sqrt{n}}$ as above. We now split the integral as

$$I' := \frac{(-1)^{n+1}}{2} \int_{|x| \leq My} f_2\left(-e^{-\pi y + \pi i x}\right) e^{\pi n y - \pi i n x} dx,$$

and

$$I'' := \frac{(-1)^{n+1}}{2} \int_{My < |x| \leq 1} f_2\left(-e^{-\pi y + \pi i x}\right) e^{\pi n y - \pi i n x} dx.$$

We first evaluate I'. Using the change of variable $v = 1 - \frac{ix}{y}$ and Theorem 10.8, we obtain

$$I' = \frac{(-1)^n y}{6i} \int_{1-Mi}^{1+Mi} e^{\frac{\pi\sqrt{n}}{3}\left(v + \frac{1}{v}\right)} dv + O\left(n^{-1} e^{\frac{2\pi\sqrt{n}}{3}}\right).$$

Now by Lemma 10.7, we obtain

$$\begin{aligned} I'' &= \frac{(-1)^n \pi}{9\sqrt{n}} \left(I_{-1}\left(\frac{2\pi\sqrt{n}}{3}\right) + O\left(e^{\frac{\pi\sqrt{n}}{3}}\right)\right) + O\left(n^{-1} e^{\frac{2\pi\sqrt{n}}{3}}\right) \\ &= \frac{\sqrt{3}}{18 n^{\frac{3}{4}}} e^{\frac{2\pi\sqrt{n}}{3}} + O\left(n^{-1} e^{\frac{2\pi\sqrt{n}}{3}}\right). \end{aligned}$$

We now turn to I'' and show that it is negligible compared to I'. Applying Corollary 10.10 and taking the absolute value of the rest of the integrand, we directly obtain that, for some $\varepsilon > 0$,

$$I'' \ll e^{\frac{2\pi\sqrt{n}}{3}(1-\varepsilon)}.$$

Thus I'' is exponentially smaller than I', and this completes the proof of (10.3). $\square$

11. The two-variable circle method: general principle

Contents

11.1. Jacobi forms and mock Jacobi forms

In this chapter we present a new two-variable generalisation of Wright's circle method. While Wright's circle method applies to modular forms and mock modular forms, our method applies to Jacobi forms and mock Jacobi forms.

11.1.1. Definitions

Let us first define Jacobi forms. For details, see [EZ85].

Intuitively, a Jacobi form is a holomorphic function $\phi(z,\tau)$ of two variables $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ which transforms like an elliptic function with respect to the first and like a modular form with respect to the second. More precisely, the definition is the following.

Definition. Let k be an integer and m a positive integer. A *Jacobi form* of weight k and index $\frac{m}{2}$ on the full modular group is a holomorphic function $\phi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ satisfying, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and all $\lambda, \mu \in \mathbb{Z}$,

1.
$$\phi\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}} \phi(z,\tau),$$

2.
$$\phi\left(z+\lambda\tau+\mu,\tau\right) = e^{-2\pi im(\lambda^2\tau+2\lambda z)}\phi(z,\tau).$$

These equations imply the periodicities $\phi(z,\tau+1)=\phi(z,\tau)$ and $\phi(z+1,\tau)=\phi(z,\tau)$.

Jacobi forms are a generalisation of modular forms in the sense that if we restrict the elliptic variable z of a Jacobi form $\phi(z,\tau)$ at torsion points $\mathbb{Q}\tau+\mathbb{Q}$, then we obtain a modular form with respect to the variable τ multiplied by a rational power of $q:=e^{2\pi i\tau}$.

Similarly, a *mock Jacobi form* is a holomorphic function $\phi:\mathbb{C}\times\mathbb{H}\to\mathbb{C}$ satisfying the same elliptic properties as Jacobi forms, and such that when we specialise the elliptic variable at torsion points, we obtain a mock modular form with respect to the other variable τ multiplied by a rational power of $q:=e^{2\pi i\tau}$. Details can be found in [Kan09] or [Zag09].

11.1.2. Examples

We now study classical examples of Jacobi and mock Jacobi forms, which will be used in our applications of the two-variable circle method of Chapters 12 and 13.

The most famous example of a Jacobi form is Jacobi's theta function, defined for $z\in\mathbb{C}$ and $\tau\in\mathbb{H}$ as

$$\theta\left(z;\tau\right):=i\zeta^{\frac{1}{2}}q^{\frac{1}{8}}\prod_{n=1}^{\infty}\left(1-q^{n}\right)\left(1-\zeta q^{n}\right)\left(1-\zeta^{-1}q^{n-1}\right), \tag{11.1}$$

where $q=e^{2\pi i\tau}$ and $\zeta=e^{2\pi iz}$. We keep these notations throughout this section.

As shown by Zwegers [Zwe02], Jacobi's theta function has the following transformation property

Lemma 11.1. *For $z\in\mathbb{C}$ and $\tau\in\mathbb{H}$, we have*

$$\theta\left(\frac{z}{\tau};-\frac{1}{\tau}\right)=-i\sqrt{-i\tau}e^{\frac{\pi iz^2}{\tau}}\theta(z;\tau).$$

We now give an example of mock Jacobi form: the Appell-Lerch sums. Following Chapter 1 of Zwegers' thesis [Zwe02], we define the following.

Definition. For $z\in\mathbb{C}$ and $\tau\in\mathbb{H}$, we define the *Mordell integral* as

$$h(z)=h(z;\tau)=\int_{-\infty}^{\infty}\frac{e^{\pi i\tau w^2-2\pi zw}}{\cosh(\pi w)}dw.$$

Definition. For $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C} \setminus (\mathbb{Z} \oplus \mathbb{Z}\tau)$, we call the expression

$$A_1(u, v; \tau) = e^{\pi i u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n^2+n}{2}} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n} \tag{11.2}$$

an *Appell-Lerch sum.*

We also define

$$\mu(u, v; \tau) := \frac{A_1(u, v; \tau)}{\theta(v; \tau)},$$

the *normalized Appell-Lerch sum.*

These functions also have interesting transformation properties.

Lemma 11.2 (Proposition 1.2 in [Zwe02]). *For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, the Mordell integral satisfies*

$$h(z) + e^{-2\pi i z - \pi i \tau} h(z + \tau) = 2\zeta^{-\frac{1}{2}} q^{-\frac{1}{8}},$$

and

$$h\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = \sqrt{-i\tau}\, e^{-\frac{\pi i z^2}{\tau}} h(z; \tau).$$

Lemma 11.3 (cf. Proposition 1.4 and 1.5 in [Zwe02]). *For $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C} \setminus (\mathbb{Z} \oplus \mathbb{Z}\tau)$, the Appell-Lerch sum satisfies*

$$\mu(-u, -v) = \mu(u, v),$$

and

$$\frac{1}{\sqrt{-i\tau}} e^{\frac{\pi i (u-v)^2}{\tau}} \mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) + \mu(u, v; \tau) = \frac{1}{2i} h(u - v; \tau),$$

or equivalently

$$-\frac{1}{\tau} e^{\frac{\pi i (u^2 - 2uv)}{\tau}} A_1\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) + A_1(u, v; \tau) = \frac{1}{2i} h(u - v; \tau)\theta(v; \tau).$$

11.2. The idea behind the method

Now that we have defined Jacobi forms and mock Jacobi forms, let us explain a method to compute the bivariate asymptotics of the coefficients of their Fourier expansion. Throughout the section we take $\tau \in \mathbb{H}$ and $w \in \mathbb{C}$, and we write $\zeta = e^{2\pi i w}$ and $q = e^{2\pi i \tau}$.

Let $(a(m,n))_{m,n}$ be a sequence of integers such that their generating function

$$J(w;\tau) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{N}} a(m,n)\zeta^m q^n,$$

is a Jacobi form or a mock Jacobi form and the series converges for $|q| < 1$ and $|\zeta| \leq 1$.

Our goal is to compute an asymptotic formula for $a(m,n)$ as m and n go to infinity. However it is not a precise definition to say that two variables go to infinity at the same time, as $m = \log n$ or $m = n$ for example both go to infinity with n but will likely not give the same asymptotics. Here we adopt the point of view of determining a range depending on n inside which m can take any value, so that the method works and a single asymptotic formula depending on n and m holds for all m as n tends to infinity. So at first we set m to be between $f(n) \leq m \leq g(n)$, where $f(n)$ and $g(n)$ are two functions of n, do all our computations with these indeterminate functions, and choose only at the end a value for $f(n)$ and $g(n)$ so that the method works.

As in the classical circle method and Wright's circle method, we want to write $a(m,n)$ as an integral of some function on some circle. However here we have two variables, so we need to introduce an auxiliary function, also defined as an integral on a circle.

We have

$$J(w;\tau) = \sum_{m\in\mathbb{Z}} \left(\sum_{n\in\mathbb{N}} a(m,n) q^n \right) \zeta^m,$$

thus by Cauchy's theorem, we have

$$J_m(\tau) := \sum_{n\in\mathbb{N}} a(m,n) q^n = \int_{\frac{-1}{2}}^{\frac{1}{2}} J(w;\tau) e^{-2\pi i m w} dw. \tag{11.3}$$

Now we can proceed as in the classical circle method by seeing $a(m,n)$ as the coefficient of q^n in the Fourier expansion of $J_m(\tau)$. So by Cauchy's theorem, we have

$$a(m,n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{J_m(\tau)}{q^{n+1}} dq,$$

where $\mathcal{C} = \left\{ q \in \mathbb{C} : |q| = e^{-\beta(n)} \right\}$ is a circle centred at the origin with radius less than 1. For now we do not choose a precise value for $\beta(n)$, it will be determined by the asymptotic behaviour of $J_m(\tau)$ close to the dominant pole.

As in Wright's circle method, we divide the circle $\mathcal{C}$ into a small major arc $\mathcal{C}_1$ centred around the dominant cusp and going from phase $-\beta(n)\alpha(m)$ to phase

$\beta(n)\alpha(m)$, and a minor arc $\mathcal{C}_2$, the complement of $\mathcal{C}_1$ in $\mathcal{C}$. Again, $\alpha(m)$ is just an indeterminate function of m for now and will be determined at the end.

If the dominant cusp of $J_m(\tau)$ is the complex root of unity q_0, let us write

$$\mathcal{C} = \left\{ q_0 e^{-\beta(n)(1+i\alpha(m)x)} : \frac{-\pi}{\beta(n)\alpha(m)} \leq x \leq \frac{\pi}{\beta(n)\alpha(m)} \right\},$$

and choose

$$\mathcal{C}_1 = \left\{ q_0 e^{-\beta(n)(1+i\alpha(m)x)}, -1 \leq x \leq 1 \right\},$$

and

$$\mathcal{C}_2 = \left\{ q_0 e^{-\beta(n)(1+i\alpha(m)x)} : 1 < |x| \leq \frac{\pi}{\beta(n)\alpha(m)} \right\}.$$

Then we can write

$$a(m,n) = I' + I'',$$

where

$$I' = \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{J_m(\tau)}{q^{n+1}} dq,$$

$$I'' = \frac{1}{2\pi i} \int_{\mathcal{C}_2} \frac{J_m(\tau)}{q^{n+1}} dq.$$

Now as in Wright's circle method, our goal is to evaluate I' and I'' and show that I'' is negligible compared to I'. To do so, we study the asymptotic behaviour of $J_m(\tau)$ close to its dominant cusp and far from it.

To obtain the asymptotic behaviour of $J_m(\tau)$ close to its dominant cusp (for $|x| \leq 1$), we study the asymptotic behaviour of $J(w,\tau)$ with w fixed and τ close to the dominant pole using the transformation properties of $J(w,\tau)$ as a Jacobi form or a mock Jacobi form. Then we integrate the asymptotic expansion of $J(w,\tau)$ with respect to w (11.3) and deduce an asymptotic formula for $J_m(\tau)$ depending on $\alpha(m)$ and $\beta(n)$, and obtain some conditions on $f(n)$ and $g(n)$ such that our computations are valid. The major difference between Jacobi forms and mock Jacobi forms here is that in the case of mock Jacobi forms, new terms appear in the transformation formulas and their integrals are often non-negligible, which usually leads to more computations than in the case of Jacobi forms.

We also determine an uniform bound for $J_m(\tau)$ far from its dominant cusp (for $|x| > 1$), using Taylor expansions for example.

In the end, as in Wright's circle method, we choose the value of $\beta(n)$ in order to obtain Bessel functions in the integral I'. Finally, we give asymptotic expansions for I' and I'', and we choose the best possible expressions for $\alpha(m)$, $f(n)$, and $g(n)$, so that all error terms are indeed negligible and that all conditions which appeared during the computation of asymptotic formulas for $J_m(\tau)$ are satisfied. This gives an asymptotic formula for $a(m,n)$ which is valid on the range $f(n) \leq m \leq g(n)$, and whose error term depends on $\alpha(m)$.

12. Application of the two-variable circle method to Jacobi forms : Asymptotics for the crank and Dyson's conjecture

Contents

12.1. Introduction

In this chapter, we apply the two-variable circle method to compute an asymptotic formula for $M(m, n)$, the number of partitions of n with crank m, whose generating function is a Jacobi form. This was done in the paper [BDar], written with Kathrin Bringmann.

Let us first recall the definition of the crank, conjectured by Dyson and discovered by Andrews and Garvan [AG88, Gar88] to explain Ramanujan's congruences.

Definition. If for a partition λ, $o(\lambda)$ denotes the number of ones in λ, and $\mu(\lambda)$ is the number of parts strictly larger than $o(\lambda)$, then the *crank* of λ is defined by

$$\text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

Let $M(m,n)$ denote the number of partitions of n with crank m.

For example the partition $5+3+3+2$ has crank 5 because it does not have 1 as a part, and $5+2+1+1$ has crank -1.

The two parameter generating function for the crank [AG88] is a Jacobi form.

Proposition 12.1. *The generating function for $M(m,n)$ is the following*

$$C(\zeta;q) := \sum_{\substack{m\in\mathbb{Z} \\ n\in\mathbb{N}}} M(m,n)\zeta^m q^n = \frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty}. \tag{12.1}$$

Remark. *The generating function above does actually not give the correct value for $M(m,0)$ and $M(m,1)$, but it does not matter here as we are only interested in the asymptotics of $M(m,n)$ when n tends to infinity.*

Therefore we can use the two-variable circle method to prove Dyson's conjecture, showing the following.

Theorem 12.2. *If $|m| \le \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$, we have as $n\to\infty$*

$$M(m,n) = \frac{\beta}{4}\text{sech}^2\left(\frac{\beta m}{2}\right)p(n)\left(1+O\left(\beta^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right), \tag{12.2}$$

with $\beta := \frac{\pi}{\sqrt{6n}}$.

Remarks.

1. For fixed m one can directly obtain asymptotic formulas since the generating function is the convolution of a modular form and a partial theta function [BM13]. The source of the difficulty of this problem is that Dyson's conjecture is a bivariate asymptotic.
2. In fact we could replace the error by $O(\beta^{\frac{1}{2}}m\alpha^2(m))$ for any $\alpha(m)$ such that $\frac{\log n}{n^{\frac{1}{4}}} = o(\alpha(m))$ for all $|m| \le \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$ and $\beta m\alpha(m) \to 0$ as $n\to\infty$. Here we chose $\alpha(m) = |m|^{-\frac{1}{3}}$ to avoid complicated expressions in the proof.

Theorem 12.2 is actually very general, as shown by the following corollary.

Corollary 12.3. *Almost all partitions satisfy Dyson's conjecture. To be more precise*

$$\sharp\left\{\lambda\vdash n : |\text{crank}(\lambda)| \le \frac{\sqrt{n}}{\pi\sqrt{6}}\log n\right\} \sim p(n). \tag{12.3}$$

We actually prove a more general theorem than Theorem 12.2 concerning the coefficients $M_k(m,n)$ defined for $k \in \mathbb{N}$ by

$$\mathcal{C}_k(\zeta;q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_k(m,n)\,\zeta^m q^n := \frac{(q)_\infty^{2-k}}{(\zeta q)_\infty (\zeta^{-1}q)_\infty}.$$

Denoting by $p_k(n)$ the number of partitions of n allowing k colors, we have

Theorem 12.4. *For k fixed and $|m| \leq \frac{1}{6\beta_k}\log n$, we have as $n \to \infty$*

$$M_k(m,n) = \frac{\beta_k}{4}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right) p_k(n)\left(1 + O\left(\beta_k^{\frac{1}{2}}|m|^{\frac{1}{3}}\right)\right),$$

with $\beta_k := \pi\sqrt{\frac{k}{6n}}$.

Remarks. 1. We note that $M(m,n) = M_1(m,n)$, so Dyson's conjecture is a particular case of Theorem 12.4.

2. Again we could replace the error by $O(\beta_k^{\frac{1}{2}} m\alpha_k^2(m))$ for any $\alpha_k(m)$ such that $\frac{\log n}{(kn)^{\frac{1}{4}}} = o(\alpha_k(m))$ for all $|m| \leq \frac{1}{6\beta_k}\log n$ and $\beta_k m\alpha_k(m) \to 0$ as $n \to \infty$.
3. The special case $k=2$ yields the birank of partitions [HL04].
4. For $k \geq 3$, the functions $C_k(\zeta;q)$ are generating functions of Betti numbers of moduli spaces of Hilbert schemes on $(k-3)-$point blow-ups of the projective plane [G̈90] (see also [BM13] and references therein). Theorem 12.4 immediately gives the limiting profile of the Betti numbers for large second Chern class of the sheaves. Recently, Hausel and Rodriguez-Villegas [HRV] also determined profiles of Betti numbers for other moduli spaces.

This chapter is organized as follows. In Section 12.2, we write $\mathcal{C}_k$ as a Jacobi form and collect properties of Euler polynomials. In Section 12.3, we determine the asymptotic behaviour of $\mathcal{C}_k$. In Section 12.4, we use the two-variable circle method to complete the proof of Theorem 12.4. In Section 12.5, we illustrate Theorem 12.2 numerically.

12.2. Preliminaries

12.2.1. Modularity of the generating functions

As usual, we employ the modularity of the generating functions $\mathcal{C}_k$. Throughout the chapter we write $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi iw}$ with $\tau \in \mathbb{H}, w \in \mathbb{C}$. We can express $\mathcal{C}_k$ as as product of a modular form and a Jacobi form,

$$\mathcal{C}_k(\zeta; q) = \frac{i\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) q^{\frac{k}{24}} \eta^{3-k}(\tau)}{\theta(w;\tau)}, \tag{12.4}$$

where η is Dedekind's eta function and θ the Jacobi theta function. We recall that their transformation formulas are given in Equation (9.1) and Lemma 11.1.

12.2.2. Euler polynomials

We will also need Euler polynomials in our computations from the next section. Let us recall a few basic facts about them. They can be defined by their generating function.

Definition. The *Euler polynomials* E_r are defined as follows

$$\frac{2e^{xt}}{e^t + 1} =: \sum_{r=0}^{\infty} E_r(x) \frac{t^r}{r!}. \tag{12.5}$$

The following lemma relates the hyperbolic secant and Euler polynomials.

Lemma 12.5. *We have*

$$-\frac{1}{2}\operatorname{sech}^2\left(\frac{t}{2}\right) = \sum_{r=0}^{\infty} E_{2r+1}(0) \frac{t^{2r}}{(2r)!}.$$

Proof: We have

$$\sum_{r\geq 0} E_{2r+1}(0) \frac{t^{2r}}{(2r)!} = \frac{d}{dt}\left(\sum_{r\geq 0} E_{2r+1}(0) \frac{t^{2r+1}}{(2r+1)!}\right).$$

By (12.5),

$$\frac{2}{e^t + 1} = \sum_{r\geq 0} E_r(0) \frac{t^r}{r!}, \tag{12.6}$$

$$\frac{2}{e^{-t} + 1} = \sum_{r\geq 0} E_r(0) \frac{(-t)^r}{r!}. \tag{12.7}$$

Thus subtracting (12.7) to (12.6), we get

$$\frac{2}{e^t + 1} - \frac{2}{e^{-t} + 1} = 2\sum_{r\geq 0} E_{2r+1}(0) \frac{t^{2r+1}}{(2r+1)!}.$$

Therefore

$$\sum_{r\geq 0} E_{2r+1}(0)\frac{t^{2r}}{(2r)!} = \frac{d}{dt}\left(\frac{1}{e^t+1} - \frac{1}{e^{-t}+1}\right)$$
$$= \frac{-e^t}{(e^t+1)^2} - \frac{e^{-t}}{(e^{-t}+1)^2}$$
$$= \frac{-2}{e^t+2+e^{-t}}$$
$$= \frac{-1}{2}\frac{4}{\left(e^{\frac{t}{2}}+e^{\frac{-t}{2}}\right)^2}$$
$$= \frac{-1}{2}\operatorname{sech}^2\left(\frac{t}{2}\right).$$

□

We also require an integral representation of Euler polynomials. To be more precise, setting for $j \in \mathbb{N}$,

$$\mathcal{E}_j := \int_0^\infty \frac{w^{2j+1}}{\sinh(\pi w)}dw, \tag{12.8}$$

we obtain the following lemma.

Lemma 12.6. *We have*

$$\mathcal{E}_j = \frac{(-1)^{j+1}E_{2j+1}(0)}{2}.$$

Proof: All poles of the integrand lie at $i\mathbb{Z}\setminus\{0\}$. Thus by the Residue Theorem

$$\mathcal{E}_j = \frac{1}{2}\int_{\mathbb{R}+\frac{i}{2}} \frac{w^{2j+1}}{\sinh(\pi w)}dw.$$

Let

$$f(T,w) := \frac{e^{2\pi i T w}}{\sinh(\pi w)}.$$

We want to show that

$$\int_{\mathbb{R}+\frac{i}{2}} f(T,w)dw = \frac{-2i}{e^{2\pi T}+1}. \tag{12.9}$$

To do this, we use the Residue Theorem again and shift the path of integration through $\mathbb{R} + \frac{3i}{2}$. Then

$$\int_{\mathbb{R}+\frac{i}{2}} f(T,w)dw - \int_{\mathbb{R}+\frac{3i}{2}} f(T,w)dw = 2\pi i \mathrm{Res}_{w=i} f(T,w) \\ = -2ie^{-2\pi T}.$$

Moreover

$$f(T, w+i) = -e^{-2\pi T} f(T,w).$$

Thus

$$\int_{\mathbb{R}+\frac{i}{2}} f(T,w)dw - \int_{\mathbb{R}+\frac{3i}{2}} f(T,w)dw = \left(1 + e^{-2\pi T}\right) \int_{\mathbb{R}+\frac{i}{2}} f(T,w)dw \\ = -2ie^{-2\pi T},$$

which gives (12.9). Inserting (12.5) in (12.9), we get

$$\sum_{r\geq 0} \frac{(2\pi i T)^r}{r!} \int_{\mathbb{R}+\frac{i}{2}} \frac{w^r}{\sinh(\pi w)} dw = -i \sum_{r\geq 0} E_r(t) \frac{(2\pi T)^r}{r!}.$$

Isolating the term in w^{2j+1} on each side we get

$$\frac{(2\pi i T)^{2j+1}}{(2j+1)!} \int_{\mathbb{R}+\frac{i}{2}} \frac{w^{2j+1}}{\sinh(\pi w)} dw = -iE_{2j+1}(0) \frac{(2\pi T)^{2j+1}}{(2j+1)!},$$

which after simplification completes the proof. □

12.3. Asymptotic behaviour of the function $\mathcal{C}_k$.

We are now ready to apply the two-variable circle method. Let us first study the asymptotic behaviour of $\mathcal{C}_k$ close to the dominant pole and far from it. Since for all $m, n \geq 0$, $M_k(-m,n) = M_k(m,n)$, we assume from now on that $m \geq 0$.

We define

$$\mathcal{C}_{m,k}(q) := \sum_{n=0}^{\infty} M_k(m,n)\, q^n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{C}_k\left(e^{2\pi i w}; q\right) e^{-2\pi i m w} dw \\ = 2 \frac{q^{\frac{k}{24}}}{\eta^k(\tau)} \int_0^{\frac{1}{2}} g\left(w;\tau\right) \cos(2\pi m w) dw,$$

where

$$g(w;\tau) := \frac{i\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right)\eta^3(\tau)}{\theta(w;\tau)}.$$

Here we used the fact that $g(-w;\tau) = g(w;\tau)$.

It turns out that the dominant pole lies at $q = 1$. Throughout the rest of the proof, let $\tau = \frac{iz}{2\pi}$, $z = \beta_k(1 + ixm^{-\frac{1}{3}})$, with $x \in \mathbb{R}$ satisfying $|x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta_k}$.

12.3.1. Bounds near the dominant pole

In this section we consider the range $|x| \leq 1$. We start by determining the asymptotic main term of g. Lemma 11.1 and the definition of θ and η immediately imply the following lemma.

Lemma 12.7. *For $0 \leq w \leq 1$ and $|x| \leq 1$, we have as $n \to \infty$*

$$g\left(w;\frac{iz}{2\pi}\right) = \frac{2\pi\sin(\pi w)}{z\sinh\left(\frac{2\pi^2 w}{z}\right)} e^{\frac{2\pi^2w^2}{z}}\left(1 + O\left(e^{-4\pi^2(1-w)\mathrm{Re}\left(\frac{1}{z}\right)}\right)\right).$$

Proof: By the transformation formulas from (9.1) and Lemma 11.1, we have

$$\begin{aligned}
g\left(w;\frac{iz}{2\pi}\right) &= \frac{i\left(\zeta^{\frac{1}{2}} - \zeta^{\frac{-1}{2}}\right)\eta^3\left(\frac{iz}{2\pi}\right)}{\theta_1\left(w;\frac{iz}{2\pi}\right)} \\
&= \frac{i\left(\zeta^{\frac{1}{2}} - \zeta^{\frac{-1}{2}}\right)\left(\frac{2\pi}{z}\right)^{\frac{3}{2}}\eta^3\left(\frac{2\pi i}{z}\right)}{i\left(\frac{2\pi}{z}\right)^{\frac{1}{2}} e^{\frac{-2\pi^2w^2}{z}}\theta_1\left(\frac{2\pi w}{iz};\frac{2\pi i}{z}\right)} \\
&= \frac{2\pi\left(\zeta^{\frac{1}{2}} - \zeta^{\frac{-1}{2}}\right)e^{\frac{2\pi^2w^2}{z}}\eta^3\left(\frac{2\pi i}{z}\right)}{z\theta_1\left(\frac{2\pi w}{iz};\frac{2\pi i}{z}\right)} \\
&= \frac{2\pi\left(\zeta^{\frac{1}{2}} - \zeta^{\frac{-1}{2}}\right)e^{\frac{2\pi^2w^2}{z}}}{iz\left(e^{\frac{2\pi^2w}{z}} - e^{\frac{-2\pi^2w}{z}}\right)}\prod_{n\geq 1}\frac{\left(1 - e^{\frac{-4\pi^2n}{z}}\right)^2}{\left(1 - e^{\frac{4\pi^2w}{z} - \frac{4\pi^2n}{z}}\right)\left(1 - e^{\frac{-4\pi^2w}{z} - \frac{4\pi^2n}{z}}\right)} \\
&= \frac{2\pi\sin(\pi w)e^{\frac{2\pi^2w^2}{z}}}{z\sinh\left(\frac{2\pi^2w}{z}\right)}\left(1 + O\left(e^{-4\pi^2\mathrm{Re}\left(\frac{1}{z}\right)(1-w)}\right)\right).
\end{aligned}$$

□

In view of Lemma 12.7 it is therefore natural to define

$$\mathcal{G}_{m,1}(z) := \frac{4\pi}{z}\int_0^{\frac{1}{2}} \frac{\sin(\pi w)}{\sinh\left(\frac{2\pi^2 w}{z}\right)} e^{\frac{2\pi^2 w^2}{z}} \cos(2\pi m w) dw,$$

$$\mathcal{G}_{m,2}(z) := 2\int_0^{\frac{1}{2}} \left(g\left(w; \frac{iz}{2\pi}\right) - \frac{2\pi \sin(\pi w)}{z \sinh\left(\frac{2\pi^2 w}{z}\right)} e^{\frac{2\pi^2 w^2}{z}} \right) \cos(2\pi m w) dw.$$

Thus

$$\mathcal{C}_{m,k}\left(q\right) = \frac{q^{\frac{k}{24}}}{\eta^k(\tau)} \left(\mathcal{G}_{m,1}(z) + \mathcal{G}_{m,2}(z)\right). \tag{12.10}$$

Recall that we want to find the asymptotic behaviour of $C_{k,m}(\tau)$ when τ is close to 0. Therefore we study the asymptotic behaviour of $G_{m,1}(z)$ and $G_{m,2}(z)$ when z is close to 0.

Before studying $G_{m,1}(z)$, we need one last lemma.

Lemma 12.8. *Assume that $|x| \leq 1$. For $m \leq \sqrt{n}\log n \frac{1}{\pi\sqrt{6k}}$, when $n \to \infty$, we have*

$$\cosh(mz) = \cosh(\beta_k m)\left(1 + O\left(\beta_k m^{\frac{2}{3}}\right)\right), \tag{12.11}$$

$$\operatorname{sech}^2\left(\frac{mz}{2}\right) = \operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\left(1 + O\left(\beta_k m^{\frac{2}{3}}\right)\right). \tag{12.12}$$

Proof: Let us start by proving (12.11). We have

$$\begin{aligned}
\cosh(mz) &= \cosh\left(\beta_k m + i\beta_k m^{\frac{2}{3}} x\right)\\
&= \cosh(\beta_k m)\cos\left(\beta_k m^{\frac{2}{3}} x\right) + i\sinh(\beta_k m)\sin\left(\beta_k m^{\frac{2}{3}} x\right)\\
&= \cosh(\beta_k m)\left(1 + O\left(\beta_k m^{\frac{2}{3}} x\right)\right) + \sinh(\beta_k m) O\left(\beta_k m^{\frac{2}{3}} x\right)\\
&= \cosh(\beta_k m)\left(1 + O\left(\beta_k m^{\frac{2}{3}}\right)\right),
\end{aligned}$$

where the third line follows from the fact that $\beta_k m^{\frac{2}{3}} \to 0$ as $n \to \infty$.

Let us now turn to (12.12). In the same way as before, we can prove that

$$\operatorname{sech}\left(\frac{mz}{2}\right) = \frac{1}{\cosh\left(\frac{mz}{2}\right)} = \frac{1}{\cosh\left(\frac{\beta_k m}{2}\right)\left(1 + O\left(\beta_k m^{\frac{2}{3}}\right)\right)}.$$

Therefore

$$\operatorname{sech}\left(\frac{mz}{2}\right) = \operatorname{sech}\left(\frac{\beta_k m}{2}\right)\left(1 + O\left(\beta_k m^{\frac{2}{3}}\right)\right),$$

and
$$\operatorname{sech}^2\left(\frac{mz}{2}\right) = \operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\left(1 + O\left(\beta_k m^{\frac{2}{3}}\right)\right).$$

□

We can now give the asymptotic behaviour of $\mathcal{G}_{m,1}(z)$ close to $z = 0$.

Lemma 12.9. *Assume that $|x| \leq 1$ and $m \leq \frac{1}{6\beta_k}\log n$. Then we have as $n \to \infty$,*

$$\mathcal{G}_{m,1}(z) = \frac{z}{4}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right) + O\left(\beta_k^2 m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\right).$$

Proof: Inserting the Taylor expansion of sin, exp, and cos, we get

$$\begin{aligned}\sin(\pi w)&e^{\frac{2\pi^2 w^2}{z}}\cos(2\pi m w)\\ &= \sum_{j,\nu,r\geq 0}\frac{(-1)^{j+\nu}}{(2j+1)!(2\nu)!r!}\pi^{2j+1}(2\pi m)^{2\nu}\left(\frac{2\pi^2}{z}\right)^r w^{2j+2\nu+2r+1}.\end{aligned}$$

This yields that

$$\mathcal{G}_{m,1}(z) = \frac{4\pi}{z}\sum_{j,\nu,r\geq 0}\frac{(-1)^{j+\nu}}{(2j+1)!(2\nu)!r!}\pi^{2j+1}(2\pi m)^{2\nu}\left(\frac{2\pi^2}{z}\right)^r \mathcal{I}_{j+\nu+r},$$

where for $\ell \in \mathbb{N}$ we define

$$\mathcal{I}_\ell := \int_0^{\frac{1}{2}} \frac{w^{2\ell+1}}{\sinh\left(\frac{2\pi^2 w}{z}\right)}dw.$$

We next relate $\mathcal{I}_\ell$ to $\mathcal{E}_\ell$ defined in (12.8). For this, we note that

$$\mathcal{I}_\ell = \int_0^\infty \frac{w^{2\ell+1}}{\sinh\left(\frac{2\pi^2 w}{z}\right)}dw - \mathcal{I}'_\ell \tag{12.13}$$

with

$$\begin{aligned}\mathcal{I}'_\ell := \int_{\frac{1}{2}}^\infty \frac{w^{2\ell+1}}{\sinh\left(\frac{2\pi^2 w}{z}\right)}dw &\ll \int_{\frac{1}{2}}^\infty w^{2\ell+1}e^{-2\pi^2 w \operatorname{Re}\left(\frac{1}{z}\right)}dw\\ &\ll \left(\operatorname{Re}\left(\frac{1}{z}\right)\right)^{-2\ell-2}\Gamma\left(2\ell+2; \pi^2 \operatorname{Re}\left(\frac{1}{z}\right)\right).\end{aligned}$$

Here $\Gamma(\alpha;x) := \int_x^\infty e^{-w}w^{\alpha-1}dw$ denotes the incomplete gamma function and throughout $g(x) \ll f(x)$ means that $g(x) = O\left(f(x)\right)$. Using that as $x \to \infty$

$$\Gamma\left(\ell;x\right) \sim x^{\ell-1}e^{-x}, \tag{12.14}$$

thus yields that

$$\mathcal{I}'_\ell \ll \left(\mathrm{Re}\left(\frac{1}{z}\right)\right)^{-1} e^{-\pi^2\mathrm{Re}\left(\frac{1}{z}\right)} \leq e^{-\pi^2\mathrm{Re}\left(\frac{1}{z}\right)}.$$

In the first summand in (12.13) we make the change of variables $w \to \frac{zw}{2\pi}$ and then shift the path of integration back to the real line by the Residue Theorem. Thus we obtain that

$$\int_0^\infty \frac{w^{2\ell+1}}{\sinh\left(\frac{2\pi^2 w}{z}\right)}dw = \left(\frac{z}{2\pi}\right)^{2\ell+2}\mathcal{E}_\ell = \left(\frac{z}{2\pi}\right)^{2\ell+2}\frac{(-1)^{\ell+1}E_{2\ell+1}(0)}{2},$$

where for the last equality we used Lemma 12.6. Thus

$$\begin{aligned}\mathcal{G}_{m,1}(z) &= \sum_{j,\nu,r\geq 0}\frac{(-1)^{r+1}}{2^{2j+r+1}(2j+1)!(2\nu)!r!}m^{2\nu}z^{2j+2\nu+r+1}\\ &\quad\times\left(E_{2j+2\nu+2r+1}(0) + O\left(|z|^{-2j-2\nu-2r-2}e^{-\pi^2\mathrm{Re}\frac{1}{z}}\right)\right)\\ &= \sum_{\nu=0}^\infty \frac{(mz)^{2\nu}}{(2\nu)!}\left(-\frac{z}{2}E_{2\nu+1}(0) + O\left(|z|^2\right)\right)\\ &= \frac{z}{4}\mathrm{sech}^2\left(\frac{mz}{2}\right) + O\left(|z|^2\cosh(mz)\right),\end{aligned}$$

where for the last equality we used Lemma 12.5. Now by Lemma 12.8,

$$\begin{aligned}G_{m,1}(z) =&\frac{z}{4}\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right)\left(1+O\left(\beta_k m^{\frac{2}{3}}\right)\right) + O\left(|z|^2\cosh(\beta_k m)\right)\\ =&\frac{z}{4}\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right) + O\left(m\beta_k^2\sqrt{1+\frac{1}{m^{\frac{2}{3}}}}\frac{1}{m^{\frac{1}{3}}}\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right)\right)\\ &+O\left(\beta_k^2\left(1+\frac{1}{m^{\frac{2}{3}}}\right)\cosh(\beta_k m)\right)\\ =&\frac{z}{4}\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right) + O\left(\beta_k^2 m^{\frac{2}{3}}\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right)\right) + O\left(\beta_k^2\cosh(\beta_k m)\right),\end{aligned}$$

where the last equality follows from the fact that $0 \leq m^{\frac{-1}{3}} \leq 1$.

For $m \geq 2$ such that $\beta_k m \to C$ for some constant $C \geq 0$ as $n \to \infty$, then

$$\beta_k^2 \cosh(\beta_k m) \ll \beta_k^2 m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{\beta_k m}{2}\right),$$

because $m^{\frac{2}{3}} \gg 1$ as $n \to \infty$.

We have

$$\cosh(\beta_k m) \leq e^{\beta_k m} \leq e^{\pi\sqrt{\frac{k}{6n}}\sqrt{n}\log(n)\frac{1}{\pi\sqrt{6k}}} = n^{\frac{1}{6}},$$

and

$$m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{\beta_k m}{2}\right) \geq m^{\frac{2}{3}} e^{-\beta_k m} \geq m^{\frac{2}{3}} n^{\frac{-1}{6}}.$$

Thus for m such that $\beta_k m \to \infty$, $m^{\frac{2}{3}} \gg n^{\frac{1}{3}}$ and

$$\beta_k^2 \cosh(\beta_k m) = O\left(\beta_k^2 m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\right).$$

This completes the proof. $\square$

We next turn to bounding $\mathcal{G}_{m,2}$.

Lemma 12.10. *Assume that $|x| \leq 1$. Then we have as $n \to \infty$*

$$\mathcal{G}_{m,2}(q) \ll \frac{1}{\beta_k} e^{-\frac{5\pi^2}{4\beta_k}}.$$

Proof: By Lemma 12.7,

$$\begin{aligned}
|G_{m,2}(z)| &\ll \int_0^{\frac{1}{2}} \left| \frac{2\pi \sin(\pi w) e^{\frac{2\pi^2 w^2}{z}}}{z \sinh\left(\frac{2\pi^2 w}{z}\right)} \cos(2\pi m w) \right| e^{-4\pi^2 \operatorname{Re}\left(\frac{1}{z}\right)(1-w)} dw \\
&\ll \frac{1}{|z|} \int_0^{\frac{1}{2}} \left| \frac{1 - e^{2\pi i w}}{1 - e^{\frac{-4\pi^2 w}{z}}} \right| e^{-2\pi^2 \operatorname{Re}\left(\frac{1}{z}\right) w + 2\pi^2 \operatorname{Re}\left(\frac{1}{z}\right) w^2 - 4\pi^2 \operatorname{Re}\left(\frac{1}{z}\right)(1-w)} dw \\
&\ll \frac{1}{|z|} \int_0^{\frac{1}{2}} \left| \frac{1 - e^{2\pi i w}}{1 - e^{\frac{-4\pi^2 w}{z}}} \right| e^{2\pi^2 \operatorname{Re}\left(\frac{1}{z}\right)(w^2 + w - 2)} dw.
\end{aligned}$$

When $w < |z|$,

$$\left| \frac{1 - e^{2\pi i w}}{1 - e^{\frac{-4\pi^2 w}{z}}} \right| \ll |z| \ll 1.$$

When $w \geq |z|$,

$$\left|\frac{1-e^{2\pi i w}}{1-e^{\frac{-4\pi^2 w}{z}}}\right| \ll \frac{1}{1-e^{-4\pi^2 \mathrm{Re}\left(\frac{1}{z}\right)w}}.$$

We have

$$\mathrm{Re}\left(\frac{1}{z}\right) = \frac{1}{\beta_k}\frac{1}{1+m^{\frac{-2}{3}}x^2},$$

and

$$|z| = \beta_k\sqrt{1+m^{\frac{-2}{3}}x^2}.$$

Therefore

$$\mathrm{Re}\left(\frac{1}{z}\right) = \frac{1}{|z|}\frac{1}{\sqrt{1+m^{\frac{-2}{3}}x^2}} \geq \frac{1}{\sqrt{2}|z|},$$

because $m^{\frac{-2}{3}}x^2 \leq 1$. Thus

$$\left|\frac{1-e^{2\pi i w}}{1-e^{\frac{-4\pi^2 w}{z}}}\right| \ll \frac{1}{1-e^{\frac{-4\pi^2 w}{\sqrt{2}|z|}}} \ll \frac{1}{1-e^{\frac{-4\pi^2}{\sqrt{2}}}}.$$

The maximum of w^2+w-2 on $[0,\frac{1}{2}]$ is $\frac{-5}{4}$. Thus

$$G_{m,2}(z) \ll \frac{1}{|z|}e^{-\frac{5}{2}\pi^2 \mathrm{Re}\left(\frac{1}{z}\right)} \ll \frac{1}{\beta_k}e^{-\frac{5\pi^2}{4\beta_k}}.$$

□

Thus $G_{m,2}(z)$ is exponentially smaller than $O\left(\beta_k^2 m^{\frac{2}{3}}\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right)\right)$.

Now we can finally give the following expression for $C_{k,m}(\tau)$ when $n \to \infty$.

Proposition 12.11. *Assume that $|x| \leq 1$. Then we have as $n \to \infty$*

$$\mathcal{C}_{m,k}(q) = \frac{z^{\frac{k}{2}+1}}{4(2\pi)^{\frac{k}{2}}}\,\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right)e^{\frac{k\pi^2}{6z}} + O\left(\beta_k^{\frac{k}{2}+2}m^{\frac{2}{3}}\,\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right)e^{\pi\sqrt{\frac{kn}{6}}}\right).$$

Proof: Recall from (12.10) that

$$\mathcal{C}_{m,k}(q) = \frac{q^{\frac{k}{24}}}{\eta^k(\tau)}\left(\mathcal{G}_{m,1}(z)+\mathcal{G}_{m,2}(z)\right).$$

Thus by Lemmas 12.9 and 12.10,

$$C_{k,m}(\tau) = \frac{q^{\frac{k}{24}}}{\eta^k(\tau)}\left(\frac{z}{4}\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right) + O\left(\beta_k^2 m^{\frac{2}{3}}\mathrm{sech}^2\left(\frac{\beta_k m}{2}\right)\right)\right).$$

Now by (9.1) we have

$$\begin{aligned}
\frac{q^{\frac{k}{24}}}{\eta^k(\tau)} = \frac{q^{\frac{k}{24}}(-i\tau)^{\frac{k}{2}}}{\eta^k\left(\frac{-1}{\tau}\right)} &=(-i\tau)^{\frac{k}{2}} e^{\frac{2k\pi i}{24}\left(\tau+\frac{1}{\tau}\right)}\left(1+O\left(e^{2\pi i\tau}\right)\right)\\
&=\left(\frac{z}{2\pi}\right)^{\frac{k}{2}} e^{\frac{2k\pi i}{24}\left(\frac{iz}{2\pi}+\frac{2\pi}{iz}\right)}\left(1+O\left(e^{-z}\right)\right)\\
&=\left(\frac{z}{2\pi}\right)^{\frac{k}{2}} e^{\frac{k\pi^2}{6z}}\left(1+O(|z|)\right)\left(1+O\left(e^{-z}\right)\right)\\
&=\left(\frac{z}{2\pi}\right)^{\frac{k}{2}} e^{\frac{k\pi^2}{6z}}\left(1+O(\beta_k)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
C_{k,m}(\tau) =& \left(\frac{z}{2\pi}\right)^{\frac{k}{2}} e^{\frac{k\pi^2}{6z}}\left(\frac{z}{4}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\right.\\
&\left.+O\left(\beta_k^2\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\right)+O\left(\beta_k^2 m^{\frac{2}{3}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\right)\right)\\
=&\left(\frac{z}{2\pi}\right)^{\frac{k}{2}} e^{\frac{k\pi^2}{6z}}\left(\frac{z}{4}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)+O\left(\beta_k^2 m^{\frac{2}{3}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\right)\right)\\
=&\frac{z^{\frac{k}{2}+1}}{4(2\pi)^{\frac{k}{2}}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)e^{\frac{k\pi^2}{6z}}\\
&+O\left(|z|^{\frac{k}{2}}\beta_k^2 m^{\frac{2}{3}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\left|e^{\frac{k\pi^2}{6\beta_k\left(1+im^{\frac{-1}{3}}x\right)}}\right|\right)\\
=&\frac{z^{\frac{k}{2}+1}}{4(2\pi)^{\frac{k}{2}}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)e^{\frac{k\pi^2}{6z}}+O\left(\beta_k^{\frac{k}{2}+2} m^{\frac{2}{3}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)e^{\pi\sqrt{\frac{kn}{6}}}\right).
\end{aligned}$$

□

12.3.2. Bounds away from the dominant pole

We next investigate the behaviour of $\mathcal{C}_{m,k}$ away from the dominant cusp $q = 1$. To be more precise, we consider the range $1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta_k}$. To do this, we use Lemma 9.2 giving an uniform bound for $P(q)$, which we proved in the application of Wright's circle method to $p(n)$ in Chapter 9.

Proposition 12.12. *Assume that $1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$. Then we have, as $n \to \infty$,*

$$|\mathcal{C}_{m,k}(q)| \ll n^{\frac{3-k}{4}} \exp\left(\pi\sqrt{\frac{kn}{6}} - \frac{\sqrt{6kn}}{8\pi}m^{-\frac{2}{3}}\right).$$

Proof: We have by definition

$$\mathcal{C}_{m,k}(q) = 2P^k(q) \int_0^{\frac{1}{2}} g(w;\tau)\cos(2\pi m w)dw.$$

The function $\mathcal{C}_k$ can also be represented as a Lerch sum. To be more precise, we have [AG88]

$$\sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} M(m,n)\,\zeta^m q^n = \frac{1-\zeta}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-\zeta q^n}.$$

Thus

$$g(w;\tau) = 1 + (1-\zeta)\sum_{n\geq 1}\frac{(-1)^n q^{\frac{n^2+n}{2}}}{1-\zeta q^n} + \left(1-\zeta^{-1}\right)\sum_{n\geq 1}\frac{(-1)^n q^{\frac{n^2+n}{2}}}{1-\zeta^{-1}q^n}.$$

So we may bound

$$g(w;\tau) \ll \sum_{n\geq 1}\frac{|q|^{\frac{n^2+n}{2}}}{1-|q|^n} \ll \frac{1}{1-|q|}\sum_{n\geq 1} e^{-\frac{\beta_k n^2}{2}} \ll \beta_k^{-\frac{3}{2}} \ll n^{\frac{3}{4}}.$$

Thus

$$|\mathcal{C}_{m,k}(q)| \ll \left|P^k(q)\right| n^{\frac{3}{4}}.$$

Using Lemma 9.2 with $v = \frac{\beta_k}{2\pi}$, $u = \frac{\beta_k m^{-\frac{1}{3}}x}{2\pi}$, and $M = m^{-\frac{1}{3}}$, yields for $1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta_k}$,

$$|P(q)| \ll n^{-\frac{1}{4}}\exp\left[\frac{2\pi}{\beta_k}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)\right].$$

Therefore

$$\begin{aligned}
|\mathcal{C}_{m,k}(q)| &\ll n^{\frac{3-k}{4}}\exp\left[\frac{2\pi k}{\beta_k}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)\right]\\
&\ll n^{\frac{3-k}{4}}\exp\left[\pi\sqrt{\frac{kn}{6}} - \frac{\sqrt{6kn}}{\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right]\\
&\ll n^{\frac{3-k}{4}}\exp\left(\pi\sqrt{\frac{kn}{6}} - \frac{\sqrt{6kn}}{8\pi}m^{-\frac{2}{3}}\right).
\end{aligned}$$

□

12.4. The Circle Method

In this section we write $M_k(m,n)$ as an integral on a circle and divide it into a major and a minor arc to complete the proof of Theorem 12.4 and thus the proof of Dyson's conjecture. We have

$$M_k(m,n) = \frac{1}{2\pi i} \int_C \frac{\mathcal{C}_{m,k}(q)}{q^{n+1}} dq, \tag{12.15}$$

where the contour is the counter-clockwise transversal of the circle $C := \{q \in \mathbb{C}\,;\, |q| = e^{-\beta_k}\}$. Recall that $z = \beta_k(1 + ixm^{-\frac{1}{3}})$. Changing variables we may write

$$M_k(m,n) = \frac{\beta_k}{2\pi m^{\frac{1}{3}}} \int\limits_{|x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta_k}} \mathcal{C}_{m,k}(e^{-z}) e^{nz} dx.$$

We split this integral into two pieces

$$M_k(m,n) = M + E$$

with

$$M := \frac{\beta_k}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq 1} \mathcal{C}_{m,k}\left(e^{-z}\right) e^{nz} dx,$$

$$E := \frac{\beta_k}{2\pi m^{\frac{1}{3}}} \int_{1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta_k}} \mathcal{C}_{m,k}\left(e^{-z}\right) e^{nz} dx.$$

In the following we show that M contributes to the asymptotic main term whereas E is part of the error term.

12.4.1. Approximating the main term

The goal of this section is to determine the asymptotic behaviour of M. We show the following.

Proposition 12.13. *We have*

$$M = \frac{\beta_k}{4} \operatorname{sech}^2\left(\frac{\beta_k m}{2}\right) p_k(n) \left(1 + O\left(\frac{m^{\frac{1}{3}}}{n^{\frac{1}{4}}}\right)\right).$$

A key step for proving this proposition is the investigation of

$$P_{s,k} := \frac{1}{2\pi i} \int\limits_{1-im^{-\frac{1}{3}}}^{1+im^{-\frac{1}{3}}} v^s e^{\pi\sqrt{\frac{kn}{6}}\left(v+\frac{1}{v}\right)} dv$$

for $s > 0$. These integrals may be related to Bessel functions. Denoting by I_s the usual I-Bessel function of order s, we have.

Lemma 12.14. *As $n \to \infty$*

$$P_{s,k} = I_{-s-1}\left(\pi\sqrt{\frac{2kn}{3}}\right) + O\left(\exp\left(\pi\sqrt{\frac{kn}{6}}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)\right)\right).$$

Proof: We set Γ to be the piecewise linear path consisting of the segments

$$\gamma_4 : \left(-\infty - \frac{im^{-\frac{1}{3}}}{2}, -1 - \frac{im^{-\frac{1}{3}}}{2}\right), \gamma_3 : \left(-1 - \frac{im^{-\frac{1}{3}}}{2}, -1 - im^{-\frac{1}{3}}\right),$$

$$\gamma_2 : \left(-1 - im^{-\frac{1}{3}}, 1 - im^{-\frac{1}{3}}\right), \gamma_1 : \left(1 - im^{-\frac{1}{3}}, 1 + im^{-\frac{1}{3}}\right),$$

then followed by the corresponding mirror images γ_2', γ_3' and γ_4'. Since $P_s = \int_{\gamma_1}$, we must show that the integrals on the other segments are bounded as claimed.

We have

$$\begin{aligned}\int_{\gamma_4} &= \frac{1}{2\pi i}\int_{-\infty}^{-1} e^{\pi\sqrt{\frac{kn}{6}}\left(t - \frac{im^{-\frac{1}{3}}}{2} + \frac{1}{t-\frac{im^{-\frac{1}{3}}}{2}}\right)}\left(t - \frac{im^{-\frac{1}{3}}}{2}\right)^s dt \\ &\ll \int_1^\infty e^{-\pi\sqrt{\frac{kn}{6}}t}\left|-t - \frac{im^{-\frac{1}{3}}}{2}\right|^s dt \\ &\ll n^{\frac{-1}{2}} e^{-\pi\sqrt{\frac{kn}{6}}},\end{aligned}$$

where the final inequality follows from a simple bound for the incomplete Gamma function.

We also have

$$\begin{aligned}\int_{\gamma_3} &= \frac{m^{\frac{-1}{3}}}{2\pi i}\int_{\frac{1}{2}}^{1} e^{-\pi\sqrt{\frac{kn}{6}}\left(1+im^{-\frac{1}{3}}t+\frac{1}{1+im^{-\frac{1}{3}}t}\right)}(-1-im^{-\frac{1}{3}}t)^s dt \\ &\ll \int_{\frac{1}{2}}^{1} e^{-\pi\sqrt{\frac{kn}{6}}\left(1+\frac{1}{1+m^{-\frac{2}{3}}t^2}\right)}|1+im^{-\frac{1}{3}}t|^s dt \\ &\ll e^{-\pi\sqrt{\frac{kn}{6}}}.\end{aligned}$$

And finally

$$\begin{aligned}
\int_{\gamma_2} &= \frac{1}{2\pi i}\int_{-1}^{1} e^{\pi\sqrt{\frac{kn}{6}}\left(t-im^{-\frac{1}{3}}+\frac{1}{t-im^{-\frac{1}{3}}}\right)}(t-im^{-\frac{1}{3}})^s dt\\
&\ll \int_{-1}^{1} e^{\pi\sqrt{\frac{kn}{6}}\left(t+\frac{t}{t^2+m^{-\frac{2}{3}}}\right)}|t-im^{-\frac{1}{3}}|^s dt\\
&\ll e^{\pi\sqrt{\frac{kn}{6}}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)}.
\end{aligned}$$

□

We now turn to the proof of Proposition 12.13.

Proof of Proposition 12.13: Using Proposition 12.11 and making a change of variables, we obtain by Lemma 12.14

$$\begin{aligned}
M =& \frac{\beta_k^{\frac{k}{2}+2}}{4(2\pi)^{\frac{k}{2}}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)P_{\frac{k}{2}+1,k} + O\left(\beta_k^{\frac{k}{2}+3}m^{\frac{1}{3}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)e^{\pi\sqrt{\frac{2kn}{3}}}\right)\\
=& \frac{\beta_k^{\frac{k}{2}+2}}{4(2\pi)^{\frac{k}{2}}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\left(I_{-\frac{k}{2}-2}\left(\pi\sqrt{\frac{2kn}{3}}\right)\right.\\
&\left.+O\left(\exp\left(\pi\sqrt{\frac{kn}{6}}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)\right)\right)\right)\\
&+O\left(\beta_k^{\frac{k}{2}+3}m^{\frac{1}{3}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)e^{\pi\sqrt{\frac{2kn}{3}}}\right).
\end{aligned}$$

Using the asymptotic expansion of the Bessel function (9.6) yields

$$\begin{aligned}
M =& \frac{\beta_k^{\frac{k}{2}+2}}{4(2\pi)^{\frac{k}{2}}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\left(\frac{e^{\pi\sqrt{\frac{2kn}{3}}}}{\pi\sqrt{2}\left(\frac{2kn}{3}\right)^{\frac{1}{4}}}+O\left(\frac{e^{\pi\sqrt{\frac{2kn}{3}}}}{n^{\frac{3}{4}}}\right)\right.\\
&\left.+O\left(\exp\left(\pi\sqrt{\frac{kn}{6}}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)\right)\right)\right)\\
&+O\left(\beta_k^{\frac{k}{2}+3}m^{\frac{1}{3}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)e^{\pi\sqrt{\frac{2kn}{3}}}\right).
\end{aligned}$$

It is not hard to see that the last error term is the dominant one. Thus

$$M = \frac{\beta_k^{\frac{k}{2}+2}}{4(2\pi)^{\frac{k}{2}}}\operatorname{sech}^2\left(\frac{\beta_k m}{2}\right)\frac{e^{\pi\sqrt{\frac{2kn}{3}}}}{\pi\sqrt{2}\left(\frac{2kn}{3}\right)^{\frac{1}{4}}}\left(1+O\left(m^{\frac{1}{3}}n^{-\frac{1}{4}}\right)\right).$$

Using that [HR18a, RZ38]

$$p_k(n) = 2\left(\frac{k}{3}\right)^{\frac{1+k}{4}} (8n)^{-\frac{3+k}{4}} e^{\pi\sqrt{\frac{2kn}{3}}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

now easily gives the claim. □

12.4.2. The error arc.

We finally bound E and show that it is exponentially smaller than M. The following proposition then immediately implies Theorem 12.4.

Proposition 12.15. *As $n \to \infty$*

$$E \ll n^{\frac{3-k}{4}} \exp\left(\pi\sqrt{\frac{2kn}{3}} - \frac{\sqrt{6kn}}{8\pi} m^{-\frac{2}{3}}\right).$$

Proof: Using Proposition 12.12, we may bound

$$\begin{aligned} E &\ll \frac{\beta_k}{m^{\frac{1}{3}}} \int_{1 \le x \le \frac{\pi m^{\frac{1}{3}}}{\beta_k}} n^{\frac{3-k}{4}} \exp\left(\pi\sqrt{\frac{kn}{6}} - \frac{\sqrt{6kn}}{8\pi} m^{-\frac{2}{3}}\right) e^{\beta_k n} dx \\ &\ll n^{\frac{3-k}{4}} \exp\left(\pi\sqrt{\frac{2kn}{3}} - \frac{\sqrt{6kn}}{8\pi} m^{-\frac{2}{3}}\right). \end{aligned}$$

□

This is exponentially smaller than M, and Theorem 12.4 (and Dyson's conjecture) is proved.

12.5. Numerical data

We illustrate our results in two tables.

n	$M(0,n)$	$\widetilde{M}(0,n)$	$\frac{M(0,n)}{\widetilde{M}(0,n)}$
20	41	~ 45	~ 0.912
50	8626	~ 9261	~ 0.931
500	$3.228743492 \cdot 10^{19}$	$\sim 3.298285542 \cdot 10^{19}$	~ 0.979
1000	$2.403603986 \cdot 10^{29}$	$\sim 2.439699707 \cdot 10^{29}$	~ 0.985

n	$M(1,n)$	$\widetilde{M}(1,n)$	$\frac{M(1,n)}{\widetilde{M}(1,n)}$
20	38	~ 44	~ 0.863
50	8541	~ 9185	~ 0.930
500	$3.226300403 \cdot 10^{19}$	$\sim 3.295574297 \cdot 10^{19}$	~ 0.979
1000	$2.402671309 \cdot 10^{29}$	$\sim 2.438696696 \cdot 10^{29}$	~ 0.985

where we set $\widetilde{M}(m,n) := \frac{\beta}{4}\mathrm{sech}^2(\frac{\beta m}{2})p(n)$.

13. Application of the two-variable circle method to mock Jacobi forms: Asymptotics for the rank

Contents

13.1. Introduction

In this chapter, we apply the two-variable circle method to $N(m, n)$, the number of partitions of n with rank m, whose generating function is a mock Jacobi form. This was done in the paper [DMar], written with Michael Mertens.

Let us recall that Dyson [Dys44] defined the rank of a partition as its largest part minus its number of parts.

Throughout the chapter, if not specified elsewise, we always assume $\tau \in \mathbb{H}$, $z \in \mathbb{R}$, $q := e^{2\pi i\tau}$, and $\zeta := e^{2\pi iz}$. With this notation, the generating function for the rank is defined as follows.

$$R(z;\tau) := \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} N(m,n)\zeta^m q^n = \frac{1-\zeta}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{3n^2+n}{2}}}{1-\zeta q^n}. \tag{13.1}$$

We prove that the same asymptotic formula that we proved for the crank in Chapter 12 also holds for the rank.

Theorem 13.1. *If* $|m| \leq \frac{\sqrt{n}\log n}{\pi\sqrt{6}}$, *we have as* $n \to \infty$

$$N(m,n) = \frac{\beta}{4}\operatorname{sech}^2\left(\frac{\beta m}{2}\right) p(n) \left(1 + O\left(\beta^{\frac{1}{2}} |m|^{\frac{1}{3}}\right)\right).$$

Remarks. 1. The case of fixed m is considered in recent work by Byungchan Kim, Eunmi Kim, and Jeehyeon Seo [KKS15].

2. As for Theorem 12.2, the error term could be replaced by $O(\beta^{\frac{1}{2}} m\alpha^2(m))$ for any $\alpha(m)$ such that $\frac{\log n}{n^{\frac{1}{4}}} = o\left(\alpha(m)\right)$ for all $|m| \leq \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$ and $\beta m\alpha(m) \to 0$ as $n \to \infty$. Here we chose $\alpha(m) = |m|^{-\frac{1}{3}}$ to avoid complicated expressions in the proof.

The rest of this chapter is organized as follows. In Section 13.2, we prove some preliminary estimates for the rank generating function using the transformation properties of mock Jacobi forms. In Section 13.3, we use these results to prove the estimates close to and far from the dominant pole, and in Section 13.4, we apply the two-variable circle method to establish our main result Theorem 13.1.

13.2. Transformation Formulae

In this section, we split $R(z;\tau)$ into several summands to determine its transformation behaviour under $\tau \mapsto -\frac{1}{\tau}$.

Lemma 13.2. *For all $\tau \in \mathbb{H}, z \in \mathbb{R}$, we have*

$$R(z;\tau) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \Bigg[\frac{i\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right)\eta^3(3\tau)}{\theta(3z;3\tau)} - \zeta^{-1}\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) A_1(3z,-\tau;3\tau) \\ -\zeta\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) A_1(3z,\tau;3\tau)\Bigg] \tag{13.2}$$

with A_1 as in (11.2).

This was first mentioned in Theorem 7.1 of [Zag09], but contained a slight typo there. To be precise, the factor i in front of the first summand was missing and the sign in front of the second and third was wrong.

Now we want to determine some asymptotic expressions for the three summands in (13.2). To do, so let us write $\tau = \frac{is}{2\pi}$, $s = \beta(1 + ixm^{\frac{-1}{3}})$ with $x \in \mathbb{R}$ satisfying $|x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$.

Lemma 13.3. *Assume that $|z| < \frac{1}{3}$. Then for $|x| \leq 1$, we have as $n \to \infty$*

$$i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} = \frac{-i\pi e^{\frac{6\pi^2 z^2}{s}}}{3s\sinh\left(\frac{2\pi^2 z}{s}\right)} \times \left(1 + O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1-3z)}\right) + O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1+3z)}\right)\right).$$

Proof: By the transformation formulae from Lemma 11.1,

$$\begin{aligned}
i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)} &= i\frac{\left(\frac{1}{\sqrt{-3i\tau}}\right)^3 \eta^3\left(-\frac{1}{3\tau}\right)}{\frac{i}{\sqrt{-3i\tau}}e^{-\pi i\frac{(3z)^2}{3\tau}}\theta\left(\frac{z}{\tau};-\frac{1}{3\tau}\right)} \\
&= i\frac{\eta^3\left(-\frac{1}{3\tau}\right)}{3\tau e^{-3\pi i\frac{z^2}{\tau}}\theta\left(\frac{z}{\tau};-\frac{1}{3\tau}\right)} \\
&= \frac{2\pi\eta^3\left(\frac{2\pi i}{3s}\right)e^{\frac{6\pi^2 z^2}{s}}}{3s\theta\left(\frac{2\pi z}{is};\frac{2\pi i}{3s}\right)} \\
&= \frac{2\pi e^{\frac{6\pi^2 z^2}{s}}e^{-\frac{\pi^2}{6s}}}{3ise^{\frac{2\pi^2 z}{s}}e^{-\frac{\pi^2}{6s}}}\prod_{k=1}^{\infty}\frac{\left(1-e^{-\frac{4\pi^2 k}{3s}}\right)^2}{\left(1-e^{\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)\left(1-e^{-\frac{4\pi^2 z}{s}-\frac{4\pi^2(k-1)}{3s}}\right)} \\
&= \frac{2\pi e^{\frac{6\pi^2 z^2}{s}}}{3ise^{\frac{2\pi^2 z}{s}}\left(1-e^{-\frac{4\pi^2 z}{s}}\right)}\prod_{k=1}^{\infty}\frac{\left(1-e^{-\frac{4\pi^2 k}{3s}}\right)^2}{\left(1-e^{\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)\left(1-e^{-\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)} \\
&= \frac{2\pi e^{\frac{6\pi^2 z^2}{s}}}{3is\left(e^{\frac{2\pi^2 z}{s}}-e^{-\frac{2\pi^2 z}{s}}\right)}\prod_{k=1}^{\infty}\frac{\left(1-e^{-\frac{4\pi^2 k}{3s}}\right)^2}{\left(1-e^{\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)\left(1-e^{-\frac{4\pi^2 z}{s}-\frac{4\pi^2 k}{3s}}\right)} \\
&= \frac{-i\pi e^{\frac{6\pi^2 z^2}{s}}}{3s\sinh\left(\frac{2\pi^2 z}{s}\right)} \\
&\times\left(1 + O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1-3z)}\right) + O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1+3z)}\right)\right).
\end{aligned}$$

□

Before estimating the two last summands of (13.2), we need two more lemmas about A_1 and h.

Lemma 13.4. *Let $z \in \mathbb{R}$ with $|z| < \frac{1}{3}$. Then for $|x| \leq 1$, we have as $n \to \infty$*

$$A_1\left(\frac{2\pi z}{is}, \mp\frac{1}{3}; \frac{2\pi i}{3s}\right) = \frac{-1}{2\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(e^{\frac{-2\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(2-3z)}\right) + O\left(e^{\frac{-2\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(2+3z)}\right).$$

Proof: In the proof of this lemma, we assume that ζ and q are such that $|\zeta q^n| < 1$ if $n > 0$ and $|\zeta q^n| > 1$ if $n < 0$. By applying the geometric series

$$\frac{1}{1-x} = \begin{cases} \sum\limits_{k=0}^{\infty} x^k & \text{if } |x| < 1, \\ -\sum\limits_{k=1}^{\infty} x^{-k} & \text{if } |x| > 1 \end{cases}$$

we find (writing $\rho = e^{\frac{2\pi i}{3}}$)

$$\zeta^{-\frac{1}{2}} A_1\left(z, \mp\tfrac{1}{3}; \tau\right) = \frac{1}{1-\zeta} + \sum_{n=1}^{\infty}(-1)^n \rho^{\mp n} q^{\frac{n^2+n}{2}} \sum_{k=0}^{\infty} \zeta^k q^{nk} - \sum_{n=1}^{\infty}(-1)^n \rho^{\pm n} q^{\frac{n^2-n}{2}} \sum_{k=1}^{\infty} \zeta^{-k} q^{(-n)\cdot(-k)}.$$

If we see the above as a power series in q, we get that when $n \to \infty$,

$$\zeta^{-\frac{1}{2}} A_1\left(z, \mp\tfrac{1}{3}; \tau\right) = \frac{1}{1-\zeta} + O(q) + O\left(\zeta^{-1} q\right).$$

Thus

$$A_1\left(z, \mp\tfrac{1}{3}; \tau\right) = \frac{-1}{\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}} + O(\zeta^{\frac{1}{2}} q) + O\left(\zeta^{-\frac{1}{2}} q\right).$$

Plugging in $\zeta = e^{\frac{4\pi^2 z}{s}}$ and $q = e^{-\frac{4\pi^2}{3s}}$ (which satisfy our condition that $|\zeta q^n| < 1$ if $n > 0$ and $|\zeta q^n| > 1$ if $n < 0$), we find:

$$\begin{aligned} A_1\left(\frac{2\pi z}{is}, \mp\frac{1}{3}; \frac{2\pi i}{3s}\right) &= \frac{-1}{e^{\frac{2\pi^2 z}{s}} - e^{\frac{-2\pi^2 z}{s}}} + O\left(e^{2\pi^2 z \operatorname{Re}\left(\frac{1}{s}\right)} e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)}\right) \\ &\quad + O\left(e^{-2\pi^2 z \operatorname{Re}\left(\frac{1}{s}\right)} e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)}\right) \\ &= \frac{-1}{2\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(e^{\frac{-2\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(2-3z)}\right) \\ &\quad + O\left(e^{\frac{-2\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(2+3z)}\right). \end{aligned}$$

□

We now turn to the Mordell integral.

Lemma 13.5. *For $|x| \le 1$, we have as $n \to \infty$ that*

$$\left| h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\right| \ll e^{\frac{-\beta}{6}}.$$

Proof: We apply Lemma 3.4 of [BMR14] with $\ell = 0$, $k = 2$, $h = \mp 1$, $u = 0$, $z = \frac{\pi}{3s}$ and $\alpha = 3z$. This gives

$$\left| h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\right| \ll e^{\frac{-\pi}{18}\operatorname{Re}\left(\frac{3s}{\pi}\right)}.$$

The result follows. □

With this, we can now prove the following estimate for the Appell-Lerch sums.

Lemma 13.6. *For $|z| \le \frac{1}{6}$ and $|x| \le 1$, as $n \to \infty$*

$$A_1(3z, \mp\tau; 3\tau) = \frac{i\pi}{3s}\frac{\zeta^{\pm 1} e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right).$$

Proof: We use the transformation properties of A_1 to obtain

$$\begin{aligned}
A_1(3z, \mp\tau; 3\tau) &= \frac{1}{2i} h(3z \pm \tau; 3\tau)\theta(\mp\tau; 3\tau) \\
&\quad + \frac{1}{3\tau} e^{\frac{\pi i(3z^2 \pm 2z\tau)}{\tau}} A_1\left(\frac{z}{\tau}, \mp\frac{1}{3}; -\frac{1}{3\tau}\right) \\
&= \frac{1}{2i} h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\theta\left(\mp\frac{is}{2\pi}; \frac{3is}{2\pi}\right) \\
&\quad + \frac{2\pi}{3is} e^{\frac{2\pi^2\left(3z^2 \pm \frac{2izs}{2\pi}\right)}{s}} A_1\left(\frac{2\pi z}{is}, \mp\frac{1}{3}; \frac{2\pi i}{3s}\right) \\
&= \pm\frac{1}{2} e^{\frac{s}{6}} h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\eta\left(\frac{is}{2\pi}\right) \\
&\quad - \frac{2\pi i}{3s} e^{\frac{6\pi^2 z^2}{s}} \zeta^{\pm 1} A_1\left(\frac{2\pi z}{is}, \mp\frac{1}{3}; \frac{2\pi i}{3s}\right),
\end{aligned}$$

by Lemmas 11.2 and 11.3. In the last equality we additionally used that

$$\theta(\mp\tau; 3\tau) = \pm i q^{-\frac{1}{6}}\eta(\tau),$$

which is easily deduced from the definition of θ in (11.1). By Lemma 11.1 and Lemma 13.5, we have

$$\left|\frac{1}{2}e^{\frac{s}{6}}h\left(3z\pm\frac{is}{2\pi};\frac{3is}{2\pi}\right)\eta\left(\frac{is}{2\pi}\right)\right|\ll e^{\frac{\beta}{6}-\frac{\beta}{6}}\left|\eta\left(\frac{is}{2\pi}\right)\right|\ll\frac{1}{|s|^{\frac{1}{2}}}e^{\frac{-\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}.$$

By Lemma 13.4,

$$-\frac{2\pi i}{3s}e^{\frac{6\pi^2z^2}{s}}\zeta^{\pm1}A_1\left(\frac{2\pi z}{is},\mp\frac{1}{3};\frac{2\pi i}{3s}\right)=\frac{\pi i}{3s}\frac{e^{\frac{6\pi^2z^2}{s}}\zeta^{\pm1}}{\sinh\left(\frac{2\pi^2z}{s}\right)}$$
$$+O\left(e^{-\pi^2\operatorname{Re}\left(\frac{1}{s}\right)\left(\frac{4}{3}-2z-6z^2\right)}\right)+O\left(e^{-\pi^2\operatorname{Re}\left(\frac{1}{s}\right)\left(\frac{4}{3}+2z-6z^2\right)}\right)$$

For $|z|\leq\frac{1}{6}$, $\frac{4}{3}-2z-6z^2>\frac{1}{6}$ and $\frac{4}{3}+2z-6z^2>\frac{1}{6}$. Therefore

$$e^{-\pi^2\operatorname{Re}\left(\frac{1}{s}\right)\left(\frac{4}{3}+2z-6z^2\right)}\ll\frac{1}{|s|^{\frac{1}{2}}}e^{\frac{-\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)},$$
$$e^{-\pi^2\operatorname{Re}\left(\frac{1}{s}\right)\left(\frac{4}{3}-2z-6z^2\right)}\ll\frac{1}{|s|^{\frac{1}{2}}}e^{\frac{-\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}.$$

Thus the dominant error term is the one coming from

$$\pm\frac{1}{2}e^{\frac{s}{6}}h\left(3z\pm\frac{is}{2\pi};\frac{3is}{2\pi}\right)\eta\left(\frac{is}{2\pi}\right).$$

The lemma follows. $\square$

13.3. Asymptotic behaviour

Since $N(m,n)=N(-m,n)$ for all m and n, we assume from now on that $m\geq0$. In this section we want to study the asymptotic behaviour of the generating function of $N(m,n)$. Let us define

$$R_m(\tau):=\int_{-\frac{1}{2}}^{\frac{1}{2}}R(z;\tau)e^{-2\pi imz}dz.$$

Let us recall that $\tau=\frac{is}{2\pi}$ and $s=\beta\left(1+ixm^{-\frac{1}{3}}\right)$ with $x\in\mathbb{R}$ satisfying $|x|\leq\frac{\pi m^{\frac{1}{3}}}{\beta}$. To simplify the forthcoming calculations, we need the following lemma.

Lemma 13.7. *It holds that*

$$R_m(\tau) = 3\frac{q^{\frac{1}{24}}}{\eta(\tau)} \int_{-\frac{1}{6}}^{\frac{1}{6}} g_m(z;\tau) e^{-2\pi i m z} dz,$$

where

$$g_m(z;\tau) := \begin{cases} -A_1(3z,\tau;3\tau)e^{3\pi i z} + A_1(3z,-\tau;3\tau)e^{-3\pi i z} & \text{for } m \equiv 0 \mod 3, \\ -A_1(3z,-\tau;3\tau)e^{-\pi i z} - i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)}e^{-\pi i z} & \text{for } m \equiv 1 \mod 3, \\ A_1(3z,\tau;3\tau)e^{\pi i z} + i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)}e^{\pi i z} & \text{for } m \equiv 2 \mod 3. \end{cases}$$

Proof: By (13.2), let us write

$$R_m(\tau) = \frac{q^{\frac{1}{24}}}{\eta(\tau)}(I_1 - I_2 - I_3),$$

where

$$\begin{aligned} I_1 &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{i\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right)\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi i m z} dz, \\ I_2 &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta^{-1}\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) A_1(3z,-\tau;3\tau) e^{-2\pi i m z} dz, \\ I_3 &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) A_1(3z,\tau;3\tau)) e^{-2\pi i m z} dz. \end{aligned}$$

First, using (11.1) and (11.2), let us notice that

$$\theta(3z+1;3\tau) = -\theta(3z;3\tau), \tag{13.3}$$
$$A_1(3z+1,\tau;3\tau) = -A_1(3z,\tau;3\tau), \tag{13.4}$$
$$A_1(3z+1,-\tau;3\tau) = -A_1(3z,-\tau;3\tau). \tag{13.5}$$

Thus by (13.3),

$$
\begin{aligned}
I_1 = & \left(\int_{-\frac{1}{2}}^{-\frac{1}{6}} + \int_{-\frac{1}{6}}^{\frac{1}{6}} + \int_{\frac{1}{6}}^{\frac{1}{2}}\right) \frac{i\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right)\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi i m z} dz \\
= & -i \int_{-\frac{1}{6}}^{\frac{1}{6}} \left(e^{\pi i (z-\frac{1}{3})} - e^{-\pi i (z-\frac{1}{3})}\right) \frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi i m (z-\frac{1}{3})} dz \\
& + i \int_{-\frac{1}{6}}^{\frac{1}{6}} \left(e^{\pi i z} - e^{-\pi i z}\right) \frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi i m z} dz \\
& - i \int_{-\frac{1}{6}}^{\frac{1}{6}} \left(e^{\pi i (z+\frac{1}{3})} - e^{-\pi i (z+\frac{1}{3})}\right) \frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi i m (z+\frac{1}{3})} dz \\
= & \int_{-\frac{1}{6}}^{\frac{1}{6}} \left[e^{\pi i z}\left(-e^{\frac{\pi i}{3}(2m-1)} + 1 - e^{\frac{\pi i}{3}(-2m+1)}\right)\right. \\
& \left. - e^{-\pi i z}\left(-e^{\frac{\pi i}{3}(2m+1)} + 1 - e^{\frac{\pi i}{3}(-2m-1)}\right)\right] \times i \frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-2\pi i m z} dz.
\end{aligned}
$$

Therefore

$$
I_1 = \begin{cases} 0 & \text{for } m \equiv 0 \mod 3, \\ -3i \int_{-\frac{1}{6}}^{\frac{1}{6}} \frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-\pi i z(2m+1)} dz & \text{for } m \equiv 1 \mod 3, \\ 3i \int_{-\frac{1}{6}}^{\frac{1}{6}} \frac{\eta^3(3\tau)}{\theta(3z;3\tau)} e^{-\pi i z(2m-1)} dz & \text{for } m \equiv 2 \mod 3. \end{cases} \tag{13.6}
$$

By the same method and using (13.4) and (13.5), we obtain

$$
I_2 = \begin{cases} -3 \int_{-\frac{1}{6}}^{\frac{1}{6}} A_1(3z, -\tau; 3\tau) e^{-\pi i z(2m+3)} dz & \text{for } m \equiv 0 \mod 3, \\ 3 \int_{-\frac{1}{6}}^{\frac{1}{6}} A_1(3z, -\tau; 3\tau) e^{-\pi i z(2m+1)} dz & \text{for } m \equiv 1 \mod 3, \\ 0 & \text{for } m \equiv 2 \mod 3, \end{cases} \tag{13.7}
$$

and

$$
I_3 = \begin{cases} 3 \int_{-\frac{1}{6}}^{\frac{1}{6}} A_1(3z, \tau; 3\tau) e^{-\pi i z(2m-3)} dz & \text{for } m \equiv 0 \mod 3, \\ 0 & \text{for } m \equiv 1 \mod 3, \\ -3 \int_{-\frac{1}{6}}^{\frac{1}{6}} A_1(3z, \tau; 3\tau) e^{-\pi i z(2m-1)} dz & \text{for } m \equiv 2 \mod 3. \end{cases} \tag{13.8}
$$

The result follows. $\square$

13.3.1. Bounds near the dominant pole

In this section we consider the range $|x| \leq 1$. We start by determining the main term of g_m.

Lemma 13.8. *For all $m \geq 0$ and $-\frac{1}{6} \leq z \leq \frac{1}{6}$, we have for $|x| \leq 1$ as $n \to \infty$*

$$g_m\left(z; \frac{is}{2\pi}\right) = \frac{2\pi \sin(\pi z) e^{\frac{6\pi^2 z^2}{s}}}{3s \sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{\frac{1}{2}}} e^{-\frac{\pi^2}{6} \operatorname{Re}\left(\frac{1}{s}\right)}\right).$$

Proof: If $m \equiv 0 \pmod 3$, we have by Lemma 13.6

$$\begin{aligned}
g_m(z;\tau) &= -A_1(3z, \tau; 3\tau) e^{3\pi i z} + A_1(3z, -\tau; 3\tau) e^{-3\pi i z} \\
&= -\frac{i\pi}{3s} \frac{e^{\pi i z} e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} + \frac{i\pi}{3s} \frac{e^{-\pi i z} e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{\frac{1}{2}}} e^{-\frac{\pi^2}{6} \operatorname{Re}\left(\frac{1}{s}\right)}\right) \\
&= \frac{i\pi}{3s} \frac{e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} \left(-e^{\pi i z} + e^{-\pi i z}\right) + O\left(\frac{1}{|s|^{\frac{1}{2}}} e^{-\frac{\pi^2}{6} \operatorname{Re}\left(\frac{1}{s}\right)}\right) \\
&= \frac{2\pi \sin(\pi z) e^{\frac{6\pi^2 z^2}{s}}}{3s \sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{\frac{1}{2}}} e^{-\frac{\pi^2}{6} \operatorname{Re}\left(\frac{1}{s}\right)}\right).
\end{aligned}$$

If $m \equiv 1 \pmod 3$, we have by Lemma 13.3 and Lemma 13.6

$$\begin{aligned}
g_m(z;\tau) &= -A_1(3z, -\tau; 3\tau) e^{-\pi i z} - i \frac{\eta^3(3\tau)}{\theta(3z; 3\tau)} e^{-\pi i z} \\
&= -\frac{i\pi}{3s} \frac{e^{\pi i z} e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6} \operatorname{Re}\left(\frac{1}{s}\right)}\right) \\
&\quad + \frac{i\pi e^{-\pi i z} e^{\frac{6\pi^2 z^2}{s}}}{3s \sinh\left(\frac{2\pi^2 z}{s}\right)} \\
&\quad \times \left(1 + O\left(e^{-\frac{4\pi^2}{3} \operatorname{Re}\left(\frac{1}{s}\right)(1-3z)}\right) + O\left(e^{-\frac{4\pi^2}{3} \operatorname{Re}\left(\frac{1}{s}\right)(1+3z)}\right)\right) \\
&= \frac{i\pi}{3s} \frac{e^{\frac{6\pi^2 z^2}{s}}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} \left(-e^{\pi i z} + e^{-\pi i z}\right) + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6} \operatorname{Re}\left(\frac{1}{s}\right)}\right) \\
&= \frac{2\pi \sin(\pi z) e^{\frac{6\pi^2 z^2}{s}}}{3s \sinh\left(\frac{2\pi^2 z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}} e^{-\frac{\pi^2}{6} \operatorname{Re}\left(\frac{1}{s}\right)}\right).
\end{aligned}$$

Finally, if $m \equiv 2 \pmod 3$, we have by Lemma 13.3 and Lemma 13.6

$$\begin{aligned}
g_m(z;\tau) &= A_1(3z,\tau;3\tau)e^{\pi iz} + i\frac{\eta^3(3\tau)}{\theta(3z;3\tau)}e^{\pi iz} \\
&= \frac{i\pi}{3s}\frac{e^{-\pi iz}e^{\frac{6\pi^2z^2}{s}}}{\sinh\left(\frac{2\pi^2z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right) \\
&\quad - \frac{i\pi e^{\pi iz}e^{\frac{6\pi^2z^2}{s}}}{3s\sinh\left(\frac{2\pi^2z}{s}\right)} \\
&\quad \times \left(1 + O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1-3z)}\right) + O\left(e^{-\frac{4\pi^2}{3}\operatorname{Re}\left(\frac{1}{s}\right)(1+3z)}\right)\right) \\
&= \frac{i\pi}{3s}\frac{e^{\frac{6\pi^2z^2}{s}}}{\sinh\left(\frac{2\pi^2z}{s}\right)}\left(e^{-\pi iz} - e^{\pi iz}\right) + O\left(\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right) \\
&= \frac{2\pi\sin(\pi z)e^{\frac{6\pi^2z^2}{s}}}{3s\sinh\left(\frac{2\pi^2z}{s}\right)} + O\left(\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{s}\right)}\right).
\end{aligned}$$

□

In view of Lemma 13.8 it is natural to define

$$\begin{aligned}
\mathcal{G}_{m,1}(s) &:= \frac{2\pi}{s}\int_{-\frac{1}{6}}^{\frac{1}{6}}\frac{\sin(\pi z)e^{\frac{6\pi^2z^2}{s}}}{\sinh\left(\frac{2\pi^2z}{s}\right)}e^{-2\pi imz}dz, \\
\mathcal{G}_{m,2}(s) &:= 3\int_{-\frac{1}{6}}^{\frac{1}{6}}\left(g_m\left(z;\frac{is}{2\pi}\right) - \frac{2\pi\sin(\pi z)e^{\frac{6\pi^2z^2}{s}}}{3s\sinh\left(\frac{2\pi^2z}{s}\right)}\right)e^{-2\pi imz}dz.
\end{aligned}$$

Thus

$$R_m(\tau) = \frac{q^{\frac{1}{24}}}{\eta(\tau)}\left(\mathcal{G}_{m,1}(s) + \mathcal{G}_{m,2}(s)\right). \tag{13.9}$$

Let us note that we can rewrite $\mathcal{G}_{m,1}(s)$ as

$$\mathcal{G}_{m,1}(s) = \frac{4\pi}{s}\int_0^{\frac{1}{6}}\frac{\sin(\pi z)e^{\frac{6\pi^2z^2}{s}}}{\sinh\left(\frac{2\pi^2z}{s}\right)}\cos(2\pi mz)dz.$$

Lemma 13.9. *Assume that $|x| \leq 1$ and $m \leq \frac{1}{6\beta} \log n$. Then we have as $n \to \infty$*

$$\mathcal{G}_{m,1}(s) = \frac{s}{4} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) + O\left(\beta^2 m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{\beta m}{2}\right)\right).$$

Proof: We use the same method as in the proof of Lemma 12.9. Inserting the Taylor expansion of $\sin(\pi z)$, $\exp\left(\frac{6\pi^2 z^2}{s}\right)$, and $\cos(2\pi m z)$ in the definition of $\mathcal{G}_{m,1}(s)$, we find that

$$\sin(\pi z) e^{\frac{6\pi^2 z^2}{s}} \cos(2\pi m z) = \sum_{j,\nu,r \geq 0} \frac{(-1)^{j+\nu}}{(2j+1)!(2\nu)!r!} \pi^{2j+1} (2\pi m)^{2\nu} \times \left(\frac{6\pi^2}{s}\right)^r z^{2j+2\nu+2r+1}.$$

This yields that

$$\mathcal{G}_{m,1}(s) = \frac{4\pi}{s} \sum_{j,\nu,r \geq 0} \frac{(-1)^{j+\nu}}{(2j+1)!(2\nu)!r!} \pi^{2j+1} (2\pi m)^{2\nu} \left(\frac{6\pi^2}{s}\right)^r \mathcal{I}_{j+\nu+r},$$

where for $\ell \in \mathbb{N}_0$ we define

$$\mathcal{I}_\ell := \int_0^{\frac{1}{6}} \frac{z^{2\ell+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz.$$

We next relate $\mathcal{I}_\ell$ to $\mathcal{E}_\ell$ defined in (12.8). For this, we note that

$$\mathcal{I}_\ell = \int_0^\infty \frac{z^{2\ell+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz - \mathcal{I}'_\ell \tag{13.10}$$

with

$$\mathcal{I}'_\ell := \int_{\frac{1}{6}}^\infty \frac{z^{2\ell+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz \ll \int_{\frac{1}{6}}^\infty z^{2\ell+1} e^{-2\pi^2 z \operatorname{Re}\left(\frac{1}{s}\right)} dz \ll \left(\operatorname{Re}\left(\frac{1}{s}\right)\right)^{-2\ell-2} \Gamma\left(2\ell+2; \frac{\pi^2}{3} \operatorname{Re}\left(\frac{1}{s}\right)\right),$$

where $\Gamma(\alpha; x) := \int_x^\infty e^{-w} w^{\alpha-1} dw$. Using that as $x \to \infty$

$$\Gamma(\ell; x) \sim x^{\ell-1} e^{-x} \tag{13.11}$$

thus yields that

$$\mathcal{I}'_\ell \ll \left(\mathrm{Re}\left(\frac{1}{s}\right)\right)^{-1} e^{-\frac{\pi^2}{3}\mathrm{Re}\left(\frac{1}{s}\right)} \le e^{-\frac{\pi^2}{3}\mathrm{Re}\left(\frac{1}{s}\right)}.$$

By a substitution in 12.6, we know that

$$\int_0^\infty \frac{z^{2\ell+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)}dz = \left(\frac{s}{2\pi}\right)^{2\ell+2}\frac{(-1)^{\ell+1}E_{2\ell+1}(0)}{2}.$$

Thus

$$\begin{aligned}\mathcal{G}_{m,1}(s) &= \sum_{j,\nu,r\ge 0}\frac{(-1)^{r+1}3^r}{2^{2j+r+1}(2j+1)!(2\nu)!r!}m^{2\nu}s^{2j+2\nu+r+1}\\ &\quad\times\left(E_{2j+2\nu+2r+1}(0) + O\left(|z|^{-2j-2\nu-2r-2}e^{-\frac{\pi^2}{3}\mathrm{Re}\left(\frac{1}{s}\right)}\right)\right)\\ &= \sum_{\nu=0}^{\infty}\frac{(ms)^{2\nu}}{(2\nu)!}\left(-\frac{s}{2}E_{2\nu+1}(0) + O\left(|s|^2\right)\right)\\ &= \frac{s}{4}\mathrm{sech}^2\left(\frac{ms}{2}\right) + O\left(|s|^2\cosh(ms)\right),\end{aligned}$$

where for the last equality we used Lemma 12.5. The end of the proof is now exactly the same as in Lemma 12.9. □

We now want to bound $\mathcal{G}_{m,2}(s)$.

Lemma 13.10. *Assume that $|x| \le 1$. Then we have as $n \to \infty$*

$$|\mathcal{G}_{m,2}(s)| \ll \frac{1}{\beta^{\frac{1}{2}}}e^{-\frac{\pi^2}{12\beta}}.$$

Proof: By Lemma 13.8, we have

$$|\mathcal{G}_{m,2}(s)| \ll \int_{-\frac{1}{6}}^{\frac{1}{6}}\left|\frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\mathrm{Re}\left(\frac{1}{s}\right)}e^{-2\pi i m z}\right|dz \ll \frac{1}{|s|^{1/2}}e^{-\frac{\pi^2}{6}\mathrm{Re}\left(\frac{1}{s}\right)}.$$

By the definition of s, we know that $\frac{1}{|s|^{1/2}} \le \frac{1}{\beta^{1/2}}$. Furthermore, as $|x| \le 1$, we have $\mathrm{Re}\left(\frac{1}{s}\right) \ge \frac{1}{2\beta}$. Thus

$$|\mathcal{G}_{m,2}(s)| \ll \frac{1}{\beta^{1/2}}e^{-\frac{\pi^2}{12\beta}}.$$

□

Combining Lemma 13.9 and Lemma 13.10, we obtain the following asymptotic estimation of $R_m(\tau)$ near the dominant pole.

Proposition 13.11. *Assume that $|x| \leq 1$. Then we have as $n \to \infty$*

$$R_m(\tau) = \frac{s^{\frac{3}{2}}}{4(2\pi)^{\frac{1}{2}}} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) e^{\frac{k\pi^2}{6s}} + O\left(\beta^{\frac{5}{2}} m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) e^{\pi\sqrt{\frac{n}{6}}}\right).$$

Proof: Recall from (13.9) that

$$R_m(\tau) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \left(\mathcal{G}_{m,1}(s) + \mathcal{G}_{m,2}(s)\right).$$

By Lemma 11.1 we see that

$$\frac{q^{\frac{1}{24}}}{\eta(\tau)} = \left(\frac{s}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\pi^2}{6s}} \left(1 + O(\beta)\right).$$

We approximate $\mathcal{G}_{m,1}$ and $\mathcal{G}_{m,2}$ using Lemma 13.9 and Lemma 13.10. The main error term comes from $\mathcal{G}_{m,1}$. We obtain

$$R_m(\tau) = \frac{s^{\frac{3}{2}}}{4(2\pi)^{\frac{1}{2}}} e^{\frac{\pi^2}{6s}} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) + O\left(s^{\frac{1}{2}} \beta^2 m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) e^{\frac{\pi^2}{6s}}\right).$$

The claim follows now using that

$$|s| \ll \beta,$$
$$\operatorname{Re}\left(\frac{1}{s}\right) \leq \frac{1}{\beta} = \frac{\sqrt{6n}}{\pi}.$$

$\square$

13.3.2. Estimates far from the dominant pole

In the previous section, we have established bounds for the behaviour of $R_m(\tau)$ close to the pole $\tau = 0$. In this section, we consider the range $1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ and use Lemma 9.2 to bound $|R_m(\tau)|$ away from $q = 1$.

Proposition 13.12. *Assume that $1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$. Then we have as $n \to \infty$*

$$|R_m(\tau)| \ll \sqrt{n} \exp\left(\pi\sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{8\pi} m^{-\frac{2}{3}}\right).$$

Proof: By (13.1), we have

$$\begin{aligned}R_m(\tau) &= P(q)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left((1-\zeta)\sum_{k\in\mathbb{Z}}\frac{(-1)^k q^{\frac{3k^2+k}{2}}}{1-\zeta q^k}\right)e^{-2\pi imz}dz\\ &= P(q)\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(1+(1-\zeta)\sum_{k\geq 1}\frac{(-1)^k q^{\frac{3k^2+k}{2}}}{1-\zeta q^k}\right.\\ &\qquad\left.+\left(1-\zeta^{-1}\right)\sum_{k\geq 1}\frac{(-1)^k q^{\frac{3k^2+k}{2}}}{1-\zeta^{-1}q^k}\right)e^{-2\pi imz}dz.\end{aligned}$$

So we may bound $|R_m(\tau)|$ when $n\to\infty$ in the following way

$$\begin{aligned}|R_m(\tau)| &\ll |P(q)|\int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{k\geq 1}\frac{|q|^{\frac{3k^2+k}{2}}}{1-|q|^k}\left|e^{-2\pi imz}\right|dz\\ &\ll |P(q)|\frac{1}{1-|q|}\sum_{k\geq 1}e^{-\beta\frac{3k^2}{2}}\\ &\ll |P(q)|\frac{1}{1-|q|}\int_{-\infty}^{\infty}e^{-\beta\frac{3x^2}{2}}dx\\ &\ll |P(q)|\frac{1}{\beta}\sqrt{\frac{2\pi}{3\beta}}\\ &\ll |P(q)|n^{\frac{3}{4}}.\end{aligned}$$

Now we use Lemma 9.2 with $v=\frac{\beta}{2\pi}$, $u=\frac{\beta m^{-\frac{1}{3}}x}{2\pi}$ and $M=m^{-\frac{1}{3}}$. We obtain that for $1\leq|x|\leq\frac{\pi m^{\frac{1}{3}}}{\beta}$,

$$|P(q)|\ll n^{-\frac{1}{4}}\exp\left[\frac{2\pi}{\beta}\left(\frac{\pi}{12}-\frac{1}{2\pi}\left(1-\frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)\right].$$

Therefore

$$\begin{aligned}|R_m(\tau)| &\ll n^{\frac{1}{2}} \exp\left[\frac{2\pi}{\beta}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)\right] \\ &\ll n^{\frac{1}{2}} \exp\left[\pi\sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right] \\ &\ll n^{\frac{1}{2}} \exp\left(\pi\sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{8\pi} m^{-\frac{2}{3}}\right).\end{aligned}$$

□

13.4. The Circle Method

In this section we write $N(m,n)$ as an integral on a circle and divide it into a major and a minor arc to complete the proof of Theorem 13.1.

Using Cauchy's theorem, we write $N(m,n)$ as as integral of its generating function $R_m(\tau)$:

$$N(m,n) = \frac{1}{2\pi i}\int_C \frac{R_m(\tau)}{q^{n+1}} dq,$$

where the contour is the counterclockwise transversal of the circle $C := \{q \in \mathbb{C} \,;\, |q| = e^{-\beta}\}$. Recall that $s = \beta(1 + ixm^{-\frac{1}{3}})$. Changing variables we may write

$$N(m,n) = \frac{\beta}{2\pi m^{\frac{1}{3}}} \int\limits_{|x| \le \frac{\pi m^{\frac{1}{3}}}{\beta}} R_m\left(\frac{is}{2\pi}\right) e^{ns} dx.$$

We split this integral into two pieces

$$N(m,n) = M + E$$

with

$$\begin{aligned}M &:= \frac{\beta}{2\pi m^{\frac{1}{3}}} \int\limits_{|x| \le 1} R_m\left(\frac{is}{2\pi}\right) e^{ns} dx, \\ E &:= \frac{\beta}{2\pi m^{\frac{1}{3}}} \int\limits_{1 \le |x| \le \frac{\pi m^{\frac{1}{3}}}{\beta}} R_m\left(\frac{is}{2\pi}\right) e^{ns} dx.\end{aligned}$$

In the following we show that M contributes to the asymptotic main term whereas E is part of the error term.

As the estimation of $R_m(\tau)$ close to the dominant pole is exactly the same as the one of $\mathcal{C}_{m,1}(q)$ in Chapter 12, the asymptotic behaviour of M here is the same as in Section 12.4.

Proposition 13.13. *We have*

$$M = \frac{\beta}{4} \operatorname{sech}^2\left(\frac{\beta m}{2}\right) p(n) \left(1 + O\left(\frac{m^{\frac{1}{3}}}{n^{\frac{1}{4}}}\right)\right).$$

Let us now turn to the integral E.

Proposition 13.14. *As $n \to \infty$*

$$E \ll n^{\frac{1}{2}} \exp\left(\pi\sqrt{\frac{2n}{3}} - \frac{\sqrt{6n}}{8\pi} m^{-\frac{2}{3}}\right).$$

Proof: Using Proposition 13.12, we may bound

$$\begin{aligned} E &\ll \frac{\beta}{m^{\frac{1}{3}}} \int_{1 \le x \le \frac{\pi m^{\frac{1}{3}}}{\beta}} n^{\frac{1}{2}} \exp\left(\pi\sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{8\pi} m^{\frac{-2}{3}}\right) e^{\beta n} dx \\ &\ll n^{\frac{1}{2}} \exp\left(\pi\sqrt{\frac{2n}{3}} - \frac{\sqrt{6n}}{8\pi} m^{\frac{-2}{3}}\right). \end{aligned}$$

□

Thus E is exponentially smaller than M. This completes the proof of Theorem 13.1.

Part IV.

An extension of q-binomial coefficients

14. An overpartition analogue of the q-binomial coefficients

Contents

14.1. Introduction

We recall that q-binomial coefficients are defined by

$$\begin{bmatrix} M+N \\ N \end{bmatrix}_q = \frac{(q)_{M+N}}{(q)_M (q)_N}.$$

In this chapter we study an overpartition analogue of the q-binomial coefficients. This was done in collaboration with Byungchan Kim in the paper [DKar]. As $\begin{bmatrix} M+N \\ N \end{bmatrix}_q$ is the generating function for the number of partitions of n fitting inside an $M \times N$ box, we study the generating function for overpartitions fitting inside an $M \times N$ box. i.e. with largest part $\leq M$ and number of parts $\leq N$, which we will call over q-binomial coefficients.

Definition. The *over q-binomial coefficients* are defined as the generating functions

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q = \sum_{n \geq 0} \overline{p}(n| \leq M \text{ parts, each} \leq N) q^n,$$

where $\overline{p}(n| \leq M$ parts, each $\leq N)$ denotes the number of overpartitions of n whose Ferrers diagram fits inside an $M \times N$ box. We will later omit the q in the notation and write $\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}$.

In Section 14.2, we give an exact expression of over q-binomial coefficients and recurrences analogous to the q-Pascal triangle (Proposition 14.2). In Section 14.3, we give various q-series identities which can be proved using over q-binomial coefficients. Finally in Section 14.4, we prove a Rogers-Ramanujan type theorem for overpartitions using over q-binomial coefficients.

14.2. Basic Properties of over q-binomial coefficients

Our first result is an exact expression for over q-binomial coefficients.

Theorem 14.1. *For positive integers M and N,*

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q = \sum_{k=0}^{\min\{M,N\}} q^{\frac{k(k+1)}{2}} \frac{(q)_{M+N-k}}{(q)_k (q)_{M-k} (q)_{N-k}}.$$

Remarks. 1. The above expression can be rewritten with q-trinomial coefficients, defined as

$$\begin{bmatrix} a+b+c \\ a,b,c \end{bmatrix}_q := \frac{(q)_{a+b+c}}{(q)_a (q)_b (q)_c}.$$

This gives

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q = \sum_{k=0}^{\min\{M,N\}} q^{\frac{k(k+1)}{2}} \begin{bmatrix} M+N-k \\ k, M-k, N-k \end{bmatrix}_q.$$

2. We have the obvious symmetry

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}}_q = \overline{\begin{bmatrix} M+N \\ M \end{bmatrix}}_q.$$

Proof of Theorem 14.1: Let $\overline{G}(M,N,k)$ be the generating function for overpartitions fitting inside an $M \times N$ rectangle and having exactly k overlined parts. Such an overpartition can be decomposed as a partition into k distinct parts, each of them being at most M, and a partition fitting inside an $M \times (N-k)$ box. By appending a partition fitting into an $(M-k) \times k$ box (generated by $\begin{bmatrix} M \\ k \end{bmatrix}_q$) to the right of the staircase partition $k + k - 1 + \cdots + 1$ (generated by $q^{\frac{k(k+1)}{2}}$) , we see that

$$q^{\frac{k(k+1)}{2}} \begin{bmatrix} M \\ k \end{bmatrix}_q$$

generates partitions into k distinct parts $\leq M$. As $\begin{bmatrix} N+M-k \\ N-k \end{bmatrix}_q$ generates the partitions fitting inside $M \times (N-k)$ box, we see that

$$\overline{G}(M,N,k) = q^{\frac{k(k+1)}{2}} \begin{bmatrix} M \\ k \end{bmatrix}_q \begin{bmatrix} N+M-k \\ N-k \end{bmatrix}_q = q^{\frac{k(k+1)}{2}} \frac{(q)_{M+N-k}}{(q)_k (q)_{M-k} (q)_{N-k}}.$$

Since $\overline{G}(M,N,k)$ is non-zero if and only if $0 \leq k \leq \min\{M,N\}$, we have

$$\overline{\begin{bmatrix} N+M \\ N \end{bmatrix}}_q = \sum_{k=0}^{\min\{M,N\}} \overline{G}(M,N,k) = \sum_{k=0}^{\min\{M,N\}} q^{\frac{k(k+1)}{2}} \frac{(q)_{M+N-k}}{(q)_k (q)_{M-k} (q)_{N-k}}.$$

□

Now we show that the over q-binomial coefficients also satisfy recurrences similar to Pascal's triangle.

Theorem 14.2. *For positive integers M and N, we have*

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}} = \overline{\begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix}} + q^N \overline{\begin{bmatrix} M+N-1 \\ N \end{bmatrix}} + q^N \overline{\begin{bmatrix} M+N-2 \\ N-1 \end{bmatrix}}, \tag{14.1}$$

$$\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}} = \overline{\begin{bmatrix} M+N-1 \\ N \end{bmatrix}} + q^M \overline{\begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix}} + q^M \overline{\begin{bmatrix} M+N-2 \\ N-1 \end{bmatrix}}. \tag{14.2}$$

We give two proofs of Theorem 14.2, one combinatorial using Ferrers diagrams, and the other one analytic using the recurrence equation for q-trinomial coefficients.

Combinatorial proof of Theorem 14.2: Let $O(M,N,n)$ denote the number of overpartitions of n fitting inside an $M \times N$ rectangle. We notice that $O(M,N,n) - O(M,N-1,n)$ is the number of overpartitions of n fitting inside an $M \times N$ rectangle having exactly N parts. Let λ be such an overpartition. If the smallest part of λ is $\overline{1}$, then by removing 1 from every part we obtain an overpartition of $n-N$ fitting inside an $(M-1) \times (N-1)$ rectangle. If the smallest part of λ is different from $\overline{1}$, by removing 1 from every part we arrive at an overpartition of $n-N$ fitting inside an $(M-1) \times N$ rectangle. Therefore, we find that

$$O(M,N,n) - O(M,N-1,n) = O(M-1,N-1,n-N) + O(M-1,N,n-N).$$

Rewriting the above identity in terms of generating functions, we obtain the first recurrence.

The second recurrence follows from a similar argument by tracking the size of the maximum part instead of the number of parts. Note that $O(M,N,n) - O(M-1,N,n)$ is the number of overpartitions of n fitting inside an $M \times N$ rectangle with largest part equal to M. If the largest part is overlined, then by removing it we obtain an overpartition of $n-M$ fitting inside a $(M-1)\times(N-1)$ rectangle. If the largest part is not overlined, then by removing it we obtain an overpartition of $n-M$ fitting inside a $M \times (N-1)$ rectangle. Therefore, we find that

$$O(M,N,n) - O(M-1,N,n) = O(M-1,N-1,n-M) + O(M,N-1,n-M).$$

Rewriting the above identity in terms of generating functions, we obtain the second recurrence. □

Analytic Proof of Theorem 14.2: A simple calculation shows that the q-trinomial coefficients satisfy the following recurrence

$$\begin{bmatrix} a+b+c \\ a,b,c \end{bmatrix} = \begin{bmatrix} a+b+c-1 \\ a-1,b,c \end{bmatrix} + q^a \begin{bmatrix} a+b+c-1 \\ a,b-1,c \end{bmatrix} + q^{a+b} \begin{bmatrix} a+b+c-1 \\ a,b,c-1 \end{bmatrix}.$$

Therefore, we find that

$$\begin{aligned}
\overline{\begin{bmatrix} M+N \\ N \end{bmatrix}} &= \sum_{k=0}^{\min\{M,N\}} q^{k(k+1)/2} \left(\begin{bmatrix} M+N-k-1 \\ N-k-1,k,M-k \end{bmatrix} \right. \\
&\qquad \left. + q^{N-k} \begin{bmatrix} M+N-k-1 \\ N-k,k-1,M-k \end{bmatrix} + q^N \begin{bmatrix} M+N-k-1 \\ N-k,k,M-k-1 \end{bmatrix} \right) \\
&= \sum_{k=0}^{\min\{M,N-1\}} q^{k(k+1)/2} \begin{bmatrix} M+N-k-1 \\ k,M-k,N-1 \end{bmatrix} \\
&\quad + \sum_{k=0}^{\min\{M-1,N-1\}} q^{k(k+1)/2+N} \begin{bmatrix} M+N-k-2 \\ k,M-1-k,N-1-k \end{bmatrix} \\
&\quad + \sum_{k=0}^{\min\{M-1,N\}} q^{k(k+1)/2+N} \begin{bmatrix} M+N-k-1 \\ k,M-1-k,N-1 \end{bmatrix} \\
&= \overline{\begin{bmatrix} M+N-1 \\ N-1 \end{bmatrix}} + q^N \overline{\begin{bmatrix} M+N-2 \\ N-1 \end{bmatrix}} + q^N \overline{\begin{bmatrix} M+N-1 \\ N \end{bmatrix}},
\end{aligned}$$

where we have made a change of variable $k \to k+1$ in the second sum. The second recurrence can be proved similarly. □

Finally, we give an asymptotic formula which we use frequently throughout this chapter.

Proposition 14.3. *Let j be a non-negative integer. We have*

$$\lim_{N\to\infty} \overline{\begin{bmatrix} N \\ j \end{bmatrix}} = \frac{(-q)_j}{(q)_j}.$$

Proof: When N tends to infinity, the restriction on the number of parts disappears. $\square$

Proposition 14.3 is useful to obtain new identities. For example, by taking a limit $N \to \infty$ in Theorem 14.1, we find that

$$\sum_{k=0}^{j} \frac{q^{k(k+1)/2}}{(q)_k (q)_{j-k}} = \frac{(-q)_j}{(q)_j},$$

which gives an alternative generating function for overpartitions into parts $\leq j$. This identity is also a special case of the finite q-binomial theorem (Theorem 3.9).

14.3. Applications

In this section, we give several q-series identities which can be proved using over q-binomial coefficients. By tracking the number of parts in the overpartitions, we prove the following.

Proposition 14.4. *Let N be a positive integer. We have*

$$\frac{(-zq)_N}{(zq)_N} = 1 + \sum_{k\geq 1} z^k q^k \left(\overline{\begin{bmatrix} N+k-1 \\ k \end{bmatrix}} + \overline{\begin{bmatrix} N+k-2 \\ k-1 \end{bmatrix}} \right).$$

Proof: Let $\overline{p}_N(n,k)$ be the number of overpartitions of n into parts $\leq N$ with k parts. Then, it is not hard to see that

$$\frac{(-zq)_N}{(zq)_N} = \sum_{n\geq 0} \sum_{k\geq 0} \overline{p}_N(n,k) z^k q^n.$$

Let λ be an overpartition counted by $\overline{p}_N(n,k)$. Discussing whether the smallest part of λ is equal to $\overline{1}$ and removing 1 from each part as in the proof of Theorem 14.2, we have

$$\overline{p}_N(n,k) = O(N-1, k, n-k) + O(N-1, k-1, n-k).$$

Thus
$$\sum_{n\geq 0}\overline{p}_N(n,k)q^n = q^k\left(\overline{\begin{bmatrix}N+k-1\\ k\end{bmatrix}}+\overline{\begin{bmatrix}N+k-2\\ k-1\end{bmatrix}}\right).$$
The claimed identity follows. □

By taking the limit as $N\to\infty$ in the above proposition, we find the following generating function.

Corollary 14.5. *Let $\overline{p}(n,k)$ be the number of overpartitions of n with k parts. Then,*
$$\sum_{n\geq 0}\sum_{k\geq 0}\overline{p}(n,k)z^kq^n = \frac{(-zq)_\infty}{(zq)_\infty} = 1+2\sum_{k\geq 1}\frac{z^kq^k(-q)_{k-1}}{(q)_k}.$$

Remark. *The above identity is a special case of the q-binomial theorem (Theorem 3.9).*

Note that
$$\sum_{k\geq 1}\frac{q^k(-q)_{k-1}}{(q)_k}\equiv\sum_{k\geq 1}\frac{q^k}{1-q^k}=\sum_{n\geq 1}\tau(n)q^n \pmod 2,$$
where $\tau(n)$ is the number of divisors of n. This recovers a well known congruence.

Corollary 14.6. *For all non-negative integers n,*
$$\overline{p}(n)\equiv 2\tau(n) \pmod 4.$$

Our next application is finding an analogue of Sylvester's identity [Syl73]:
$$\begin{aligned}(-xq)_N = 1&+\sum_{j\geq 1}\begin{bmatrix}N+1-j\\ j\end{bmatrix}(-xq;q)_{j-1}x^jq^{3j(j-1)/2}\\ &+\sum_{j\geq 1}\begin{bmatrix}N-j\\ j\end{bmatrix}(-xq;q)_{j-1}x^{j+1}q^{3j(j+1)/2}.\end{aligned}$$

We define $\overline{S}(N;x;q)$ as
$$\overline{S}(N;x,q) := 1+\sum_{j\geq 1}\left(\overline{\begin{bmatrix}N-1\\ j-1\end{bmatrix}}\frac{(-xq)_{j-1}}{(xq)_{j-1}}x^jq^{j^2}+\overline{\begin{bmatrix}N\\ j\end{bmatrix}}\frac{(-xq)_j}{(xq)_j}x^jq^{j^2}\right).$$

Then we have the following identity.

Theorem 14.7. *Let N be a positive integer. We have*

$$\overline{S}(N;x;q) = \frac{(-xq)_N}{(xq)_N}.$$

Proof: Let us consider an overpartition into parts $\leq N$, generated by $\frac{(-xq)_N}{(xq)_N}$. The variable j counts the size of the Durfee square of the overpartition. The Durfee square is generated by $x^j q^{j^2}$. Then either the corner at the bottom right of the Durfee square is overlined or it is not. If it is overlined, then we have an overpartition generated by $\overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}}$ to the right of the Durfee square, and an overpartition generated by $\frac{(-xq)_{j-1}}{(xq)_{j-1}}$ under it. If it is not overlined, then we have an overpartition generated by $\overline{\begin{bmatrix} N \\ j \end{bmatrix}}$ to the right of the Durfee square, and an overpartition generated by $\frac{(-xq)_j}{(xq)_j}$ under it. These two cases correspond to the two sums in $\overline{S}(N;x;q)$. $\square$

By taking the limit as $N \to \infty$ and using Proposition 14.3, we obtain the following identity:

Corollary 14.8. *We have*

$$\frac{(-xq)_\infty}{(xq)_\infty} = 1 + \sum_{j\geq 1} \left(\frac{(-q)_{j-1}}{(q)_{j-1}} \frac{(-xq)_{j-1}}{(xq)_{j-1}} x^j q^{j^2} + \frac{(-q)_j}{(q)_j} \frac{(-xq)_j}{(xq)_j} x^j q^{j^2} \right).$$

In particular, by setting $x = -1$ we obtain a well known theta function identity.

$$\frac{(q)_\infty}{(-q)_\infty} = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}.$$

As another application, we obtain the overpartition rank generating function. Recall that Dyson [Dys44] defined the rank of a partition as the difference between the size of its largest part and its number of parts. For an overpartition, we can define a rank in the same way [BL07]. Let $\overline{N}(m,n)$ be the number of overpartitions of n with rank m. Then, we can express the generating function in terms of over q-binomial coefficients.

Theorem 14.9. *Let m be a non-negative integer. We have*

$$\overline{N}_m(q) := \sum_{n\geq 0} \overline{N}(m,n) q^n$$

$$= 2q^{m+1} + \sum_{k\geq 2} q^{2k+m-1}\left(\overline{\begin{bmatrix}2k+m-2\\k-1\end{bmatrix}} + \overline{\begin{bmatrix}2k+m-3\\k-1\end{bmatrix}}\right.$$
$$\left. + \overline{\begin{bmatrix}2k+m-3\\k-2\end{bmatrix}} + \overline{\begin{bmatrix}2k+m-4\\k-2\end{bmatrix}}\right).$$

Proof: If there is only one part in the overpartition, $m+1$ and $\overline{m+1}$ are the only two such overpartitions with rank m, which corresponds to $2q^{m+1}$. Now we assume that an overpartition has at least two parts and the rank of the overpartition is m. Under this assumption, the largest part would be $m+k$ and the number of parts would be k. This corresponds to q^{m+2k-1} in the summation. Now the first sum counts the case where the largest part is not overlined and there is no $\overline{1}$. The second sum counts the case where the largest part is overlined and there is no $\overline{1}$. The third sum counts the case where the largest part is not overlined and the smallest part is $\overline{1}$. The last sum corresponds to the case where the largest part is overlined and the smallest part is $\overline{1}$. $\square$

Comparing this with the known generating function for $\overline{N}_m(q)$ [Lov05a, Proposition 3.2]

$$\overline{N}_m(q) = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^{n-1}q^{n^2+|m|n}(1-q^n)}{1+q^n},$$

we derive the following identity.

Corollary 14.10. *Let m be a non-negative integer. We have*

$$2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^{n-1}q^{n^2+|m|n}(1-q^n)}{1+q^n}$$
$$= 2q^{1+m} + \sum_{k\geq 2} q^{2k+m-1}\left(\overline{\begin{bmatrix}2k+m-2\\k-1\end{bmatrix}} + \overline{\begin{bmatrix}2k+m-3\\k-1\end{bmatrix}}\right.$$
$$\left. + \overline{\begin{bmatrix}2k+m-3\\k-2\end{bmatrix}} + \overline{\begin{bmatrix}2k+m-4\\k-2\end{bmatrix}}\right).$$

14.4. Proof of a Rogers-Ramanujan type identity

We showed in the last section that we can establish various identities by employing over q-binomial coefficients. Here, we highlight that over q-binomial coefficients can be used to derive a Rogers-Ramanujan type theorem for overpartitions. Recall that the first Rogers-Ramanujan identity is given by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

The left-hand side can be interpreted as the generating function for partitions with a difference ≥ 2 between two successive parts, and the right-hand side as the generating function for partitions into parts $\equiv 1, 4 \pmod 5$. Motivated by a proof of Andrews which uses recurrence relations of q-binomial coefficients [And88, And89], we find a Rogers-Ramanujan type identity for overpartitions.

Theorem 14.11. *Let $A(n)$ be the number of overpartitions $\lambda_1 + \cdots + \lambda_\ell$ of n satisfying the following difference conditions:*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 1, & \text{if } \lambda_i \text{ is not overlined,} \\ 2, & \text{if } \lambda_i \text{ is overlined.} \end{cases}$$

If there are ℓ parts in the overpartition, we define $\lambda_{\ell+1} = 0$.

Let $B(n)$ be the number of overpartitions of n with non-overlined parts $\equiv 2 \pmod 4$ and $C(n)$ be the number of partitions of n into parts $\not\equiv 0 \pmod 4$.

For all non-negative integers n,

$$A(n) = B(n) = C(n).$$

Remark. *This is a special case of [Lov03, Theorem 1.2], which has been generalised by Chen, Sang, and Shi [CSS13]. While the previous results are obtained by employing Bailey chains, here we use the recurrence formulas satisfied by the over q-binomial coefficients.*

By Theorems 3.1 and 3.3, we have

$$\sum_{n\geq 0} B(n) q^n = \frac{(-q;q)_\infty}{(q^2;q^4)_\infty},$$

and

$$\sum_{n\geq 0} C(n) q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}.$$

Moreover

$$(q;q)_\infty (-q;q)_\infty = (q^2;q^2)_\infty = (q^2;q^4)_\infty (q^4;q^4)_\infty,$$

so $B(n) = C(n)$. Thus the important equality in Theorem 14.11 is $A(n) = B(n)$. Let us now prove it.

We first define two functions

$$\overline{D}(N, x; q) := \sum_{j \geq 0} \overline{\begin{bmatrix} N \\ j \end{bmatrix}} x^j q^{j(j+1)/2},$$

and

$$\overline{C}(N, x; q) := \sum_{j \geq 0} \overline{\begin{bmatrix} N \\ j \end{bmatrix}} \frac{(xq)_j}{(-xq)_j} (-1)^j x^{2j} \left(q^{j(2j+1)} - xq^{(j+1)(2j+1)} \right).$$

The following observation is the key for obtaining a Rogers-Ramanujan type identity.

Theorem 14.12. *Let N be a positive integer. We have*

$$\frac{(xq)_N}{(-xq)_N} \overline{D}(N, x; q) - \overline{C}(N, x; q) \in x^2 q^{N+3} \cdot \mathbb{Z}[[x, q]].$$

By taking the limit as $N \to \infty$, we obtain the following corollary.

Corollary 14.13. *We have*

$$\frac{(xq)_\infty}{(-xq)_\infty} \overline{D}(\infty, x; q) = \overline{C}(\infty, x; q).$$

In particular, the case $x = 1$ is a Rogers-Ramanujan type identity, where we applied Lemma 14.3 to evaluate the limit.

Corollary 14.14. *We have*

$$\frac{(q)_\infty}{(-q)_\infty} \overline{D}(\infty, 1; q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(2n+1)} = (q, q^3, q^4; q^4)_\infty.$$

Proof of Theorem 14.11: After multiplying both sides by $\frac{(-q)_\infty}{(q)_\infty}$, and by the definitions, we obtain that

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}(-q)_k}{(q)_k} = \frac{(-q)_\infty}{(q^2; q^4)_\infty} = \frac{1}{(q, q^2, q^3; q^4)_\infty}.$$

A basic partition theoretic interpretation of the above identity gives the desired result. $\square$

Now we turn to proving Theorem 14.12. Let $\overline{g}(N,x) := \frac{(xq)_N}{(-xq)_N}\overline{D}(N,x)$. The key idea of the proof is that $\overline{g}(N,x)$ and $\overline{C}(N,x)$ satisfy the same recurrence (up to a high power of q times a polynomial in x and q) as follows.

Lemma 14.15.

$$\overline{g}(N,x) = (1-xq)\overline{g}(N-1,xq) + \frac{(xq)_2}{(-xq)_2}xq^2\overline{g}(N-2,xq^2).$$

Lemma 14.16.

$$\overline{C}(N,x) - (1-xq)\overline{C}(N-1,xq) - \frac{(xq)_2}{(-xq)_2}xq^2\overline{C}(N-2,xq^2) \in x^2q^{N+3}\cdot\mathbb{Z}[[x,q]]. \tag{14.3}$$

Theorem 14.12 follows immediately from these two recurrences and an induction over N. We now need to prove these lemmas.

Proof of Lemma 14.15: By applying the first recurrence in Theorem 14.2, we find that

$$\begin{aligned}
\overline{D}(N,x) &= \sum_{j\geq 0}\left(\overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}} + q^j\overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}} + q^j\overline{\begin{bmatrix} N-2 \\ j-1 \end{bmatrix}}\right)x^jq^{j(j+1)/2} \\
&= \sum_{j\geq 0}\overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}}x^{j+1}q^{(j+1)(j+2)/2} + \overline{D}(N-1,xq) \\
&+ \sum_{j\geq 0}\overline{\begin{bmatrix} N-2 \\ j \end{bmatrix}}x^{j+1}q^{(j+1)(j+4)/2} \\
&= (1+xq)\overline{D}(N-1,xq) + xq^2\overline{D}(N-2,xq^2),
\end{aligned}$$

where we replace $j-1$ by j in the first and third sums for the second identity. After multiplying by $\frac{(xq)_N}{(-xq)_N}$ we get the desired recurrence. □

Proof of Lemma 14.16: We calculate each term in (14.3). By the definition of $\overline{C}$, we find that

$$\begin{aligned}
(1-xq)\overline{C}(N-1,xq) = (1+xq)&\sum_{j\geq 0}\overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}}\frac{(xq)_{j+1}}{(-xq)_{j+1}} \\
&\times (-1)^jx^{2j}\left(q^{2j^2+3j} - xq^{2j^2+5j+2}\right).
\end{aligned}$$

Expanding and making the change of variable $j \to j-1$ in the fourth sum, we get

$$\begin{aligned}(1-xq)\overline{C}(N-1,xq) &= \sum_{j\geq 0} \overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^j x^{2j} q^{2j^2+3j} \\ &+ \sum_{j\geq 0} \overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^{j+1} x^{2j+1} q^{2j^2+5j+2} \\ &+ \sum_{j\geq 0} \overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^j x^{2j+1} q^{2j^2+3j+1} \\ &+ \sum_{j\geq 1} \overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}} \frac{(xq)_{j}}{(-xq)_{j}} (-1)^j x^{2j} q^{2j^2+j}.\end{aligned} \tag{14.4}$$

By the change of variable $j \to j-1$ in the definition of $\frac{(xq)_2}{(-xq)_2} xq^2 \overline{C}(N-2, xq^2)$, we obtain

$$\begin{aligned}\frac{(xq)_2}{(-xq)_2} xq^2 \overline{C}(N-2,xq^2) &= \sum_{j\geq 1} \overline{\begin{bmatrix} N-2 \\ j-1 \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^{j+1} x^{2j-1} q^{2j^2+j-1} \\ &+ \sum_{j\geq 1} \overline{\begin{bmatrix} N-2 \\ j-1 \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^j x^{2j} q^{2j^2+3j}.\end{aligned} \tag{14.5}$$

Using the first recurrence (14.1) on the first sum in (14.4) and the second sum in (14.5) and extracting the term $j = 0$ in the first and third sums of (14.4) leads to

$$\begin{aligned}&(1-xq)\overline{C}(N-1,xq) + \frac{(xq)_2}{(-xq)_2} xq^2 \overline{C}(N-2,xq^2) \\ &= 1 - xq + \sum_{j\geq 1} \overline{\begin{bmatrix} N \\ j \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^j x^{2j} q^{2j^2+2j} \\ &\quad + \sum_{j\geq 1} \overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^{j+1} x^{2j} q^{2j^2+2j} \\ &\quad + \sum_{j\geq 0} \overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^{j+1} x^{2j+1} q^{2j^2+5j+2} \\ &\quad + \sum_{j\geq 1} \overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}} \frac{(xq)_{j+1}}{(-xq)_{j+1}} (-1)^j x^{2j+1} q^{2j^2+3j+1}\end{aligned} \tag{14.6}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}}\frac{(xq)_j}{(-xq)_j}(-1)^j x^{2j} q^{2j^2+j}$$
$$+\sum_{j\geq 1}\overline{\begin{bmatrix} N-2 \\ j-1 \end{bmatrix}}\frac{(xq)_{j+1}}{(-xq)_{j+1}}(-1)^{j+1} x^{2j-1} q^{2j^2+j-1}.$$

Now we want to write both $\overline{C}(N,x)$ and

$$(1-xq)\overline{C}(N-1,xq)+\frac{(xq)_2}{(-xq)_2}xq^2\overline{C}(N-2,xq^2)$$

as sums involving the product $\frac{(xq)_j}{(-xq)_{j+1}}$ to be able to make cancellations. We have

$$\overline{C}(N,x)=\sum_{j\geq 0}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}\left(1+xq^{j+1}\right)(-1)^j x^{2j} q^{2j^2+j}$$
$$+\sum_{j\geq 0}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}\left(1+xq^{j+1}\right)(-1)^{j+1} x^{2j+1} q^{2j^2+3j+1}.$$

Extracting the term $j=0$ of each sum and expanding, we get

$$\begin{aligned}\overline{C}(N,x)=1-xq&+\sum_{j\geq 1}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^j x^{2j} q^{2j^2+j}\\&+\sum_{j\geq 1}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^j x^{2j+1} q^{2j^2+2j+1}\\&+\sum_{j\geq 1}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1} x^{2j+1} q^{2j^2+3j+1}\\&+\sum_{j\geq 1}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1} x^{2j+2} q^{2j^2+4j+2}.\end{aligned} \tag{14.7}$$

Rewriting all the sums in (14.6) except the third one in terms of $\frac{(xq)_j}{(-xq)_{j+1}}$ leads to

$$(1-xq)\overline{C}(N-1,xq)+\frac{(xq)_2}{(-xq)_2}xq^2\overline{C}(N-2,xq^2) \tag{14.8}$$
$$=1-xq+\sum_{j\geq 1}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^j x^{2j} q^{2j^2+2j}$$

$$
\begin{aligned}
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j+1}q^{2j^2+3j+1} \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j}q^{2j^2+2j} \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j+1}q^{2j^2+3j+1} \\
&+\sum_{j\geq 0}\overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}}\frac{(xq)_{j+1}}{(-xq)_{j+1}}(-1)^{j+1}x^{2j+1}q^{2j^2+5j+2} \qquad (14.9) \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j+1}q^{2j^2+3j+1} \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N-1 \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j+2}q^{2j^2+4j+2} \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j}q^{2j^2+j} \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N-1 \\ j-1 \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j+1}q^{2j^2+2j+1} \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N-2 \\ j-1 \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j-1}q^{2j^2+j-1} \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N-2 \\ j-1 \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j}q^{2j^2+2j}.
\end{aligned}
$$

Subtracting (14.7) from (14.9) and noting that the third sum of (14.7) cancels with the second sum of (14.9) we obtain

$$
\begin{aligned}
&\overline{C}(N,x)-(1-xq)\overline{C}(N-1,xq)-\frac{(xq)_2}{(-xq)_2}xq^2\overline{C}(N-2,xq^2) \\
&=\sum_{j\geq 1}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j}q^{2j^2+j} \qquad (14.10) \\
&+\sum_{j\geq 1}\overline{\begin{bmatrix} N \\ j \end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j+1}q^{2j^2+2j+1} \qquad (14.11)
\end{aligned}
$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N\\ j\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j+2}q^{2j^2+4j+2} \tag{14.12}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N\\ j\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j}q^{2j^2+2j} \tag{14.13}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j}q^{2j^2+2j} \tag{14.14}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j+1}q^{2j^2+3j+1} \tag{14.15}$$

$$+\sum_{j\geq 0}\overline{\begin{bmatrix}N-1\\ j\end{bmatrix}}\frac{(xq)_{j+1}}{(-xq)_{j+1}}(-1)^{j}x^{2j+1}q^{2j^2+5j+2} \tag{14.16}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j+1}q^{2j^2+3j+1} \tag{14.17}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j+2}q^{2j^2+4j+2} \tag{14.18}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j}q^{2j^2+j} \tag{14.19}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j+1}q^{2j^2+2j+1} \tag{14.20}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N-2\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j-1}q^{2j^2+j-1} \tag{14.21}$$

$$+\sum_{j\geq 1}\overline{\begin{bmatrix}N-2\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j}q^{2j^2+2j}. \tag{14.22}$$

By the second recurrence (14.2), we observe that the sum (14.15) is equal to

$$\sum_{j\geq 1}\overline{\begin{bmatrix}N-2\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j+1}q^{2j^2+3j+1}+O\left(x^3q^{N+5}\right), \tag{14.23}$$

where we define $f(x,q)=O(x^kq^\ell)$ to mean that $f(x,q)\in x^kq^\ell\mathbb{Z}[[x,q]]$. Thus by the first recurrence (14.1), the sum of (14.11), (14.15), (14.17) and (14.20) is equal to $O\left(x^3q^{N+5}\right)$. Furthermore, by the second recurrence (14.2), the sum of (14.12) and (14.18) is $O\left(x^4q^{N+7}\right)$. Finally again by the second recurrence (14.2), the sum (14.13) is equal to

$$\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j+1}x^{2j}q^{2j^2+2j}+O\left(x^2q^{N+3}\right).$$

Thus by the first recurrence (14.1), the sum of (14.10), (14.13), (14.19) and (14.22) is equal to $O\left(x^2q^{N+3}\right)$.

Hence we are left with the following

$$\begin{aligned}
&\overline{C}(N,x)-(1-xq)\overline{C}(N-1,xq)-\frac{(xq)_2}{(-xq)_2}xq^2\overline{C}(N-2,xq^2)\\
&=\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j}q^{2j^2+2j}\\
&+\sum_{j\geq 0}\overline{\begin{bmatrix}N-1\\ j\end{bmatrix}}\frac{(xq)_{j+1}}{(-xq)_{j+1}}(-1)^{j}x^{2j+1}q^{2j^2+5j+2}\\
&+\sum_{j\geq 1}\overline{\begin{bmatrix}N-2\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j-1}q^{2j^2+j-1}\\
&+O\left(x^2q^{N+3}\right).
\end{aligned}$$

By the second recurrence (14.2), the third sum is equal to

$$\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j+1}}(-1)^{j}x^{2j-1}q^{2j^2+j-1}+O\left(x^3q^{N+7}\right).$$

Factorising it with the first sum we get

$$\begin{aligned}
&\overline{C}(N,x)-(1-xq)\overline{C}(N-1,xq)-\frac{(xq)_2}{(-xq)_2}xq^2\overline{C}(N-2,xq^2)\\
&=\sum_{j\geq 0}\overline{\begin{bmatrix}N-1\\ j\end{bmatrix}}\frac{(xq)_{j+1}}{(-xq)_{j+1}}(-1)^{j}x^{2j+1}q^{2j^2+5j+2}\\
&+\sum_{j\geq 1}\overline{\begin{bmatrix}N-1\\ j-1\end{bmatrix}}\frac{(xq)_j}{(-xq)_{j}}(-1)^{j}x^{2j-1}q^{2j^2+j-1}\\
&+O\left(x^2q^{N+3}\right).
\end{aligned}$$

Now by a simple change of variable $j\to j-1$ we see that the two sums are cancelled, and this completes the proof. □

14.5. Concluding Remarks

The purpose of this chapter was emphasizing combinatorial motivations and roles of recurrence formulas of over q-binomial coefficients. In this sense, the finite form of the identities are the main objects of this chapter. The limiting version of the identities can also be proved using well-known transformation formulas in the theory of basic hypergeometric series : Corollary 14.8 can be obtained by setting $a = xq$, $b = -q$, and $c, d \to \infty$ in the very-well-poised ${}_6\phi_5$ summation [GR04, (II.20)]. Corollary 14.13 can also be proved using the ${}_8\phi_7$ summation and Heine's transformation. By setting $a = xq$, $b = -q$, and $c, d, e, f \to \infty$ in the ${}_8\phi_7$ summation [GR04, (III.23)], we find that

$$\overline{C}(\infty, x; q) = (xq)_\infty \lim_{e,f\to\infty} {}_2\phi_1(e, f; -xq; q, xq^2/ef).$$

By employing Heine's transformation [GR04, (III.2)], we can derive that the limit in the above equation is the same as

$$\frac{1}{(-xq)_\infty}\overline{D}(\infty, x; q).$$

A sequence $(a_1, a_2, \ldots, a_n)$ is said to be *unimodal* if

$$a_1 \le a_2 \le \cdots \le a_k \ge a_{k+1} \ge \cdots \ge a_n,$$

for some $1 \le k \le n$. A polynomial is *unimodal* if the sequence of its coefficients is unimodal. The classical q-binomial coefficients are known to be unimodal. Computer experiments lead us to conjecture that over q-binomial coefficients seem to satisfy the same property.

Conjecture 14.17. *Over q-binomial coefficients are unimodal.*

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